4F3 – Predictive Control

Lecture 2

Digital State Space Control Theory

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Outline Of Course

- Introduction to predictive control
- Digital state space control theory
- Unconstrained predictive control
- Predictive control with constraints
- Set-point tracking and offset-free control
- Stability and feasibility in predictive control – Dr Jan Maciejowski
- Case study by industrial speaker – Dr Paul Austin

Outline Of Lecture 2

- Review of linear algebra
- Digital (sampled-data) control
- Discrete-time models of sampled-data systems
- Stability of discrete-time state-space systems
- Reachability and observability
- Stabilizability and detectability
- State feedback control via pole placement
- Observer design via pole placement
- Output feedback control and the separation principle

Recommended Books

- Linear Algebra:
  - G. Strang. *Linear Algebra and Its Applications*, 3rd ed., Thomson Learning (1988)
  - D.C. Lay. *Linear Algebra and Its Applications,* 3rd ed., Addison Wesley (2003)
- Digital Control:
  - G.F. Franklin et al. *Digital Control of Dynamic Systems*, 3rd ed., Addison Wesley (1998)
  - K.J. Åström and B. Wittenmark. *Computer Controlled Systems,* 3rd ed., Prentice Hall (1997)

Linear Independence and Matrix Rank

- The vectors $v_1,\ldots,v_n$ are ***linearly independent*** if $c_1v_1+\cdots+c_nv_n = 0$ only happens when $c_1=\cdots=c_n= 0$
- The ***rank*** of a matrix A is defined as the number of linearly independent columns
- **Fact**: The number of independent columns equals the number of independent rows (*column rank* = *row rank*), i.e.

$$\text{rank } A = \text{rank } A^T$$

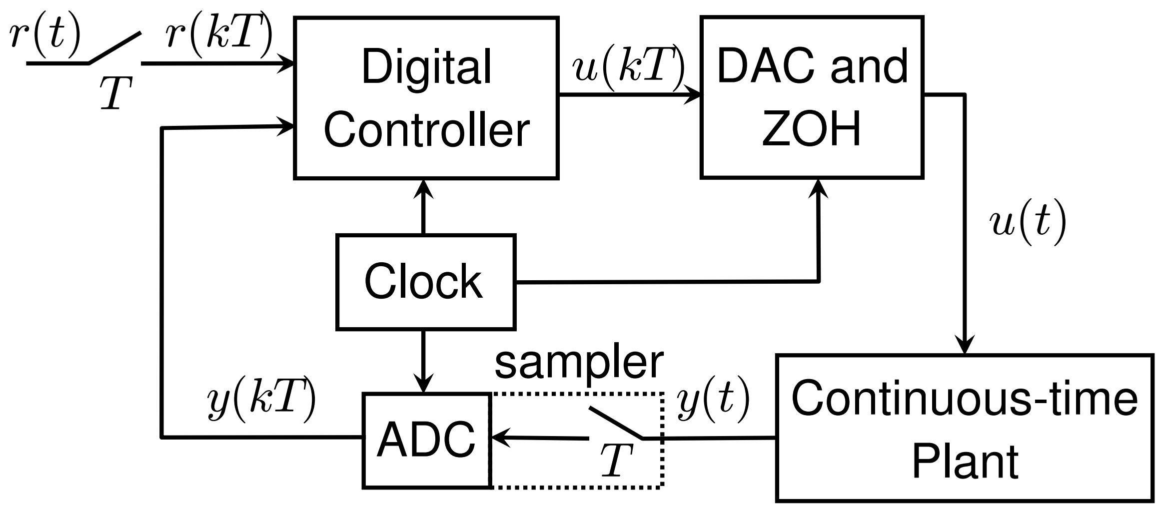
Existence Of Solutions To $Ax = b$

- Set of r linear equalities $Ax = b$ with c unknowns
  - Matrix A has r rows and c columns
  - Column vector b has length r
- **Facts**:
  - A solution exists for ***a given*** $b \Leftrightarrow \text{rank } A = \text{rank}(A \;\; b)$
  - A solution exists for ***every*** $b \Leftrightarrow \text{rank } A = r$ (possible only if $r \leq c$)
  - The solution, ***if it exists***, is ***unique*** $\Leftrightarrow \text{rank } A = c$ (possible only if $r \geq c$)
  - If A is ***square*** and ***invertible***, then the solution exists, is unique and given by $x = A^{-1}b$

Eigenvalues And Eigenvectors

- Let A be a *square* $n \times n$ matrix and $Ax = \lambda x$:
  - Scalar λ is an ***eigenvalue*** of A
  - Vector $x \neq 0$ is an ***eigenvector*** of A
- $\lambda_i(A)$ denotes i^{th} eigenvalue of A
- Largest absolute value of set of eigenvalues is the ***spectral radius*** of A, i.e. $\rho(A) := \max\{|\lambda_1(A)|, \ldots, |\lambda_n(A)|\}$
- $\det(\lambda I - A)$ is the ***characteristic polynomial*** of A
- **Facts**:
  - The eigenvalues are the roots of the characteristic polynomial:
    $$\lambda \text{ is an eigenvalue of } A \Leftrightarrow \det(\lambda I - A) = 0$$
  - Each eigenvalue has a corresponding non-zero eigenvector x:
    $$(\lambda I - A)x = 0 \text{ or } Ax = \lambda x$$

Digital (Sampled-Data) Control



- T is the sample period, k is the sample number, t denotes time
- $s(t)$ is a continuous-time signal and $s(kT)$ is a discrete-time signal
- We will also use $s(k)$ to denote the value of signal s at time $t = kT$
- $s(k+1)$ is then the value of signal $s(\cdot)$ at time $t = (k+1)T$

Linear Continuous-Time Systems

- Linear continuous-time (CT) state-space system:

$$\dot{x} = Fx + Gu$$
$$y = Cx + Du$$
$$z = Hx$$

	Column vector name	*Size*
x	States	n
u	Controlled inputs/manipulated variables (MV)	m
y	Outputs/measured variables	p
z	Controlled variables (CV) – often $H = C$	q

Solution Of Linear CT Systems

- Given:
  - Initial state $x(t_0)$ at time t_0
  - Input trajectory $u(\cdot)$ over time interval $[t_0,t]$, i.e. $u:[t_0,t] \to \mathbb{R}^m$
- It is possible to show that the solution of the CT system at time t is given by

$$x(t) = e^{F(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{F(t-s)}Gu(s)ds$$

Discrete Models of Sampled-Data Systems

- With zero-order hold (ZOH) at the DAC:
  - $u(t) = u(kT)$ for all $t \in [kT, kT+T)$
- Make the substitutions $t = kT+T$, $t_0 = kT$ and $\eta = kT+T-s$
- We then get that the discrete-time (DT) state-space model

$$\begin{aligned} x(kT+T) &= Ax(kT) + Bu(kT) \\ y(kT) &= Cx(kT) + Du(kT) \\ z(kT) &= Hx(kT) \end{aligned}$$

  is an ***exact*** representation of the sampled CT system if

$$A = e^{FT}, \qquad B = \int_0^T e^{F\eta} d\eta \, G$$

Discrete Models of Sampled-Data Systems

- In the literature on sampled-data systems, one often finds the notation:
  - $F \leftarrow A$
  - $G \leftarrow B$
  - $A \leftarrow \Phi$
  - $B \leftarrow \Gamma$
- Notation that we will use is consistent with literature on predictive control
- Drop T in DT model and use $x(k)$, rather than $x(kT)$
- Often use x to denote ***current state*** $x(k)$
- Throughout the course we will assume that $D = 0$

Discrete Models of Sampled-Data Systems

- With a ZOH at the DAC, it is possible to get an exact DT model if CT system is
  - linear, or
  - linear with input saturation (common nonlinearity)
- In general, it is *not possible* to get an *exact* DT model for an *arbitrary nonlinear* CT system:

$$\dot{x} = f(x,u)$$
$$y = h(x,u)$$

- With nonlinear models, use DT approximations or numerical integration to compute $x(kT)$

Solution Of Linear DT Systems

- Given:
  - State $x(k)$ at discrete time k
  - Finite input sequence $u(k)$,…,$u(k+N-1)$
- Solution of DT model at discrete-time $k+N$ is

$$x(k+N) = A^N x(k) + \sum_{i=0}^{N-1} A^i Bu(k+N-1-i)$$

- We will return to this in Lecture 3

Stability Of Nonlinear DT Systems

- Consider the DT system, where $f : \mathbb{R}^n \to \mathbb{R}^n$

$$x(k+1) = f(x(k))$$

- Let $x(k)$ denote the solution of the system at discrete-time instant k if the initial state is $x(0)$ at discrete-time instant $k = 0$
- A state x is called an ***equilibrium*** if $x = f(x)$
  - Differs from definition of equilibrium for CT systems

Stability Of Nonlinear DT Systems

- An equilibrium x is ***stable*** if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|x(0) - x\| < \delta \Rightarrow \|x(k) - x\| < \varepsilon \text{ for all } k \geq 0$$

- An equilibrium x is ***locally asymptotically stable*** if it is stable *and* there exists an $M > 0$ such that

$$\|x(0) - x\| < M \Rightarrow \lim_{k\to\infty} x(k) = x$$

- The equilibrium is ***globally asymptotically stable*** if the above holds for all $M > 0$
- Analogous to definitions for CT systems

Stability Of Linear DT Systems

- Consider the linear DT system with *zero input* $u(k)=0, \forall k \geq 0$:

$$x(k+1) = Ax(k)$$

- Clearly, x is an equilibrium if $x = Ax$
- **Fact**:
  - The origin is globally asymptotically stable if and only if all e/values of A are strictly inside the unit disk, i.e. $\rho(A) < 1$
- We say the matrix A is ***stable*** if and only if $\rho(A) < 1$
  - Differs from one for linear CT systems – see 3F2
- Sampled-data systems with ZOH:
  - E/values of A are $e^{\lambda_i(F)T}$, $i = 1,\ldots,n$
  - CT system stable $\Rightarrow$ DT model stable

Reachability

- (A,B) is ***reachable*** if it is possible to find an input sequence such that an arbitrary state can be reached from any initial state in *finite time*
  - Analogous to concept of controllability of the CT system (F,G) – see 3F2
- Note:
  - (F,G) controllable $\not\Rightarrow$ (A,B) reachable
  - However, this is usually not a problem

Reachability

- The ***reachability matrix*** of (A,B) is defined as:

$$\mathcal{C}(A,B) := \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}$$

- **Equivalent statements**:
  - (A,B) is reachable
  - E/values of $A+BK$ can be freely assigned (with restriction that complex e/values are in conjugate pairs) by suitable choice of K
  - $\text{rank } \mathcal{C}(A,B) = n$

Observability

- (C,A) is ***observable*** if the initial state can be determined from a *finite* sequence of inputs and outputs
  - Analogous to concept of observability of the CT pair (C,F) – see 3F2
- Note:
  - (C,F) observable $\nRightarrow$ (C,A) observable
  - However, this is usually not a problem

Observability

- The ***observability matrix*** of the pair (C,A) is defined as:

$$\mathcal{O}(C,A) := \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

- **Equivalent statements**:
  - (C,A) is observable
  - E/values of $A+LC$ can be freely assigned (with restriction that complex e/values are in conjugate pairs) by suitable choice of L
  - $\mathrm{rank}\ \mathcal{O}(C,A) = n$
  - (A^T,C^T) is reachable

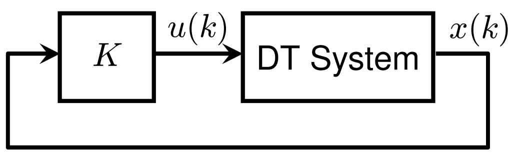
Stabilizability And Detectability

- The notions of controllability and observability can be relaxed:
  - The pair (A,B) is said to be ***stabilizable*** if there exists a K such that $A+BK$ is stable
  - The pair (C,A) is said to be ***detectable*** if there exists an L such that $A+LC$ is stable
- Note:
  - reachable/observable $\Rightarrow$ stabilizable/detectable
  - stabilizable/detectable $\not\Rightarrow$ reachable/observable

Stabilizability And Detectability

- For DT systems, let $\Lambda := \{ \lambda_i(A) : |\lambda_i(A)| \geq 1 \}$ be the e/values of A that are on or outside the unit circle
- (A,B) is ***stabilizable*** if and only if for all $\lambda \in \Lambda$, the matrix
$$\begin{pmatrix} A - \lambda I & B \end{pmatrix}$$
is full *row* rank
- (C,A) is ***detectable*** if and only if for all $\lambda \in \Lambda$, the matrix
$$\begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$$
is full *column* rank
- For CT systems, the conditions are the same, but we let $\Lambda := \{ \lambda_i(A) \mid \mathrm{Re}(\lambda_i(A)) \geq 0 \}$ be the e/values of A that are on or to the right of the imaginary line

State Feedback And Regulation



- Given the DT system $x(k+1) = Ax(k) + Bu(k)$
- Measurements of all of the states are available ($C = I$)
- All states are controlled variables ($C = H$)
- Aim:
  - Choose feedback matrix gain $K \in \mathbb{R}^{m \times n}$ such that
  - The control law $u(k) = Kx(k)$ ***regulates*** the states of the system around the origin
- Closed-loop system is $x(k+1) = (A+BK)x(k)$
- Origin of closed-loop system is globally asymptotically stable if and only if $A+BK$ stable

State Feedback And Pole Placement

- Recall that we can place the e/values of $A+BK$ arbitrarily if and only if (A,B) is reachable

- If (A,B) is stabilizable, but not reachable, then procedure is slightly more complicated (cannot place e/values arbitrarily)

State Feedback And Pole Placement

- Recipe if (A,B) is reachable:
  - Given desired e/values $p_1,\ldots,p_n$, with each $|p_i| < 1$ and complex e/values in conjugate pairs
  - Compute coefficients of polynomial
  $$p(\lambda) = (\lambda - p_1)(\cdots)(\lambda - p_n)$$
  - Compute characteristic polynomial $\det(\lambda I - A - BK)$
  - Equate coefficients of characteristic polynomial with those of $p(\lambda)$
  - We now have n equalities with $m \times n$ unknowns
  - Solve for K

Observer Design

- Often measurements of all states not available ($C \neq I$)
- States can be estimated by using an ***observer*** (also called an ***estimator)***
- The DT system equations are:

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k)$$

- The equations for the ***observer*** are:

$$\hat{x}(k|k) = \hat{x}(k|k-1) + L[\hat{y}(k|k-1) - y(k)]$$
$$\hat{x}(k+1|k) = A\hat{x}(k|k) + Bu(k)$$
$$\hat{y}(k|k-1) = C\hat{x}(k|k-1)$$

- Notation: $\hat{s}(i|j)$ is an ***estimate*** of signal $s(\cdot)$ at time i given measurements up to time j

Observer Design

- Eliminate variables in ***observer equations*** to get

$$\hat{x}(k|k) = (A + LCA)\hat{x}(k-1|k-1) + (B + LCB)u(k-1) - Ly(k)$$

- Observer is stable if and only if $A+LCA$ is stable
- If we define the ***state estimation error*** as

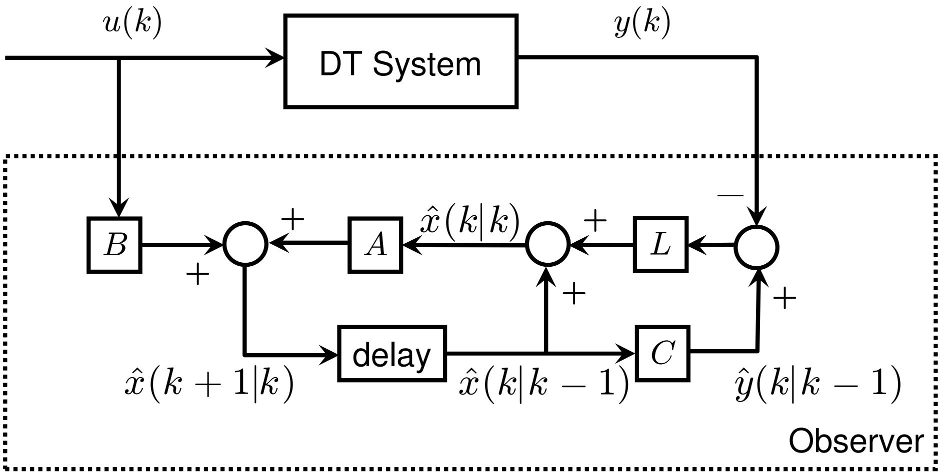
$$e(k) := x(k) - \hat{x}(k|k)$$

then we have

$$e(k+1) = (A + LCA)e(k)$$

- Error converges to zero if and only if $A+LCA$ stable

Observer Design



$$\hat{x}(k|k) = \hat{x}(k|k-1) + L[\hat{y}(k|k-1) - y(k)]$$
$$\hat{x}(k+1|k) = A\hat{x}(k|k) + Bu(k)$$
$$\hat{y}(k|k-1) = C\hat{x}(k|k-1)$$

Observer Design And Pole Placement

- Recall we can place e/values of $A+LC$ arbitrarily if and only if (C,A) is observable
- Similarly, we can place e/values of $A+LCA$ arbitrarily if and only if (CA,A) observable:
  - (C,A) observable ***and*** all $\lambda_i(A) \neq 0 \Rightarrow (CA,A)$ observable
  - Place e/values of $A+LCA$ arbitrarily by suitable choice of L
  - L can be computed as in state feedback pole placement
  - Equate coefficients of $\det(\lambda I - A - LCA)$ with those of $p(\lambda)$
- (C,A) detectable $\Rightarrow (CA,A)$ detectable
  - If (CA,A) is detectable, but not observable, then observer design is slightly more complicated (e/values cannot be placed arbitrarily)

Observer Design And Pole Placement

- Warning:
  - There are many types of observers in the literature
  - Be careful when choosing and implementing one
- In practice, we often use a ***Kalman Filter*** for an observer:
  - Assumes noise is white with known covariance matrices
  - Minimises mean square state estimation error

Output Feedback And Regulation

- The DT system equations are:

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k)$$

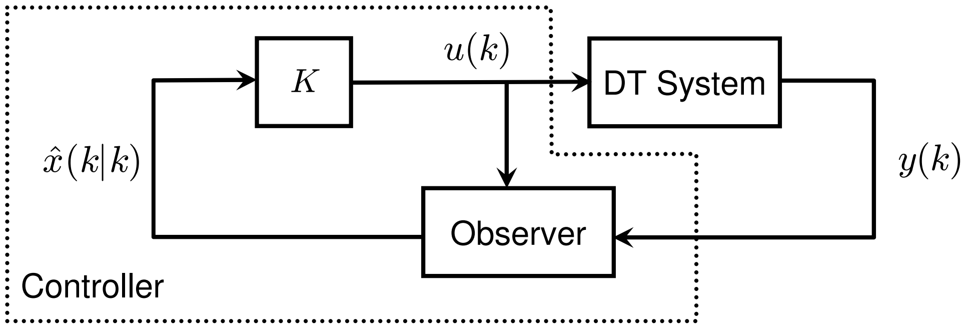
- The states cannot be measured ($C \neq I$)
- Regulate the states of the system around the origin
- It seems intuitive to try the control law

$$u(k) = K\hat{x}(k|k)$$

- The estimate of the state is obtained from

$$\begin{aligned}\hat{x}(k|k) =& (A + LCA)\hat{x}(k-1|k-1) \\ &+ (B + LCB)u(k-1) - Ly(k)\end{aligned}$$

Output Feedback And Regulation



The Separation Principle

- Define the state estimator error as before:

$$e(k) := x(k) - \hat{x}(k|k)$$

- The ***closed-loop system*** now becomes:

$$x(k+1) = (A + BK)x(k) - BKe(k)$$

$$e(k+1) = (A + LCA)e(k)$$

- The e/values of the closed-loop system are:
  - E/values of $A+BK$ – from state feedback design
  - E/values of $A+LCA$ – from observer design
- Special case of the so-called ***separation principle:***
  - Controller and observer can be designed separately, yet used together

A Final Word On Confusing Notation

- In the literature, one often finds the following notation:
  - $K \leftarrow -L$
  - $L \leftarrow -K$
- The use of a minus sign is confusing – we all know that, depending on the system, a stabilising feedback gain can be negative *or* positive!

Summary

- Linear algebra – importance of rank and e/values
- Sampled-data control
  - Sample CT system's output with ADC
  - Control computed then implemented with DAC
- Need a DT model of sampled CT system
  - Can obtain exact DT models if ZOH at DAC
- Linear DT system asymptotically stable $\Leftrightarrow \rho(A) < 1$
- Reachable/stabilizable $\Rightarrow$ State feedback control
- Observable/detectable $\Rightarrow$ Design an observer
- Stabilizable + detectable $\Rightarrow$ Output feedback control