
4F3 – Predictive Control

Lecture 5

Set-point Tracking and Offset-free Control

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Outline Of Course

- Introduction to predictive control
- Digital state space control theory
- Unconstrained predictive control
- Predictive control with constraints
- Set-point tracking and offset-free control
- Stability and feasibility in predictive control – Dr Jan Maciejowski
- Case study by industrial speaker – Dr Paul Austin

Outline Of Lecture 5

- The **set-point tracking** problem
 - Without constraints
 - **Offset-free control** in the presence of **constant disturbances**
 - Without constraints
 - The internal model principle
 - Set-point tracking and offset-free control with **constraints**
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The Set-point Tracking Problem

- Up to now, we have been concerned with the **regulation** problem only:
 - Regulate the states and inputs around the origin, e.g. linear control laws of the form $u(k) = Kx(k)$
 - In practice, we often want to track some time-varying **set-point/reference** signal
 - Aircraft landing by auto-pilot
 - Robot arm with a pre-specified trajectory
 - We will consider the tracking of **piecewise constant** reference signals
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Linear Discrete-Time Systems

- Linear DT state-space system:

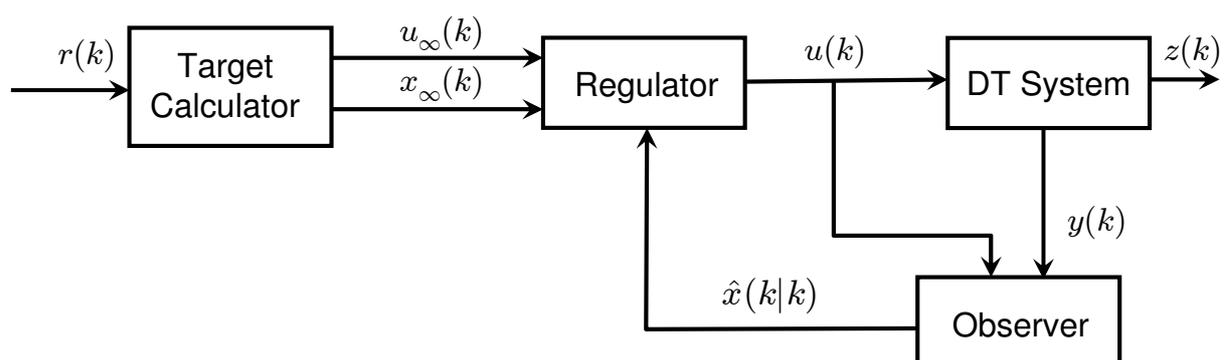
$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k)$$

$$z(k) = Hx(k)$$

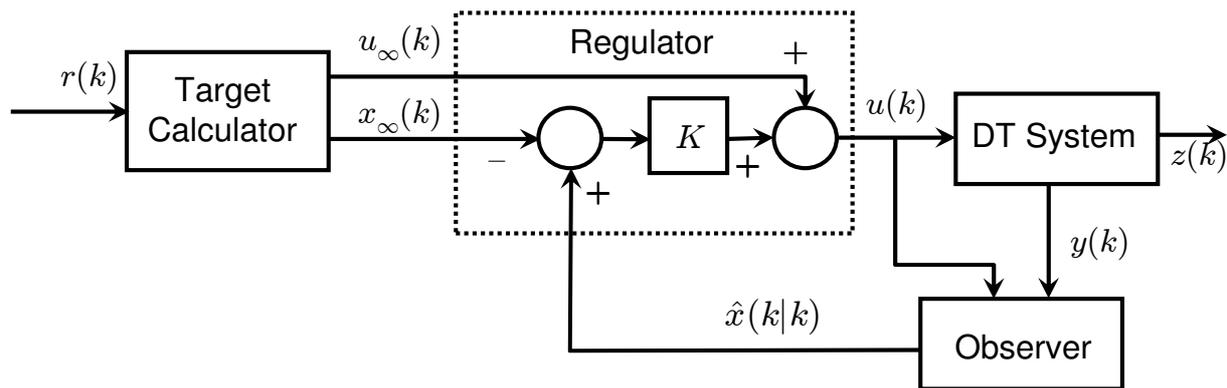
	<i>Variable</i>	<i>Size</i>
x	States	n
u	Controlled inputs/manipulated variables (MV)	m
y	Outputs/measured variables	p
z	Controlled variables (CV)	q

Set-point Tracking: A General Framework



- Control system so that **controlled variable** $z(k) \rightarrow r(k)$ as $k \rightarrow \infty$ if the **reference sequence** $r(\cdot)$ tends to some constant value
- Future values of the reference sequence $r(\cdot)$ are not known at time k
- At **each time** instant k , given the **current value** of the reference $r(k)$:
 - Target calculator** computes **target input** $u_\infty(k)$ and **target state** $x_\infty(k)$
 - Regulator** controls the system around the target input $u_\infty(k)$ and target state $x_\infty(k)$ (if $r(k) = 0$, a natural choice would be $x_\infty(k) = 0$ and $u_\infty(k) = 0$)

Linear Controller, No Constraints



- Given a linear state feedback control gain $K \in \mathbb{R}^{m \times n}$ (e.g. an unconstrained LQR or RHC gain), we will consider the control law:

$$u(k) = u_{\infty}(k) + K(\hat{x}(k|k) - x_{\infty}(k))$$

- If $A+BK$ and the observer are stable, then the closed-loop system is stable if $r(\cdot)$ (more precisely, $x_{\infty}(\cdot)$ and $u_{\infty}(\cdot)$) is constant
- The problem is how to design the **target calculator** to guarantee the targets $x_{\infty}(k)$ and $u_{\infty}(k)$ are computed such that $z(k) \rightarrow r(k)$ as $k \rightarrow \infty$

Choice Of Controlled Variables

- We don't always have all the outputs as controlled variables (i.e. $z(k) \neq y(k)$ and $H \neq C$)
- One cannot attempt to control all outputs or states without offset:
 - Cannot have zero forward velocity in a car and have the car follow a track around a course
 - Cannot have an aeroplane hover (unless it's a Harrier)
- In practice, we often choose the controlled variable (CV) $z(k)$ as a linear combination of a subset of the states and/or outputs
- We therefore ask the question:

Which choices of H are allowed?

Existence Of Solutions To $Ax = b$

- Set of r linear equalities $Ax = b$ with c unknowns
 - Matrix A has r rows and c columns
 - Column vector b has length r
- **Facts:**
 - A solution exists for **a given** $b \Leftrightarrow \text{rank } A = \text{rank}(A \ b)$
 - A solution exists for **every** $b \Leftrightarrow \text{rank } A = r$ (possible only if $r \leq c$)
 - The solution, **if it exists**, is **unique** $\Leftrightarrow \text{rank } A = c$ (possible only if $r \geq c$)
 - If A is **square** and **invertible**, then the solution exists, is unique and given by $x = A^{-1}b$

Existence Of Target Equilibrium Pair

- Recall that the state x is an **equilibrium** if $x = f(x)$
- For a linear DT system, the pair (x_∞, u_∞) will be called an **(offset-free) target equilibrium pair** if

$$x_\infty = Ax_\infty + Bu_\infty$$

and

$$Hx_\infty = r$$

- Rearranging the above gives:

$$\begin{pmatrix} I - A & -B \\ H & 0 \end{pmatrix} \begin{pmatrix} x_\infty \\ u_\infty \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}$$

- A target equilibrium pair exists if a solution to the above set of linear equalities exists
- Depending on the choice of H and the current value of the reference r , a target equilibrium pair may or may not exist

Existence Of Target Equilibrium Pair

- H is chosen such that a target equilibrium pair exists for any choice of r :

- **Fact:** A **sufficient** condition for guaranteeing **existence** of a target equilibrium pair for **any** choice of r is that

$$\begin{pmatrix} I - A & -B \\ H & 0 \end{pmatrix}$$

is **full row rank**

- This implies that H has to be **full row rank** and number of inputs \geq number of CVs

Target Calculation: No Constraints

- At each time instant, the **target calculator** solves the following set of linear equalities:

$$\begin{pmatrix} I - A & -B \\ H & 0 \end{pmatrix} \begin{pmatrix} x_\infty \\ u_\infty \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}$$

- Note that if a solution exists, then it may or may not be unique
- If the target calculator computes x_∞ and u_∞ as above, then it is easy to show that $z(k) \rightarrow r(k)$ as $k \rightarrow \infty$ if
 - the sequence $r(\cdot)$ tends to a **constant value**
 - the control is given by

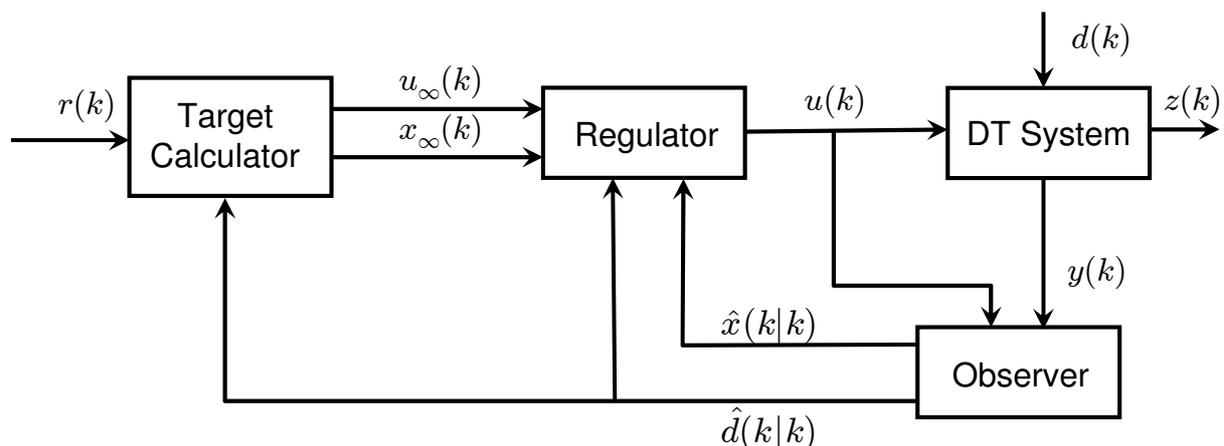
$$u(k) = u_\infty(k) + K(\hat{x}(k|k) - x_\infty(k))$$

- $A+BK$ and the observer are **stable**

Disturbances And Integral Control

- In practise, there is always **plant-model mismatch** or **disturbances** acting on the system
- Recall from your 2nd year course on SISO systems that **integral control** (PI vs. P) allows one to **reject constant disturbances**
- We will show how to achieve the equivalent of integral control in MIMO systems if there are **constant, unmeasured disturbances**

Offset-free Control



- Control system so that the controlled variable $z(k) \rightarrow r(k)$ as $k \rightarrow \infty$ if the **reference sequence** $r(\cdot)$ and **disturbance sequence** $d(\cdot)$ tend to constant values
- Future values of the reference sequence $r(\cdot)$ and the disturbance sequence $d(\cdot)$ are not known at time k
- Note that the target calculator *and* regulator require the current estimate of the disturbance $\hat{d}(k|k)$

Constant Disturbance Model

- We assume the following model with a **constant disturbance** acting on the states and outputs:

$$x(k+1) = Ax(k) + Bu(k) + B_d d(k)$$

$$d(k+1) = d(k)$$

$$y(k) = Cx(k) + C_d d(k)$$

$$z(k) = Hx(k) + H_d d(k)$$

- The disturbance $d(k) \in \mathbb{R}^l$, the matrices $B_d \in \mathbb{R}^{n \times l}$, $C_d \in \mathbb{R}^{p \times l}$ and $H_d \in \mathbb{R}^{q \times l}$
- H_d arises because the controlled variables are linear combinations of outputs and/or states

The Augmented System

- We augment the state of the system with the disturbance to get the **augmented system**:

$$\begin{pmatrix} x(k+1) \\ d(k+1) \end{pmatrix} = \begin{pmatrix} A & B_d \\ 0 & I \end{pmatrix} \begin{pmatrix} x(k) \\ d(k) \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u(k)$$

$$y(k) = (C \quad C_d) \begin{pmatrix} x(k) \\ d(k) \end{pmatrix}$$

$$z(k) = (H \quad H_d) \begin{pmatrix} x(k) \\ d(k) \end{pmatrix}$$

Estimating The State And Disturbance

- The augmented system is, in general, not guaranteed to be detectable for arbitrary B_d and C_d :

- **Fact:** The **augmented system** is **detectable** if and only if (C, A) is detectable **and**

$$\begin{pmatrix} I - A & -B_d \\ C & C_d \end{pmatrix}$$

is **full column rank**

- This implies that
 - number of disturbances \leq number of outputs
- If the above conditions hold, then we can design a stable observer or Kalman filter for the *augmented system* in order to compute $\hat{x}(k|k)$ and $\hat{d}(k|k)$

Existence Of Target Equilibrium Pair: Constant Disturbance

- We proceed along the same lines as before
- For a linear DT system, the pair (x_∞, u_∞) will be called an **(offset-free) target equilibrium pair** if

$$x_\infty = Ax_\infty + Bu_\infty + B_d \hat{d}$$

and

$$Hx_\infty + H_d \hat{d} = r$$

- Rearranging the above gives:

$$\begin{pmatrix} I - A & -B \\ H & 0 \end{pmatrix} \begin{pmatrix} x_\infty \\ u_\infty \end{pmatrix} = \begin{pmatrix} B_d \hat{d} \\ r - H_d \hat{d} \end{pmatrix}$$

- Note estimate of disturbance on RHS

Existence Of Target Equilibrium Pair: Constant Disturbance

- H is chosen such that a target equilibrium pair exists for any choice of r and \hat{d}
 - **Fact:** A **sufficient** condition for guaranteeing **existence** of a target equilibrium pair for **any** choice of r and \hat{d} is that

$$\begin{pmatrix} I - A & -B \\ H & 0 \end{pmatrix}$$

is **full row rank**

- This implies that H has to be **full row rank** and number of inputs \geq number of CVs

Target Calculation: No Constraints And Constant Disturbance

- At each time instant, the **target calculator** solves the following set of linear equalities:

$$\begin{pmatrix} I - A & -B \\ H & 0 \end{pmatrix} \begin{pmatrix} x_\infty \\ u_\infty \end{pmatrix} = \begin{pmatrix} B_d \hat{d} \\ r - H_d \hat{d} \end{pmatrix}$$

- Note that if a solution exists, then it may or may not be unique
- **WARNING:** Unlike the disturbance-free case, offset-free control is not yet guaranteed. Offset-free control is highly dependent on the choice of observer and controller.
- However, we can derive a result that is independent of the exact choice of observer and controller.

Main Result On Offset-free Control

- As before, let:

- The sequences $r(\cdot)$ and $d(\cdot)$ tend to **constant** values
- The **detectability** conditions on page 17 hold
- The **observer** is **stable**
- The **existence** condition on page 19 holds
- The control input is given by

$$u(k) = u_{\infty}(k) + K(\hat{x}(k|k) - x_{\infty}(k))$$

where $x_{\infty}(k)$ and $u_{\infty}(k)$ satisfy the set of equalities on p. 20

- $A+BK$ is **stable**
- **Deceivingly simple-looking amazing fact:**
 - If above conditions hold *and* **number of disturbances = number of outputs**, then $z(k) \rightarrow r(k)$ as $k \rightarrow \infty$

On The Choice Of Disturbance Model

- The choice of B_d and C_d is often not obvious
- The performance can be greatly affected by a proper/improper choice of disturbance model
- Physical insight into the process is often used to choose a disturbance model
- Commonly-used disturbance models are:
 - $B_d = B$ and $C_d = 0 \Rightarrow$ **Input disturbance** only
 - $B_d = 0$ and $C_d = I \Rightarrow$ **Output disturbance** only
- Any B_d and C_d is allowed, provided that the conditions listed on the previous page are satisfied

The Internal Model Principle

- One can guarantee offset-free control even if the actual plant dynamics (A_p, B_p, C_p) differ from those in the model (A, B, C) , provided the closed-loop matrix $A_p + B_p K$ is stable
- This is an example of a more general phenomenon, known as the **Internal Model Principle**, which (roughly) states:

*A controller that guarantees **offset-free control** even when the system parameters are perturbed, must incorporate a **model** of the dynamic structure of the **disturbance** as well as the **reference signals***
- In our case, the **observer** contains a model of the **disturbance** and the **target calculator** contains a model of the **disturbance and reference**

Target Calculation With Constraints

- For the sake of simplicity, let the constraints be given by
$$u_{\text{low}} \leq u_s \leq u_{\text{high}}, \quad s = 0, \dots, N - 1$$
$$y_{\text{low}} \leq y_s \leq y_{\text{high}}, \quad s = 1, \dots, N$$
- As before, let:
 - The sequences $r(\cdot)$ and $d(\cdot)$ tend to **constant** values
 - The **detectability** conditions on page 17 hold
 - The **observer** is **stable**
 - The **existence** condition on page 19 holds
 - The **number of disturbances** = **number of outputs** (page 21)
- but add the condition that
 - There exists **target equilibrium pairs** that **satisfy the constraints** for all values of $r(k)$ and $\hat{d}(k|k)$

Target Calculation With Constraints

At **each time instant**, given the

- current **reference** r and
- current estimate of the **disturbance** \hat{d}
- the **target calculator** computes the **target equilibrium pair** (x_∞, u_∞) by solving the following **Quadratic Program (QP)**:

$$\min_{x_\infty, u_\infty} (1/2)(u_\infty - \bar{u})^T (u_\infty - \bar{u})$$

subject to the constraints

$$\begin{pmatrix} I - A & -B \\ H & 0 \end{pmatrix} \begin{pmatrix} x_\infty \\ u_\infty \end{pmatrix} = \begin{pmatrix} B_d \hat{d} \\ r - H_d \hat{d} \end{pmatrix}$$

$$u_{\text{low}} \leq u_\infty \leq u_{\text{high}}$$

$$y_{\text{low}} \leq Cx_\infty + C_d \hat{d} \leq y_{\text{high}}$$

- \bar{u} is the **ideal steady-state** for the inputs

Offset-free Predictive Control With Constraints

- **Given:** $x_\infty, u_\infty, \hat{x}$ and \hat{d}
- Compute a **finite** input sequence u_0, u_1, \dots, u_{N-1} , which minimises

$$\sum_{s=0}^{N-1} [(x_s - x_\infty)^T Q (x_s - x_\infty) + (u_s - u_\infty)^T R (u_s - u_\infty)]$$

subject to the constraints

$$+ (x_N - x_\infty)^T P (x_N - x_\infty)$$

$$x_0 = \hat{x}$$

$$x_{s+1} = Ax_s + Bu_s + B_d \hat{d}, \quad s = 0, \dots, N-1$$

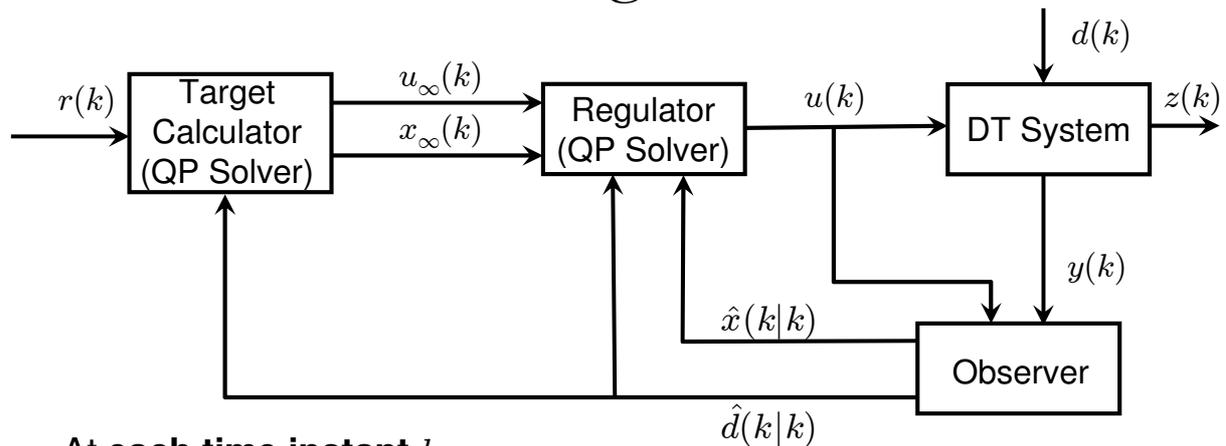
$$y_s = Cx_s + C_d \hat{d}, \quad s = 1, \dots, N$$

$$u_{\text{low}} \leq u_s \leq u_{\text{high}}, \quad s = 0, \dots, N-1$$

$$y_{\text{low}} \leq y_s \leq y_{\text{high}}, \quad s = 1, \dots, N$$

- As in Lectures 3 and 4, the **optimal input sequence** can be found by **solving** an appropriately-defined **QP**

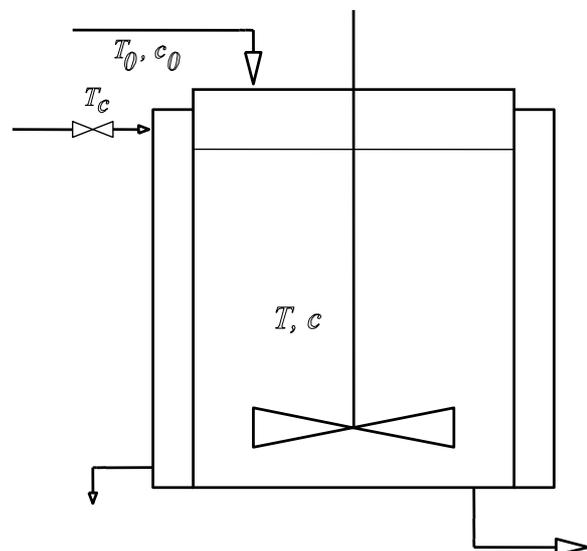
Offset-free Receding Horizon Control



- At each time instant k ,
 - Take **measurement** $y(k)$ and compute **estimates** $\hat{x}(k|k)$ and $\hat{d}(k|k)$
 - Compute **target equilibrium pair** $x_\infty(k)$ and $u_\infty(k)$ by solving QP on p. 25
 - Compute **optimal input sequence** by solving QP on p. 26
 - Implement only **first input** in optimal sequence, i.e. $u(k) = u_0^*(x(k))$
- If the QPs are always **feasible** and the closed-loop system is **stable**, then $z(k) \rightarrow r(k)$ as $k \rightarrow \infty$

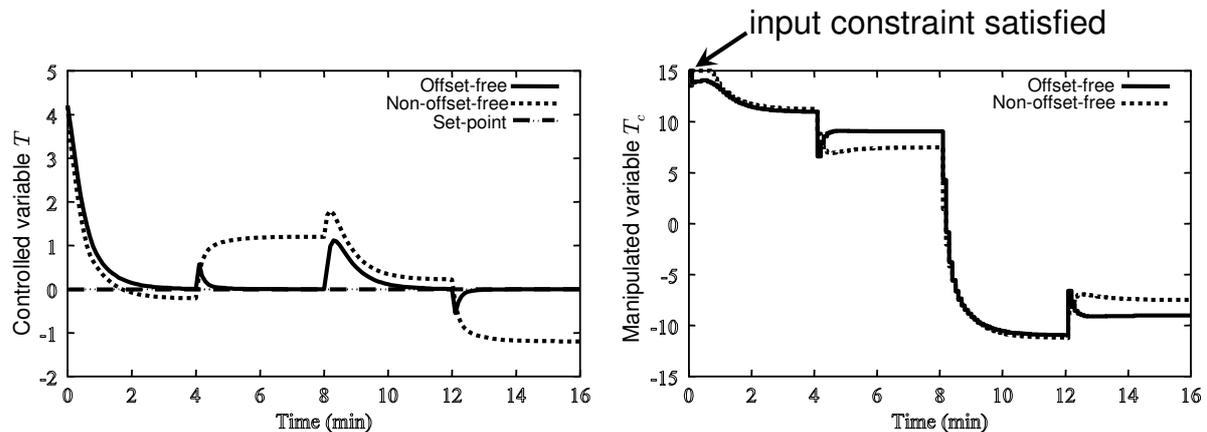
Example: CSTR

- Continuous stirred-tank reactor (CSTR)
- States:
 - Concentration c
 - Temperature T
- T is controlled variable (CV)
- T_c is manipulated variable (MV)
- Input constraints:
 - $-15 \leq T_c \leq 15$
- State constraints:
 - $-0.5 \leq c \leq 0.5$
 - $-5 \leq T \leq 5$
- Disturbances c_0 and T_0



Example: CSTR

- Comparison with non-offset-free RHC
- Disturbances change at $t = 0, 4, 8, 12$



Summary

- **Set-point tracking problem:**
 - Compute a **target pair** $x_\infty(k)$ and $u_\infty(k)$ for the current set-point
 - **Regulate** state and input around new **target pair**
- **Offset-free control** in presence of **constant disturbances:**
 - Augment original model with **disturbance model**
 - Estimate disturbance using **augmented model**
 - Compute target pair based on current set-point **and** disturbance
- **Internal model principle:**
 - Good control requires a **model** of the disturbance **and** reference
- **Predictive control with constraints:** At each time instant:
 - Include constraints in **target calculator** and solve a QP
 - **Include disturbance** in prediction of states and constraints
 - **Penalise deviations** of state and input from **target pair**
 - Compute optimal input sequence by solving a QP
 - Implement only **first input** in optimal input sequence