Engineering Tripos Part IIB **Module 4F3**

Nonlinear and Predictive Control
Solutions to Examples Paper 4F3/2

1. Before proceeding, note that $x_1 = Ax_0 + Bu_0$ and

$$x_2 = Ax_1 + Bu_1 = A^2x_0 + ABu_0 + Bu_1$$

The constraints can be rewritten:

$$\begin{aligned} -1 \leq u_0 &\leq 5 \\ -1 \leq u_1 &\leq 5 \\ \begin{pmatrix} -2 \\ -1 \end{pmatrix} \leq x_1 &\leq \begin{pmatrix} 4 \\ 3 \end{pmatrix} \\ \begin{pmatrix} -2 \\ -1 \end{pmatrix} \leq x_2 &\leq \begin{pmatrix} 4 \\ 3 \end{pmatrix} \end{aligned}$$

is equivalent to

$$\begin{aligned} u_0 &\leq 5 \\ u_1 &\leq 5 \\ -u_0 &\leq 1 \\ -u_1 &\leq 1 \\ x_1 &\leq \begin{pmatrix} 4 \\ 3 \end{pmatrix} \\ x_2 &\leq \begin{pmatrix} 4 \\ 3 \end{pmatrix} \\ -x_1 &\leq \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ -x_2 &\leq \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{aligned}$$

is equivalent to

$$\begin{aligned}
u_0 &\leq 5 \\
u_1 &\leq 5 \\
-u_0 &\leq 1 \\
-u_1 &\leq 1 \\
Ax_0 + Bu_0 &\leq \begin{pmatrix} 4 \\ 3 \end{pmatrix} \\
A^2x_0 + ABu_0 + Bu_1 &\leq \begin{pmatrix} 4 \\ 3 \end{pmatrix} \\
-Ax_0 - Bu_0 &\leq \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
-A^2x_0 - ABu_0 - Bu_1 &\leq \begin{pmatrix} 2 \\ 1 \end{pmatrix}
\end{aligned}$$

is equivalent to

$$\begin{aligned}
u_0 &\leq 5 \\
u_1 &\leq 5 \\
-u_0 &\leq 1 \\
-u_1 &\leq 1 \\
Bu_0 &\leq \begin{pmatrix} 4 \\ 3 \end{pmatrix} - Ax_0 \\
ABu_0 + Bu_1 &\leq \begin{pmatrix} 4 \\ 3 \end{pmatrix} - A^2x_0 \\
-Bu_0 &\leq \begin{pmatrix} 2 \\ 1 \end{pmatrix} + Ax_0 \\
-ABu_0 - Bu_1 &\leq \begin{pmatrix} 2 \\ 1 \end{pmatrix} + A^2x_0
\end{aligned}$$

is equivalent to

$$
\begin{aligned}
u_0 &\le 5 \\
u_1 &\le 5 \\
-u_0 &\le 1 \\
-u_1 &\le 1 \\
\begin{pmatrix} 0.5 \\ 1 \end{pmatrix} u_0 &\le \begin{pmatrix} 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x_0 \\
\begin{pmatrix} 1.5 \\ 1 \end{pmatrix} u_0 + \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} u_1 &\le \begin{pmatrix} 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} x_0 \\
- \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} u_0 &\le \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x_0 \\
- \begin{pmatrix} 1.5 \\ 1 \end{pmatrix} u_0 - \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} u_1 &\le \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} x_0
\end{aligned}
$$

is equivalent to

$$
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
-1 & 0 \\
0 & -1 \\
0.5 & 0 \\
1 & 0 \\
1.5 & 0.5 \\
1 & 1 \\
-0.5 & 0 \\
-1 & 0 \\
-1.5 & -0.5 \\
-1 & -1
\end{pmatrix}
\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \le
\begin{pmatrix} 5 \\ 5 \\ 1 \\ 1 \\ 4 \\ 3 \\ 4 \\ 3 \\ 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} +
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
-1 & -1 \\
0 & -1 \\
-1 & -2 \\
0 & -1 \\
1 & 1 \\
0 & 1 \\
1 & 2 \\
0 & 1
\end{pmatrix} x_0
$$

The result is completed by comparing terms.

Note that any solution is allowed if the same rows of J, c and W are interchanged.

2. The constraints are equivalent to:

$$
\begin{aligned}
\begin{pmatrix} -1 \\ -2 \end{pmatrix} &\le u_0 - u_{-1} \le \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\
\begin{pmatrix} -1 \\ -2 \end{pmatrix} &\le u_1 - u_0 \le \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\
\begin{pmatrix} -1 \\ -2 \end{pmatrix} &\le u_2 - u_1 \le \begin{pmatrix} 3 \\ 1 \end{pmatrix}
\end{aligned}
$$

which is equivalent to:

$$u_0 - u_{-1} \leq \binom{3}{1}$$
$$-u_0 + u_{-1} \leq \binom{1}{2}$$
$$u_1 - u_0 \leq \binom{3}{1}$$
$$-u_1 + u_0 \leq \binom{1}{2}$$
$$u_2 - u_1 \leq \binom{3}{1}$$
$$-u_2 + u_1 \leq \binom{1}{2}$$

which is equivalent to

$$u_0 \leq \binom{3}{1} + u_{-1}$$
$$-u_0 \leq \binom{1}{2} - u_{-1}$$
$$-u_0 + u_1 \leq \binom{3}{1}$$
$$u_0 - u_1 \leq \binom{1}{2}$$
$$-u_1 + u_2 \leq \binom{3}{1}$$
$$+u_1 - u_2 \leq \binom{1}{2}$$

Since the number of inputs $m = 2$, the above is equivalent to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} \leq \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} u_{-1}$$

The result is completed by comparing terms.

Note that any solution is allowed if the same rows of J, c and W are interchanged.

3. (a) Note that (C, A) is detectable with the given value of C — see Question 4 in Examples Paper 4F3/3. Hence, in order to test whether the augmented system is detectable, all we need to do is check whether the matrix

$$\begin{pmatrix} I - A & -B_d \\ C & C_d \end{pmatrix}$$

is full column rank.

i. By inspection,

$$\text{rank} \begin{pmatrix} I - A & -B_d \\ C & C_d \end{pmatrix} = \text{rank} \begin{pmatrix} 0 & -1 & -0.5 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} = 3$$

The augmented system is detectable.

ii. By inspection,

$$\text{rank} \begin{pmatrix} I - A & -B_d \\ C & C_d \end{pmatrix} = \text{rank} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 2 < 3$$

The augmented system is not detectable.

iii. By inspection,

$$\text{rank} \begin{pmatrix} I - A & -B_d \\ C & C_d \end{pmatrix} = \text{rank} \begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 3 < 4$$

The augmented system is not detectable.

(b) Note that, since we have an input disturbance, $B_d = B$.
Offset-free control is possible if the set of linear equalities

$$\begin{pmatrix} I-A & -B \\ H & 0 \end{pmatrix} \begin{pmatrix} x_\infty \\ u_\infty \end{pmatrix} = \begin{pmatrix} B_d d \\ r - H_d d \end{pmatrix}$$

has a solution. This can be done by testing a certain number of rank conditions.

i. By inspection,

$$\text{rank} \begin{pmatrix} I-A & -B \\ H & 0 \end{pmatrix} = \text{rank} \begin{pmatrix} 0 & -1 & -0.5 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} = 3$$

This implies that the matrix on the LHS has an inverse, so a solution exists for any RHS, hence offset-free control is possible for any r and d.

ii. By inspection,

$$\text{rank} \begin{pmatrix} I-A & -B \\ H & 0 \end{pmatrix} = \text{rank} \begin{pmatrix} 0 & -1 & -0.5 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 3$$

and

$$\text{rank} \begin{pmatrix} I-A & -B & B_d d \\ H & 0 & r - H_d d \end{pmatrix} = \text{rank} \begin{pmatrix} 0 & -1 & -0.5 & 0.5 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = 3$$

Since the rank of the augmented matrix is the same as the rank of the matrix on the LHS, this implies that the RHS is in the range of the matrix on the LHS, which implies that a solution exists. Hence, offset-free control is possible for this specific value of r and d.

iii. Note that

$$\det \begin{pmatrix} I-A & -B & B_d d \\ H & 0 & r - H_d d \end{pmatrix} = \det \begin{pmatrix} 0 & -1 & -0.5 & 0.5 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} = 1$$

Since the augmented matrix is square and the determinant is non-zero, this implies that the augmented matrix is full rank, i.e.

$$\text{rank} \begin{pmatrix} I-A & -B & B_d d \\ H & 0 & r - H_d d \end{pmatrix} = \text{rank} \begin{pmatrix} 0 & -1 & -0.5 & 0.5 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} = 4$$

Since the rank of the augmented matrix is greater than the rank of the matrix on the LHS, this implies that the RHS is not in the range of the matrix on the LHS, which implies that a solution does not exist. Hence, offset-free control is not possible for this value of r and d.

4. In the following, the identity matrix I and zero matrix/vector 0 have appropriate dimensions.

   (a) We are interested in translating the following constrained optimisation problem to the general form of a QP:

$$\min_{x_\infty, u_\infty} \frac{1}{2}(u_\infty - \bar{u})^T(u_\infty - \bar{u})$$

subject to the constraints

$$\begin{pmatrix} I-A & -B \\ H & 0 \end{pmatrix}\begin{pmatrix} x_\infty \\ u_\infty \end{pmatrix} = \begin{pmatrix} B_d\hat{d} \\ r - H_d\hat{d} \end{pmatrix}$$

$$u_{\text{low}} \le u_\infty \le u_{\text{high}}$$

$$y_{\text{low}} \le Cx_\infty + C_d\hat{d} \le y_{\text{high}}$$

As suggested, let $\theta := \begin{pmatrix} x_\infty^T & u_\infty^T \end{pmatrix}^T$.
First, we note that the scalar $u_\infty^T\bar{u} = \bar{u}^T u_\infty$ and rewrite the cost as

$$\begin{aligned}
\frac{1}{2}(u_\infty - \bar{u})^T(u_\infty - \bar{u}) &= \frac{1}{2}\left(u_\infty^T u_\infty - u_\infty^T\bar{u} - \bar{u}^T u_\infty + \bar{u}^T\bar{u}\right) \\
&= \frac{1}{2}\left(u_\infty^T u_\infty - 2\bar{u}^T u_\infty + \bar{u}^T\bar{u}\right) \\
&= \frac{1}{2}\begin{pmatrix} x_\infty^T & u_\infty^T \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} x_\infty \\ u_\infty \end{pmatrix} - \begin{pmatrix} 0 & \bar{u}^T \end{pmatrix}\begin{pmatrix} x_\infty \\ u_\infty \end{pmatrix} + \frac{1}{2}\bar{u}^T\bar{u}
\end{aligned}$$

After comparison with the general form of the QP, it follows that

$$G = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \qquad f = \begin{pmatrix} 0 \\ -\bar{u} \end{pmatrix}, \qquad e = \frac{1}{2}\bar{u}^T\bar{u}.$$

By comparing the equality constraints with the general form of the QP, it follows that

$$D = \begin{pmatrix} I-A & -B \\ H & 0 \end{pmatrix}, \qquad g = \begin{pmatrix} B_d\hat{d} \\ r - H_d\hat{d} \end{pmatrix}.$$

We can rewrite the inequality constraints as

$$\begin{aligned}
u_\infty &\le u_{\text{high}} \\
-u_\infty &\le -u_{\text{low}} \\
Cx_\infty &\le y_{\text{high}} - C_d\hat{d} \\
-Cx_\infty &\le -y_{\text{low}} + C_d\hat{d}
\end{aligned}$$

which is equivalent to

$$\begin{pmatrix} 0 & I \\ 0 & -I \\ C & 0 \\ -C & 0 \end{pmatrix} \begin{pmatrix} x_\infty \\ u_\infty \end{pmatrix} \leq \begin{pmatrix} u_{\text{high}} \\ -u_{\text{low}} \\ y_{\text{high}} - C_d\hat{d} \\ -y_{\text{low}} + C_d\hat{d} \end{pmatrix}$$

By comparing the inequality constraints with the general form of the QP, it follows that

$$E = \begin{pmatrix} 0 & I \\ 0 & -I \\ C & 0 \\ -C & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} u_{\text{high}} \\ -u_{\text{low}} \\ y_{\text{high}} - C_d\hat{d} \\ -y_{\text{low}} + C_d\hat{d} \end{pmatrix}.$$

In the above, note that g and h are functions of the current setpoint r and estimate of the disturbance $\hat{d}$. Hence, the optimal value of the decision variable θ is a function of r and $\hat{d}$.

Note also that e is not a function of any of the decision variables. Hence, the value of e does not affect the optimal solution and can actually be set to zero at all times.

(b) We are interested in translating the following constrained optimisation problem to the general form of a QP:

$$\min_{u_0,\dots,u_{N-1}} \sum_{s=0}^{N-1} [(x_s - x_\infty)^T Q (x_s - x_\infty) + (u_s - u_\infty)^T R (u_s - u_\infty)] \\ + (x_N - x_\infty)^T P (x_N - x_\infty)$$

subject to the constraints

$$\begin{aligned} x_0 &= \hat{x} \\ x_{s+1} &= Ax_s + Bu_s + B_d\hat{d}, \quad s = 0, \dots, N-1 \\ y_s &= Cx_s + C_d\hat{d}, \quad s = 1, \dots, N \\ u_{\text{low}} \leq u_s &\leq u_{\text{high}}, \quad s = 0, \dots, N-1 \\ y_{\text{low}} \leq y_s &\leq y_{\text{high}}, \quad s = 1, \dots, N \end{aligned}$$

As suggested, we first introduce the change of variables

$$\begin{aligned} v_s &= u_s - u_\infty, \quad s = 0, \dots, N-1 \\ w_s &= x_s - x_\infty, \quad s = 1, \dots, N \end{aligned}$$

and let the decision variable be

$$\theta := \begin{pmatrix} v_0^T & w_1^T & v_1^T & w_2^T & v_2^T & \cdots & w_{N-1}^T & v_{N-1}^T & w_N^T \end{pmatrix}^T.$$

In other words, the optimisation problem now becomes

$$\min_{\theta} \; (\hat{x} - x_\infty)^T Q (\hat{x} - x_\infty) + v_0^T R v_0 + \sum_{s=1}^{N-1} w_s^T Q w_s + v_s^T R v_s + w_N^T P w_N$$

subject to the constraints

$$\begin{aligned}
w_1 + x_\infty &= A\hat{x} + B(v_0 + u_\infty) + B_d\hat{d} \\
w_{s+1} + x_\infty &= A(w_s + x_\infty) + B(v_s + u_\infty) + B_d\hat{d}, \quad s = 1, \dots, N-1 \\
u_{\text{low}} \le v_s &+ u_\infty \le u_{\text{high}}, \quad s = 0, \dots, N-1 \\
y_{\text{low}} \le C(w_s &+ x_\infty) + C_d\hat{d} \le y_{\text{high}}, \quad s = 1, \dots, N
\end{aligned}$$

Note that, in the above, the constraint $x_0 = \hat{x}$ was first substituted into the constraint $x_1 = Ax_0 + Bu_0 + B_d\hat{d}$ before making the change of variables.
By inspection of the cost, one gets that

$$G = 2\begin{pmatrix} R & 0 & 0 & 0 & \cdots & & \\ 0 & Q & 0 & 0 & \cdots & & \\ 0 & 0 & R & 0 & \cdots & & \\ 0 & 0 & 0 & Q & \cdots & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \\ & & & & & R & 0 \\ & & & & & 0 & P \end{pmatrix}, \quad f = 0, \quad e = (\hat{x} - x_\infty)^T Q (\hat{x} - x_\infty).$$

Next, we consider rewriting the constraints in the form

$$\begin{aligned}
-Bv_0 + w_1 &= A\hat{x} - x_\infty + Bu_\infty + B_d\hat{d} \\
-Aw_s - Bv_s + w_{s+1} &= (A - I)x_\infty + Bu_\infty + B_d\hat{d}, \quad s = 1, \dots, N-1 \\
v_s &\le u_{\text{high}} - u_\infty, \quad s = 0, \dots, N-1 \\
-v_s &\le -u_{\text{low}} + u_\infty, \quad s = 0, \dots, N-1 \\
Cw_s &\le y_{\text{high}} - Cx_\infty - C_d\hat{d}, \quad s = 1, \dots, N \\
-Cw_s &\le -y_{\text{low}} + Cx_\infty + C_d\hat{d}, \quad s = 1, \dots, N
\end{aligned}$$

By inspection of the equality constraints, one gets that

$$D = \begin{pmatrix} -B & I & 0 & 0 & 0 & 0 & \cdots \\ 0 & -A & -B & I & 0 & 0 & \cdots \\ 0 & 0 & 0 & -A & -B & I & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$g = \begin{pmatrix} A\hat{x} - x_\infty + Bu_\infty + B_d\hat{d} \\ (A-I)x_\infty + Bu_\infty + B_d\hat{d} \\ (A-I)x_\infty + Bu_\infty + B_d\hat{d} \\ \vdots \end{pmatrix}$$

Similarly, by inspection of the inequality constraints, one gets that

$$E = \begin{pmatrix} I & 0 & 0 & 0 & \cdots \\ -I & 0 & 0 & 0 & \cdots \\ 0 & C & 0 & 0 & \cdots \\ 0 & -C & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & \cdots \\ 0 & 0 & -I & 0 & \cdots \\ 0 & 0 & 0 & C & \cdots \\ 0 & 0 & 0 & -C & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad h = \begin{pmatrix} u_{\text{high}} - u_\infty \\ -u_{\text{low}} + u_\infty \\ y_{\text{high}} - Cx_\infty - C_d\hat{d} \\ -y_{\text{low}} + Cx_\infty + C_d\hat{d} \\ u_{\text{high}} - u_\infty \\ -u_{\text{low}} + u_\infty \\ y_{\text{high}} - Cx_\infty - C_d\hat{d} \\ -y_{\text{low}} + Cx_\infty + C_d\hat{d} \\ \vdots \end{pmatrix}$$

In the above, note that h is a function of u_∞, x_∞ and $\hat{d}$ and that g is a function of $\hat{x}$, u_∞, x_∞ and $\hat{d}$, hence the optimal value of the decision variable θ is a function of $\hat{x}$, u_∞, x_∞ and $\hat{d}$.

Note also that e is not a function of any of the decision variables. Hence, the value of e does not affect the optimal solution and can actually be set to zero at all times.

5. Note that there is only one state and one input in this example, hence K, P, etc., are all scalars.

   (a) Firstly, the control gain K has to be chosen such that the closed-loop system $x(k+1) = (A+BK)x(k)$ is stable, i.e.

$$\begin{aligned} \rho(A+BK) < 1 &\Leftrightarrow \rho(2+3K) < 1 \\ &\Leftrightarrow |2+3K| < 1 \\ &\Leftrightarrow -1 < 2+3K < 1 \\ &\Leftrightarrow -1 < K < -\frac{1}{3}. \end{aligned}$$

   Next, recall that the terminal cost has to be a Control Lyapunov function associated with the terminal controller, i.e. P has to be strictly positive if it is a scalar (recall that P has to be positive definite if P is a matrix) and

$$\begin{aligned} &(A+BK)^T P(A+BK) - P \leq -Q - K^T RK \\ \Leftrightarrow\ &P(2+3K)^2 - P \leq -10 - 2K^2 \\ \Leftrightarrow\ &P(9K^2 + 12K + 3) \leq -10 - 2K^2. \end{aligned}$$

In order to proceed, note that the function $f(K) = 9K^2 + 12K + 3$ has roots at -1 and $-1/3$ and that $f(K)$ is strictly less than zero over the range $-1 < K < -1/3$, hence

$$P(9K^2 + 12K + 3) \leq -10 - 2K^2 \Leftrightarrow P \geq -\frac{10 + 2K^2}{9K^2 + 12K + 3}.$$

Next, note that $-(10+2K^2)/(9K^2+12K+3)$ is also strictly greater than zero over the range $-1 < K < -1/3$. Combining this with the fact that P has to be strictly positive, it follows that

$$P \geq -\frac{10 + 2K^2}{9K^2 + 12K + 3}$$

is sufficient for the above inequality to hold if $-1 < K < -1/3$.

(b) The lower constraint on the state implies that $-8 \leq c$ and the upper constraint implies that $c \leq 10$. Combining this with the assumption that $c > 0$, it follows that $0 < c \leq 8$.

(c) We require that $-3 \leq Kx_N \leq 2$ for all x_N that satisfy $-c \leq x_N \leq c$. Since K and x_N are scalars, we only need to consider the upper and lower admissible values of the terminal state; if the input constraints are satisfied at the extreme values of the set of admissible terminal states, then the input constraints are also satisfied at intermediate values for the terminal state.

Taking the upper admissible value for the terminal state, it follows that $-3 \leq Kc \leq 2$ and the lower admissible value for the terminal state, it follows that $-3 \leq -Kc \leq 2$ or, equivalently, $3 \geq Kc \geq -2$.

Combining all of the above constraints, it follows that

$$\max\{-2, -3\} \leq Kc \leq \min\{2, 3\} \Leftrightarrow -2 \leq Kc \leq 2 \Leftrightarrow |Kc| \leq 2.$$

Under the assumption that $K \neq 0$ (if $K = 0$ then any positive value for c is admissible) , it follows that

$$0 < c \leq 2/|K|.$$

(d) We require that $-c \leq (A + BK)x_N \leq c$ for all x_N that satisfy $-c \leq x_N \leq c$. As with with part (c), since A, B and K are scalars, we need only check the extreme values of the terminal state for the system $x(k+1) = (2 + 3K)x(k)$.

Taking the upper admissible value for the terminal state, we require that $-c \leq (2+3K)c \leq c$ and the lower admissible value for the terminal state, we require that $-c \leq -(2 + 3K)c \leq c$ or, equivalently, $c \geq (2 + 3K)c \geq -c$.

Combining all of the above constraints, it follows that

$$-c \leq (2 + 3K)c \leq c \Leftrightarrow -1 \leq 2 + 3K \leq 1 \Leftrightarrow -1 \leq K \leq -\frac{1}{3}.$$

(e) Combining parts (a) and (d), it is clear that only values for K in the range $-1 < K < -1/3$ are allowed.

From part (c) it follows that, for a given c, we have to ensure that

$$|K| \leq 2/c.$$

If $c \geq 6$, then this implies that K also has to satisfy $|K| < 1/3$. This is in conflict with the constraint that $-1 < K < -1/3$, which implies that $1/3 < |K| < 1$. Hence, only values of c less than 6 are allowed, i.e.

$$0 < c < 6.$$

6. (a) $V^*(Ax+Bu_0^*(x))$ is the optimal cost if $Ax+Bu_0^*(x)$ is the current state. $\tilde{U}(x)$ is only a *candidate* input sequence and not necessarily the optimal input sequence if the current state is $Ax + Bu_0^*(x)$. An arbitrary input sequence will result in a cost that is greater or equal to the optimal cost, hence

$$V^*(Ax + Bu_0^*(x)) \leq V(Ax + Bu_0^*(x), \tilde{U}(x)).$$

(b) Since $x = x_0^*(x)$, it follows that

$$V^*(x) = x^T Qx + u_0^*(x)^T Ru_0^*(x) + x_N^*(x)^T Px_N^*(x) + \sum_{s=1}^{N-1} x_s^*(x)^T Qx_s^*(x) + u_s^*(x)^T Ru_s^*(x)$$

If we now define

$$\begin{aligned} w_0 &:= Ax + Bu_0^*(x) \\ w_{s+1} &:= Aw_s + Bv_s, \quad s = 0, \dots, N-1 \end{aligned}$$

where

$$\begin{aligned} v_s &:= u_{s+1}^*(x), \quad s = 0, \dots, N-2, \\ v_{N-1} &:= Kx_N^*(x), \end{aligned}$$

Hence,

$$\begin{aligned} & V(Ax + Bu_0^*(x), \tilde{U}(x)) = V\left(w_0, \begin{pmatrix} v_0^T & \cdots & v_{N-1}^T \end{pmatrix}^T\right) \\ &= w_N^T P w_N + \sum_{s=0}^{N-1} w_s^T Q w_s + v_s^T R v_s \\ &= w_N^T P w_N + w_{N-1}^T Q w_{N-1} + v_{N-1}^T R v_{N-1} + \sum_{s=0}^{N-2} w_s^T Q w_s + v_s^T R v_s \end{aligned}$$

Note that it is possible to verify that, with the above definitions,

$$\begin{aligned} w_s &= x^*_{s+1}(x), \quad s = 0, \ldots, N-1 \\ w_N &= (A + BK)x^*_N(x) \end{aligned}$$

Hence, after making the suitable substitutions, it follows that

$$\begin{aligned} &V(Ax + Bu^*_0(x), \tilde{U}(x)) \\ &= ((A + BK)x^*_N(x))^T P(A + BK)x^*_N(x) + x^*_N(x)^T Q x^*_N(x) \\ &\qquad + x^*_N(x)^T K^T RK x^*_N(x) + \sum_{s=0}^{N-2} x^*_{s+1}(x)^T Q x^*_{s+1}(x) + u^*_{s+1}(x)^T R u^*_{s+1}(x) \\ &= (Ax^*_N(x) + BKx^*_N(x))^T P(Ax^*_N(x) + BKx^*_N(x)) \\ &\qquad + x^*_N(x)^T Q x^*_N(x) + x^*_N(x)^T K^T RK x^*_N(x) \\ &\qquad\quad + \sum_{s=1}^{N-1} x^*_s(x)^T Q x^*_s(x) + u^*_s(x)^T R u^*_s(x) \end{aligned}$$

The claim that

$$V(Ax + Bu^*_0(x), \tilde{U}(x)) = V^*(x) + \ell(x)$$

follows by inspection.

(c) If Q and R are positive definite, then $x'Qx > 0$ for all $x \neq 0$ and $u^*_0(x)^T R u^*_0(x) \geq 0$ for all x, hence

$$\ell_1(x) := -x^T Q x - u^*_0(x)^T R u^*_0(x) \leq -x^T Q x < 0 \text{ for all } x \neq 0.$$

Next, note that

$$\begin{aligned} \ell_2(x) := &-x^*_N(x)^T P x^*_N(x) + x^*_N(x)^T Q x^*_N(x) + x^*_N(x)^T K^T RK x^*_N(x) \\ &+ (Ax^*_N(x) + BKx^*_N(x))^T P(Ax^*_N(x) + BKx^*_N(x)) \\ = &\; x^*_N(x)^T \left[(A + BK)^T P(A + BK) - P + Q + K^T RK\right] x^*_N(x). \end{aligned}$$

Hence, if

$$(A + BK)^T P(A + BK) - P + Q + K^T RK \leq 0$$

or, equivalently,

$$(A + BK)^T P(A + BK) - P \leq -Q - K^T RK.$$

then

$$\ell_2(x) \leq 0 \text{ for all } x.$$

Hence,

$$\ell(x) = \ell_1(x) + \ell_2(x) < 0 \text{ for all } x \neq 0.$$

We can now proceed to show that $V^*(\cdot)$ is a Lyapunov function for the closed-loop system. Combining part (a) with part (b) we get

$$V^*(Ax + Bu_0^*(x)) \leq V(Ax + Bu_0^*(x), \tilde{U}(x)) = V^*(x) + \ell(x)$$

Since $\ell(x) < 0$ for all $x \neq 0$, it follows that $V(Ax + Bu_0^*(x), \tilde{U}(x)) < V^*(x)$, hence

$$V^*(Ax + Bu_0^*(x)) < V^*(x) \text{ for all } x \neq 0.$$

Since P, Q and R are all positive definite, it follows that $V^*(0) = 0$ (since $U^*(0) = 0$) and

$$V^*(x) \geq x^T Q x > 0 \text{ for all } x \neq 0.$$

(Note that it is also important to show that $V^*(\cdot)$ satisfies certain continuity conditions. However, the proof of this is not straightforward. For this course, it is sufficient to just assume that $V^*(\cdot)$ has the required continuity properties.)
$V^*(\cdot)$ satisfies all the conditions to be a Lyapunov function for the closed-loop system

$$x(k+1) = Ax(k) + Bu_0^*(x(k)).$$

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