Passive Network Synthesis Revisited

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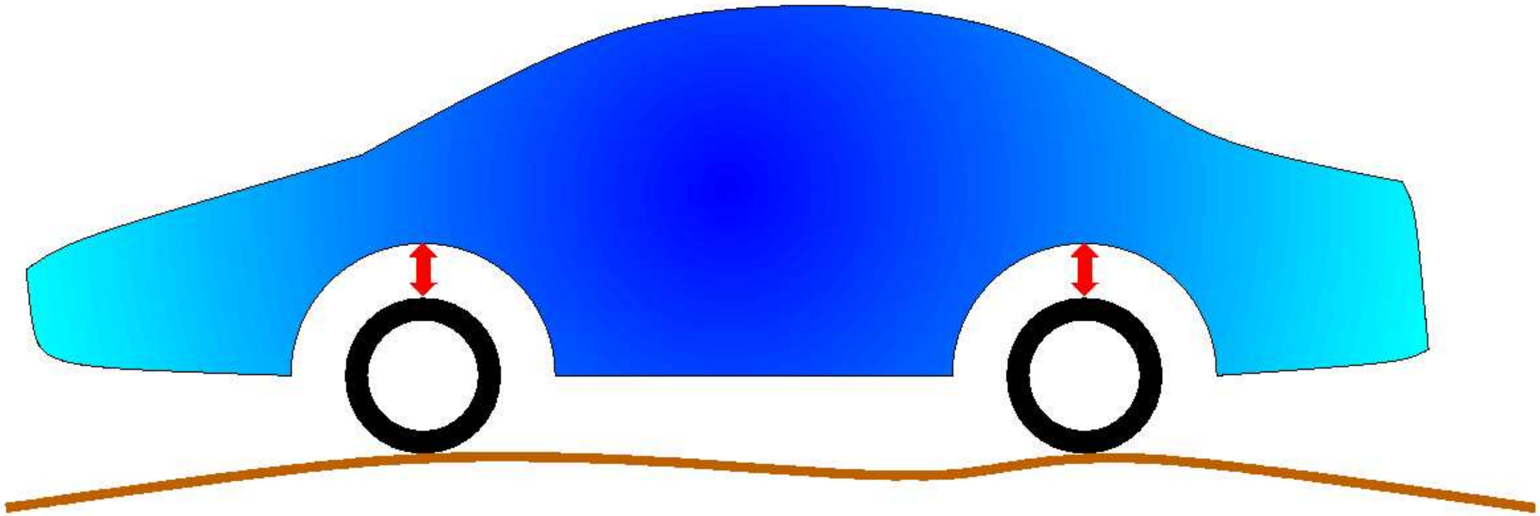
Mathematical Theory of Networks and Systems (MTNS)
17th International Symposium
Kyoto International Conference Hall
Kyoto, Japan, July 24-28, 2006

Semi-Plenary Lecture

Outline of Talk

1. Motivating example (vehicle suspension).
2. A new mechanical element.
3. Positive-real functions and Brune synthesis.
4. Bott-Duffin method.
5. Darlington synthesis.
6. Minimum reactance synthesis.
7. Synthesis of resistive n-ports.
8. Vehicle suspension.
9. Synthesis with restricted complexity.
10. Motorcycle steering instabilities.

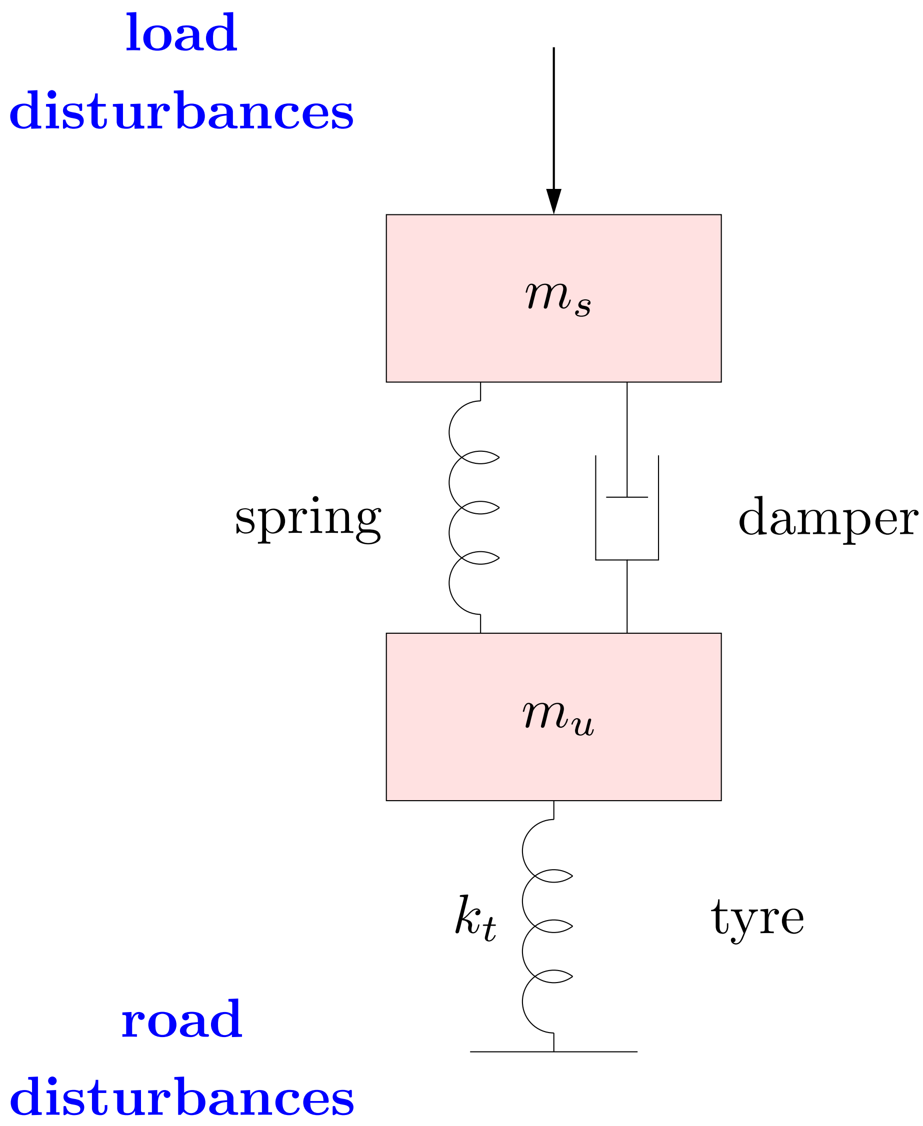
Motivating Example – Vehicle Suspension



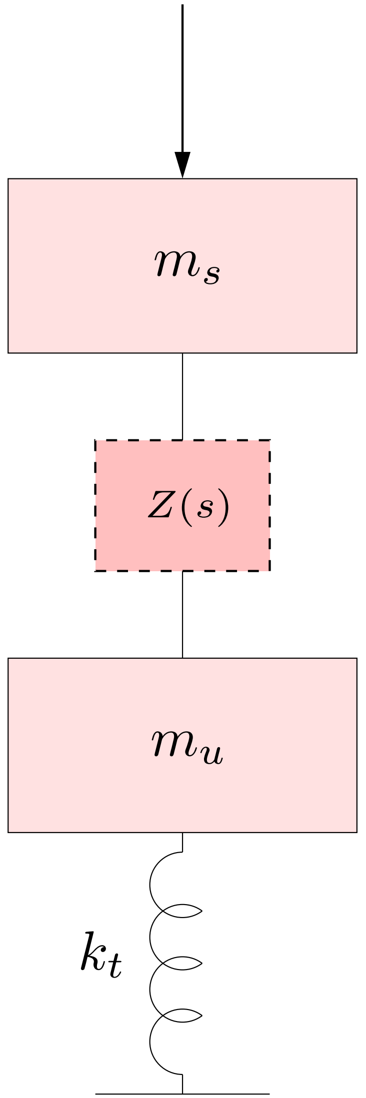
Performance objectives

1. Control vehicle body in the face of variable loads.
2. Insulate effect of road undulations (ride).
3. Minimise roll, pitch under braking, acceleration and cornering (handling).

Quarter-car Vehicle Model (conventional suspension)



The Most General Passive Vehicle Suspension



Replace the spring and damper with a general positive-real impedance $Z(s)$.

But is $Z(s)$ physically realisable?

Electrical-Mechanical Analogies

1. Force-Voltage Analogy.

voltage	$\leftrightarrow$	force
current	$\leftrightarrow$	velocity

Oldest analogy historically, cf. electromotive force.

2. Force-Current Analogy.

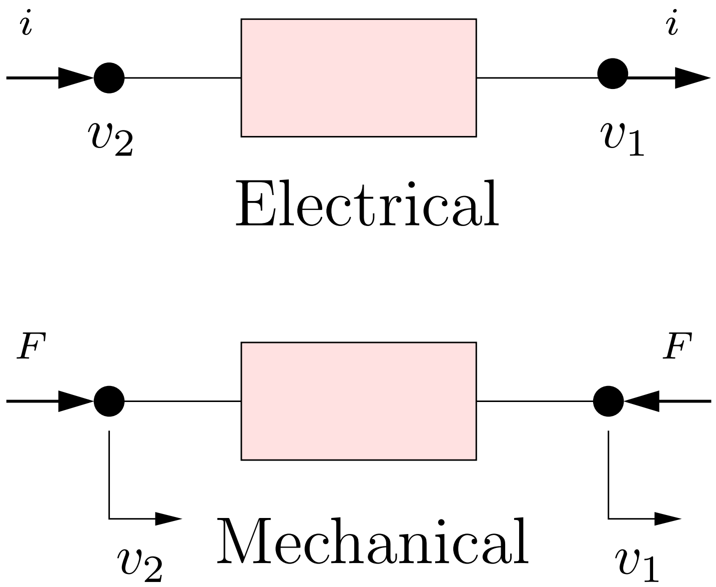
current	$\leftrightarrow$	force
voltage	$\leftrightarrow$	velocity
electrical ground	$\leftrightarrow$	mechanical ground

Independently proposed by: Darrieus (1929), Hähnle (1932), Firestone (1933).

Respects circuit "topology", e.g. terminals, through- and across-variables.

Standard Element Correspondences (Force-Current Analogy)

$v = Ri$	resistor	$\leftrightarrow$	damper	$cv = F$
$v = L\frac{di}{dt}$	inductor	$\leftrightarrow$	spring	$kv = \frac{dF}{dt}$
$C\frac{dv}{dt} = i$	capacitor	$\leftrightarrow$	mass	$m\frac{dv}{dt} = F$



What are the **terminals** of the mass element?

The Exceptional Nature of the Mass Element

Newton's Second Law gives the following network interpretation of the mass element:

- One terminal is the centre of mass,
- Other terminal is a fixed point in the inertial frame.

Hence, the mass element is analogous to a **grounded** capacitor.

Standard network symbol for the mass element:

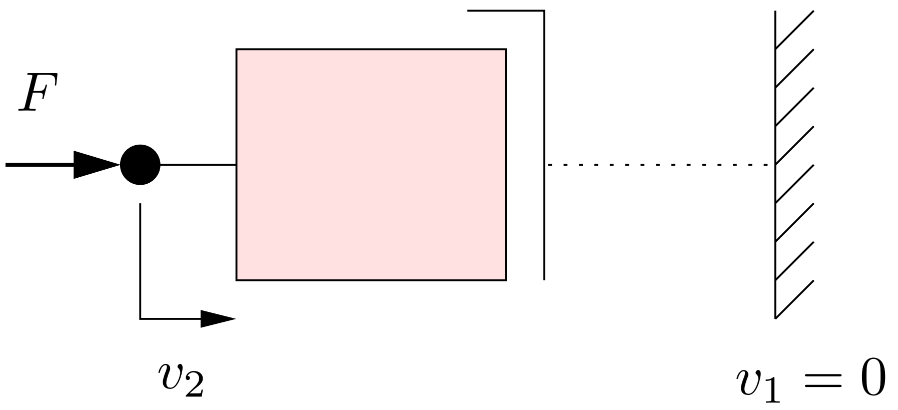
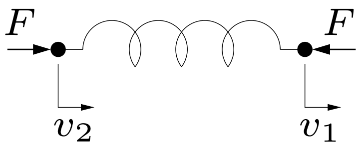
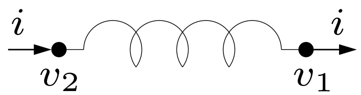
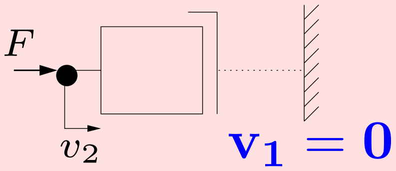
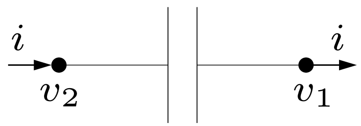
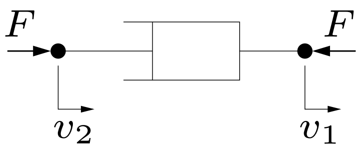
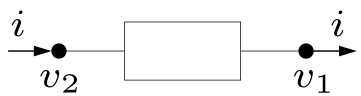


Table of usual correspondences

Mechanical	Electrical
 spring	 inductor
 mass	 capacitor
 damper	 resistor

Consequences for network synthesis

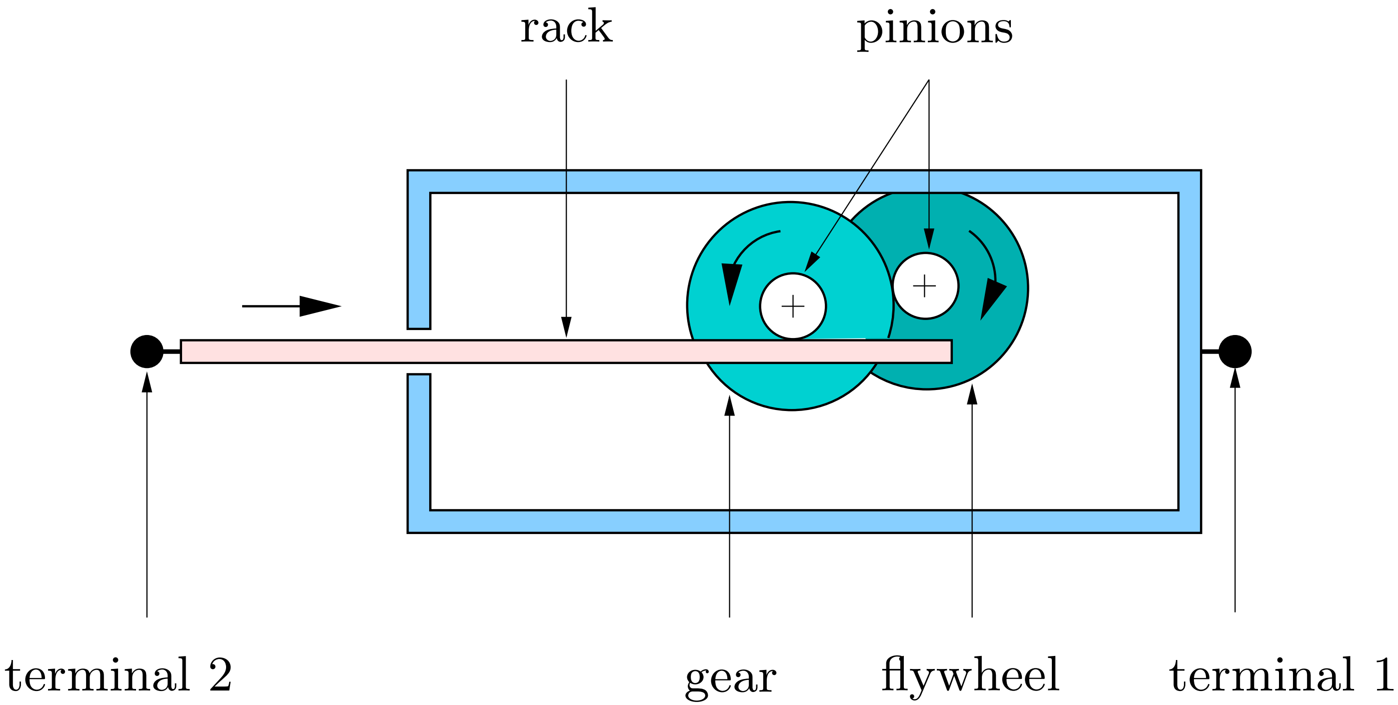
Two major problems with the use of the mass element for **synthesis** of "black-box" mechanical impedances:

- An electrical circuit with ungrounded capacitors will not have a direct mechanical analogue,
- Possibility of unreasonably large masses being required.

Question

Is it possible to construct a physical device such that the relative acceleration between its endpoints is proportional to the applied force?

One method of realisation



Suppose the flywheel of mass m rotates by α radians per meter of relative displacement between the terminals. Then:

$$\mathbf{F} = (\mathbf{m}\alpha^{\mathbf{2}})\,(\dot{\mathbf{v}}_{\mathbf{2}} - \dot{\mathbf{v}}_{\mathbf{1}})$$

(Assumes mass of gears, housing etc is negligible.)

The Ideal Inerter

We define the Ideal Inerter to be a mechanical one-port device such that the equal and opposite force applied at the nodes is proportional to the relative acceleration between the nodes, i.e.

$$F = b(\dot{v}_2 - \dot{v}_1).$$

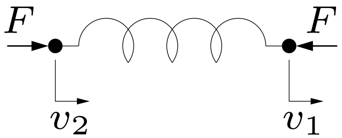
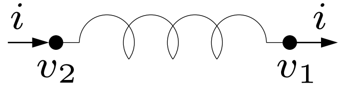
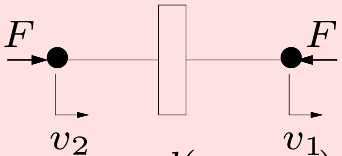
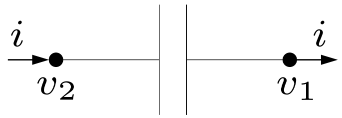
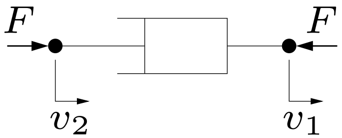
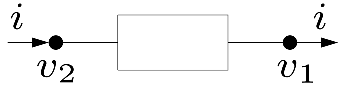
We call the constant b the **inertance** and its units are kilograms.

The ideal inerter can be approximated in the same sense that real springs, dampers, inductors, etc approximate their mathematical ideals.

We can assume its mass is small.

M.C. Smith, Synthesis of Mechanical Networks: The Inerter,
IEEE Trans. on Automat. Contr., **47** (2002), pp. 1648–1662.

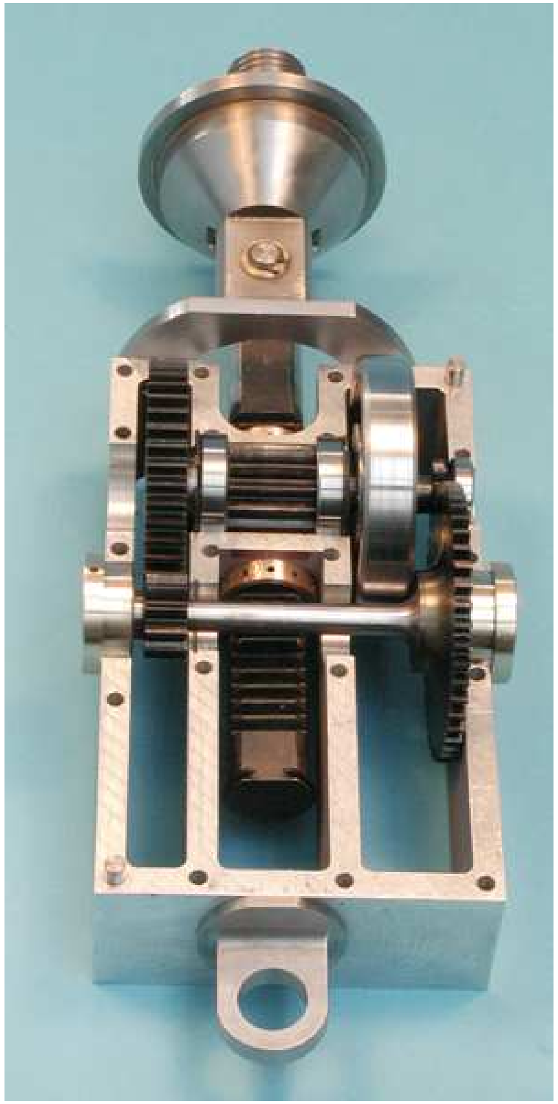
A new correspondence for network synthesis

Mechanical	Electrical
 $Y(s) = \frac{k}{s}$ $\frac{dF}{dt} = k(v_2 - v_1)$ spring	 $Y(s) = \frac{1}{Ls}$ $\frac{di}{dt} = \frac{1}{L}(v_2 - v_1)$ inductor
 $Y(s) = bs$ $F = b\frac{d(v_2 - v_1)}{dt}$ inerter	 $Y(s) = Cs$ $i = C\frac{d(v_2 - v_1)}{dt}$ capacitor
 $Y(s) = c$ $F = c(v_2 - v_1)$ damper	 $Y(s) = \frac{1}{R}$ $i = \frac{1}{R}(v_2 - v_1)$ resistor

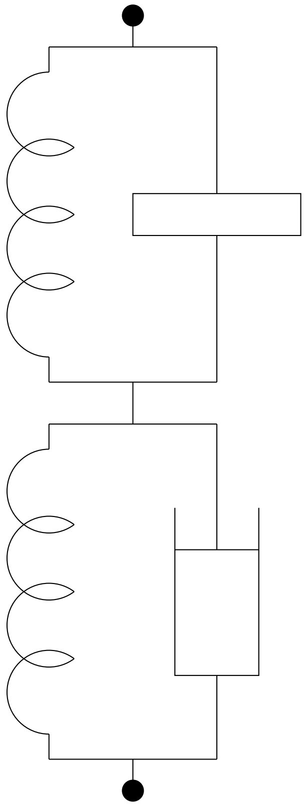
$$Y(s) = \text{admittance} = \frac{1}{\text{impedance}}$$

Rack and pinion inerter
made at
Cambridge University
Engineering Department

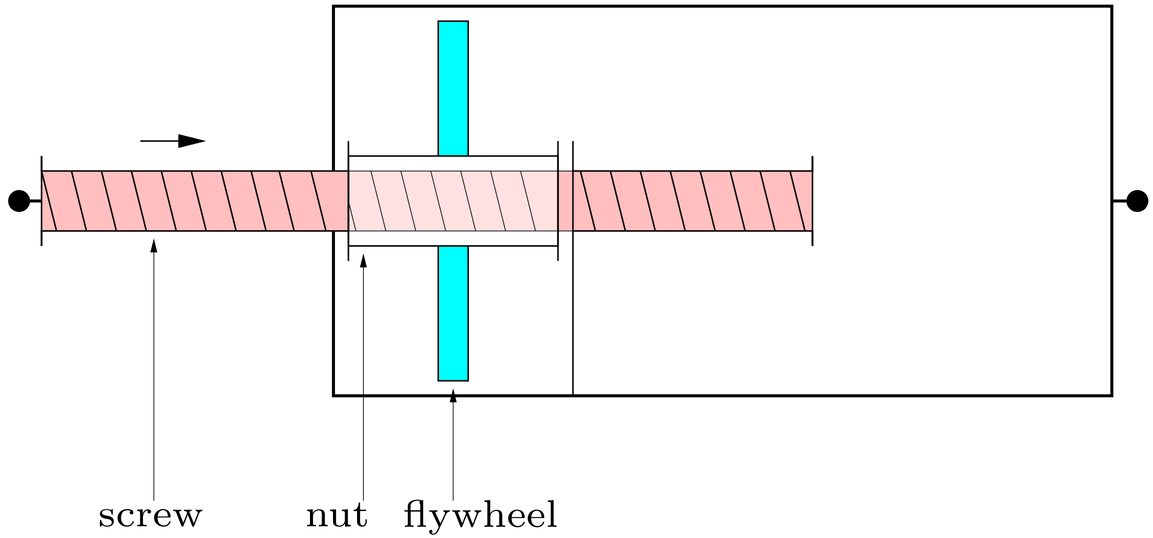
mass $\approx$ 3.5 kg
inertance $\approx$ 725 kg
stroke $\approx$ 80 mm

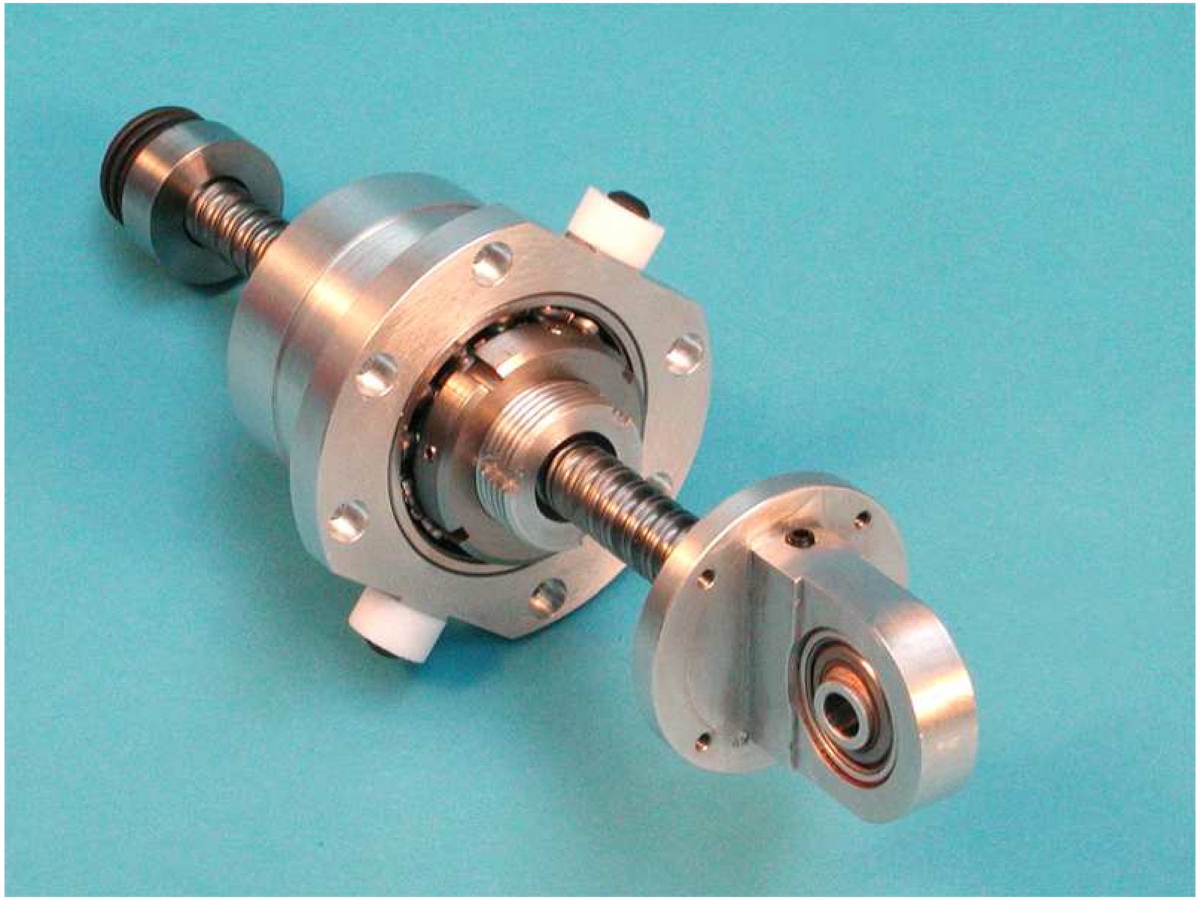


Damper-inerter series arrangement with centring springs



Alternative Realisation of the Inerter

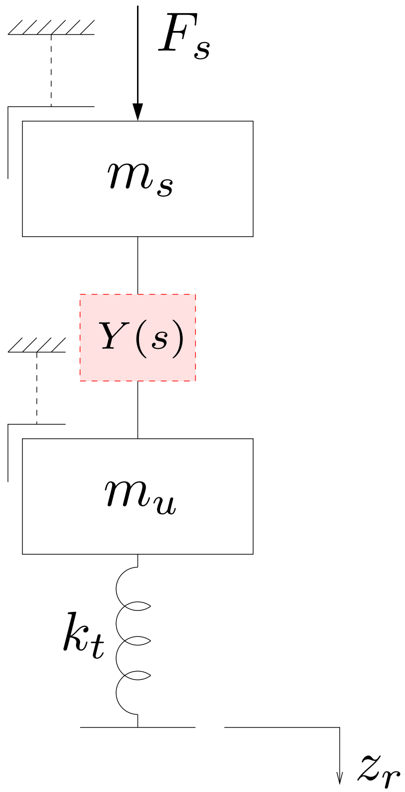


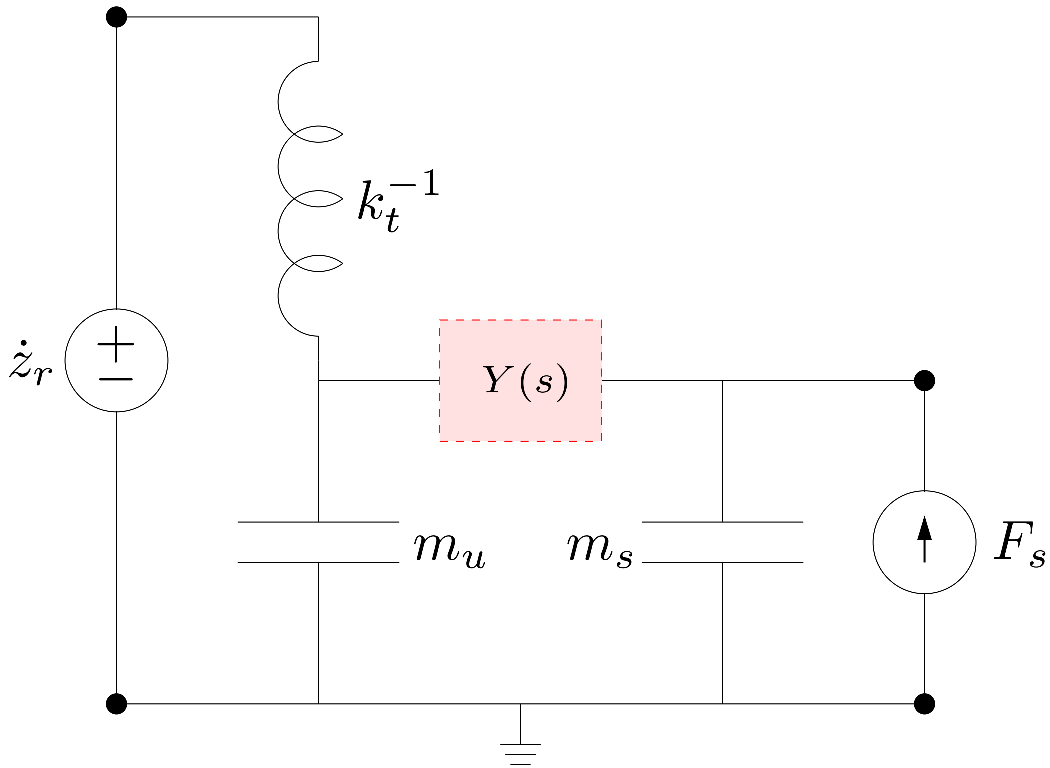


Ballscrew inerter made at Cambridge University Engineering Department

Mass $\approx$ 1 kg, Inertance (adjustable) = 60–180 kg

Electrical equivalent of quarter car model





$$Y(s) = \text{Admittance} = \frac{\text{Force}}{\text{Velocity}} = \frac{\text{Current}}{\text{Voltage}}$$

Positive-real functions

Definition. A function $Z(s)$ is defined to be **positive-real** if one of the following two equivalent conditions is satisfied:

1. $Z(s)$ is analytic and $Z(s) + Z(s)^* \geq 0$ in $\mathrm{Re(s)} > 0$.
2. $Z(s)$ is analytic in $\mathrm{Re(s)} > 0$, $Z(j\omega) + Z(j\omega)^* \geq 0$ for all ω at which $Z(j\omega)$ is finite, and any poles of $Z(s)$ on the imaginary axis or at infinity are simple and have a positive residue.

Passivity Defined

Definition. A network is **passive** if for all admissible v, i which are square integrable on $(-\infty, T]$,

$$\int_{-\infty}^{T} v(t)i(t)\,dt \geq 0.$$

Proposition. Consider a one-port electrical network for which the impedance $Z(s)$ exists and is real-rational. The network is passive if and only if $Z(s)$ is positive-real.

R.W. Newcomb, Linear Multiport Synthesis, McGraw-Hill, 1966.
B.D.O. Anderson and S. Vongpanitlerd, Network Analysis and Synthesis, Prentice-Hall, 1973.

SYNTHESIS OF A FINITE TWO-TERMINAL NETWORK WHOSE DRIVING-POINT IMPEDANCE IS A PRESCRIBED FUNCTION OF FREQUENCY

By Otto Brune [1]

Contents

Part I. Introduction

1. *Statement of the Problem*

In the well known methods of analysing the performance of linear passive electrical networks with lumped network elements it is usual to derive from the given structure of the network a scalar function $Z(\lambda)$ known as the impedance function of the network; this function determines completely the performance

[1] Containing the principal results of a research submitted for a doctor's degree in the Department of Electrical Engineering, Massachusetts Institute of Technology. The author is indebted to Dr. W. Cauer who suggested this research.

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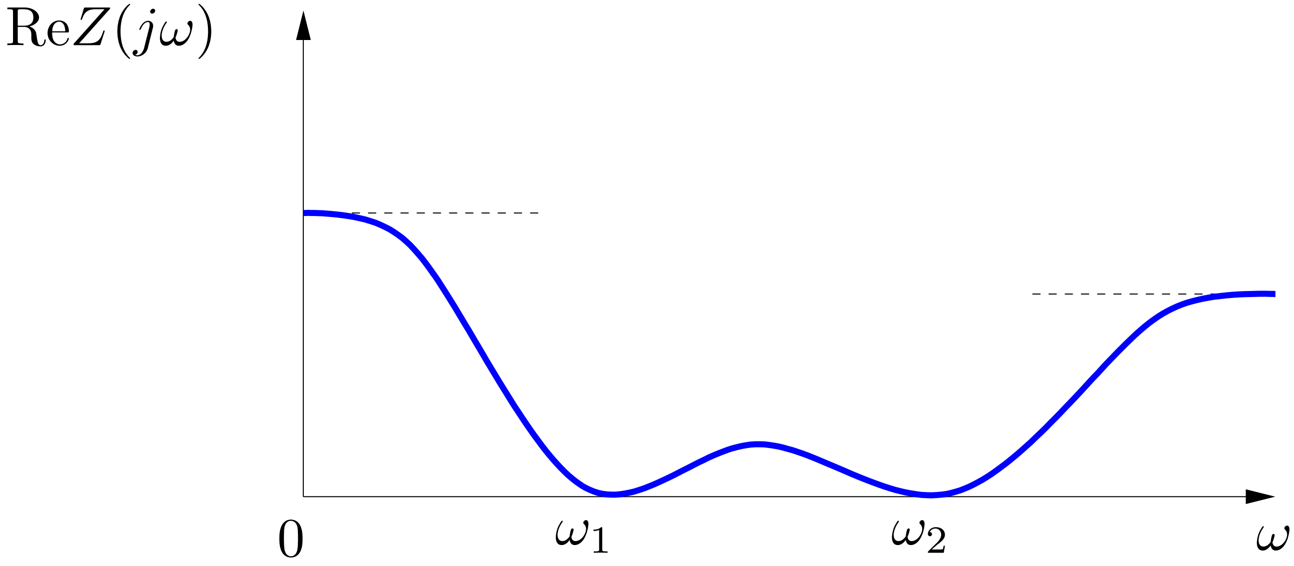
O. Brune showed that any (rational) positive-real function could be realised as the impedance or admittance of a network comprising resistors, capacitors, inductors *and transformers.* (1931)

Minimum functions

A **minimum function** $Z(s)$ is a positive-real function with no poles or zeros on $j\mathbb{R} \cup \{\infty\}$ and with the real part of $Z(j\omega)$ equal to 0 at one or more frequencies.



Foster preamble for a positive-real $Z(s)$

Removal of poles/zeros on $j\mathbb{R} \cup \{\infty\}$. e.g.

$$\frac{s^2+s+1}{s+1} \quad = \quad s + \frac{1}{s+1}$$

$$\uparrow$$

lossless

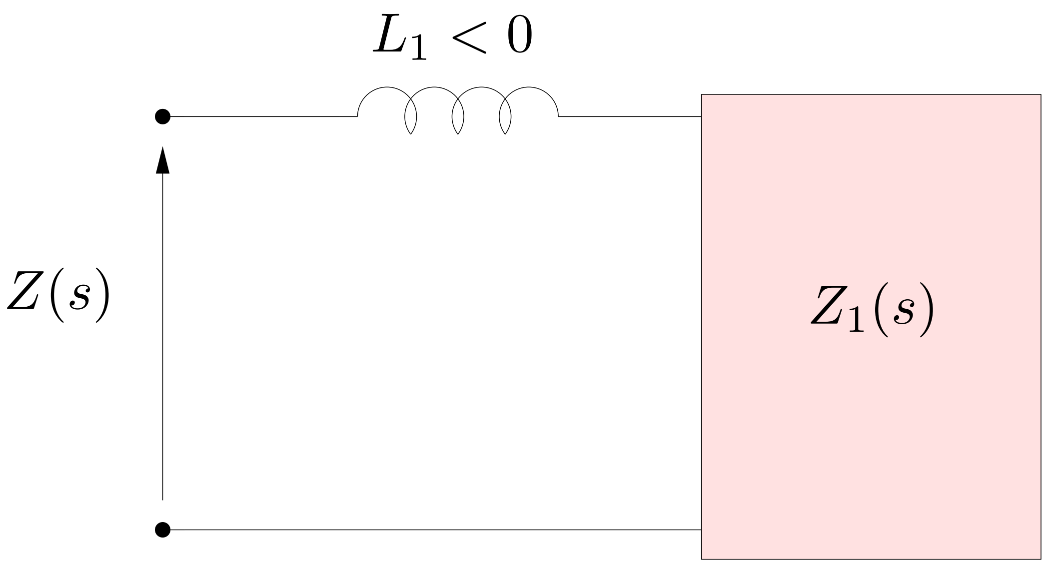
$$\downarrow$$

$$\frac{s^2+1}{s^2+2s+1} \quad = \quad \left(\frac{2s}{s^2+1} + 1 \right)^{-1}$$

Can always reduce a positive-real $Z(s)$ to a minimum function.

The Brune cycle

Let $Z(s)$ be a minimum function with $Z(j\omega_1) = jX_1$ $(\omega_1 > 0)$.
Write $L_1 = X_1/\omega_1$ and $Z_1(s) = Z(s) - L_1 s$.
Case 1. $(L_1 < 0)$



(**negative inductor!**)

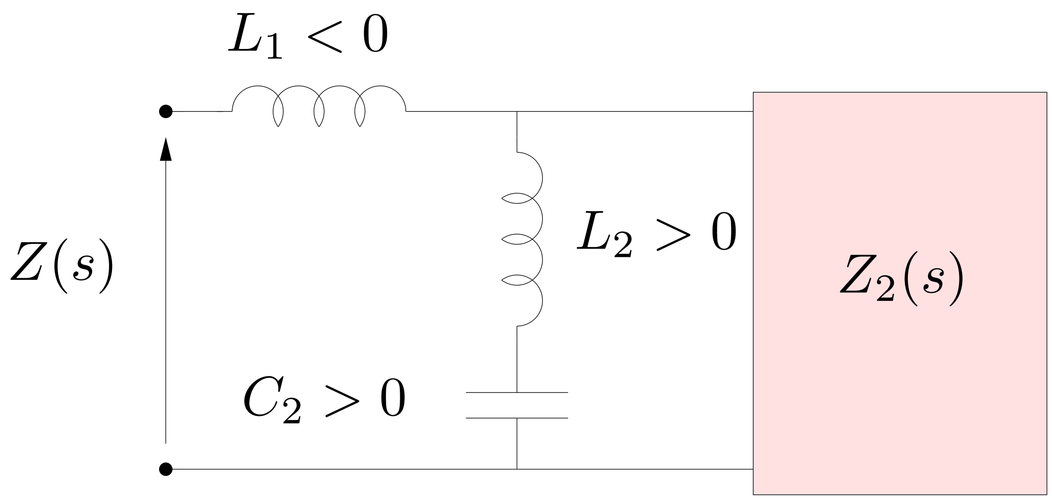
$Z_1(s)$ is positive-real. Let $Y_1(s) = 1/Z_1(s)$. Therefore, we can write

$$Y_2(s) = Y_1(s) - \frac{2K_1 s}{s^2 + \omega_1^2}$$

for some $K_1 > 0$.

The Brune cycle (cont.)

Then:



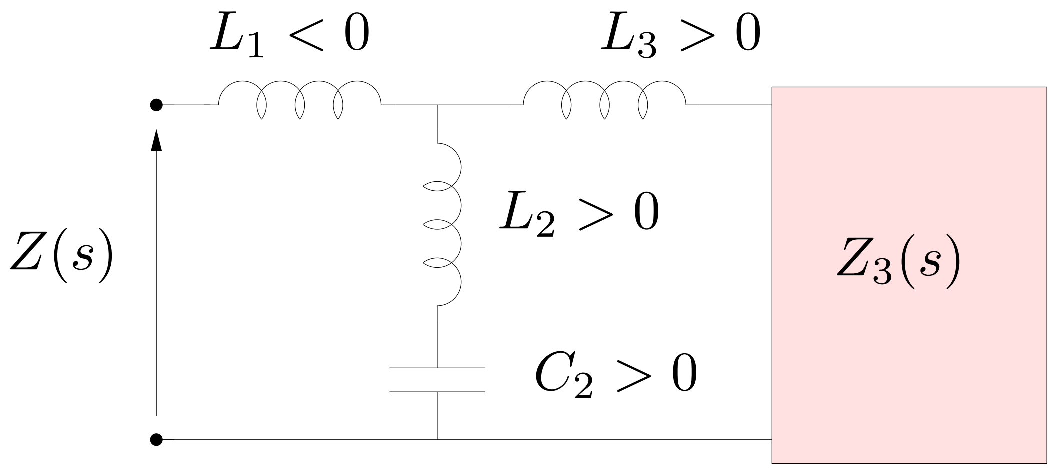
where $L_2 = 1/2K_1$, $C_2 = 2K_1/\omega_1^2$ and $Z_2 = 1/Y_2$.
Straightforward calculation shows that

$$Z_2(s) = sL_3 + \underset{\substack{\uparrow \\ \text{proper}}}{Z_3(s)}$$

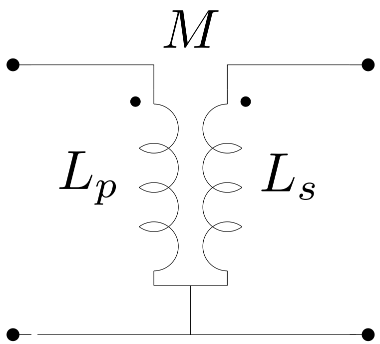
where $L_3 = -L_1/(1 + 2K_1L_1)$. Since $Z_2(s)$ is positive-real, $L_3 > 0$ and $Z_3(s)$ is positive-real.

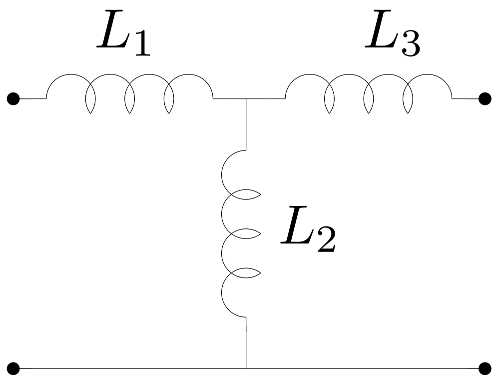
The Brune cycle (cont.)

Then:



To remove negative inductor:



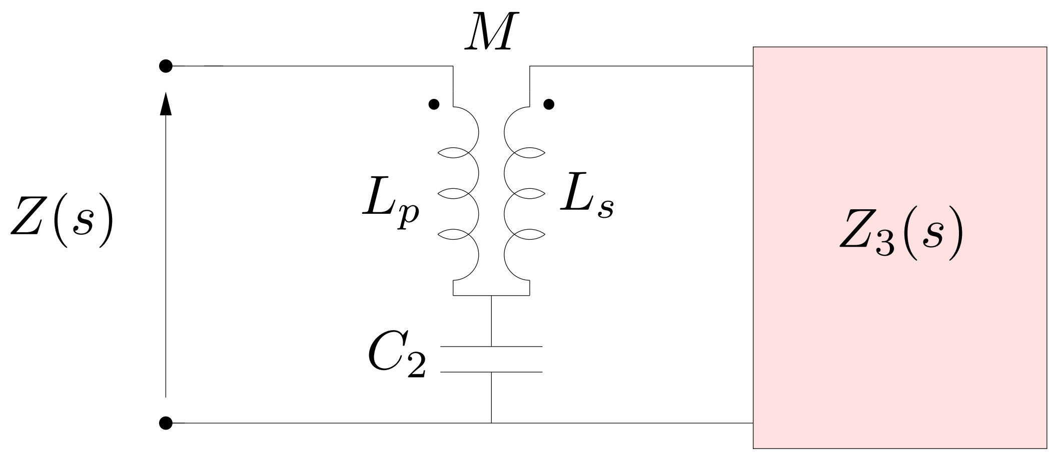


$$L_p = L_1 + L_2$$
$$L_s = L_2 + L_3$$
$$M = L_2$$

Some algebra shows that: $L_p, L_s > 0$ and $\frac{M^2}{L_p L_s} = 1$ (unity coupling coefficient).

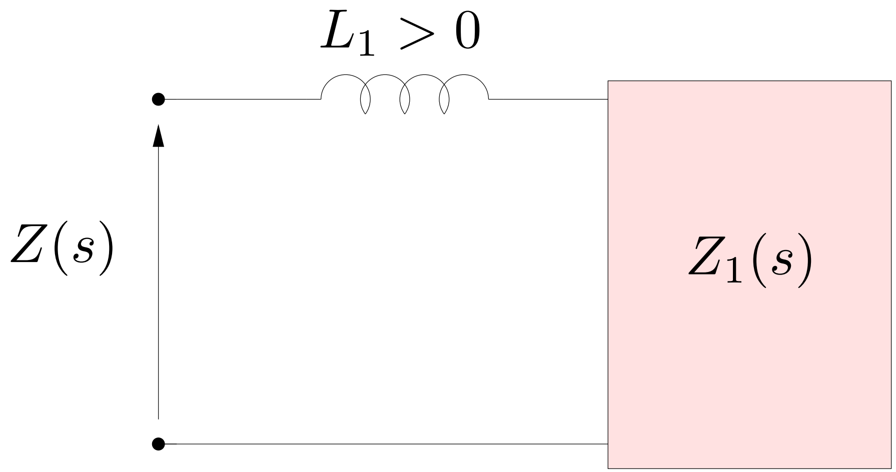
The Brune cycle (cont.)

Realisation for completed cycle:



The Brune cycle (cont.)

Case 2. ($L_1 > 0$). As before $Z_1(s) = Z(s) - L_1 s$



(**no need for negative inductor!**)

Problem: $Z_1(s)$ is not positive-real!

Let's press on and hope for the best!!

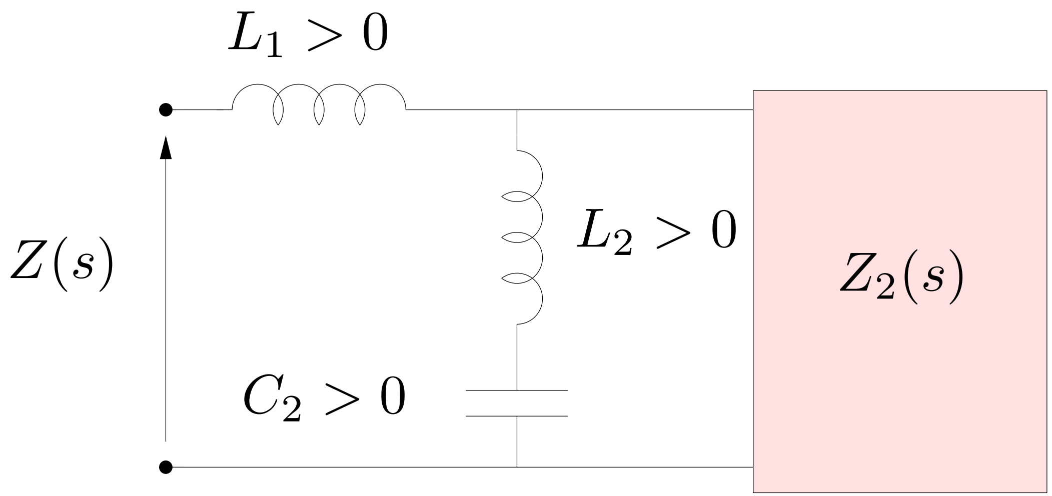
As before let $Y_1 = 1/Z_1$ and write

$$Y_2(s) = Y_1(s) - \frac{2K_1 s}{s^2 + \omega_1^2}.$$

Despite the fact that Y_1 is not positive-real we can show that $K_1 > 0$.

The Brune cycle (cont.)

Hence:



But still $Z_2(s)$ is not positive-real. Again we can check that

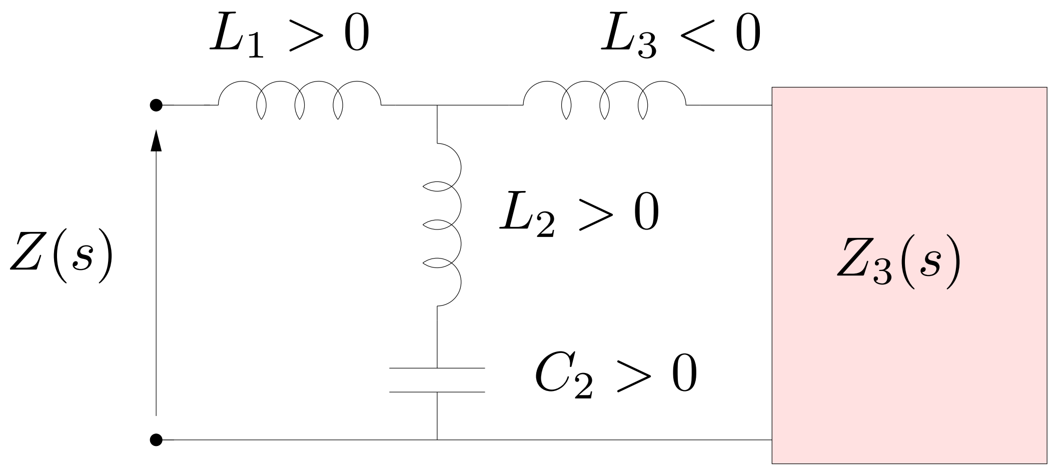
$$Z_2(s) = sL_3 + \underset{\substack{\uparrow \\ \text{proper}}}{Z_3(s)}$$

where $L_3 = -L_1/(1 + 2K_1L_1)$.

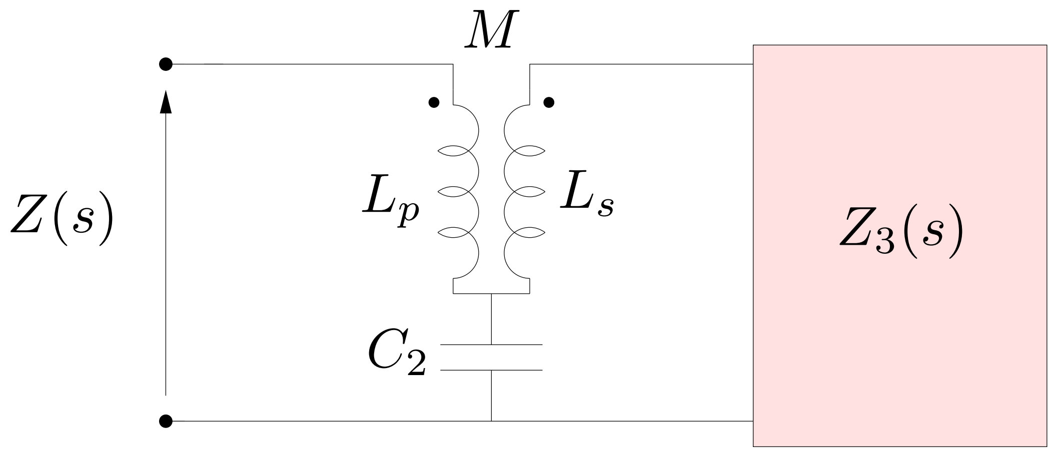
This time $L_3 < 0$ and $Z_3(s)$ is positive-real.

The Brune cycle (cont.)

So:



As before we can transform to:



where $L_p, L_s > 0$ and $\frac{M^2}{L_p L_s} = 1$ (unity coupling coefficient).

Letters to the Editor

The Ordering Reaction in Co-Pt Alloys†

J. B. NEWKIRK,* A. H. GEISLER,* AND D. L. MARTIN**

March 2, 1949

AN ordering reaction can occur in binary alloys of cobalt and platinum whose composition is near 50 atomic percent. The maximum temperature of order is about 825°C for the 50 atomic percent alloy and lower for those off this composition. No other reaction occurs below the maximum temperature of order. The unit cell is face-centered cubic above this temperature and ordered face-centered tetragonal below. In some respects this reaction has for its prototype the one found in the CuAu alloy.

Evidence is given which indicates that at certain temperatures and compositions the ordering reaction proceeds through a two-phase stage that by holding within a measurable temperature range discreet regions of order and of disorder may be caused to exist together in equilibrium.

On the basis of preliminary evidence, it appears that at an early stage of the ordering process, coherency between regions of order and of disorder may exist. Lattice straining, induced as a consequence of this, may account for the unusual physical properties which develop during the course of the ordering process. Thus, the process may resemble that of solid solution percipitation (aging) in its effect on certain physical properties.

Further study of the alloy is in progress.

† This letter is part of the Special Section on the Pittsburgh X-Ray and Electron Diffraction Conference which appears on pages 725–746 of this issue.

* Graduate Student, Department of Metallurgical Engineering, Carnegie Institute of Technology, Pittsburgh 13, Pennsylvania.
** Research Associate, Research Laboratory, General Electric Company, Schenectady, New York.

Impedance Synthesis without Use of Transformers

R. BOTT AND R. J. DUFFIN
Department of Mathematics, Carnegie Institute of Technology, Pittsburgh, Pennsylvania

December 13, 1948

LET $Z(s)$ be termed a B(rune) function if: (1) it is a rational function; (2) it is real for real s; and (3) the real part of Z is positive when the real part of s is positive. In his significant thesis, O. Brune[1] shows that the driving-point impedance of a passive network is a B function of the complex frequency variable s. Conversely, he shows that any B function can be realized by some passive network and gives rules for constructing such a network. In this synthesis he is forced to employ transformers with perfect coupling. It is recognized by Brune and others that the introduction of perfect transformers is objectionable from an engineering point of view. Prior to Brune, R. M. Foster[2] had shown how to synthesize the driving-point impedance of networks containing no resistors by simple series-parallel combinations of inductors and capacitors. This note gives a similar synthesis of an arbitrary impedance by series-parallel combinations of inductors, resistors, and capacitors.

A B function can be expressed as the ratio of two polynomials without common factor. Let the "rank" be the sum of the degrees of these polynomials. Obviously any B function of rank O can be synthesized. Suppose, then, it has been shown that all B functions of rank lower than n can be synthesized, and let $Z(s)$ be a B function of rank n. Brune gives four rules for carrying out a mathematical induction to a B function of lower rank:

(a) If Z has a pole on the imaginary axis, then Z can be synthesized by a parallel resonant element in series with an impedance Z' of lower rank; $Z=1/(cs+1/ls)+Z'$ where l^{-1}, $c\geq 0$.

(b) If Z has a zero on the imaginary axis, then Z can be synthesized by a series resonant element in parallel with an impedance Z' of lower rank; $1/Z=1/(ls+1/cs)+1/Z'$ where l, $c^{-1}\geq 0$.

(c) If the real part of Z does not vanish on the imaginary axis, $Z=r+Z_0$ where r is a positive constant (to be interpreted as resistance) and Z_0 is a B function of no greater rank than Z.

Brune's fourth rule, (d), which employs the perfect transformer, we replace by the following procedure:

(d′) If none of these reductions are possible, there exists a $w>0$ such that $Z(iw)$ is purely imaginary. First assume that $Z(iw)=iwL$ with $L>0$. We now make use of a key theorem discovered by P. I. Richards.[3] Let k be a positive number, and let

$$R(s)=\frac{kZ(s)-sZ(k)}{kZ(k)-sZ(s)}. \quad (1)$$

Then $R(s)$ is a B function whose rank does not exceed the rank of $Z(s)$. Richards states this theorem for $k=1$; the above form is an obvious modification, because $Z(ks)$ is also a B function. Let k satisfy the equation $L=Z(k)/k$. This is clearly always possible, because the function on the right varies from ∞ to 0 as k varies from 0 to ∞. With this choice of k, clearly $R(iw)=0$. Solving (1) for Z gives

$$\begin{aligned} Z(s)&=(1/Z(k)R(s)+s/kZ(k))^{-1}+(k/Z(k)s+R(s)/Z(k))^{-1} \\ &=(1/Z_1(s)+Cs)^{-1}+(1/Ls+1/Z_2)^{-1}. \end{aligned} \quad (2)$$

Here $Z_1(s)=kLR(s)$, $Z_2(s)=kL/R(s)$, $C=1/k^2L$. Since Z_1 is a B function with a zero on the imaginary axis, it can be synthesized. Likewise, Z_2 is a B function with a pole on the imaginary axis and can be synthesized. $Z(s)$ is therefore synthesized by two networks in series. The first network consists of the impedance Z_1 in parallel with a capacitor C, and the second network consists of the impedance Z_2 in parallel with an inductor L. In the case that $Z(iw) = -iwL$, similar considerations applied to the function $1/Z$ show that Z is synthesized by two networks in parallel. The synthesized network finally resulting has the configuration of a tree whose branches are ladder networks.

Richards[4] has sought necessary and sufficient conditions for the driving-point impedance of resistor-transmission-line circuits by means of an ingenious transformation of the Brune theory. The perfect transformers, which are again found to be objectionable, may be dispensed with by the above procedure.

1 O. Brune, J. Math. and Phys. 10, 191–236 (1931).
2 R. M. Foster, Bell Syst. Tech. J. 3, 259 (1924).
3 P. I. Richards, Duke Math. J. 14, 777–786 (1947).
4 P. I. Richards, Proc. I.R.E. 36, 217–220 (1948).

An Improvement in the Shadow-Cast Replica Technique

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May 6, 1949

WILLIAMS and Backus[1] have recently discussed in full detail the shadow-cast replica technique of electron microscopy. In the course of an investigation of the reactions of proteins in thin films,[2,3] we have performed some experiments with this technique embodying an improvement which we wish to report.

In this technique, a thin film of a metal such as chromium or uranium is deposited at an oblique angle onto the surface to be examined, by evaporation in a high vacuum. One method of removing this replica from the surface involves first, the deposition of a thin film (about 1000A) of parlodion on top of the metal film,

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R. Bott and R.J. Duffin showed that transformers were unnecessary in the synthesis of positive-real functions. (1949)

Richards's transformation

Theorem. If $Z(s)$ is positive-real then

$$R(s) = \frac{kZ(s) - sZ(k)}{kZ(k) - sZ(s)}$$

is positive-real for any $k > 0$.

Proof.

$$\begin{aligned} Z(s) \text{ is p.r.} \quad &\Rightarrow \quad Y(s) = \frac{Z(s) - Z(k)}{Z(s) + Z(k)} \text{ is b.r. and } Y(k) = 0 \\ &\Rightarrow \quad Y'(s) = \frac{k+s}{k-s} Y(s) \quad \text{is b.r.} \\ &\Rightarrow \quad Z'(s) = \frac{1 + Y'(s)}{1 - Y'(s)} \quad \text{is p.r.} \end{aligned}$$

$R(s) = Z'(s)$ after simplification.

Bott-Duffin construction (cont.)

Idea: use Richards's transformation to eliminate transformers from Brune cycle.

As before, let $Z(s)$ be a minimum function with $Z(j\omega_1) = jX_1$ $(\omega_1 > 0)$.

Write $L_1 = X_1/\omega_1$.

Case 1. $(L_1 > 0)$

Since $Z(s)$ is a minimum function we can always find a k s.t. $L_1 = Z(k)/k$.

Therefore:

$$R(s) = \frac{kZ(s) - sZ(k)}{kZ(k) - sZ(s)}$$

has a zero at $s = j\omega_1$.

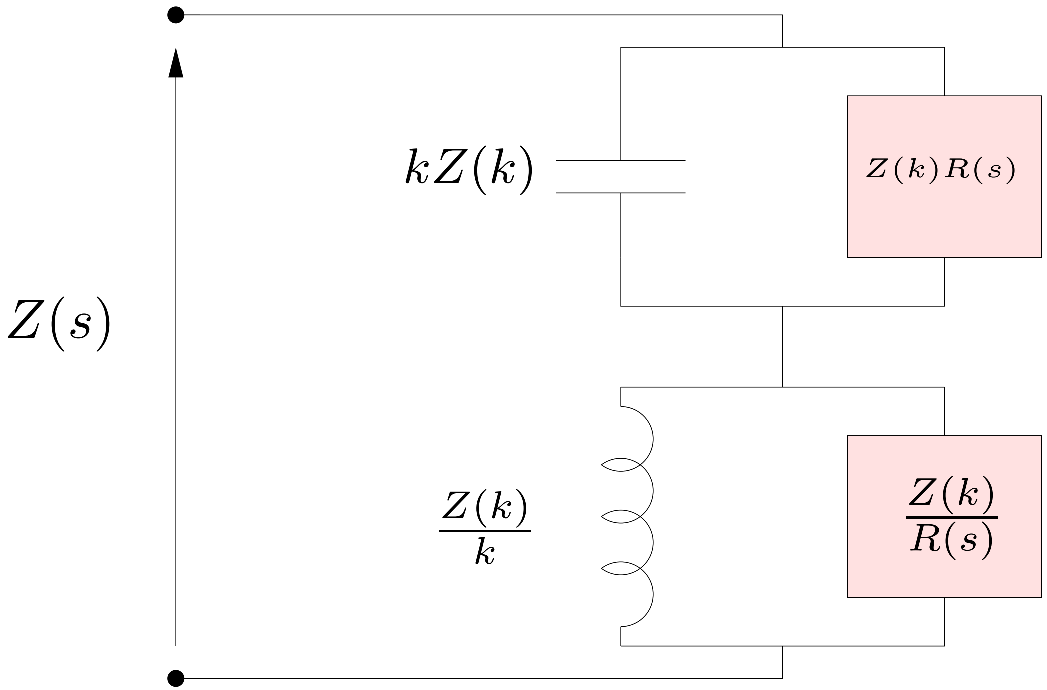
Bott-Duffin construction (cont.)

We now write:

$$
\begin{aligned}
Z(s) &= \frac{kZ(k)R(s) + Z(k)s}{k + sR(s)} \\
&= \frac{kZ(k)R(s)}{k + sR(s)} + \frac{Z(k)s}{k + sR(s)} \\
&= \frac{1}{\frac{1}{Z(k)R(s)} + \frac{s}{kZ(k)}} + \frac{1}{\frac{k}{Z(k)s} + \frac{R(s)}{Z(k)}}.
\end{aligned}
$$

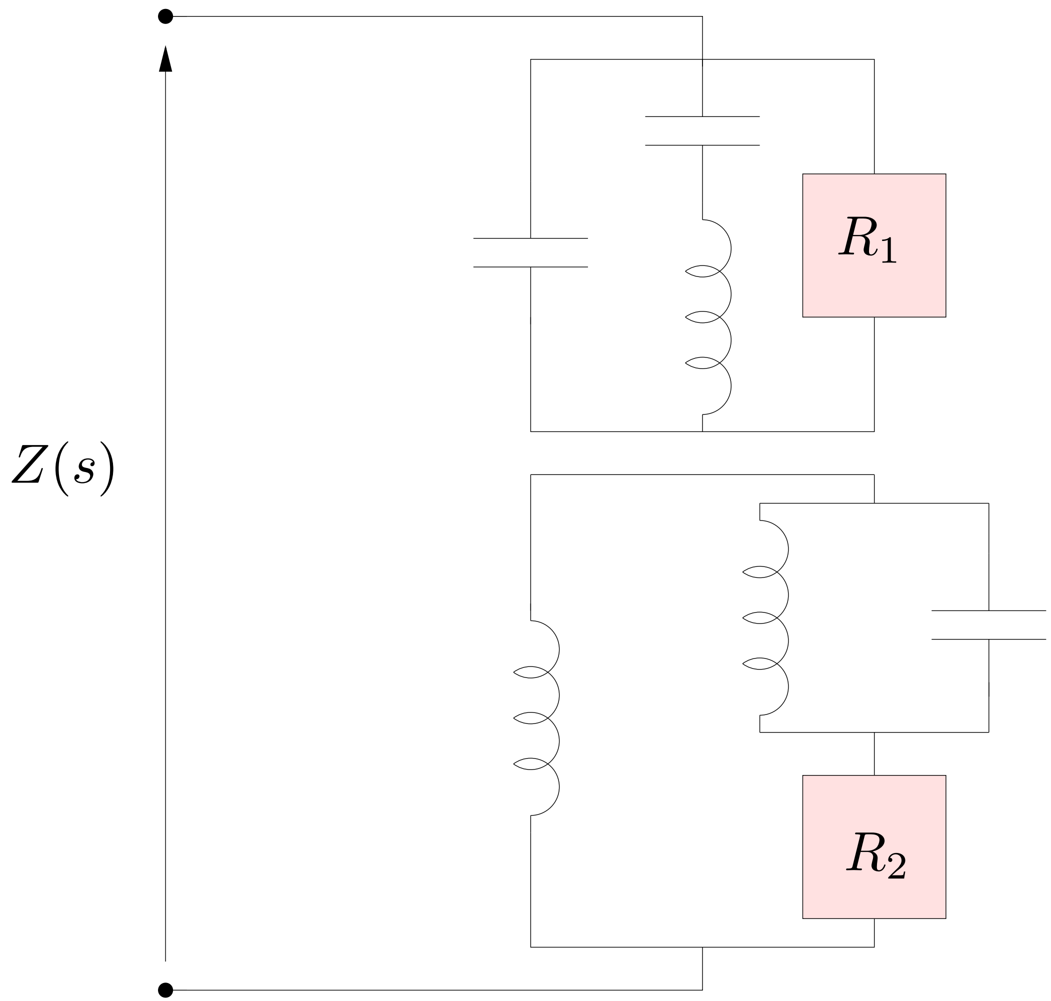
Bott-Duffin construction (cont.)

$$Z(s) = \frac{1}{\frac{1}{Z(k)R(s)} + \frac{s}{kZ(k)}} + \frac{1}{\frac{k}{Z(k)s} + \frac{R(s)}{Z(k)}}$$



Bott-Duffin construction (cont.)

We can write: $\frac{1}{Z(k)R(s)} = \text{const} \times \frac{s}{s^2+\omega_1^2} + \frac{1}{R_1(s)}$ etc.



Example — restricted degree

Proposition. Consider the real-rational function

$$Y_b(s) = k\frac{a_0 s^2 + a_1 s + 1}{s(d_0 s^2 + d_1 s + 1)}$$

where d_0, $d_1 \geq 0$ and $k > 0$. Then $Y_b(s)$ is positive real if only if the following three inequalities hold:

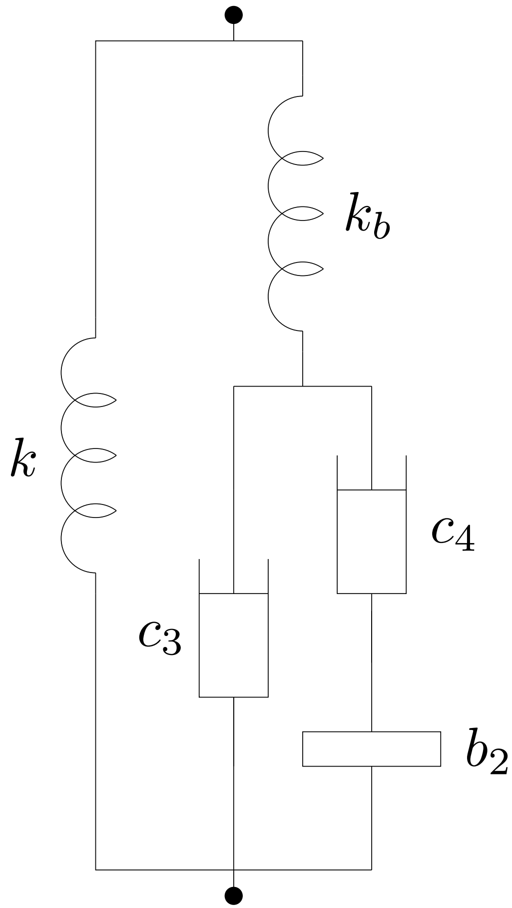
$$\begin{aligned} \beta_1 &:= a_0 d_1 - a_1 d_0 \geq 0, \\ \beta_2 &:= a_0 - d_0 \geq 0, \\ \beta_3 &:= a_1 - d_1 \geq 0. \end{aligned}$$

BRUNE REALISATION PROCEDURE FOR $Y_b(s)$

Foster preamble always sufficient to complete the realisation if β_1, $\beta_2 > 0$. (No Brune or Bott-Duffin cycle is required).

A continued fraction expansion is obtained:

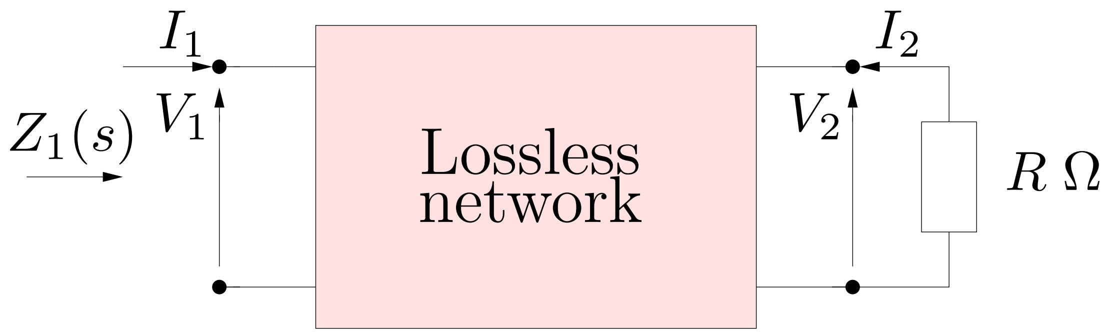
$$
\begin{aligned}
Y_b(s) &= k\frac{a_0 s^2 + a_1 s + 1}{s(d_0 s^2 + d_1 s + 1)} \\
&= \frac{k}{s} + \cfrac{1}{\cfrac{s}{k_b} + \cfrac{1}{c_3 + \cfrac{1}{\cfrac{1}{c_4} + \cfrac{1}{b_2 s}}}}
\end{aligned}
$$



where $k_b = \dfrac{k\beta_2}{d_0}$, $c_3 = k\beta_3$, $c_4 = \dfrac{k\beta_4}{\beta_1}$, $b_2 = \dfrac{k\beta_4}{\beta_2}$ and $\beta_4 := \beta_2^2 - \beta_1\beta_3$.

Darlington Synthesis

Realisation in Darlington form: a lossless two-port terminated in a single resistor.



For a lossless two-port with impedance:

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{pmatrix}$$

we find

$$Z_1(s) = Z_{11} \frac{R^{-1}(Z_{11}Z_{22} - Z_{12}^2)/Z_{11} + 1}{R^{-1}Z_{22} + 1}.$$

Writing

$$Z_1 = \frac{m_1 + n_1}{m_2 + n_2} = \frac{n_1}{m_2} \frac{m_1/n_1 + 1}{n_2/m_2 + 1},$$

where m_1, m_2 are polynomials of even powers of s and n_1, n_2 are polynomials of odd powers of s, suggests the identification:

$$Z_{11} = \frac{n_1}{m_2}, \quad Z_{22} = R\frac{n_2}{m_2}, \quad Z_{12} = \sqrt{R}\frac{\sqrt{n_1 n_2 - m_1 m_2}}{m_2}.$$

Augmentation factors are necessary to ensure a rational square root.

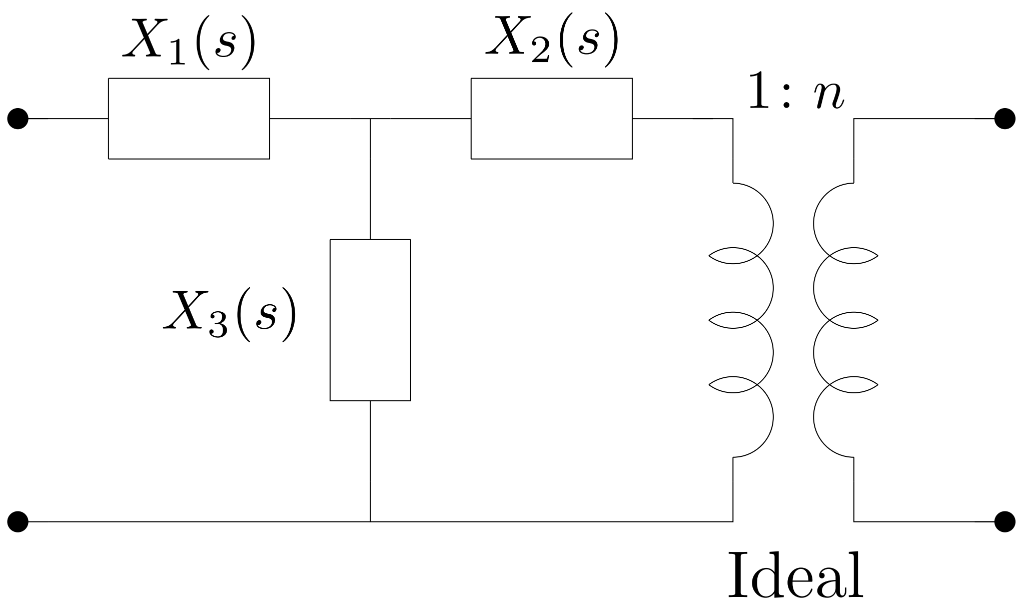
Once $Z(s)$ has been found, we then write:

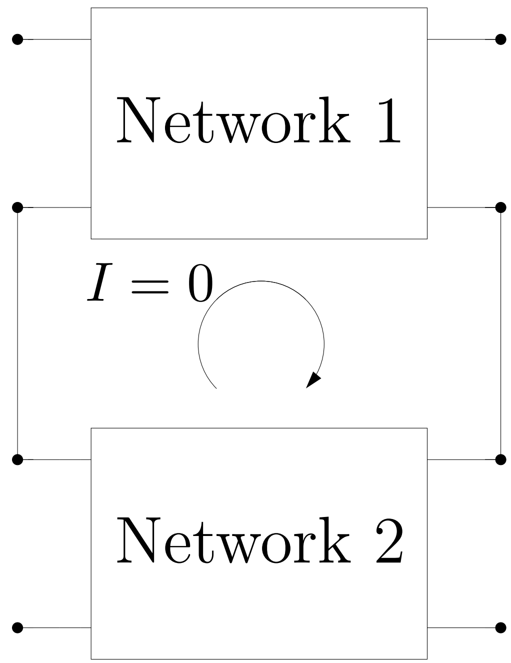
$$Z(s) = sC_1 + \frac{s}{s^2 + \alpha^2}C_2 + \cdots$$

where C_1 and C_2 are non-negative definite constant matrices.

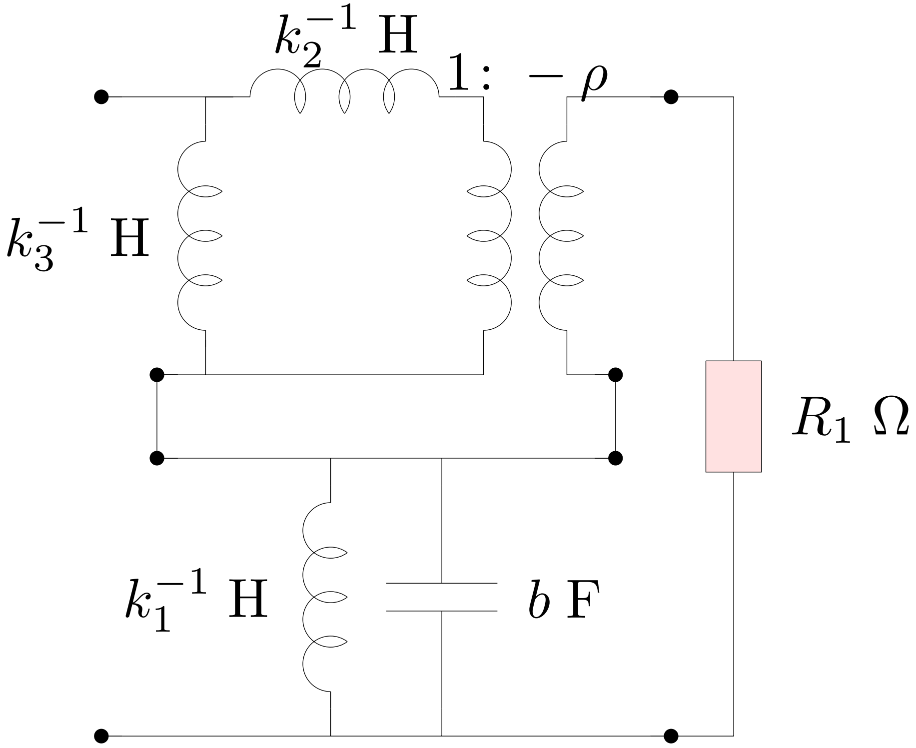
Darlington Synthesis (cont.)

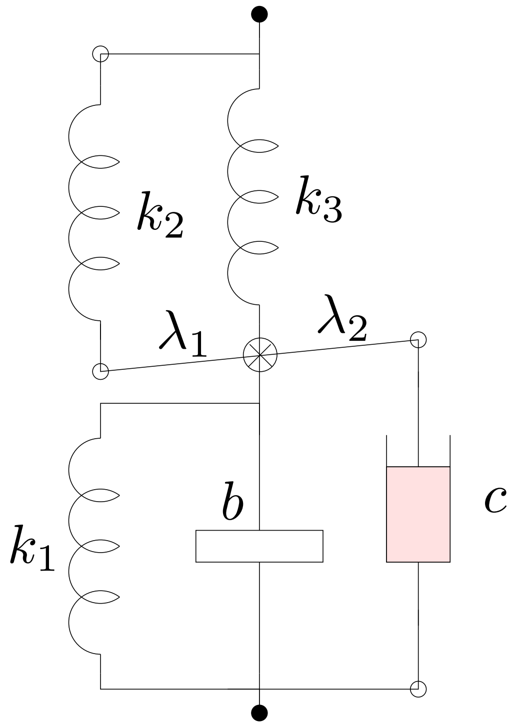
Each term in the sum is realised in the form of a T-circuit and a series connection of all the elementary two-ports is then made:



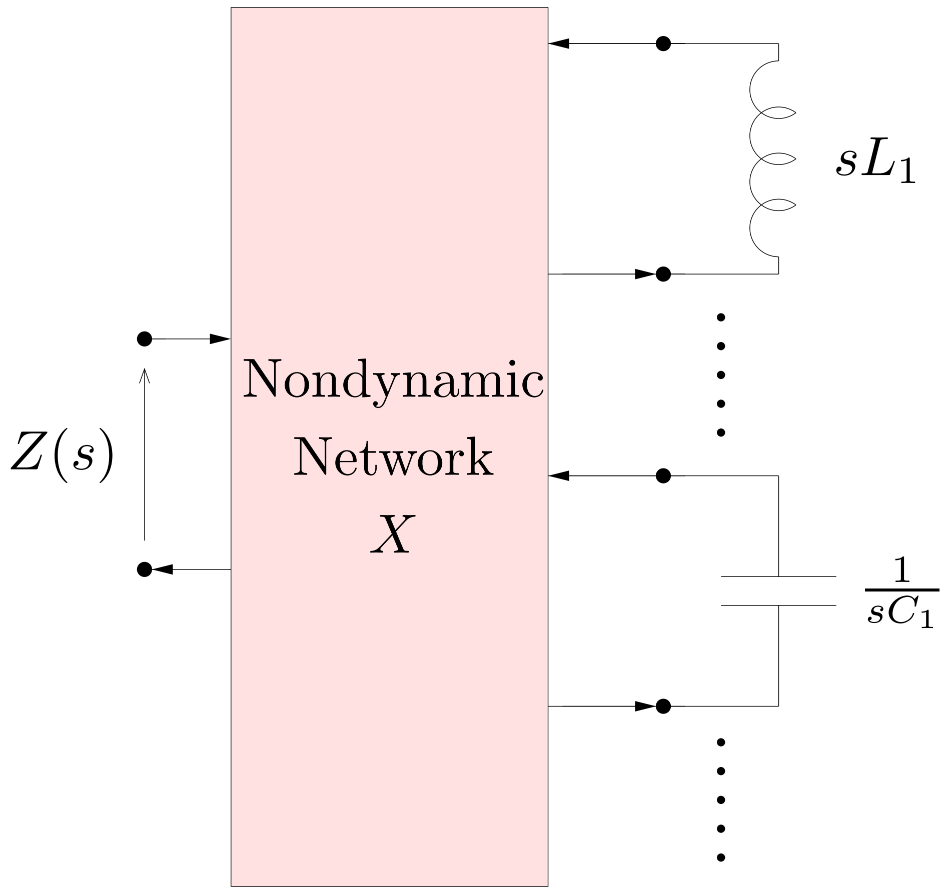


Electrical and mechanical realisations of the admittance $Y_b(s)$





MINIMUM REACTANCE SYNTHESIS



Let $L_1 = \cdots = C_1 = \cdots = 1$. If

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

is the hybrid matrix of X, i.e.

$$\begin{pmatrix} \underline{v}_1 \\ \underline{i}_2 \end{pmatrix} = M \begin{pmatrix} \underline{i}_1 \\ \underline{v}_2 \end{pmatrix},$$

then

$$Z(s) = M_{11} - M_{12}(sI + M_{22})^{-1} M_{21}.$$

D.C. Youla and P. Tissi, "N-Port Synthesis via Reactance Extraction, Part I", *IEEE International Convention Record*, 183–205, 1966.

Minimum reactance synthesis

Conversely, if we can find a state-space realisation $Z(s) = C(sI - A)^{-1}B + D$ such that the constant matrix

$$M = \begin{bmatrix} D & -C \\ B & -A \end{bmatrix}$$

has the properties

$$\begin{aligned} M + M' &\geq 0, \\ \mathrm{diag}\{I, \Sigma\}M &= M\mathrm{diag}\{I, \Sigma\} \end{aligned}$$

where Σ is a diagonal matrix with diagonal entries $+1$ or -1. Then M is the hybrid matrix of the nondynamic network terminated with inductors or capacitors, which realises $Z(s)$.

A construction is possible using the positive-real lemma and matrix factorisations.

B.D.O. Anderson and S. Vongpanitlerd, Network Analysis and Synthesis, Prentice-Hall, 1973.

Synthesis of resistive n-ports

Let R be a symmetric $n \times n$ matrix.

A necessary and sufficient condition for R to be realisable as the driving-point impedance of a network comprising resistors and transformers only is that it is non-negative definite.

No necessary and sufficient condition is known in the case that transformers are not available.

A general necessary condition is known: that the matrix is paramount.[1]

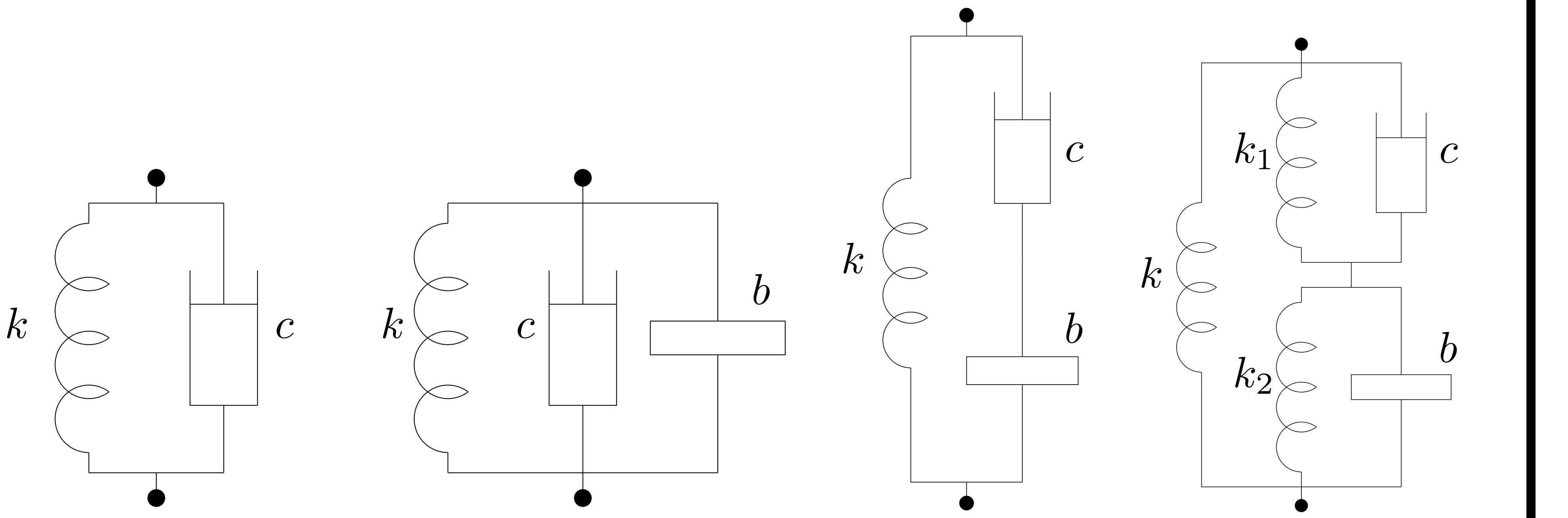
A matrix is defined to be the *paramount* if each principal minor of the matrix is not less than the absolute value of any minor built from the same rows.

It is also known that paramountcy is sufficient for the case of $n \leq 3$.[2]

[1] I. Cederbaum, "Conditions for the Impedance and Admittance Matrices of n-ports without Ideal Transformers", *IEE Monograph No. 276R*, 245–251, 1958.

[2] P. Slepian and L. Weinberg, "Synthesis applications of paramount and dominant matrices", Proc. National Electron. Conf., vol. 14, Chicago, Illinois, Oct. 611-630, 1958.

Simple Suspension Struts



Layout S1 Layout S2 Layout S3 Layout S4

parallel **series**

Performance Measures

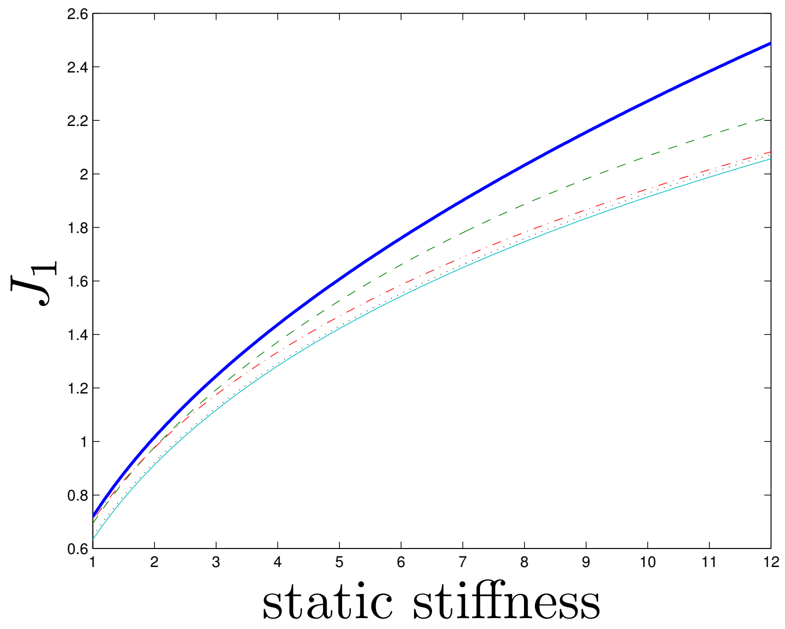
Assume:

$$\text{Road Profile Spectrum} = \kappa |n|^{-2} \qquad (\text{m}^3/\text{cycle})$$

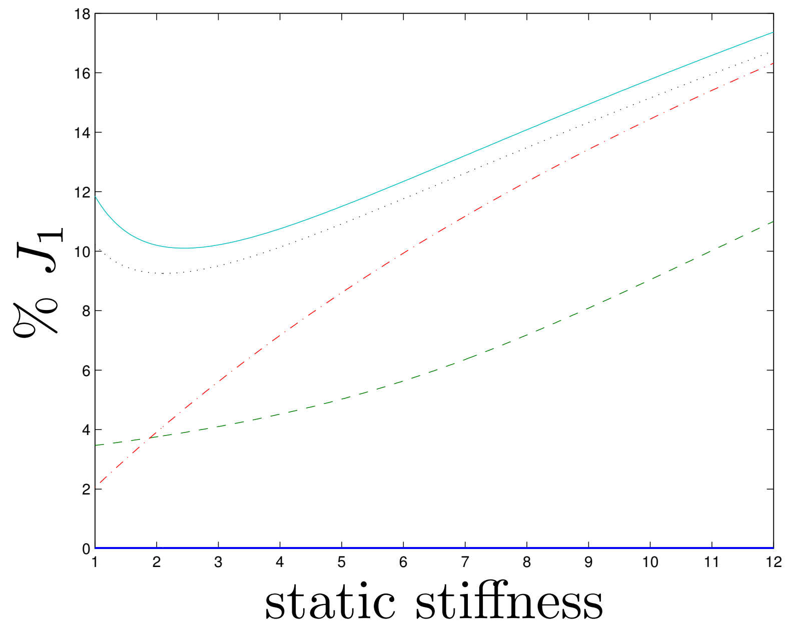
where $\kappa = 5 \times 10^{-7}\ \text{m}^3\text{cycle}^{-1}$ = road roughness parameter. Define:

$$\begin{aligned} J_1 &= E\left[\ddot{z}_s^2(t)\right] \qquad \textbf{ride comfort} \\ &= \text{r.m.s. body vertical acceleration} \end{aligned}$$

Optimisation of J_1 (ride comfort)



(a) Optimal J_1

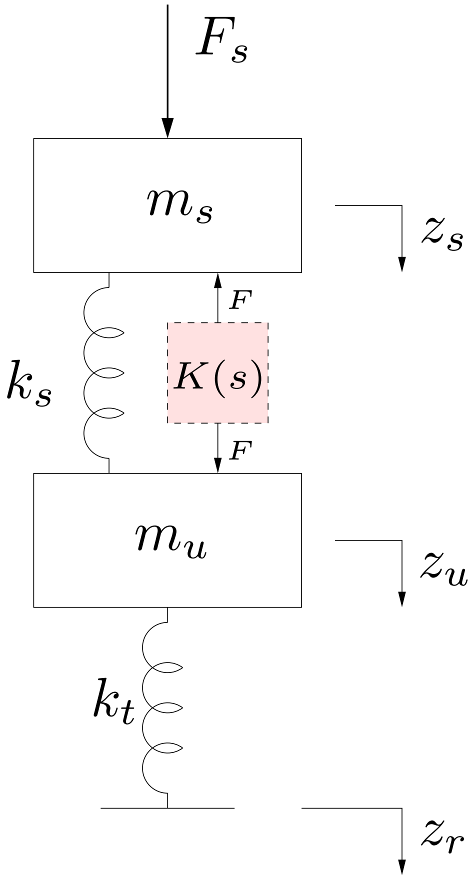


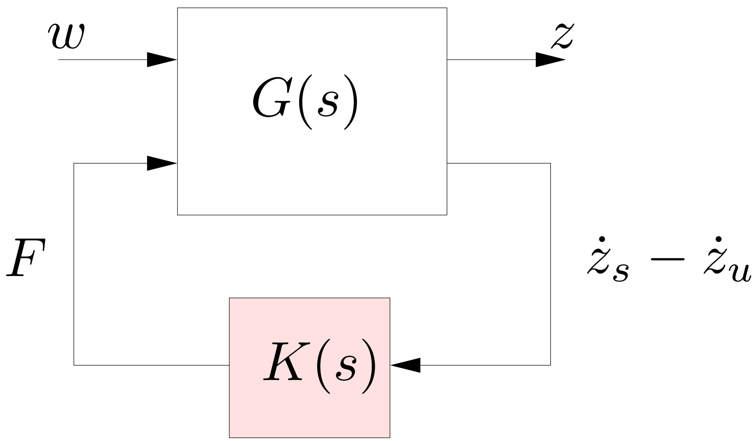
(b) Percentage improvement in J_1

Key: layout S1 (**bold**), layout S2 (**dashed**), layout S3 (**dot-dashed**), and layout S4 (**solid**).

M.C. Smith and F-C. Wang, 2004, Performance Benefits in Passive Vehicle Suspensions Employing Inerters, *Vehicle System Dynamics*, **42**, 235–257.

Control synthesis formulation





Ride comfort: $w = z_r$, $z = \dot{z}_s$

Performance measure: $J_1 = \text{const.} \times \|T_{\hat{z}_r \to s\hat{z}_s}\|_2$

Bilinear Matrix Inequality (BMI) formulation

Let $K(s) = C_k(sI - A_k)^{-1}B_k + D_k$ and $T_{\hat{z}_r \to s\hat{z}_s} = C_{cl}(sI - A_{cl})^{-1}B_{cl}$.

Theorem. There exists a positive real controller $K(s)$ such that $\|T_{\hat{z}_r \to s\hat{z}_s}\|_2 < \nu$ and A_{cl} is stable, if and only if the following problem is feasible for some $X_{cl} > 0$, $X_k > 0$, Q, ν^2 and A_k, B_k, C_k, D_k of compatible dimensions:

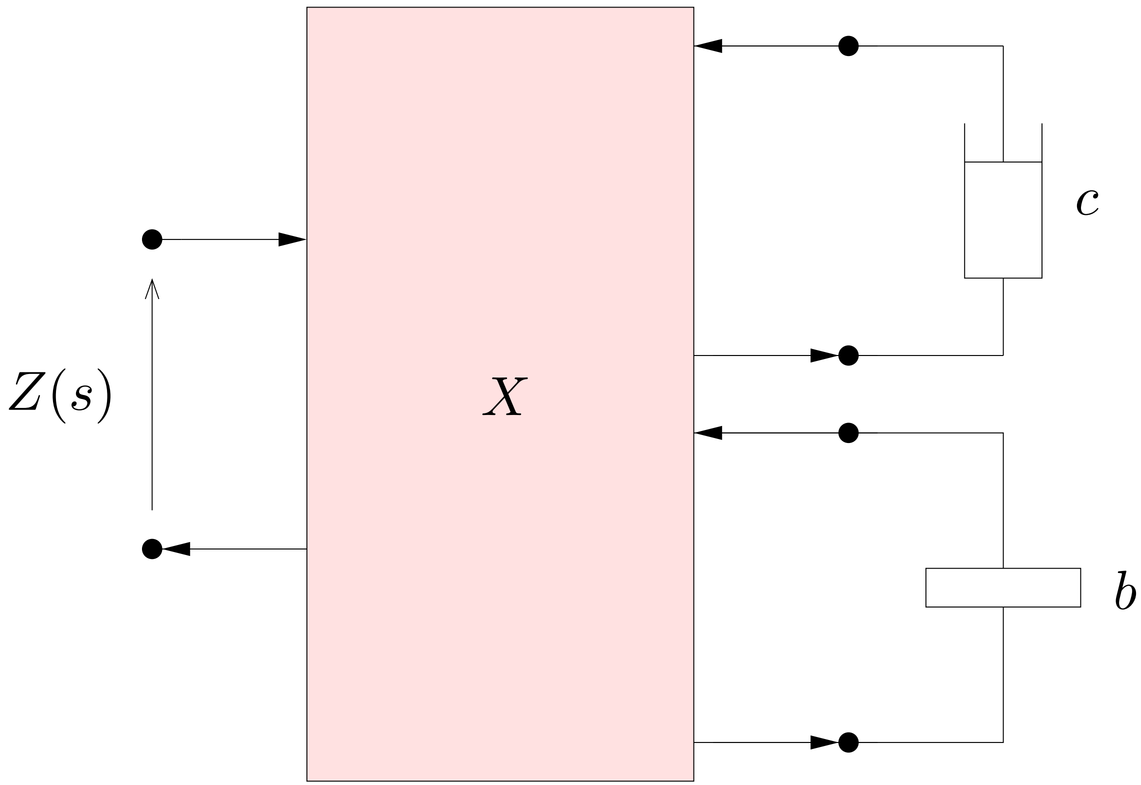
$$\begin{bmatrix} A_{cl}^T X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} \\ B_{cl}^T X_{cl} & -I \end{bmatrix} < 0, \ \begin{bmatrix} X_{cl} & C_{cl}^T \\ C_{cl} & Q \end{bmatrix} > 0, \ \mathrm{tr}(Q) < \nu^2,$$

$$\begin{bmatrix} A_k^T X_k + X_k A_k & X_k B_k - C_k^T \\ B_k^T X_k - C_k & -D_k^T - D_k \end{bmatrix} < 0.$$

C. Papageorgiou and M.C. Smith, 2006, Positive real synthesis using matrix inequalities for mechanical networks: application to vehicle suspension, *IEEE Trans. on Contr. Syst. Tech.*, **14**, 423–435.

A special problem

What class of positive-real functions $Z(s)$ can be realised using one damper, one inerter, any number of springs and no transformers?



Leads to the question: when can X be realised as a network of springs?

THEOREM. Let

$$Y(s) = \frac{(R_2R_3 - R_6^2)\,s^3 + R_3\,s^2 + R_2\,s + 1}{s(\det R\,s^3 + (R_1R_3 - R_5^2)\,s^2 + (R_1R_2 - R_4^2)\,s + R_1)}, \qquad (1)$$

where $R := \begin{bmatrix} R_1 & R_4 & R_5 \\ R_4 & R_2 & R_6 \\ R_5 & R_6 & R_3 \end{bmatrix}$ is non-negative definite.

A positive-real function $Y(s)$ can be realised as the driving-point admittance of a network comprising one damper, one inerter, any number of springs and no transformers if and only if $Y(s)$ can be written in the form of (1) and there exists an invertible diagonal matrix $D = \mathrm{diag}\{1, x, y\}$ such that DRD is paramount.

An explicit set of inequalities can be found which are necessary and sufficient for the existence of x and y.

M.Z.Q. Chen and M.C. Smith, Mechanical networks comprising one damper and one inerter, in preparation.

Collaboration with Imperial College

Application to motorcycle stability.

At high speed motorcycles can experience significant steering instabilities.

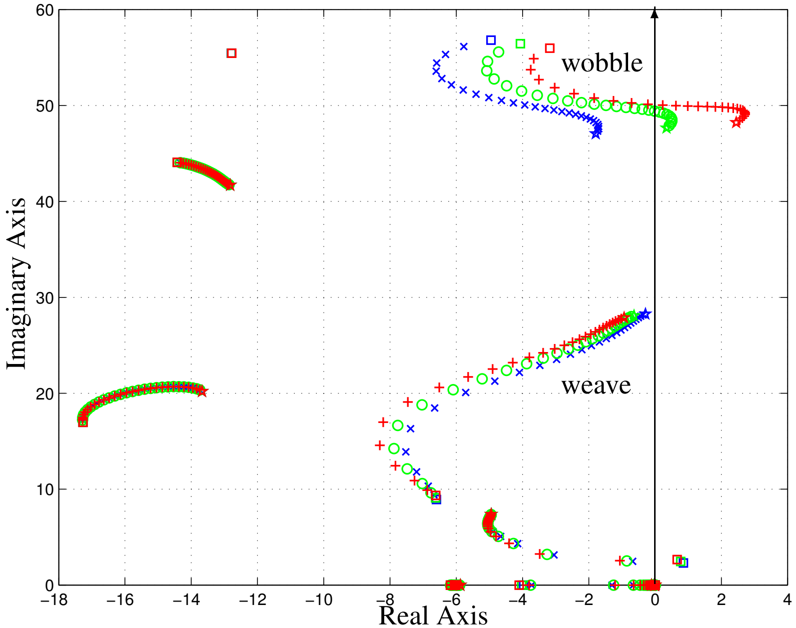
Observe: Paul Orritt at the 1999 Manx Grand Prix

Weave and Wobble oscillations

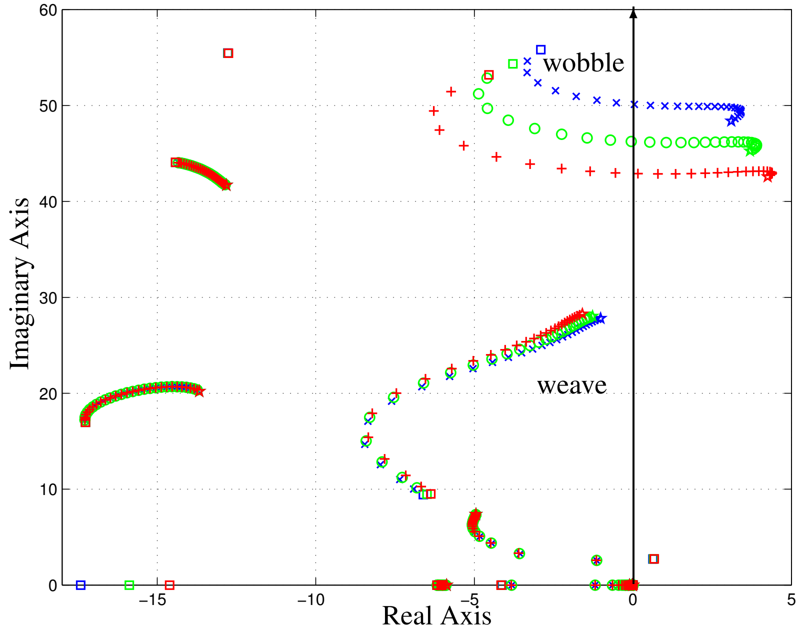
Steering dampers improve wobble (6–9 Hz) and worsen weave (2–4 Hz).

Simulations show that steering inerters have, roughly, the opposite effect to the damper. Root-loci (with speed the varied parameter):

Steering damper root-locus



Steering inerter root-locus



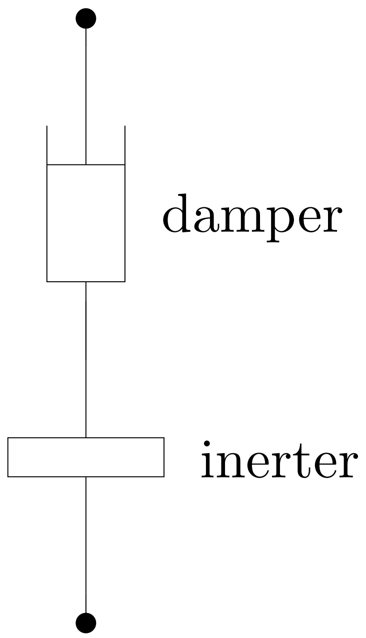
Can the advantages be combined?

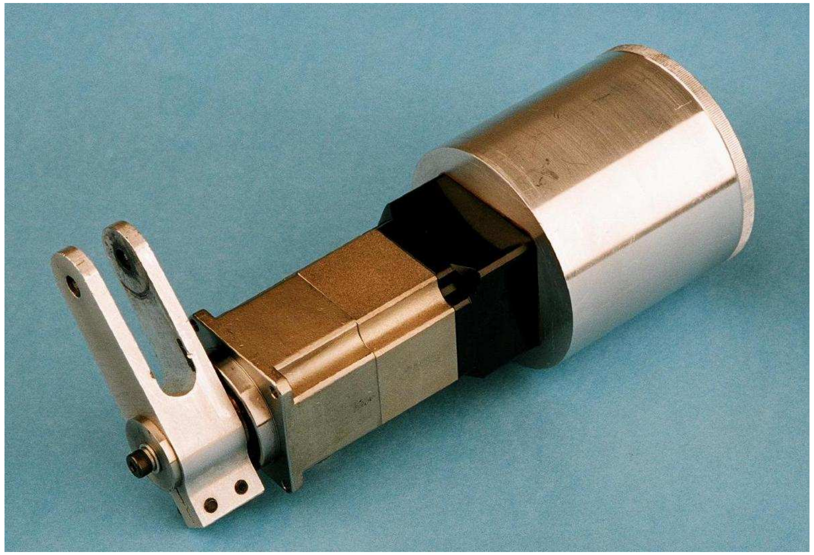
S. Evangelou, D.J.N. Limebeer, R.S. Sharp and M.C. Smith, 2006, Steering compensation for high-performance motorcycles, *Transactions of ASME, J. of Applied Mechanics*, **73**, to appear.

SOLUTION — A STEERING COMPENSATOR

... consisting of a network of dampers, inerters and springs.

Needs to behave like an inerter at weave frequencies and like a damper at wobble frequencies.





Prototype designed by N.E. Houghton and manufactured in the Cambridge University Engineering Department.

Conclusion

- A new mechanical element called the "inerter" was introduced which is the true network dual of the spring.
- The inerter allows classical electrical network synthesis to be mapped exactly onto mechanical networks.
- Applications of the inerter: vehicle suspension, motorcycle steering and vibration absorption.
- Economy of realisation is an important problem for mechanical network synthesis.
- The problem of minimal realisation of positive-real functions remains unsolved.

Acknowledgements

Research students	Michael Chen, Frank Scheibe
Design Engineer	Neil Houghton
Technicians	Alistair Ross, Barry Puddifoot, John Beavis
Collaborations	Fu-Cheng Wang (National Taiwan University)
	Christos Papageorgiou (University of Cyprus)
	David Limebeer, Simos Evangelou, Robin Sharp (Imperial College)