Engineering Tripos Part IIB **Module 4F3**

Nonlinear and Predictive Control

Examples Paper 4F3/2

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1. Consider the system

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k)$$

with

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}.$$

The following constraints on the state and input are given:

$$-1 \leq u_s \leq 5, \quad s = 0, 1$$
$$\begin{pmatrix} -2 \\ -1 \end{pmatrix} \leq x_s \leq \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \quad s = 1, 2$$

Compute the matrices J, c and W such that the constraints can be rewritten in the form:

$$JU \leq c + Wx(k),$$

where the current state $x_0 = x(k)$ and the column vector

$$U := \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

2. The following constraints on the rate of change of the inputs are given:

$$\begin{pmatrix} -1 \\ -2 \end{pmatrix} \leq \Delta u_s \leq \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad s = 0, 1, \ldots, N-1$$

where

$$\Delta u_s := u_s - u_{s-1}, \quad s = 0, 1, \ldots, N-1$$

and $u_{-1} := u(k-1)$ is the input at the previous sample instant.

For a horizon length $N = 3$, compute the matrices J, c and W such that the constraints can be rewritten in the form:

$$JU \leq c + Wu(k-1),$$

where the column vector $U := \begin{pmatrix} u_0^T & \cdots & u_{N-1}^T \end{pmatrix}^T$.

3. A constant, unmeasured disturbance is acting on the system considered in Question 1 with $C := \begin{pmatrix} 1 & 0 \end{pmatrix}$. The effect of the disturbance is modelled by augmenting the system as follows:

$$\begin{aligned}
\begin{pmatrix} x(k+1) \\ d(k+1) \end{pmatrix} &= \begin{pmatrix} A & B_d \\ 0 & I \end{pmatrix} \begin{pmatrix} x(k) \\ d(k) \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u(k) \\
y(k) &= \begin{pmatrix} C & C_d \end{pmatrix} \begin{pmatrix} x(k) \\ d(k) \end{pmatrix} \\
z(k) &= \begin{pmatrix} H & H_d \end{pmatrix} \begin{pmatrix} x(k) \\ d(k) \end{pmatrix}
\end{aligned}$$

In the following, I_2 is a 2×2 identity matrix and $0_{2\times 1}$ is a 2×1 zero vector:

(a) Is the augmented system detectable if the disturbance model is given by:

i. $B_d = B$ and $C_d = 0$, i.e. an input disturbance?

ii. $B_d = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$ and $C_d = 1$, i.e. an output disturbance?

iii. $B_d = I_2$ and $C_d = \begin{pmatrix} 0 & 0 \end{pmatrix}$, i.e. independent state disturbances?

(b) An input disturbance of size $d(k) = 1$ is acting on the system for all $k \geq 0$, Is it possible to have offset-free control of the controlled variable $z(\cdot)$ if we choose:

i. $H = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $H_d = 0$ and $r(k) = c$ for all $k \geq 0$, where c is any scalar?

ii. $H = I_2$, $H_d = 0_{2\times 1}$ and $r(k) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ for all $k \geq 0$?

iii. $H = I_2$, $H_d = 0_{2\times 1}$ and $r(k) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$ for all $k \geq 0$?

4. A more general form of a quadratic program (QP) than the one introduced on page 30 in Lecture 4 has linear equality constraints in addition to inequality constraints, i.e. it has the form:

$$\min_{\theta} \frac{1}{2}\theta^T G \theta + f^T \theta + e$$

subject to

$$\begin{aligned}
D\theta &= g \\
E\theta &\leq h
\end{aligned}$$

where θ, f, g and h are column vectors, G, D and E are matrices of compatible dimensions and e is a scalar.

(a) Show that the target calculation problem on page 25 of Lecture 5 can be written as a QP in the above form.

Hint: Let the decision variable be $\theta := \begin{pmatrix} x_\infty^T & u_\infty^T \end{pmatrix}^T$.

(b) Show that the regulation problem on page 26 of Lecture 5 can be written as a QP in the above form.

Hint: It is easiest to make a change of variables $v_s = u_s - u_\infty$ for $s = 0, \ldots, N-1$ and $w_s = x_s - x_\infty$ for $s = 1, \ldots, N$ and let the decision variable be

$$\theta := \begin{pmatrix} v_0^T & w_1^T & v_1^T & w_2^T & v_2^T & \cdots & w_{N-1}^T & v_{N-1}^T & w_N^T \end{pmatrix}^T.$$

5. Consider the problem of designing a state feedback receding horizon controller for the system $x(k+1) = 2x(k) + 3u(k)$.

   The constraints on the state and input are

$$\begin{aligned} &-3 \le u_s \le 2, \quad s = 0, 1, \ldots, N-1 \\ &-8 \le x_s \le 10, \quad s = 1, 2, \ldots, N-1 \\ &-c \le x_N \le c \end{aligned}$$

   where c is a strictly positive scalar.

   (a) Which range of values for the weight P in the terminal cost $x_N^T P x_N$ and gain K in the terminal control law $u = Kx$ are sufficient to ensure that the closed-loop system is stabilized by a receding horizon controller, where the state and input weights are $Q = 10$ and $R = 2$, respectively?

   (b) For which values of c is the terminal constraint state-admissible?

   (c) Given a terminal control law $u = Kx$, for which values of c is the terminal constraint input-admissible?

   (d) For which values of K is the terminal constraint invariant for the closed-loop system $x(k+1) = (A + BK)x(k)$?

   (e) Based on your answers to parts (a)–(d), for which values of c can one choose the terminal weight P such that the closed-loop system is stabilized by a constrained receding horizon controller, while also guaranteeing that the constraints are satisfied for all time?

6. Let the cost function be defined as

$$V(x, U) := x_N^T P x_N + \sum_{s=0}^{N-1} x_s^T Q x_s + u_s^T R u_s$$

   where $x_0 = x = x(k)$ is the current state and

$$x_{s+1} = Ax_s + Bu_s, \; s = 0, 1, \ldots, N-1.$$

   Let the value function be

$$V^*(x) := \min_U V(x, U)$$

and the optimal input sequence be

$$\begin{aligned} U^*(x) &:= \begin{pmatrix} u_0^*(x)^T & u_1^*(x)^T & \cdots & u_{N-1}^*(x)^T \end{pmatrix}^T \\ &:= \arg\min_U V(x, U). \end{aligned}$$

Consider also the input sequence that is constructed by shifting the optimal input sequence and appending it as follows:

$$\tilde{U}(x) := \begin{pmatrix} u_1^*(x)^T & \cdots & u_{N-1}^*(x)^T & (Kx_N^*(x))^T \end{pmatrix}^T,$$

where the terminal control law $u = Kx$ is chosen such that $A + BK$ is stable and the predicted optimal state trajectory is defined as

$$\begin{aligned} x_0^*(x) &= x \\ x_{s+1}^*(x) &= Ax_s^*(x) + Bu_s^*(x), \; s = 0, 1, \ldots, N-1. \end{aligned}$$

The optimal input sequence is implemented in a receding horizon fashion, i.e. the closed-loop system is given by

$$x(k+1) = Ax(k) + Bu_0^*(x(k)).$$

(a) Why is $V(Ax + Bu_0^*(x), \tilde{U}(x))$ an upper bound for $V^*(Ax + Bu_0^*(x))$?

(b) By noting that $V^*(x) = V(x, U^*(x))$, show that

$$V(Ax + Bu_0^*(x), \tilde{U}(x)) = V^*(x) + \ell(x)$$

where

$$\begin{aligned} \ell(x) := &- x^T Q x - u_0^*(x)^T R u_0^*(x) - x_N^*(x)^T P x_N^*(x) \\ &+ x_N^*(x)^T Q x_N^*(x) + x_N^*(x)^T K^T R K x_N^*(x) \\ &+ (Ax_N^*(x) + BKx_N^*(x))^T P (Ax_N^*(x) + BKx_N^*(x)). \end{aligned}$$

(c) Show that $\ell(x) < 0$ for all $x \neq 0$ if Q and R are positive definite and P and K are chosen to satisfy

$$(A + BK)^T P (A + BK) - P \leq -Q - K^T R K.$$

Hence, show that $V^*(\cdot)$ is a Lyapunov function for the closed-loop system if P is also positive definite.

Answers

1.

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 0.5 & 0 \\ 1 & 0 \\ 1.5 & 0.5 \\ 1 & 1 \\ -0.5 & 0 \\ -1 & 0 \\ -1.5 & -0.5 \\ -1 & -1 \end{pmatrix}, \qquad c = \begin{pmatrix} 5 \\ 5 \\ 1 \\ 1 \\ 4 \\ 3 \\ 4 \\ 3 \\ 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \qquad W = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & -1 \\ 0 & -1 \\ -1 & -2 \\ 0 & -1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}$$

2.

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}, \qquad c = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

3. (a) (i) Yes. (ii) No. (iii) No.
   (b) (i) Yes. (ii) Yes. (iii) No.

5. (a) $-1 < K < -\frac{1}{3}$ and $P \geq -(10 + 2K^2)/(9K^2 + 12K + 3)$
   (b) $0 < c \leq 8$
   (c) $0 < c \leq 2/|K|$
   (d) $-1 \leq K \leq -\frac{1}{3}$
   (e) $0 < c < 6$

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