Engineering Tripos Part IIB **Module 4F3**

Nonlinear and Predictive Control

Examples Paper 4F3/1

1. List a number of potential advantages and disadvantages of Predictive Control, compared to some of the control synthesis methods you may have encountered in your studies so far.

2. Give a brief description and graphical illustration of the receding horizon principle. Explain why, in receding horizon control, the closed-loop trajectory may not correlate with the predictions made at each time instant, even if there are no disturbances or plant-model mismatch.

3. (a) Suppose the only solution to the set of linear equalities $Ax = 0$ (r equalities in c unknowns) is $x = 0$. What is the rank and why?

   (b) If $Ax = b$ always has at least one solution, show that the only solution to $A^T y = 0$ is $y = 0$.

   (c) Give examples of matrices A for which the number of solutions to $Ax = b$ is

      i. 0 or 1, depending on b;
      ii. ∞, independent of b;
      iii. 0 or ∞, depending on b;
      iv. 1, regardless of b.

4. Consider the system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

with

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}.$$

   (a) Compute the state feedback control gain K by hand so that the poles of the closed-loop system $x(k+1) = (A + BK)x(k)$ are at $\lambda = 0.8 \pm j0.25$.

   (b) Show that (C, A) is not observable if $C = (0\ 1)$, but that it is observable if $C = (1\ 0)$. Explain your results, stating why the observability or lack of it occurred.

5. Consider the system $x(k+1) = Ax(k) + Bu(k)$ with

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 0.5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

   (a) Show that the system is open-loop unstable (in other words, if $u(k) = 0$).

   (b) Show that the system is stabilisable, but not reachable. Explain your results.

   (c) If we denote the controller gain $K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$, for what range of k_1 and k_2 is the closed-loop system $x(k+1) = (A + BK)x(k)$ stable?

6. Recall the observer equations on page 27 of Lecture 2:

$$\begin{aligned} \hat{x}(k|k) &= \hat{x}(k|k-1) + L[\hat{y}(k|k-1) - y(k)] \\ \hat{x}(k+1|k) &= A\hat{x}(k|k) + Bu(k) \\ \hat{y}(k|k-1) &= C\hat{x}(k|k-1) \end{aligned}$$

   (a) Show that the observer equations can be written as on page 28, i.e.

$$\hat{x}(k|k) = (A + LCA)\hat{x}(k-1|k-1) + (B + LCB)u(k-1) - Ly(k).$$

   If we define the state estimation error as $e(k) := x(k) - \hat{x}(k|k)$, show that the error dynamics is given by:

$$e(k+1) = (A + LCA)e(k).$$

   (b) Show that an alternative expression for the observer is:

$$\hat{x}(k+1|k) = (A + ALC)\hat{x}(k|k-1) + Bu(k) - ALy(k).$$

   If we define the state estimation error as $e(k) := x(k) - \hat{x}(k|k-1)$, show that the error dynamics is given by:

$$e(k+1) = (A + ALC)e(k).$$

   (c) Show that if A is invertible and $A + LCA$ is stable, then $A + ALC$ is stable. What does this observation imply about the relation between the expressions for the observer and error dynamics in parts (a) and (b)?

   Hint: Recall that a square matrix X is said to be *similar* to another square matrix Y if there exists a nonsingular matrix S such that $X = S^{-1}YS$. Recall also that the eigenvalues of similar matrices are equal.

7. Show that if P is positive definite and Q is positive semi-definite, then $P + R^TQR$ is positive definite and invertible for any given R (of appropriate dimensions).

8. We will verify the claims made on pages 20 and 21 of Lecture 3. Consider the quadratic function:

$$q(U) = \frac{1}{2}U^T G U + U^T f + c,$$

where G is a symmetric matrix, f is a vector and c is a scalar.

   (a) Show that the gradient (written as a column vector) is

   $$\nabla_U q(U) = GU + f.$$

   (b) Why does the function $q(\cdot)$ have a unique stationary point if G is positive definite?

   (c) Show that

   $$q(U) = q(\bar{U}) + \left[\nabla_U q(\bar{U})\right]^T (U - \bar{U}) + \frac{1}{2}(U - \bar{U})^T G (U - \bar{U})$$

   for any $U \neq \bar{U}$.

   (d) Show that if $\bar{U}$ is a stationary point of $q(\cdot)$ and G is positive definite, then show that $q(\bar{U})$ is the global minimum of $q(\cdot)$.

9. Consider the system in Question 4 and let the control horizon $N = 2$ and the weights $P = 2I$, $Q = I$ and $R = 1$.

   Using the procedure described in Lecture 3, compute the (unconstrained) RHC state feedback gain K_{RHC} by hand. Is the matrix $A + BK_{\text{RHC}}$ stable?

10. Let $P = Q = \alpha I$ and $R = \beta I$, where α and β are strictly positive scalars. Show that, for an arbitrary system $x(k+1) = Ax(k) + Bu(k)$, the value for the RHC gain matrix K_{RHC} does not change if we change α and β while keeping the ratio $\alpha : \beta$ and horizon length N constant.

    Hint: Examine how α and β enter in the expressions for F and G in Lecture 3.

Answers

4. (a) $K = \begin{pmatrix}-0.1 & -0.35\end{pmatrix}$.

5. (c) $-3 < k_1 < -1$ and for any choice of k_2.

9. $K_{\text{RHC}} = \begin{pmatrix}-0.4762 & -1.0476\end{pmatrix}$. The matrix $A + BK_{\text{RHC}}$ is stable.

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