Engineering Tripos Part IIB **Module 4F3**

Nonlinear and Predictive Control

Solutions to Examples Paper 4F3/1

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1. Covered in the lecture notes.

2. First part of question covered in the lecture notes.

   The reason why, even in the absence of disturbances, the closed-loop trajectory may not correlate with the predictions made at each time instant, is because at a given sample instant, one can see further into the future than at the previous sample instant. The last part of the input trajectory computed at the previous sample instant may therefore not be equal to the first part of the optimal input trajectory at the current sample instant.

3. (a) $\text{rank}(A) = c$, because we are told the solution exists *and* is unique — the solution, if it exists, is unique if and only if A is full column rank. Note also that this implies that $r \geq c$.

   (b) Since a solution to $Ax = b$ exists for any choice of b, it follows that A has full row rank, hence A^T has full column rank. Following the same reasoning as in the solution to Question 3(a), it follows that the solution to $A^T y = 0$ is unique ($y = 0$ is trivially a solution to $A^T y = 0$).

   (c) There are no unique solutions to this question. However, here are some simple examples:

   i.
   $$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

   $Ax = b$ has no solution if $b = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$, because $\text{rank}(A) = 2 < \text{rank}\begin{pmatrix} A & b \end{pmatrix} = 3$.

   $Ax = b$ has exactly one solution if $b = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$, namely $x = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$, because $\text{rank}(A) = \text{rank}\begin{pmatrix} A & b \end{pmatrix}$ (existence) and A is full column rank (uniqueness).

   ii.
   $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

   For any choice of vector b, $Ax = b$ has the solution $x = \begin{pmatrix} b^T & c \end{pmatrix}^T$, where c is any scalar.

iii.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

No solution exists to $Ax = b$ if $b = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$, because $\text{rank}(A) = 1 < \text{rank}\begin{pmatrix} A & b \end{pmatrix} = 2$.

$Ax = b$ has the solution $x = \begin{pmatrix} 1 & c \end{pmatrix}^T$, where c is any scalar, if $b = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$.

iv.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

A is square and invertible, hence the solution to $Ax = b$ exists, is unique and (in this case) is given by $x = b$.

4. (a) Let $K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$, then

$$A + BK = \begin{pmatrix} 1 + 0.5k_1 & 1 + 0.5k_2 \\ k_1 & 1 + k_2 \end{pmatrix},$$

Recall that the eigenvalues of $A + BK$ are the roots of the characteristic polynomial of $A + BK$, i.e. all λ such that $\det(\lambda I - A - BK) = 0$. Hence,

$$\lambda I - A - BK = \begin{pmatrix} \lambda - 1 - 0.5k_1 & -1 - 0.5k_2 \\ -k_1 & \lambda - 1 - k_2 \end{pmatrix}$$

and

$$\det(\lambda I - A - BK) = \lambda^2 + \lambda(-2 - 0.5k_1 - k_2) + (1 - 0.5k_1 + k_2).$$

The desired characteristic polynomial is

$$p(\lambda) = (\lambda - 0.8 - j0.25)(\lambda - 0.8 + j0.25) = \lambda^2 - 1.6\lambda + 0.7025.$$

Equating coefficients gives

$$0.5k_1 - k_2 = 0.2975 \text{ and } 0.5k_1 + k_2 = -0.4.$$

Solving these equations one obtains

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0.5 & -1 \\ 0.5 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0.2975 \\ -0.4 \end{pmatrix} = \begin{pmatrix} -0.1025 \\ -0.34875 \end{pmatrix} \approx \begin{pmatrix} -0.1 \\ -0.35 \end{pmatrix},$$

hence $K = \begin{pmatrix} -0.1025 & -0.34875 \end{pmatrix} \approx \begin{pmatrix} -0.1 & -0.35 \end{pmatrix}$.

(b) Since the number of states $n = 2$, the observability matrix is

$$\mathcal{O}(C, A) = \begin{pmatrix} C \\ CA \end{pmatrix}.$$

If $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$, then

$$\operatorname{rank}(\mathcal{O}(C, A)) = \operatorname{rank} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = 1 < 2 = n,$$

hence (C, A) is not observable if $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$.
If $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$, then

$$\operatorname{rank}(\mathcal{O}(C, A)) = \operatorname{rank} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 2 = n,$$

hence (C, A) is observable if $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$.
By examining the A matrix, one can see why lack of observability occurs in the former case, but not in the latter case. Note that, due to the zero in the second row and first column of A, the current value of the first state does not affect the value of the second state at the next sample instant. However, due to the non-zero value in the first row and second column of A, the current value of the second state does affect the value of the first state at the next sample instant.
If $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$, then one only measures the second state and, since the current value of the first state does not affect the value of the second state at the next sample instant, one cannot ever hope to estimate the value of the first state by only measuring the second state.
However, if $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$, then one measures the first state and, since the value of the second state does affect the value of the first state at the next sample instant, one can estimate the value of the second state after a number of sample instants by only measuring the first state.
As a further exercise: Show that, if $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$ in this example, then one only needs a measurement of the output at two consecutive sample instants in order to compute the exact value of both states at these sample instants (this is known in the literature as "dead-beat" estimation).

5. (a) The eigenvalues of A are 2 and 0.5 (recall that it is easy to show that the eigenvalues of a triangular matrix are equal to the diagonal components of the matrix). Since the first eigenvalue is outside the unit disk, the system is open-loop unstable.

   (b) Let $K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$, then

$$A + BK = \begin{pmatrix} 2 + k_1 & 1 + k_2 \\ 0 & 0.5 \end{pmatrix},$$

By inspection, the eigenvalues of $A + BK$ are $2 + k_1$ and 0.5. Since 0.5 is already in the unit disk, all one has to do to ensure that the closed-loop system is stable, is to choose k_1 such that $-1 < 2+k_1 < 1$. Note also that any value of k_2 is allowed, since the eigenvalues are not dependent on k_2. Hence, the system is stabilizable because there exists a K such that the eigenvalues of $A + BK$ are inside the unit disk.

As can be seen, since one of the eigenvalues of $A + BK$ is always at 0.5, one cannot assign the poles of the closed-loop system arbitrarily, hence the system is not reachable. This is also easily verified by considering the reachability matrix. Since the number of states $n = 2$, the reachability matrix is

$$\mathcal{C}(A, B) = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}.$$

By inspection, $\text{rank}(\mathcal{C}(A, B)) = 1 < 2 = n$, hence the system is not reachable.

(c) It follows immediately from the above that k_1 has to satisfy $-3 < k_1 < -1$ and that any value of k_2 is allowed.

6. Let

$$\hat{x}(k|k) = \hat{x}(k|k-1) + L[\hat{y}(k|k-1) - y(k)] \tag{1}$$
$$\hat{x}(k+1|k) = A\hat{x}(k|k) + Bu(k) \tag{2}$$
$$\hat{y}(k|k-1) = C\hat{x}(k|k-1) \tag{3}$$

(a) By substituting k with $k-1$ in (2), it follows that

$$\hat{x}(k|k-1) = A\hat{x}(k-1|k-1) + Bu(k-1). \tag{4}$$

By substituting (4) into (3), we get

$$\hat{y}(k|k-1) = CA\hat{x}(k-1|k-1) + CBu(k-1). \tag{5}$$

By substituting (4) and (5) into (1) and collecting terms, it follows that

$$\hat{x}(k|k) = (A + LCA)\hat{x}(k-1|k-1) + (B + LCB)u(k-1) - Ly(k).$$

Recalling that $x(k+1) = Ax(k) + Bu(k)$, $y(k) = Cx(k)$ and by substituting k

with $k+1$ where necessary, it follows that

$$\begin{aligned}
e(k+1) &= x(k+1) - \hat{x}(k+1|k+1) \\
&= Ax(k) + Bu(k) - \hat{x}(k+1|k+1) \\
&= Ax(k) + Bu(k) \\
&\quad - (A + LCA)\hat{x}(k|k) - (B + LCB)u(k) + Ly(k+1) \\
&= Ax(k) - (A + LCA)\hat{x}(k|k) - LCBu(k) + Ly(k+1) \\
&= Ax(k) - (A + LCA)\hat{x}(k|k) - LCBu(k) + LCx(k+1) \\
&= Ax(k) - (A + LCA)\hat{x}(k|k) - LCBu(k) \\
&\quad + LC(Ax(k) + Bu(k)) \\
&= (A + LCA)x(k) - (A + LCA)\hat{x}(k|k) \\
&= (A + LCA)(x(k) - \hat{x}(k|k)) \\
&= (A + LCA)e(k)
\end{aligned}$$

(b) By substituting (3) into (1) and collecting terms we get

$$\hat{x}(k|k) = (I + LC)\hat{x}(k|k-1) - Ly(k). \tag{6}$$

By substituting (6) into (2), it follows that

$$\hat{x}(k+1|k) = (A + ALC)\hat{x}(k|k-1) + Bu(k) - ALy(k).$$

Recalling that $x(k+1) = Ax(k) + Bu(k)$, $y(k) = Cx(k)$ and by substituting k with $k+1$ where necessary, it follows that

$$\begin{aligned}
e(k+1) &= x(k+1) - \hat{x}(k+1|k) \\
&= Ax(k) + Bu(k) - \hat{x}(k+1|k) \\
&= Ax(k) + Bu(k) - (A + ALC)\hat{x}(k|k-1) - Bu(k) + ALy(k) \\
&= Ax(k) - (A + ALC)\hat{x}(k|k-1) + ALCx(k) \\
&= (A + ALC)x(k) - (A + ALC)\hat{x}(k|k-1) \\
&= (A + ALC)(x(k) - \hat{x}(k|k-1)) \\
&= (A + ALC)e(k)
\end{aligned}$$

(c) If A is invertible, then A^{-1} exists. It follows that

$$A^{-1}(A + ALC)A = (I + LC)A = A + LCA.$$

This implies that $A + ALC$ and $A + LCA$ are similar, hence the eigenvalues of $A + ALC$ are equal to the eigenvalues of $A + LCA$. Hence, if $A + LCA$ is stable, then so is $A + ALC$, and vice versa.

This result implies that if the observer and error dynamics in part (a) are stable, then the observer and error dynamics in part (b) are stable, and vice versa.

NB: It is worth noting that A need not be invertible in order for the eigenvalues of $A + ALC$ to be equal to the eigenvalues for $A + LCA$. However, showing that this is true is slightly more complicated than above.

7. Since P is positive definite, $x^T Px > 0$ for all $x \neq 0$. Since Q is positive semi-definite, $y^T Qy \geq 0$ for all y. Hence, given any R of suitable dimensions, $x^T R^T QRx \geq 0$ for all x (by letting $y = Rx$ in $y^T Qy$).

   Putting these two observations together, one gets that $x^T(P + R^T QR)x = x^T Px + x^T R^T QRx = x^T Px + y^T Qy > 0$ for all $x \neq 0$ and all y such that $y = Rx$ (note that $x \neq 0$ does not imply that $y \neq 0$ for arbitrary R). Hence, $P + R^T QR$ is positive definite and invertible for any given R of suitable dimensions.

8. In this question, it is important to verify and use the fact that G is symmetric, i.e. $G = G^T$. This is because it follows, from the definition of a positive definite matrix, that P, Q and R are symmetric[1], hence Ω and Ψ are symmetric, which implies that $G = 2(\Psi + \Gamma^T \Omega \Gamma)$ is symmetric.

   (a) Let U be a vector in $\mathbb{R}^t$ (in the lecture notes, $t = Nm$), i.e.

   $$U = \begin{pmatrix} u_{\{1\}} & \cdots & u_{\{t\}} \end{pmatrix}^T,$$

   where $u_{\{i\}}$ denotes the i^{th} component of U (not to be confused with u_i, which is the predicted input at time $k + i$). Similarly, let the matrix

   $$G = \begin{pmatrix} g_{\{1,1\}} & \cdots & g_{\{1,t\}} \\ \vdots & \ddots & \vdots \\ g_{\{t,1\}} & \cdots & g_{\{t,t\}} \end{pmatrix}$$

   and the vector

   $$f = \begin{pmatrix} f_{\{1\}} & \cdots & f_{\{t\}} \end{pmatrix}^T.$$

   It follows that one can write

   $$q(U) = \frac{1}{2} \begin{pmatrix} u_{\{1\}} & \cdots & u_{\{t\}} \end{pmatrix} \begin{pmatrix} g_{\{1,1\}}u_{\{1\}} + \cdots + g_{\{1,t\}}u_{\{t\}} \\ \vdots \\ g_{\{t,1\}}u_{\{1\}} + \cdots + g_{\{t,t\}}u_{\{t\}} \end{pmatrix} + f_{\{1\}}u_{\{1\}} + \cdots + f_{\{t\}}u_{\{t\}} + c$$

[1] Confusingly, the literature is often inconsistent on the definition of a positive definite matrix. Some authors do not require a positive definite matrix to be symmetric. However, the definition that seems to be most commonly adopted requires the matrix to be symmetric (or Hermitian if it is a complex matrix).

or, equivalently,

$$\begin{aligned}
q(U) &= c + \frac{1}{2}\sum_{i=1}^{t}\left(g_{\{i,1\}}u_{\{1\}} + \cdots + g_{\{i,t\}}u_{\{t\}} + 2f_{\{i\}}\right)u_{\{i\}} \\
&= c + \frac{1}{2}\sum_{i=1}^{t}\left(2f_{\{i\}}u_{\{i\}} + \sum_{j=1}^{t} g_{\{i,j\}}u_{\{i\}}u_{\{j\}}\right) \\
&= c + \frac{1}{2}\sum_{i=1}^{t}\left(2f_{\{i\}}u_{\{i\}} + g_{\{i,i\}}u_{\{i\}}^2 + \sum_{j=1,\ j\neq i}^{t} g_{\{i,j\}}u_{\{i\}}u_{\{j\}}\right)
\end{aligned}$$

Taking special care when computing the partial derivatives gives

$$\frac{\partial q(U)}{\partial u_{\{i\}}} = \frac{1}{2}\left(2f_{\{i\}} + 2g_{\{i,i\}}u_{\{i\}} + \sum_{j=1,\ j\neq i}^{t}\left(g_{\{i,j\}} + g_{\{j,i\}}\right)u_{\{j\}}\right).$$

However, since $G = G^T$, it follows that $g_{\{i,j\}} = g_{\{j,i\}}$, hence

$$\begin{aligned}
\frac{\partial q(U)}{\partial u_{\{i\}}} &= f_{\{i\}} + g_{\{i,i\}}u_{\{i\}} + \sum_{j=1,\ j\neq i}^{t} g_{\{i,j\}}u_{\{j\}} \\
&= f_{\{i\}} + \sum_{j=1}^{t} g_{\{i,j\}}u_{\{j\}}.
\end{aligned}$$

Recall that the definition of the gradient of $q(\cdot)$ is

$$\nabla_U q(U) = \begin{pmatrix} \frac{\partial q(U)}{\partial u_{\{1\}}} \\ \vdots \\ \frac{\partial q(U)}{\partial u_{\{t\}}} \end{pmatrix}.$$

Hence,

$$\begin{aligned}
\nabla_U q(U) &= \begin{pmatrix} \sum_{j=1}^{t} g_{\{1,j\}}u_{\{j\}} \\ \vdots \\ \sum_{j=1}^{t} g_{\{t,j\}}u_{\{j\}} \end{pmatrix} + \begin{pmatrix} f_{\{1\}} \\ \vdots \\ f_{\{t\}} \end{pmatrix} \\
&= \begin{pmatrix} g_{\{1,1\}} & \cdots & g_{\{1,t\}} \\ \vdots & \ddots & \vdots \\ g_{\{t,1\}} & \cdots & g_{\{t,t\}} \end{pmatrix} \begin{pmatrix} u_{\{1\}} \\ \vdots \\ u_{\{t\}} \end{pmatrix} + \begin{pmatrix} f_{\{1\}} \\ \vdots \\ f_{\{t\}} \end{pmatrix} \\
&= GU + f
\end{aligned}$$

(b) A stationary point of a function occurs when the gradient of the function is zero, i.e. when $\nabla_U q(U) = GU + f = 0$ or, equivalently, when $GU = -f$. If G is positive definite, then G^{-1} exists and the set of linear equations $GU = -f$ has a unique solution, hence $q(\cdot)$ has a unique stationary point.

(c) By noting that $U^T G\bar{U}$ is a scalar and $G = G^T$, hence

$$U^T G\bar{U} = \bar{U}^T GU.$$

This fact is used to show that

$$\begin{aligned}
\frac{1}{2}(U-\bar{U})^T G(U-\bar{U}) &= \frac{1}{2}(U-\bar{U})^T G(U-\bar{U}) \\
&= \frac{1}{2}(U^T - \bar{U}^T)(GU - G\bar{U}) \\
&= \frac{1}{2}(U^T GU - U^T G\bar{U} - \bar{U}^T GU + \bar{U}^T G\bar{U}) \\
&= \frac{1}{2}(U^T GU - 2U^T G\bar{U} + \bar{U}^T G\bar{U}) \\
&= \frac{1}{2}U^T GU - U^T G\bar{U} + \frac{1}{2}\bar{U}^T G\bar{U}
\end{aligned}$$

Next, note that

$$\begin{aligned}
\left[\nabla_U q(\bar{U})\right]^T (U-\bar{U}) &= (U-\bar{U})^T \left[\nabla_U q(\bar{U})\right] \\
&= (U^T - \bar{U}^T)(G\bar{U} + f) \\
&= U^T G\bar{U} + U^T f - \bar{U}^T G\bar{U} - \bar{U}^T f
\end{aligned}$$

It follows trivially that

$$q(\bar{U}) = \frac{1}{2}\bar{U}^T G\bar{U} + \bar{U}^T f + c$$

The result follows by adding the above three equations together to get

$$\begin{aligned}
q(\bar{U}) &+ \left[\nabla_U q(\bar{U})\right]^T (U-\bar{U}) + \frac{1}{2}(U-\bar{U})^T G(U-\bar{U}) \\
&= \frac{1}{2}U^T GU + U^T f + c \\
&= q(U)
\end{aligned}$$

(d) Since $\bar{U}$ is a stationary point, $\nabla_U q(\bar{U}) = G\bar{U} + f = 0$, hence

$$\left[\nabla_U q(\bar{U})\right]^T (U-\bar{U}) = 0.$$

From part (c) above, it follows that

$$\begin{aligned}
q(U) - q(\bar{U}) &= \left[\nabla_U q(\bar{U})\right]^T (U-\bar{U}) + \frac{1}{2}(U-\bar{U})^T G(U-\bar{U}) \\
&= \frac{1}{2}(U-\bar{U})^T G(U-\bar{U})
\end{aligned}$$

Since G is positive definite, $\frac{1}{2}(U-\bar{U})^T G(U-\bar{U}) > 0$ for all $U-\bar{U} \neq 0$, hence

$$q(U) - q(\bar{U}) > 0$$

for all $U - \bar{U} \neq 0$ or, equivalently,

$$q(U) > q(\bar{U})$$

for all $U \neq \bar{U}$. This implies that $q(\bar{U})$ is strictly less than $q(U)$ for all $U \neq \bar{U}$, hence $\bar{U}$ is a global minimiser of $q(\cdot)$ if G is positive definite.

9. First, compute the prediction matrices

$$\Phi = \begin{pmatrix} A \\ A^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } \Gamma = \begin{pmatrix} B & 0 \\ AB & B \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 1 & 0 \\ 1.5 & 0.5 \\ 1 & 1 \end{pmatrix}.$$

Next, compute the weight matrices

$$\Omega = \begin{pmatrix} Q & 0 \\ 0 & P \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \text{ and } \Psi = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the cost matrices

$$G = 2(\Psi + \Gamma^T \Omega \Gamma) = \begin{pmatrix} 17.5 & 7 \\ 7 & 7 \end{pmatrix} \text{ and } F = 2\Gamma^T \Omega \Phi = \begin{pmatrix} 7 & 19 \\ 2 & 8 \end{pmatrix}.$$

The optimal input sequence is given by

$$U^*(x) = -G^{-1}Fx = \begin{pmatrix} -0.4762 & -1.0476 \\ 0.1905 & -0.0952 \end{pmatrix} x.$$

Since there is only one input $m = 1$, the receding horizon control law is

$$K_{\text{RHC}} = -\begin{pmatrix} 1 & 0 \end{pmatrix} G^{-1}F = \begin{pmatrix} -0.4762 & -1.0476. \end{pmatrix}$$

The closed-loop matrix is

$$A + BK_{\text{RHC}} = \begin{pmatrix} 0.7619 & 0.4762 \\ -0.4762 & -0.0476 \end{pmatrix}$$

whose eigenvalues are $0.3571 \pm j0.2508$. These eigenvalues are inside the unit disk, i.e. $\rho(A + BK_{\text{RHC}}) = 0.4363 < 1$, hence $A + BK_{\text{RHC}}$ is stable.

10. By inspection, it follows that

$$\Omega = \alpha I \text{ and } \Psi = \beta I$$

where I represents the identity matrix of appropriate dimensions. Hence,

$$G = 2(\Psi + \Gamma^T \Omega \Gamma) = 2(\beta I + \alpha \Gamma^T \Gamma) \text{ and } F = 2\Gamma^T \Omega \Phi = 2\alpha \Gamma^T \Phi.$$

The receding horizon control law is given by $K_{\text{RHC}} = -\begin{pmatrix} I_m & 0 \end{pmatrix} G^{-1} F$. Consider now

$$\begin{aligned} G^{-1}F &= (\beta I + \alpha \Gamma^T \Gamma)^{-1} \alpha \Gamma^T \Phi \\ &= \frac{\beta}{\beta}(\beta I + \alpha \Gamma^T \Gamma)^{-1} \alpha \Gamma^T \Phi \\ &= \left(\beta^{-1}\beta I + \beta^{-1}\alpha \Gamma^T \Gamma\right)^{-1} \frac{\alpha}{\beta} \Gamma^T \Phi \\ &= \frac{\alpha}{\beta}\left(I + \frac{\alpha}{\beta}\Gamma^T \Gamma\right)^{-1} \Gamma^T \Phi \end{aligned}$$

This implies that if we change α and β, but keep the ratio $\alpha : \beta$ the same, then the receding horizon control law $K_{\text{RHC}} = -\begin{pmatrix} I_m & 0 \end{pmatrix} G^{-1} F$ does not change. In other words, it is not the *absolute* values of the weights P, Q and R that determine the control law, but the *ratios* amongst the respective components.

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