A Balanced View of System Identification

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Abstract

A new form of balanced realisation appropriate to minimum-phase systems is developed and ...

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I hereby declare that this dissertation is not substantially the same as any that I have submitted for a degree or diploma or any other qualification at any other University. I further state that no part of this dissertation has already been or is being concurrently submitted for any such degree, diploma or other qualification. This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration.

Chapter 1

Introduction and Overview

1.1 Introduction

System identification forms part of the statistical sciences and systems theory which describe the construction and estimation of models of processes which evolve over time. The unidirectional flow of time imposes special constraints on the models used, such as causality, stability or Markov properties.

One useful type of model for deterministic and stochastic processes (discussed in Chapter 2) is a state-space model, because it captures the directional nature of time and incorporates the concept of a summary of the past behaviour of the system relevant to the future, i.e. the state.

One particular type of state-space realisation for asymptotically stable systems (introduced by Moore [92]) and widely applied in the control literature is the Lyapunov balanced realisation. These realisations allow implementation of a simple model reduction scheme by neglecting parts of the state seen as least important.

Several state-space models may represent the same process (shown in §2.2.1) but a canonical form may be used to avoid this non-unique representation (see §4.1 for details). Balanced realisations provide a starting point for the construction of canonical forms, as demonstrated by Maciejowski and Ober [81, 82, 96, 95, 98] and [90], and are thus an appropriate representation for system identification.

This thesis continues the efforts of Maciejowski and Ober in the application of balanced realisations to system identification and considers aspects of modelling systems and their estimation in connection with balanced realisations. The thesis provides a balancing scheme and a canonical form for minimum-phase systems. By exploiting relationships between several canonical forms it is established that the parameters of this canonical form may be estimated from observed data. Algorithms for such estimation are examined in some detail in Chapters 6 and 7.

1.2 Identification Schema

An approach to system identification is described here as four steps required to delineate some of the important tasks in the modelling procedure. This method cannot specify which aspects of the process under consideration are to be modelled or evaluate the adequacy of the results of the modelling procedure since these tasks require situation-specific knowledge. The general steps described may be repeated as more knowledge of the system is aquired. However, use of the results from previous steps may introduce statistical bias and so more care is then required in interpreting the final results.

This scheme assumes that the aim of the identification is a parametric description of the system under study. This is the usual approach, but non-parametric methods may also be applied in this area. One point of note is that the methods described in Chapters 6 and 7 are essentially non-parametric methods from which parametric information may be extracted. It is also assumed that the system is to be described by a set of differential equations. However, much of the following schema would hold for other descriptions e.g. partial-differential or integro-differential equations. Finally, the identification schema provides a rationale for the order of the Chapters in this thesis.

The schema consists of model class selection, parametrisation of the model class chosen, estimation of the initial parameter values, and finally minimisation of an appropriate loss function to obtain parameters best describing the available data.

1) **Model Class Selection**

Model class selection is merely the specification of a mathematical description of a set of models. Specification of a set of models requires incorporation of all of the *a priori* information available about the system to be identified. If little information is available in advance, only gross properties of the system may be specified e.g. stability, hyperstability, minimality of phase, losslessness. If more is known this should be used to restrict these broad classes to some subset e.g. if a bound is available as the frequency response of the process, or the static gain is known.

No particular application is motivating this study and therefore the approach taken here is to examine general classes of system. Ober, in the papers mentioned above, has already shown that many generic classes used in control theory may be described using variants of balanced realisations.

For the purposes of the predictor error method (see §1.4) it is also useful to provide a description of the set of predictors for the models in the selected set.

2) **Parametrisation**

Definition 1.1 (Parametrisation) *A* parametrisation *of a set M is a bijective map*

$$\rho : A \to M,$$

where $A \subseteq \mathbb{Z}^k \times \mathbb{R}^l$ for some $k, l \in \mathbb{N}$.

■

Having found a class of models and predictors for the system under consideration, it is useful to be able to describe these systems in terms of a set of real and integer parameters. Parametrisations provide such a convenient representation and facilitate the optimization procedure in 4). A parametrisation may also aid the understanding of the underlying system, because the parameters themselves may correspond to important properties of the system, as shown in Chapters 3, 4 and 5. This interpretative function is described in §5.2.1, which in turn is the basis for the estimation schemes described in Chapters 6 and 7.

3) **Estimation of Initial Parameter Values**

Except in the simplest of cases (e.g. ARX models, see §7.1.1), or where very strong *a priori* information is available (see Åström [9]), immediate resort to a numerical optimization procedure may not produce useful results due to the existence of local minima of the prescribed objective function (see §1.4). It is therefore useful to obtain initial estimates of the parameters from some other method, such as those described in Chapters 6 and 7, before passing to stage 4).

Hendry's paper [57] contains results showing that a consistent initial estimation followed by some maximum likelihood equivalent method has the same asymptotic properties as the ideal maximum likelihood scheme i.e. asymptotic efficiency. It is this consistency property which is pursued in Chapters 6 and 7 for the initial estimation procedures they present.

4) **Minimization of the Loss Function**

Finally, it is attempted to find the model, in the class specified in 1) above, which best fits the data. The measurement of goodness of fit will be given in terms of a loss functional such as the likelihood or predictor error (see §1.4). The main peculiarity of these numerical optimisation problems is that the loss functional, for finite data, is a random variable and is thus dependent upon the realisation observed. A straightforward guide to numerical optimization problems is found in Soderstrom and Stoica [116] Chapter 7.

It is assumed in this thesis that the modeller has, *a priori*, very little structural information regarding the process being modelled, and therefore has to search among a large class of possible models.

1.3 Plan of the Thesis

The material in this thesis corresponds approximately to the preceding schema: Chapters 2 and 3 describe the model classes to be used later, Chapter 4 provides the parametrisations of these models and Chapters 6 and 7 provides an approach to initial estimation schema. Chapter 5 describes some key relationships particular to this approach.

Chapter 2 presents most of the technical material and notation needed in the subsequent Chapters. A brief exposition of the theory of deterministic systems and stationary stochastic processes is presented, and the important relationships with Hankel operators are emphasized. The close relationship between the state-space realisations and Hankel operators is also drawn out, in particular the relationship between the spectral properties of Hankel operators with symmetric symbols and the state-space realisation of the symbols.

Special attention is given to the canonical correlation Hankel operator associated with a time-series. Various decompositions of this operator are obtained which are basic to much of the work in Chapters 6 and 7. The relationships between this Hankel operator and some quantities found in information theory are also discussed.

In Chapter 3 the first step in the identification schema is taken. After a brief discussion of the various possible model types and some of their respective merits, model classes are derived and described for two commonly used classes of stochastic process and associated predictor processes.

Chapter 4 introduces a novel balancing schema for minimum-phase transfer functions. Its close relationship to that class of systems is demonstrated, and the standard results on existence, uniqueness and invariance of the balancing scheme are proved. The classes of symmetric positive-real and symmetric stable transfer functions are also considered. Canonical forms for these classes are derived which incorporate some of the important characteristics of the symmetry induced in the state-space. A similar stepwise methodology for devising canonical forms is applied to all three classes.

While the canonical form for minimum-phase systems exhibits no symmetry of its own, it is possible to show a relationship between the canonical form of the minimum-phase spectral factor and the spectral summand of a symmetric spectral-density. This shows how the sign symmetry matrix S appears in a simplified canonical form for the minimum-phase transfer function. This relationship is described in §4.4.4.

Previously defined balancing schemes such as that proposed by Moore [92] have an in-built model reduction scheme. What facilitates this is a property discussed in §4.3.4 named *reducibility*. In Proposition 4.28 this property is demonstrated for the minimum-phase balancing scheme. In §4.5 relative $\mathcal{L}^\infty$-error bounds are obtained for this model reduction scheme by relating the balancing scheme to the stochastic balancing method (see that section for references).

The results of Chapter 5 complement many of the familiar (e.g. Faurre [31]) relationships between spectral factors and spectral summands. It is shown that each of the spectral summands, the left and right spectral factors and the phase factors (for both discrete and continuous time) may be simultaneously balanced, although each with respect to a different balancing scheme. The spectral factors are balanced as described in Chapter 4. They are closely related one to another and this may be expressed in terms of the state-space realisations of the various transfer functions. In the case where the spectral-density is symmetric these state-space relationships may be exploited to show that all of the realisations share the same symmetry properties.

In conjunction with the relationships between state realisations and Hankel operators, the relationships between the various canonical forms allow their parameters to be connected to input-output properties of linear systems. In particular, these connections may be exploited to show that the parameters of the canonical forms of Chapter 4 are related to properties of the spectrum and Schmidt vectors of the canonical correlation Hankel operator derived from the underlying stochastic process.

Chapter 6 presents an attempt to construct a complete algorithm for the implementation of the third step of the proposed identification schema for the time-series models described in Chapter 3. Motivated by the relationships derived in Chapter 5 the algorithm is based upon the estimation of the canonical correlation Hankel operator, which should provide some information about the model parameters.

Chapter 6 reviews some necessary spectral approximation results and describes three algorithms of sequentially increasing implementability. A variety of results are obtained for the algorithms, including the consistency of the idealised and finite-section algorithms. However, it has not been possible to derive consistency for the fully implementable algorithm due to the relative weakness of available results on spectral factorization algorithms. Some results on the behaviour of Wilson's algorithm for spectral factorization are presented, but these are not sufficient for the required purpose.

Chapter 7 describes and analyses an initial estimation scheme analogous to that of Chapter 6, but for the more general class of ARMAX systems (see Chapter 3 for the definition). This algorithm avoids spectral factorization by using some powerful results of Ljung and Wahlberg regarding the asymptotic properties of the ARX estimation method.

In essence the methods described in Chapters 6 and 7 are non-parametric methods from which parametric information may be obtained via the relationships established in Chapter 5.

The Appendices contain the details of two non-symmetric canonical forms related to those treated in the main text, and the calculations of higher order moments.

1.4 Remaining Issues

The important issues of parameter optimization and model order selection are not covered in the remainder of this thesis. Parameter optimization is used to give the best available representation of the data with respect to some relevant criterion. Model order selection determines how complex the best model should be.

1.4.1 Parameter Optimization

The issue of parameter optimization, mentioned as 4) above and not covered in this thesis, arises from the consideration of the best specification of the model in terms of some measure of "goodness of fit" or "suitability of representation." In many cases, the mathematical problem derived lacks any analytical solution, and so the model is only implicitly defined by this optimization problem. More specifically, both of the main established methods for approaching system identification (the Maximum Likelihood method and the Minimum Predictor-Error Method) have solutions which may only be stated implicitly as solutions to optimizations.

The Maximum Likelihood method has been applied to various dynamical systems since Mann and Wald [83] investigated autoregressive models. There is a huge literature on these methods and almost any textbook contains a discussion of them (e.g. Hannan and Deistler [46]). As with all maximum likelihood approaches the model chosen from some parametrised class is that which maximises the likelihood function at the given realisation of the observed series. This likelihood function, in the Gaussian case, may be written as

$$L_N\left((y(i))_{i=1}^N ; (u(i))_{i=1}^N , \theta\right) = \frac{1}{N}\sum_{i=1}^N \log\det \Sigma_{i|i-1}(\theta) + \mathrm{tr}\,(y(i)-\hat{y}(i))\,\Sigma_{i|i-1}(\theta)^{-1}(y(i)-\hat{y}(i))^T \tag{1.1}$$

where $y(i)$ is the observed series, $\hat{y}(i)$ is the one-step-ahead predictor (optimal in least squares sense) given the model specified by θ, $\Sigma_{i|i-1}(\theta)$ is the covariance of the predictor error, and $u(i)$ is a deterministic input sequence. Let $\tilde{\theta}_N$ be the parameter vector at which the minimum of L_N occurs (assuming that such exists).

Apart from the specifically statistical concerns of whether the $\tilde{\theta}_N$ specified by this method converges to the true value (again assuming this exists) and the asymptotic distribution of these parameters, $\tilde{\theta}_N$ must be obtained at each step via some optimization procedure. As well as the direct approach of optimizing the above functional, many indirect and more computationally efficient methods have been suggested. Most notable are those by Mayne and Firoozan [88], Åström and Mayne [8] and Hannan and Rissanen [48].

The Minimum Predictor-Error method chooses the best predictor from a parametrised set such that it minimises

$$L'_N\left((y(i))_{i=1}^N ; (u(i))_{i=1}^N , \theta\right) = \frac{1}{N}\sum_{i=1}^N \|y(i)-\hat{y}(i;\theta)\|_2^2 . \tag{1.2}$$

As an alternative to attempts to approximate the optima of these two problems, approximate problems which yield exact analytic solutions may be constructed e.g. the Whittle likelihood or Hankel norm techniques e.g. Gombani & Pavon [40].

The methods involved in the direct solution of these problems (Maximum Likelihood method and the Minimum Predictor-Error method) are, in the main, straightforward applications of well-known optimization techniques. The optimisaion technique chosen depends largely upon the smoothness of the criterion and of the parametrisation. The parametrisations derived in later Chapters may be used to provide optimisation algorithms to solve these problems. However, such results are not original and would deal with concerns not considered elsewhere in this thesis e.g. numerical efficiency and accuracy. For this reason these results are not pursued.

1.4.2 Model Order Estimation

The remaining issue of model order estimation is usually addressed via an information criterion or via a series of hypothesis tests.

Information criteria are loss functions formed from L_N above (see (1.1)) by the addition of a term proportional to a function $\alpha(N)$ of the number of data points, and the number of continuous parameters involved in the model. This extra term may be considered a penalty term on the number of parameters used in the model. It has been shown for various functions α that the associated optimization schemes are strongly consistent i.e. they are asymptotically minimised at the true model order (if such exists). These results and many others are discussed in Veres [119], which provides an excellent summary of this area.

While information criteria may provide asymptotic strong consistency of the order estimate, for finite data they are not necessarily the most powerful tests available. The alternative is to use a form of hypothesis test based on statistics such as the Lagrange multiplier, the likelihood ratio or Pierce-Box-Ljung statistics (see p150 of Veres [119]). A related approach, which might be usefully connected to the approach taken in Chapters 6 and 7 involves testing how many canonical correlation coefficients are non-zero (see Tsay and Tiao [117]). While occasionally useful, such methods have the drawback that there is no natural candidate for an algorithm combining a sequence of such tests to provide an overall consistent scheme. This point is discussed in Akaike [2].

Chapter 2

Deterministic and Stochastic Models

In this Chapter some of the relevant theory of deterministic system theory and stationary stochastic processes is reviewed. For a fuller treatment which lays the appropriate groundwork Caines [16] is an excellent source, especially the Appendices.

The major technical tool used in the following sections is Hilbert space. Young's book [135] is an excellent introduction to this concept from a system theoretic point-of-view. This is a rich and interesting area, but most of the interesting subtleties are avoided in what follows by the assumption that the systems under consideration have rational transfer functions.

2.1 Preliminaries

Before proceeding with the major concerns of this Chapter, it is necessary briefly to introduce some of the notation used subsequently. A full list of symbols is found in Appendix C, and full treatment of the functional analytic and operator theoretic concepts used below may be found in Kato [75] or Dunford & Schwartz [26, 27, 28].

$\mathbb{Z}, \mathbb{Z}_+, \mathbb{Z}_-, \mathbb{R}, \mathbb{C}, \mathbb{C}_-, \mathbb{C}_+, \mathbb{T}, \mathbb{D}, i\mathbb{R}$ are respectively the set of integers, positive integers, non-positive integers, real line, complex plane, open left half-plane, open right half-plane, unit circle, open unit disc and imaginary axis. $\mathbb{R}^{p\times q}$ and $\mathbb{C}^{p\times q}$ are the sets of real and complex $p \times q$ matrices. If A is such a matrix: its (i,j)th element will be denoted $[A]_{ij}$; its transpose by A^T; inverse by A^{-1}; and conjugate transpose by A^*. A^{-T} and A^{-*} are self-evident combinations of these basic operations. If A is square, i.e. $p = q$, then tr A denotes the trace of A, and det A denotes its determinant. $0_{p\times q}$ and I_p are the zero matrix and the identity matrix of appropriate dimension.

In the case where A is an operator (all operators are assumed linear) then A^{-1} is the algebraic inverse (where this exists) and A^* is the adjoint.

The eigenvalues of a matrix (or compact operator) A will be ordered in decreasing order of magnitude and increasing phase. The *spectrum* (in this case merely the set of eigenvalues) will be denoted $\lambda(A) := (\lambda(A)_k)_{k=1}^N$ where N is the number of eigenvalues of A (possibly infinite) counting multiplicities, with each eigenvalue appearing a number of times equal to its multiplicity.

The *singular values* of a matrix (or compact operator) A will be ordered in decreasing order of magnitude and are defined as

$$\sigma_k(A) := \sqrt{\lambda_k(A^*A)}.$$

The norms on matrices will be

$$\begin{aligned}
\|A\|_1 &:= \sum_{i=1}^{p}\sum_{j=1}^{q}\left|[A]_{ij}\right| \\
\|A\|_2 &:= \left(\sum_{i=1}^{p}\sum_{j=1}^{q}\left|[A]_{ij}\right|^2\right)^{\frac{1}{2}} \\
\|A\|_p &:= \left(\sum_{i=1}^{p}\sum_{j=1}^{q}\left|[A]_{ij}\right|^p\right)^{\frac{1}{p}}
\end{aligned}$$

and

$$\|A\|_\sigma := \sigma_1(A).$$

For operators in general

$$\|A\|_{op} := \sigma_1(A),$$

which, of course is redundant. The purpose of this redundant notation is to make clear whether the object under consideration is being considered as a matrix or operator.

A sequence of objects e.g. vectors, matrices or random variables, with symbol y, and elements indexed by the set I will be written $(y(t))_{t\in I}$.

The evolution of discrete-time systems considered below will be represented by elements from the sequence spaces

$$l^2_{p\times q} := \left\{ u = (u(t))_{t\in\mathbb{Z}} \mid u(t) \in \mathbb{R}^{p\times q}, \text{ and } \left\|(u(t))_{t\in\mathbb{Z}}\right\|_2 < \infty \right\} \tag{2.1}$$

where

$$\begin{aligned} \left\|(u(t))_{t\in\mathbb{Z}}\right\|_2^2 &:= \sum_{t=-\infty}^{\infty} \|u(t)\|_2^2 \\ &= \sum_{t=-\infty}^{\infty} \operatorname{tr} u(t)u(t)^*, \end{aligned} \tag{2.2}$$

which is the norm induced by the inner product given, for $(u(t))_{t\in\mathbb{Z}}, (v(t))_{t\in\mathbb{Z}} \in l^2_{p\times q}$, by

$$\left\langle (u(t))_{t\in\mathbb{Z}}, (v(t))_{t\in\mathbb{Z}} \right\rangle_{l^2_{p\times q}} := \sum_{t=-\infty}^{\infty} \operatorname{tr} u(t)v(t)^*.$$

Analogously, the spaces $l^r_{p\times q}$, $r \in [1,\infty)$ may be defined in terms of the sequence norms

$$\left\|(u(t))_{t\in\mathbb{Z}}\right\|_r := \left(\sum_{t=-\infty}^{\infty} \|u(t)\|_r^r \right)^{\frac{1}{r}} \tag{2.3}$$

although these lack the inner-product structure of $l^2_{p\times q}$. Moreover, $l^r_{p\times q}(I)$ may be defined for any countable set I in a similar manner.

Given $u = (u(t))_{t\in\mathbb{Z}} \in l^s_{p\times q}$, $s \in [1,\infty)$, and $v = (v(t))_{t\in\mathbb{Z}} \in l^2_{q\times r}$ then the *convolution* of these sequences, written

$$y = u * v$$

is a sequence $y = (y(t))_{t\in\mathbb{Z}}$ given by

$$y(t) = \sum_{s=-\infty}^{\infty} u(s)v(t-s). \tag{2.4}$$

where this sum converges. The sequence $(y(t))_{t\in\mathbb{Z}}$ will not in general lie in $l^2_{p\times r}$.

An important space for representing discrete-time linear systems is

$$\mathcal{L}^\infty_{p\times q}(\mathbb{T}) := \left\{ f : \mathbb{T} \to \mathbb{C}^{p\times q} \mid f \text{ is measurable }, \ \bar{f}(e^{i\theta}) = f(e^{-i\theta}) \text{ and } \|f\|_\infty < \infty \right\} \tag{2.5}$$

where

$$\|f\|_\infty := \operatorname{ess}\sup_{z\in\mathbb{T}} \sigma_1(f(z)).$$

If $f \in \mathcal{L}^\infty_{p\times q}(\mathbb{T})$ then it has a Fourier representation (see Champeney [17] for an extensive treatment of such representations)

$$f(e^{i\theta}) = \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ik\theta}$$

where $\hat{f} = \left(\hat{f}(k)\right)_{k\in\mathbb{Z}} \in l^2_{p\times q}$ and $\hat{f} * u \in l^2_{p\times r}$, $\forall u \in l^2_{q\times r}$, $\forall r \in \mathbb{N}$ (see Caines [16] pp.74–77 for a system theoretic application of this result).

Another space used to represent systems is the Hardy space

$$\mathcal{H}^{\infty}_{p\times q}(\mathbb{D}) := \left\{ f : \mathbb{D} \to \mathbb{C}^{p\times q} \mid f \text{ is analytic in } \mathbb{D},\ \bar{f(z)} = f(\bar{z}) \text{ and } \|f\|_{\infty} < \infty \right\} \tag{2.6}$$

where the appropriate norm in this circumstance is

$$\|f\|_{\infty} := \sup_{0\leqslant r<1} \|f_r\|_{\infty}$$

with $f_r : \mathbb{T} \to \mathbb{C}^{p\times q}$ and $f_r(e^{i\theta}) = f(re^{i\theta})$. Other Hardy spaces may be defined by

$$\mathcal{H}^{s}_{p\times q}(\mathbb{D}) := \left\{ f : \mathbb{D} \to \mathbb{C}^{p\times q} \mid f \text{ is analytic in } \mathbb{D},\ \bar{f(z)} = f(\bar{z}) \text{ and } \|f\|_{s} < \infty \right\} \tag{2.7}$$

where

$$\|f\|_{s} := \sup_{0\leqslant r<1} \|f_r\|_{s}.$$

For continuous-time systems, their evolution will be described by elements of the space of square Lebesgue-integrable functions

$$\mathcal{L}^2_{p\times q}(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{R}^{p\times q} \mid f \text{ is integrable and } \|f\|_2 < \infty \right\} \tag{2.8}$$

where

$$\|f\|_2 := \left(\int_{-\infty}^{\infty} \|f(t)\|_2^2 \, dt \right)^{\frac{1}{2}}$$

which is the norm induced by the inner-product given, for $f,\ g \in \mathcal{L}^2_{p\times q}(\mathbb{R})$, by

$$\langle f, g \rangle_{\mathcal{L}^2_{p\times q}(\mathbb{R})} := \operatorname{tr} \int_{-\infty}^{\infty} f(t)^* g(t) \, dt. \tag{2.9}$$

The variety of 2-norms will be distinguished by the objects upon which they operate.

Such spaces may be defined analogously using other domains e.g. $\mathcal{L}^2_{p\times q}(i\mathbb{R})$, $\mathcal{L}^2_{p\times q}([0,\infty))$ and $\mathcal{L}^2_{p\times q}((-\infty, 0])$. The space $\mathcal{L}^2_{p\times q}(\mathbb{T})$ is defined in a similar manner, but with the integral taken round the unit circle. Hence

$$\mathcal{L}^2_{p\times q}(\mathbb{T}) := \left\{ f : \mathbb{R} \to \mathbb{C}^{p\times q} \mid f \text{ is integrable},\ \bar{f(z)} = f(\bar{z}), \text{ and } \|f\|_2 < \infty \right\} \tag{2.10}$$

where $\|f\|_2$ is that induced by

$$\langle f, g \rangle_{\mathcal{L}^2_{p\times q}(\mathbb{T})} := \frac{1}{2\pi} \operatorname{tr} \int_{0}^{2\pi} f(e^{i\theta})^* g(e^{i\theta}) \, d\theta. \tag{2.11}$$

The spaces $\mathcal{L}^r_{p\times q}$ may be defined in terms of the norms

$$\|f\|_r := \left(\int_{-\infty}^{\infty} \|f(t)\|_r^r \, dt \right)^{\frac{1}{r}} \tag{2.12}$$

though these lack the inner-product structure. The spaces $\mathcal{L}^p_{p\times q}(\mathbb{T})$ are defined in an analogous manner.

Given $u \in \mathcal{L}^2_{p\times q}(\mathbb{R})$ and $v \in \mathcal{L}^2_{q\times r}(\mathbb{R})$ then the *convolution* of these functions, written

$$y = u * v$$

is a function $y : \mathbb{R} \to \mathbb{R}^{p\times r}$ which, from Theorem 8.14 Rudin [113], is almost everywhere defined by

$$y(t) = \int_{-\infty}^{\infty} u(s) v(t-s) \, ds. \tag{2.13}$$

This function is not necessarily an element of $\mathcal{L}^2_{p\times r}(\mathbb{R})$.

Continuous-time systems will be represented by elements of

$$\mathcal{L}^\infty_{p\times q}(i\mathbb{R}) := \left\{ f : i\mathbb{R} \to \mathbb{C}^{p\times q} \mid f \text{ is measurable, } \bar{f(i\omega)} = f(-i\omega) \text{ and } \|f\|_\infty < \infty \right\} \tag{2.14}$$

where

$$\|f\|_\infty := \text{ess} \sup_{z\in i\mathbb{R}} \sigma_1(f(z)).$$

As in the discrete-time case, there is a corresponding Hardy space associated with $\mathcal{L}^\infty_{p\times q}(i\mathbb{R})$ namely

$$\mathcal{H}^\infty_{p\times q}(\mathbb{C}_+) := \left\{ f : \mathbb{C}_+ \to \mathbb{C}^{p\times q} \mid f \text{ is analytic in } \mathbb{C}_+,\ \bar{f(z)} = f(\bar{z}), \text{ and } \|f\|_\infty < \infty \right\} \tag{2.15}$$

where

$$\|f\|_\infty := \sup_{x>0} \left\| f_{(x)} \right\|_\infty \tag{2.16}$$

where $f_{(x)} : i\mathbb{R} \to \mathbb{C}^{p\times q}$ and $f_{(x)}(iy) := f(x+iy)$. Other Hardy spaces may be defined by

$$\mathcal{H}^s_{p\times q}(\mathbb{C}_+) := \left\{ f : \mathbb{C}_+ \to \mathbb{C}^{p\times q} \mid f \text{ is analytic in } \mathbb{C}_+ \text{ and } \|f\|_s < \infty \right\} \tag{2.17}$$

where

$$\|f\|_s := \sup_{x>0} \left\| f_{(x)} \right\|_s.$$

Hardy spaces on the left half-plane may be defined *mutatis mutandis.*

Finally, the *Laplace transform* relates the time domain description to a *frequency domain* description of the system in terms of element of $\mathcal{L}^2(i\mathbb{R})$

$$\begin{aligned} \text{Lap}_{p\times q} : \mathcal{L}^2_{p\times q}(-\infty,\infty) &\to \mathcal{L}^2_{p\times q}(i\mathbb{R}) \\ \text{Lap}_{p\times q} : \hat{f}(t) &\mapsto f(s) = \int_{-\infty}^{\infty} e^{-st}\hat{f}(-t)\,dt \\ &= \int_{-\infty}^{\infty} e^{st}\hat{f}(t)\,dt \end{aligned} \tag{2.18}$$

This has a different sign from the Laplace transform found in many engineering texts. It is a unitary isomorphism.

2.2 Deterministic Models

The following subsections consider systems for which the influence of a known external input sequence completely determines the output. This, of course, ignores the effects of unknown inputs, disturbances and errors. These will be dealt with by the models in the next section.

The main purpose of this section is to establish what type of signals are being modelled, the type of models of the system's action, and the variety of representations of those systems i.e. convolution, operator, and state-space representations.

The system theory presented below is well known, with many good references available. A compact and thorough presentation from a less operator based point-of-view is found in Appendix B of Caines [16]. Definitions of common system theoretic terms used below may be found there.

2.2.1 Discrete-Time Models

A *deterministic system* will be taken to be a system such that the output is specified by a function on the set of possible inputs.

The systems considered here will have input sequences $u = (u(t))_{t\in\mathbb{Z}}$ in the Hilbert space $l^2_{m\times 1}$.

Let the *shift operator* or *time advance operator* on $l^2_{p\times q}$ be defined as

$$\begin{aligned} S_{p\times q} : l^2_{p\times q} &\rightarrow l^2_{p\times q} \\ S_{p\times q} : (u(t))_{t\in\mathbb{Z}} &\mapsto (v(t))_{t\in\mathbb{Z}} \end{aligned}$$

where

$$v(t) = u(t+1).$$

A *linear, time-invariant system* with m inputs and p outputs is a bounded linear operator

$$T : l^2_{m\times 1} \rightarrow l^2_{p\times 1}$$

which intertwines the shift operator i.e.

$$S_{p\times 1}T = TS_{m\times 1}.$$

It is well known that such an operator is a *convolution operator* (e.g. see Chapter 9 Theorem 1.1 p.235 Foias and Frazho [32] for this result) which is specified by some sequence $\hat{\phi} = \left(\hat{\phi}(t)\right)_{t\in\mathbb{Z}}$, $\hat{\phi}(t) \in \mathbb{C}^{p\times m}$ by

$$\begin{aligned} \mathbf{C}_{\hat{\phi}} : l^2_{m\times 1} &\rightarrow l^2_{p\times 1} \\ \mathbf{C}_{\hat{\phi}} : u &\mapsto \hat{\phi} * u. \end{aligned}$$

For such an operator to be bounded, it is known that $\hat{\phi}$ must be the Fourier sequence of

$$\phi(e^{i\theta}) = \sum_{k=-\infty}^{\infty} \hat{\phi}(k)e^{ik\theta} \in \mathcal{L}^{\infty}_{m\times p}(\mathbb{T}).$$

This symbol ϕ is called the *transfer function* of the system, and is sometimes referred to as the system itself. The symbol is part of the *frequency domain* description of the system which will be considered in §2.2.4.

To introduce the notion of non-anticipativeness, define the truncation operators

$$\begin{aligned} \mathbf{Trunc}^s_{p\times q} : l^2_{p\times q} &\rightarrow l^2_{p\times q} \\ \mathbf{Trunc}^s_{p\times q} : (u(t))_{t\in\mathbb{Z}} &\mapsto (y(t))_{t\in\mathbb{Z}} \end{aligned}$$

where

$$y(t) = \begin{cases} u(t) & t \leqslant s \\ 0_{p\times q} & t > s \end{cases}.$$

A linear time-invariant system T is said to be *non-anticipative* if

$$\mathbf{Trunc}^s_{p\times 1} T \mathbf{Trunc}^s_{m\times 1} = \mathbf{Trunc}^s_{p\times 1} T, \ \forall s \in \mathbb{Z}.$$

This may be interpreted as implying that the input sequence beyond time s is irrelevant to the output on or before time s. This condition implies, from (2.4), that the symbol of the system is of the form

$$\phi(e^{i\theta}) = \sum_{k=0}^{\infty} \hat{\phi}(k)e^{ik\theta}. \tag{2.19}$$

Note 2.1 *Such a transfer function is often termed* stable. *This property has been obtained directly from the assumption of the boundedness of the operator T.*

■

In this case the symbol may be extended to a function inside the unit disc

$$\phi(z) = \sum_{k=0}^{\infty} \hat{\phi}(k) z^k$$

which is easily seen to be analytic and bounded in the $\mathcal{H}^{\infty}_{m\times q}$-norm.

If the symbol ϕ of a linear, time-invariant, non-anticipative system is a rational matrix then the system is said to be *finite-dimensional.* In this case it has been shown (Chapter 10 Kalman *et al* [73]) that the symbol has a *state-space realisation of order* n

$$\begin{aligned} \phi(z) &= C(z^{-1}I_n - A)^{-1}B + D \\ &= D + C\left(\sum_{k=0}^{\infty} z^k A^k\right) B \end{aligned} \tag{2.20}$$

for some $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$, and is written $\phi \stackrel{s}{=} (A, B, C, D)$. At the end of this section and in the next this notation is extended to include other classes of symbols. From this it is clear that

$$\hat{\phi}(k) = \begin{cases} CA^{k-1}B & k > 0 \\ D & k = 0 \\ 0_{m\times p} & k < 0 \end{cases}$$

and it may be shown that the relationship between the input and output of the system may be represented by a set of recursive equations

$$\left.\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}\right\} \tag{2.21}$$

where $x = (x(n))_{n\in\mathbb{Z}} \in l^2_{n\times 1}$. Such a representation is called a *state-space representation* of the system. The auxiliary sequence $(x(n))_{n\in\mathbb{Z}}$ is known as the *state* of the system. The state of the system summarises all that is necessary to generate the output sequence $(y(t))_{t=s}^{\infty}$ from the input sequence $(u(t))_{t=s}^{\infty}$ using the formula (2.21).

It is clear that several state-space realisations may exist for any given transfer function. If two state-space realisations $\mathcal{P} = (A, B, C, D)$ and $\mathcal{Q} = (\bar{A}, \bar{B}, \bar{C}, \bar{D})$ are realisations of the same transfer function, then they are said to be *input-output* equivalent, written $\mathcal{P} \sim \mathcal{Q}$. This defines an equivalence relation on quadruples $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ for all possible $n \in \mathbb{N}$.

The relation $\sim$ is of fundamental interest when approaching proper rational matrices via state-space realisations, and for one class of realisations it has a particularly clear characterization. A pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ is said to be *controllable* if

$$\operatorname{rank}[B, AB, \ldots, A^{n-1}B] = n$$

and $(C, A) \in \mathbb{R}^{p\times n} \times \mathbb{R}^{n\times n}$ is *observable* if

$$\operatorname{rank}[C^T, A^TC^T, \ldots, {A^{n-1}}^T C^T]^T = n.$$

A quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ is *minimal* if (A, B) is controllable and (C, A) is observable. For more detailed discussion of these notions refer to one of the standard texts such as Wonham [132] or Kailath [72].

Hence define

$$\mathrm{L}_{\mathrm{n}}^{\mathrm{p,m}} := \left\{(A,B,C,D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m} \mid (A,B,C,D) \text{ is minimal}\right\}.$$

Also, let $\mathrm{TL}_{\mathrm{n}}^{\mathrm{p,m}}$ denote the set of $p\times m$ rational matrices with realisations in $\mathrm{L}_{\mathrm{n}}^{\mathrm{p,m}}$. Thus equation (2.20) defines a mapping $\mathrm{L}_{\mathrm{n}}^{\mathrm{p,m}} \to \mathrm{TL}_{\mathrm{n}}^{\mathrm{p,m}}$. Further to the above, it has also been shown, e.g. in Chapter 10 [73], that each $p\times m$ proper rational matrix has a minimal realisation $(A,B,C,D) \in \mathrm{L}_{\mathrm{n}}^{\mathrm{p,m}}$, for some $n \in \mathbb{N}$, and moreover, that n is the least integer such that any realisation of that order exists, and a realisation is of order n if and only if it is minimal. The *McMillan degree* of a proper rational matrix may thus be defined as the order of a minimal realisation. Therefore $\mathrm{L}_{\mathrm{n}}^{\mathrm{p,m}}$ is the set of all minimal realisations of transfer functions in $\mathrm{TL}_{\mathrm{n}}^{\mathrm{p,m}}$, which is the set of all $p\times m$ proper rational matrices of McMillan degree n.

From (2.20) it is clear that the image G of $\mathcal{Q} = (A,B,C,D)$ is preserved by the action

$$(A,B,C,D) \mapsto (TAT^{-1}, TB, CT^{-1}, D), \quad T \in \mathrm{GL}_n(\mathbb{R}). \tag{2.22}$$

This action of $T \in \mathrm{GL}_n(\mathbb{R})$ (the set of all $n\times n$ invertible real matrices) on $\mathrm{L}_{\mathrm{n}}^{\mathrm{p,m}}$ is called a *state-space transformation by T* and shall be denoted by

$$T\mathcal{Q}T^{-1} = T(A,B,C,D)T^{-1} := (TAT^{-1}, TB, CT^{-1}, D).$$

The importance of this action resides in the fact that its orbits in $\mathrm{L}_{\mathrm{n}}^{\mathrm{p,m}}$ and the $\sim$-equivalence classes in $\mathrm{L}_{\mathrm{n}}^{\mathrm{p,m}}$ coincide. Furthermore, the stabilizer of any element of $\mathrm{L}_{\mathrm{n}}^{\mathrm{p,m}}$ is the trivial group $\{I_p\}$. This is the content of the State Space Isomorphism Theorem (e.g. Chapter 10 [73], §6.2 [72]).

From the analyticity of ϕ, and its boundedness on $\mathbb{T}$, it is clear that $C(z^{-1}I_n - A)^{-1}B + D$ may have no poles in $\bar{\mathbb{D}}$ (the closure of $\mathbb{D}$) and hence that the spectrum of A is contained in $\mathbb{D}$. Any square matrix with $\lambda(A) \subseteq \mathbb{D}$ is said to be *discrete-time stable* (merely stable if the context makes clear the remaining qualification).

Let

$$\mathrm{DC}_{\mathrm{n}}^{\mathrm{p,m}} := \left\{(A,B,C,D) \in \mathrm{L}_{\mathrm{n}}^{\mathrm{p,m}} \mid \lambda(A) \subseteq \mathbb{D}\right\},$$

and $\mathrm{TDC}_{\mathrm{n}}^{\mathrm{p,m}}$ be the set of rational functions with realisations in $\mathrm{DC}_{\mathrm{n}}^{\mathrm{p,m}}$.

Some particular realisations of the transfer functions of linear, time-invariant, non-anticipative, finite-dimensional systems are of considerable importance, as will be shown in Proposition 2.19 and further discussed in §4.2. This classification was first introduced to the control literature by Moore in [92] where the existence and uniqueness properties are discussed (see Proposition 4.11 for a similar result in a different context).

Definition 2.2 (Discrete-Time Lyapunov Balancing [92]) *Given $(A,B,C,D) \in \mathrm{DC}_{\mathrm{n}}^{\mathrm{p,m}}$ and solutions P and Q to*

$$\begin{aligned} R_{DL}((A,B,C,D);P) &:= APA^T - P + BB^T = 0 &(2.23)\\ R_{DL}^D((A,B,C,D);Q) &:= A^TQA - Q + C^TC = 0 &(2.24)\end{aligned}$$

then (A,B,C,D) is said to be discrete-time Lyapunov balanced *with* Lyapunov Gramian *P if $P = Q = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_k I_{n_k}) = \mathrm{diag}(\mu_1, \ldots, \mu_n)$ where $\sigma_1 > \ldots > \sigma_k$ and $\mu_1 \geqslant \ldots \geqslant \mu_n$.*

The μ_i are the Lyapunov characteristic values *or* Hankel singular values *and σ_i has multiplicity n_i.*

■

Lemma 2.3 *If P and Q satisfy (2.23) and (2.24) respectively then*

$$\begin{aligned} P &= \sum_{k=0}^{\infty} A^k B B^T A^{T^k} \ \textit{and} \\ Q &= \sum_{k=0}^{\infty} A^{T^k} C^T C A^k. \end{aligned}$$

■

If $S \in \mathbb{R}^{n \times n}$ is a *sign matrix* i.e. $S = \operatorname{diag}(s_1, \ldots, s_n)$ where $s_i \in \{1, -1\}$, $i = 1, \ldots, n$, and

$$(A^T, C^T, B^T, D^T) = S(A, B, C, D)S$$

then (A, B, C, D) is said to be a *sign-symmetric realisation*. This notion will be seen to be equally applicable to continuous-time state-space realisations as introduced in the next section.

For systems which are *completely anticipative*, that is

$$\left(I_{l^2_{p\times 1}} - \mathbf{Trunc}^s_{p\times 1}\right) T \left(I_{l^2_{m\times 1}} - \mathbf{Trunc}^s_{m\times 1}\right) = \left(I_{l^2_{p\times 1}} - \mathbf{Trunc}^s_{p\times 1}\right) T$$

then the symbol is such that

$$\phi(e^{i\theta}) = \sum_{k=-\infty}^{-1} \hat{\phi}(k) e^{ik\theta} \tag{2.25}$$

which is analytic outside the unit circle, and hence the extension to the complement of $\bar{\mathbb{D}}$

$$\phi(z) = \sum_{k=-\infty}^{-1} \hat{\phi}(k) z^k$$

may be represented by

$$\phi(z) = C(zI_n - A)^{-1} B$$

in the rational case, with A discrete-time stable. This implies that anti-stable functions also have realisations in $\mathrm{DC}^{\mathrm{p,m}}_{\mathrm{n}}$, and hence may be balanced as in Definition 2.2. Equations similar to (2.21) may also be derived in this case, excepting that the equations generate the output sequence backwards in the time-index.

Note 2.4 *There is an implicit understanding that $\phi \overset{s}{=} (A, B, C, D)$ is interpreted differently for non-anticipative and completely anticipative systems.*

■

2.2.2 Continuous-Time Models

The theory for continuous-time systems may be developed in a similar manner to the previous section. However, as such systems are not used extensively in what follows, this section is even more cursory than the preceding section.

A linear, time-invariant, non-anticipative, finite-dimensional, continuous-time, deterministic system with m inputs and p outputs will be taken to be an integral convolution operator T such that

$$\begin{aligned} T : \mathcal{L}^2_{m\times 1}(\mathbb{R}) &\to \mathcal{L}^2_{p\times 1}(\mathbb{R}) \\ T : f &\mapsto \hat{\phi} * f \end{aligned}$$

where $\phi = \mathrm{Lap}_{m\times p}\hat{\phi}$ is a rational element of $\mathcal{H}^\infty_{m\times p}(\mathbb{C}_+)$, and is termed the symbol, or *transfer function* of the system. Since ϕ is rational, the results of Chapter 10 Kalman *et al* [73] mentioned above imply the existence of a minimal *continuous-time state-space realisation of some order* n

$$\phi(s) \;=\; C(sI_n - A)^{-1}B + D \tag{2.26}$$

for $(A, B, C, D) \in \mathrm{L}_\mathrm{n}^{\mathrm{p,m}}$ with A *continuous-time stable* i.e. $\lambda(A) \subseteq \mathbb{C}_-$. This will be written $\phi \stackrel{s}{=} (A, B, C, D)$. The output of the system may be represented by the following set of differential equations

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}\right\} \tag{2.27}$$

which is called a *state-space representation* of the system. The auxiliary function $x(t) \in \mathcal{L}^2_{n\times 1}(\mathbb{R})$ is known as the *state* of the system. The state of the system summarises all that is necessary to generate the output function $y(t)$ from the input function $u(t)$ using the equations (2.27).

Let

$$\mathrm{C}_\mathrm{n}^{\mathrm{p,m}} := \{(A, B, C, D) \in \mathrm{L}_\mathrm{n}^{\mathrm{p,m}} \mid \lambda(A) \subseteq \mathbb{C}_-\},$$

and $\mathrm{TC}_\mathrm{n}^{\mathrm{p,m}}$ be the set of rational functions with realisations in $\mathrm{C}_\mathrm{n}^{\mathrm{p,m}}$. A form of balanced realisation similar to that defined in the previous subsection may be defined for this class.

Definition 2.5 (Continuous-Time Lyapunov Balancing) *Given* $(A, B, C, D) \in \mathrm{C}_\mathrm{n}^{\mathrm{p,m}}$ *with solutions* P *and* Q *to*

$$R_L((A, B, C, D); P) \;:=\; AP + PA^T + BB^T = 0 \tag{2.28}$$

$$R_L^D((A, B, C, D); Q) \;:=\; A^TQ + QA + C^TC = 0 \tag{2.29}$$

then (A, B, C, D) *is said to be* continuous-time Lyapunov balanced *with* Lyapunov Gramian P *if* $P = Q = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_k I_{n_k}) = \mathrm{diag}(\mu_1, \ldots, \mu_n)$ *where* $\sigma_1 > \ldots > \sigma_k$ *and* $\mu_1 \geqslant \ldots \geqslant \mu_n$.

The μ_i *are the* Lyapunov characteristic values *or* Hankel singular values *and* σ_i *has multiplicity* n_i.

■

The symbol of a *completely anticipative* continuous-time system is analytic in the left half-plane, and may be represented by

$$\phi(s) = C(-sI_n - A)^{-1}B + D$$

with A continuous-time stable. This will also be denoted by $\phi \stackrel{s}{=} (A, B, C, D)$.

2.2.3 Relation Between Discrete and Continuous-Time Models

The relationship between functions defined on the unit circle and on the imaginary axis is a simple Möebius transformation somewhat ambiguously termed "the bilinear transform"

$$\begin{aligned} M : \mathbb{C} &\;\to\; \mathbb{C} \\ M : z &\;\mapsto\; \frac{1-z}{1+z}. \end{aligned} \tag{2.30}$$

The following result summarises the important properties of this mapping when used as a "change of variables."

Proposition 2.6 (The Bilinear Transform, e.g. Partington [103]) *1)* $M(z) = \frac{1-z}{1+z}$ *is a self inverse bijection from* $\mathbb{D}$ *to* $\mathbb{C}_+$.

2)

$$g(z) \in \mathcal{L}^r_{p\times q}(\mathbb{T}) \Leftrightarrow G(s) = \frac{g(M(s))}{\sqrt[r]{\pi(1+s)^2}} \in \mathcal{L}^r_{p\times q}(i\mathbb{R}),$$

and moreover,

$$\|g\|_r = \|G\|_r\,;\ 1 \leqslant r < \infty.$$

3) For $r = \infty$ *then* $G(s) = g(M(s))$ *with* $\|g\|_\infty = \|G\|_\infty$.

4) g *is analytic in* $\mathbb{D}$ *iff* G *is analytic in* $\mathbb{C}_+$.

■

Corollary 2.6.1 *The mapping*

$$\begin{aligned} V_r :\ & \mathcal{L}^r_{p\times q}(\mathbb{T}) && \to \mathcal{L}^r_{p\times q}(i\mathbb{R}) \\ V_r :\ & \quad g && \mapsto (V_r g)(s) = G(s) = \frac{g(M(s))}{\sqrt[r]{\pi(1+s)^2}} \end{aligned} \tag{2.31}$$

maps $\mathcal{H}^r_{p\times q}(\mathbb{D})$ *to* $\mathcal{H}^r_{p\times q}(\mathbb{C}_+)$. *The inverse of this mapping is*

$$\begin{aligned} V_r^{-1} :\ & \mathcal{L}^r_{p\times q}(i\mathbb{R}) && \to \mathcal{L}^r_{p\times q}(\mathbb{R}) \\ V_r^{-1} :\ & \quad G && \mapsto (V_r^{-1} G)(z) = g(z) = \sqrt[r]{\frac{\pi}{(1+z)^2}}\, G(M(z)) \end{aligned} \tag{2.32}$$

■

Proposition 2.7 (State-Space Properties Under Bilinear Transform e.g. Glover [35]) *If*

$$g(z) = C(z^{-1}I_n - A)^{-1}B + D$$

and $-1 \notin \lambda(A)$ *then* $G(s) = g(M(s))$ *has realisation*

$$(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = \left((I+A)^{-1}(A-I), \sqrt{2}(I+A)^{-1}B, \sqrt{2}C(I+A)^{-1}, D - C(I+A)^{-1}B\right).$$

Also, if

$$G(s) = \tilde{C}\left(sI_n - \tilde{A}\right)^{-1}\tilde{B} + \tilde{D}$$

and $1 \notin \lambda(A)$ *then* $g(z) = G(M(z)) = C(z^{-1}I_n - A)^{-1}B + D$ *where*

$$(A, B, C, D) = \left((I-\tilde{A})^{-1}(I+\tilde{A}), \sqrt{2}(I-\tilde{A})^{-1}\tilde{B}, \sqrt{2}\tilde{C}(I-\tilde{A})^{-1}, \tilde{D} + \tilde{C}(I-\tilde{A})^{-1}\tilde{B}\right).$$

The following also hold:

1) (A, B, C, D) *is sign-symmetric realisation with respect to the sign matrix* S *if and only if* $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ *is sign-symmetric realisation with respect to* S.

2) $(A, B, C, D) \in \mathrm{DC}^{\mathrm{p,m}}_{\mathrm{n}}$ *if and only if* $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \in \mathrm{C}^{\mathrm{p,m}}_{\mathrm{n}}$.

3) (A, B, C, D) *is discrete-time Lyapunov balanced if and only if* $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ *is continuous-time Lyapunov balanced. Moreover, the Gramians arising in each case are identical.*

Proof : The formulae given here are widely available in the control literature.

1) follows by a simple calculation.

2) follows from the Spectral Mapping Theorem i.e.

$$\lambda(M(A)) = M(\lambda(A)).$$

3) This is shown in Glover [35]. Also see Lemma 4.9.

■

2.2.4 Hankel Operators

The theory of Hankel operators is a rich one with many applications. In this thesis two connections will be made: with dynamical systems, as will be explored in this subsection; and with stochastic processes in §2.3. These connections have been the impetus for many of the results in this area e.g. [55, 24, 56, 55, 54, 109, 111, 118, 133, 34, 1, 105, 106]. In recent years particular attention has been paid by the control community to the application and extension of such results. Particular note must be given to the works based upon Hankel-Norm approximation, mainly in Glover [35].

To show clearly the relationship between linear systems and Hankel operators, it is necessary to elucidate the connection between some of the spaces defined in §2.1.

The subspaces of $l^2_{p\times q}$ composed of those sequences which are zero for $t > 0$ and of those sequences which are zero for $t \leqslant 0$ are isomorphic, in a natural way, to the spaces $l^2_{p\times q}(\mathbb{Z}_+)$ and $l^2_{p\times q}(\mathbb{Z}_-)$ and will subsequently be identified with them. Hence it is possible to state that

$$l^2_{p\times q} = l^2_{p\times q}(\mathbb{Z}_-) \bigoplus l^2_{p\times q}(\mathbb{Z}_+)$$

where this is taken as an internal orthogonal direct sum, by the above identification.

One further space of interest is

$$\mathcal{H}^2_{p\times q}(\bar{\mathbb{D}}^{\mathsf{C}}) := \left\{ f : \bar{\mathbb{D}}^{\mathsf{C}} \to \mathbb{C}^{p\times q} \mid f \text{ is analytic in } \bar{\mathbb{D}}^{\mathsf{C}},\ \bar{f(z)} = f(\bar{z}) \text{ and } \|f\|_2 < \infty \right\}$$

where

$$\|f\|_2 := \sup_{1<r<\infty} \|f_r\|_2$$

with f_r as in §2.1 and $\bar{\mathbb{D}}^{\mathsf{C}}$ is the complement of the closure of $\mathbb{D}$.

For any $f \in \mathcal{H}^2_{p\times q}(\mathbb{D})$ or $\mathcal{H}^2_{p\times q}(\bar{\mathbb{D}}^{\mathsf{C}})$, a boundary value function $\tilde{f} : \mathbb{T} \to \mathbb{C}^{p\times q}$ may be defined by taking non-tangential limits, e.g. see Rudin [113] Theorem 17.11. $\tilde{f}$ is easily shown to be an element of $\mathcal{L}^2_{p\times q}(\mathbb{T})$ of the same norm as f. Hence by this process $\mathcal{H}^2_{p\times q}(\mathbb{D})$ and $\mathcal{H}^2_{p\times q}(\bar{\mathbb{D}}^{\mathsf{C}})$ may be identified with subspaces of $\mathcal{L}^2_{p\times q}(\mathbb{T})$, and by consideration of their power-series representations (cf. (2.19) and (2.25)) it is clear that

$$\mathcal{L}^2_{p\times q}(\mathbb{T}) = \mathcal{H}^2_{p\times q}(\mathbb{D}) \bigoplus \mathcal{H}^2_{p\times q}(\bar{\mathbb{D}}^{\mathsf{C}}).$$

The *Hilbert isomorphism* (more commonly known as the z-transform in the engineering literature)

$$\begin{aligned} \text{Hilb} : l^2_{p\times q} &\to \mathcal{L}^2_{p\times q}(\mathbb{T}) \\ \text{Hilb} : (u(t))_{t\in\mathbb{Z}} &\mapsto \sum_{t=-\infty}^{\infty} u(t)e^{-it\theta} \end{aligned} \tag{2.33}$$

respects the orthogonal decompositions above, and so

$$\begin{aligned}\text{Hilb}\left(l^2_{p\times q}(\mathbb{Z}_-)\right) &= \mathcal{H}^2_{p\times q}(\mathbb{D})\\ \text{Hilb}\left(l^2_{p\times q}(\mathbb{Z}_+)\right) &= \mathcal{H}^2_{p\times q}(\bar{\mathbb{D}}^{\mathbf{C}}).\end{aligned}$$

An *orthogonal projection* onto a closed subspace H_1 of a Hilbert space H_2 is defined as

$$\begin{aligned}\mathbf{P}_{H_1} : H_2 &\rightarrow H_2\\ \mathbf{P}_{H_1} : h &\mapsto x\end{aligned}$$

where $h = x + y$, $x \in H_1$ and $y \in H_1^{\perp}$.

Lemma 2.8

$$\mathbf{P}_{\mathcal{H}^2_{p\times q}(\bar{\mathbb{D}}^{\mathbf{C}})}\text{Hilb}u = \text{Hilb}\mathbf{P}_{l^2_{p\times q}(\mathbb{Z}_+)}u, \quad \forall u \in l^2_{p\times q}$$

Proof : Take $u = (u(t))_{t\in\mathbb{Z}} \in l^2_{p\times q}$, then

$$\begin{aligned}\left(\mathbf{P}_{\mathcal{H}^2_{p\times q}(\bar{\mathbb{D}}^{\mathbf{C}})}\text{Hilb}\right)u &= \mathbf{P}_{\mathcal{H}^2_{p\times q}(\bar{\mathbb{D}}^{\mathbf{C}})}\sum_{t=-\infty}^{\infty} u(t)z^{-t}\\ &= \sum_{t=-\infty}^{-1} u(t)z^{-t}\\ &= \text{Hilb}\,(u(t))_{t\in\mathbb{Z}_+}\\ &= \left(\text{Hilb}\mathbf{P}_{l^2_{p\times q}(\mathbb{Z}_+)}\right)u\end{aligned}$$

■

The projections $\mathbf{P}_{\mathcal{H}^2_{p\times q}(\mathbb{D})}$ and $\mathbf{P}_{\mathcal{H}^2_{p\times q}(\bar{\mathbb{D}}^{\mathbf{C}})}$ will be denoted $\mathbf{P}_+$ and $\mathbf{P}_-$. These are known as the *Riesz Projections*. Also, let $[f]_+$ and $[f]_-$ be $\mathbf{P}_+f$ and $\mathbf{P}_-f$ respectively.

The Hilbert isomorphism allows an alternative representation of a linear system, called the *frequency domain* description. This is a representation of the operator

$$\text{Hilb}\,T\,\text{Hilb}^{-1} : \mathcal{L}^2_{m\times 1}(\mathbb{T}) \rightarrow \mathcal{L}^2_{p\times 1}(\mathbb{T}).$$

The following Lemma shows that the time-domain convolution operator corresponds to a frequency-domain *multiplication operator*, that is, for $\phi \in \mathcal{L}^{\infty}_{p\times q}(\mathbb{T})$

$$\begin{aligned}\mathbf{M}_{\phi} : \mathcal{L}^2_{m\times 1}(\mathbb{T}) &\rightarrow \mathcal{L}^2_{p\times 1}(\mathbb{T})\\ \mathbf{M}_{\phi} : f &\mapsto \phi f.\end{aligned}$$

Lemma 2.9 (Relationship between Time and Frequency Domain Systems)

$$\text{Hilb}\mathbf{C}_{\hat{\phi}} = \mathbf{M}_{\phi}\text{Hilb}$$

where $\phi \in \mathcal{L}^{\infty}_{p\times q}(\mathbb{T})$ *and* $\hat{\phi} = \text{Hilb}^{-1}\phi$.

Proof : Note that if $\phi(e^{i\theta}) = \sum_{k=-\infty}^{\infty}\hat{\phi}(k)e^{ik\theta}$, $u(e^{i\theta}) = \sum_{k=-\infty}^{\infty}\hat{u}(k)e^{ik\theta}$, $\hat{\phi} = \left(\hat{\phi}(t)\right)_{t\in\mathbb{Z}}$ and $\hat{u} = (\hat{u}(t))_{t\in\mathbb{Z}}$ then

$$\begin{aligned}(\phi u)\left(e^{i\theta}\right) &= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\hat{\phi}(k)\hat{u}(l)e^{i(k+l)\theta}\\ &= \sum_{t=-\infty}^{\infty}\hat{y}(t)e^{it\theta}\end{aligned}$$

where

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} \hat{\phi}(k)\hat{u}(t-k).$$

Hence,

$$\begin{aligned}
\mathrm{Hilb}\mathbf{C}_{\hat{\phi}}\hat{u} &= \mathrm{Hilb}\left(\hat{\phi} * \hat{u}\right) \\
&= \mathrm{Hilb}\hat{\phi}\mathrm{Hilb}\hat{u} \\
&= \phi u \\
&= \mathbf{M}_{\phi}\mathrm{Hilb}\hat{u}.
\end{aligned}$$

■

This result, in combination with the notion of systems introduced in §2.2.1, shows how the frequency domain description of a system may be used in a parallel fashion to the time-domain version.

Hankel operators may be considered in system theoretic terms to express the relationship of the past of the input sequence to the future of the output sequence of a system.

Definition 2.10 (Hankel Operator) *A* discrete-time, frequency domain Hankel operator with symbol $\phi \in \mathcal{L}^{\infty}_{p\times q}(\mathbb{T})$ *is a linear mapping*

$$\begin{aligned}
\mathbf{H}_{\phi} : \mathcal{H}^2_{q\times 1}(\mathbb{D}) &\rightarrow \mathcal{H}^2_{p\times 1}(\bar{\mathbb{D}}^{\mathsf{C}}) \\
\mathbf{H}_{\phi} : f &\mapsto \mathbf{P}_{-}\phi f.
\end{aligned}$$

■

There are at least three other closely related forms of Hankel operator used commonly, namely continuous-time frequency domain, discrete-time time-domain and continuous-time time-domain operators. Continuous-time time-domain have been used extensively in the study of integral operators e.g. see Gohberg & Krein [39].

The objects introduced in the following result will be shown to have important applications in system theory, and are the basis of part of the estimation scheme described in Chapter 6.

Theorem 2.11 (Schmidt Expansion of a Compact Operator e.g. Partington [104]) *A Hilbert space operator $T : H_1 \rightarrow H_2$ is compact if and only if there exist orthonormal sequences of elements in H_1 and H_2 respectively, $(u_i)_{i=1}^{\infty}$ and $(v_i)_{i=1}^{\infty}$ and a sequence of real positive numbers converging to zero $(\sigma_i)_{i=1}^{\infty}$ such that*

$$Tx = \sum_{i=1}^{\infty} \sigma_i \left\langle x, u_i \right\rangle_{H_1} v_i, \;\; \forall x \in H_1.$$

Furthermore, the σ_i are unique, and the spaces spanned by the Schmidt vectors v_i corresponding to distinct σ_i are unique.

Proof : Theorem 1.2 [104] presents a short elegant proof. ■

Definition 2.12 (Schmidt Vectors) *The pairs (u_i, v_i), $i \in \mathbb{N}$ are called the* Schmidt pairs *of T and are individually known as* Schmidt vectors. *The σ_i are the* singular values *of the operator.*

■

Corollary 2.12.1 (Alternative Definition of Schmidt Vectors) *Any sequence of positive real numbers which converge to zero, and set of vectors $(u_i, v_i)_{i\in\mathbb{N}}$ are the singular values and Schmidt vectors of an operator T if they satisfy the following pair of complementary equations*

$$\begin{aligned} Tu_i &= \sigma_i v_i \\ T^* v_i &= \sigma_i u_i. \end{aligned}$$

■

Lemma 2.13 (Schmidt Vectors of a Rational Hankel Operator) *Let $\phi \in \mathcal{L}^\infty_{p\times q}(\mathbb{T})$ have $[\phi]_- = C(zI_n - A)^{-1}B$. Also let $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times q} \times \mathbb{R}^{p\times n}$ be discrete-time Lyapunov balanced with Gramian $\Sigma = \mathrm{diag}(\mu_1, \ldots, \mu_n)$.*

The Hankel operator with symbol ϕ, namely $\mathbf{H}_\phi$, has rank n with singular values $(\mu_i)_{i=1}^n$ and n Schmidt pairs (u_i, v_i) $i = 1, \ldots, n$, given by

$$\begin{aligned} u_i(z) &= B^T(z^{-1}I_n - A^T)^{-1}A^T F_i + B^T F_i \\ &\stackrel{s}{=} (A^T, A^T F_i, B^T, B^T F_i) \end{aligned} \tag{2.34}$$

$$\begin{aligned} v_i(z) &= C(zI_n - A)^{-1}F_i \\ &\stackrel{s}{=} (A, F_i, C, 0) \end{aligned} \tag{2.35}$$

where $F_i \in \mathbb{R}^{n\times 1}$ is zero except for the ith row which is arbitrary.

Proof : The rank of $\mathbf{H}_\phi$ follows from Kronecker's Theorem e.g. [104]. Utilizing Lemma 2.3, then

$$\begin{aligned} \mathbf{H}_\phi u_i &= \mathbf{P}_- \mathbf{M}_\phi u_i \\ &= \mathbf{P}_- C(zI_n - A)^{-1}B\left(B^T(z^{-1}I_n - A^T)^{-1}A^T F_i + B^T F_i\right) \\ &= \mathbf{P}_-\left[C(zI_n - A)^{-1}BB^T(z^{-1}I_n - A^T)^{-1}A^T F_i + C(zI_n - A)^{-1}BB^T F_i\right] \\ &= C\mathbf{P}_-\left[\sum_{k=0}^\infty\sum_{l=0}^\infty z^{l-k}A^k BB^T A^{T^l} A^T F_i + (zI_n - A)^{-1}BB^T F_i\right] \\ &= C\mathbf{P}_-\left[\sum_{t=-\infty}^\infty z^t \sum_{l=0}^\infty A^{l-t}BB^T A^{T^l} A^T F_i + (zI_n - A)^{-1}BB^T F_i\right] \\ &= C\sum_{t=-\infty}^{-1} z^t A^{-t}\Sigma A^T F_i + C(zI_n - A)^{-1}BB^T F_i \\ &= C(zI_n - A)^{-1}\left[A\Sigma A^T + BB^T\right]F_i \\ &= \mu_i C(zI_n - A)^{-1}F_i \text{ from (2.24)} \\ &= \mu_i v_i \end{aligned}$$

$$\begin{aligned} \mathbf{H}^*_\phi v_i &= \mathbf{P}_+\mathbf{M}^*_\phi v_i \\ &= \mathbf{P}_+\mathbf{M}_{\phi^*} v_i \\ &= \mathbf{P}_+ B^T(z^{-1}I_n - A^T)^{-1}C^T C(zI_n - A)^{-1}F_i \\ &= B^T\mathbf{P}_+\sum_{k=0}^\infty\sum_{l=0}^\infty z^{k-l}A^{T^k}C^T CA^l F_i \end{aligned}$$

$$\begin{aligned}
&= B^T \sum_{t=0}^{\infty} z^t A^{T^t} \sum_{l=0}^{\infty} A^{T^l} C^T C A^l F_i \\
&= B^T \sum_{t=0}^{\infty} z^t A^{T^t} \Sigma F_i \\
&= B^T (z^{-1} I_n - A^T)^{-1} A^T \Sigma F_i + B^T \Sigma F_i \\
&= \mu_i (B^T (z^{-1} I_n - A^T)^{-1} A^T F_i + B^T F_i) \\
&= \mu_i u_i
\end{aligned}$$

Hence, from Corollary 2.12.1, the pairs (u_i, v_i) are the Schmidt pairs of $\mathbf{H}_\phi$, and μ_i are its singular values. ∎

Of course, where repeated characteristic values occur these Schmidt vectors are not unique, although they do provide an orthogonal basis for the appropriate vector space of Schmidt vectors.

Lemma 2.14 (Image of Hankel Operator) *If $\phi \in \mathcal{L}^\infty_{p\times q}(\mathbb{T})$ such that $[\phi]_- = \sum_{j=-n}^{-1} \hat{\phi}_j z^j$ then*

$$\mathrm{im}\mathbf{H}_\phi \leqslant \bigvee_{\substack{i=1 \\ A \in \mathbb{R}^{p\times p}}}^{n} A z^{-i}.$$

Proof :

$$\begin{aligned}
\mathbf{H}_\phi A z^k &= \mathbf{P}_- \phi A z^k \text{ if } k \geqslant 0 \\
&= \mathbf{P}_- \sum_{j=-n}^{-1} \hat{\phi}_j z^{j+k} \\
&= \sum_{j=-n}^{-(k+1)} \hat{\phi}_j z^{j+k} \in \bigvee_{\substack{i=1 \\ A \in \mathbb{R}^{p\times p}}}^{n} A z^{-i}
\end{aligned}$$

∎

The next result shows that more information may be obtained from the spectrum of a Hankel operator if the symbol has a rational symmetric symbol. However, to achieve this the operator must be adjusted slightly so that it is an *endomorphism*, that is a mapping with the same domain and range. Let End(H) denote the set of all bounded linear endomorphisms on H.

Definition 2.15 (Flip Operator) *The operators*

$$\begin{aligned}
R_{p\times q} : \mathcal{L}^2_{p\times q}(\mathbb{T}) &\to \mathcal{L}^2_{p\times q}(\mathbb{T}) \\
R_{p\times q} : \sum_{t=-\infty}^{\infty} A_t e^{it\theta} &\mapsto \sum_{t=-\infty}^{\infty} A_t e^{-i(t+1)\theta}
\end{aligned}$$

are called flip operators *and, following the usual conventions,*

$$\begin{aligned}
R_{p\times q}\left(\mathcal{H}^2_{p\times q}(\mathbb{D})\right) &= \mathcal{H}^2_{p\times q}(\bar{\mathbb{D}}^{\complement}) \\
R_{p\times q}\left(\mathcal{H}^2_{p\times q}(\bar{\mathbb{D}}^{\complement})\right) &= \mathcal{H}^2_{p\times q}(\mathbb{D}).
\end{aligned}$$

The subscripts will generally be suppressed, being obvious from the context.

∎

Lemma 2.16 (Properties of the Flip Operator)

$$\begin{aligned} R_{p\times q}\mathbf{P}_+ &= \mathbf{P}_- R_{p\times q} \\ R_{p\times q}\mathbf{P}_- &= \mathbf{P}_+ R_{p\times q} \\ R_{p\times q}\mathbf{M}_{\phi(e^{i\theta})} &= \mathbf{M}_{\phi(e^{-i\theta})} R_{p\times q} \\ R^*_{p\times q} &= R_{p\times q} \end{aligned}$$

■

Lemma 2.17 (Conditions for Self-Adjoint Hankel Operator)

$$R\mathbf{H}_\phi \textit{ is self-adjoint} \Leftrightarrow \phi \textit{ is symmetric}$$

Proof :

$$\begin{aligned} (R\mathbf{H}_\phi)^* &= (R\mathbf{P}_-\mathbf{M}_\phi\mathbf{P}_+)^* \\ &= \mathbf{P}_+^*\mathbf{M}_\phi^*\mathbf{P}_-^* R^* \\ &= \mathbf{P}_+\mathbf{M}_{\phi^*}\mathbf{P}_- R \\ &= \mathbf{P}_+\mathbf{M}_{\phi^*} R\mathbf{P}_+ \\ &= \mathbf{P}_+ R\mathbf{M}_{\phi^*(e^{-i\theta})}\mathbf{P}_+ \\ &= R\mathbf{P}_-\mathbf{M}_{\phi^*(e^{-i\theta})}\mathbf{P}_+ \end{aligned}$$

■

Lemma 2.18 (Action of R on Schmidt Vectors) *Given v_i as in (2.35) then*

$$R(v_i(z)) = C(z^{-1}I_n - A)^{-1}AF_i + CF_i.$$

Proof :

$$\begin{aligned} Rv_i(z) &= RC(zI_n - A)^{-1}F_i \\ &= RC\sum_{k=0}^{\infty} z^{-(k+1)}A^kF_i \\ &= C\sum_{k=0}^{\infty} z^kA^kF_i \\ &= C(z^{-1}I_n - A)^{-1}AF_i + CF_i. \end{aligned}$$

■

Proposition 2.19 (Eigenvectors of Hankel Operators) *Under the same conditions as in Lemma 2.13, and assuming that (A, B, C) is sign-symmetric with respect to $S = \mathrm{diag}(s_1, \ldots, s_n)$ then u_i is an eigenvector of $R\mathbf{H}_\phi$ with eigenvalue $s_i\mu_i$, that is*

$$R\mathbf{H}_\phi u_i = s_i\mu_i u_i.$$

Proof : From equations (2.34) and (2.35) and the symmetry hypothesis then:

$$\begin{aligned} R\mathbf{H}_\phi u_i &= \mu_i Rv_i \\ &= \mu_i(C(z^{-1}I_n - A)^{-1}AF_i + CF_i) \\ &= \mu_i(B^TS(z^{-1}I_n - SA^TS)^{-1}SA^TSF_i + B^TSF_i) \\ &= \mu_i(B^T(z^{-1}I_n - A^T)^{-1}A^TSF_i + B^TSF_i) \\ &= s_i\mu_i(B^T(z^{-1}I_n - A^T)^{-1}A^TF_i + B^TF_i) \\ &= s_i\mu_i u_i. \end{aligned}$$

■

2.3 Stochastic Models

In the preceding section the relationship between inputs and outputs of a linear system was described. For any real system this does not exhaust the possible description of its behaviour and there remain several ways in which a model may be inadequate. The system may exhibit non-linear behaviour of many sorts or may vary over time. Even if the system is truly linear and time-invariant, it may be too complex to represent it fully or it may not be appropriate to model it in such a manner. This may arise when the system is infinite-dimensional or of very high McMillan degree.

This section seeks to represent some of these remaining unmodelled effects. The type of explanation being sought is not a causal explanation, and this is reflected in the lack of time-directedness in the models described below.

Despite the application of more complicated models e.g. non-linear, time-varying, infinite-dimensional models aspects of the behaviour of any real system may remain unexplained. There may be many reasons for this including the action of other unmodelled inputs to the system. Also, there may be certain random effects upon the system, and intrinsic uncertainty about the actual behaviour of the system.

Finally, any model makes assumptions about the nature of the object/system under consideration (e.g. continuity of media, particulate nature of matter etc.) which may deviate from the underlying actuality in quite radical ways. In some cases these assumptions could completely disable the model from explaining certain phenomena e.g. lasers.

All of the probabilistic notions to be introduced below may be found in any standard text on stochastic processes. Particularly recommended are Grimmett & Stirzaker or Doob [45, 25] and the first few Chapters of Caines [16].

Throughout this thesis it will be assumed that all of the random variables mentioned will be defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F}$ is a σ-field of events in Ω on which $\mathbb{P}$ is a probability measure.

2.3.1 Time-Domain Representations

In the input-output situation described in §2.2 the information of significance is the mapping between inputs and outputs. However, for the random processes described below there is no simple notion of inputs and outputs. Thus where the important object of consideration is the symbol of a deterministic system, in the stochastic case the important object will be the underlying space generated by the process. This is what the first part of the following subsection constructs. However, it is possible to generate a Hankel operator from this underlying space, the construction of which will be described in the §2.3.2.

In what follows it is only the difference in time between two events that is of interest, and not the associated direction. That is, it is the correlation structure of the stochastic process that is of principal concern. While time-oriented dynamical models may be constructed (e.g. the innovations representations, see Theorem 2.23) there is no *a priori* preference or distinction between the forward and backward representations. This is appropriate because the processes are being used to describe phenomena without seeking causal explanation. It would seem that causality is the very motivation for the concept of directedness.

A *discrete-time p-component stochastic process* will be taken to be a sequence $(y(t))_{t\in\mathbb{Z}}$ where

$$y(t) \in \mathcal{L}^2\left((\Omega,\mathcal{F},\mathbb{P}),\mathbb{R}^{p\times 1}\right) := \left\{ f:\Omega\to\mathbb{R}^{p\times 1} \;\middle|\; f \text{ is measurable and } \int_\Omega \|f(\omega)\|_2^2 \, d\mathbb{P}(\omega) < \infty \right\}. \tag{2.36}$$

The components of each vector will be denoted by

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}.$$

All of the processes considered here will be 2nd order stationary as defined below.

Definition 2.20 (r-th Order Stationarity) *A stochastic process* $(y(t))_{t\in\mathbb{Z}}$ *will be said to be* rth order stationary *if*

$$\mathbb{E}\prod_{j=1}^{r} y_{i_j}(t_j) = \mathbb{E}\prod_{j=1}^{r} y_{i_j}(t_j+\tau),\ \ \forall i_1,\ldots,i_r \in \{1,\ldots,r\}, \forall t_1,\ldots,t_r,\tau \in \mathbb{Z}$$

where

$$\mathbb{E}(f) := \int_\Omega f(\omega)\, d\mathbb{P}(\omega) \tag{2.37}$$

is the expectation operator*. In this case the following rth-order moment are defined*

$$\gamma_{i_1\ldots i_r}(t_1,\ldots,t_{r-1}) := \mathbb{E}y_{i_1}(0)\prod_{j=2}^{r} y_{i_j}(t_{j-1}). \tag{2.38}$$

A wide sense stationary process *is a 2nd-order stationary process with zero mean, that is*

$$\mathbb{E}y(t) = 0_{p\times 1},\ \ \forall t \in \mathbb{Z}.$$

In this case the following is defined

$$\Gamma_y(k) := \mathbb{E}(y(0)y^T(k)).$$

This may be considered as a form of outer product, and provides the Gramian structure as described in Wiener & Masani [125, 126] and [86].

■

Note 2.21 *Stationary processes, that is processes whose finite-dimensional joint distributions demonstrate a shift invariance property similar to that found in Definition 2.20, will not be dealt with in this thesis. For a treatment of these processes see Rozanov [112] and Caines [16] Appendix D.*

■

It is clear that a process provides a basis for a subspace of $\mathcal{L}^2\left((\Omega,\mathcal{F},\mathbb{P}),\mathbb{R}^{p\times 1}\right)$. However, for a variety of reasons, mainly historic e.g. see Masani & Wiener [86, 85], a richer subspace is associated. Hence, let h_t^y be the closed linear span of the p components of $y(t)$, which will be denoted

$$h_t^y := \bigvee\{y_1(t),\ldots,y_p(t)\}.$$

Also let

$$
\begin{aligned}
h^y &:= \bigvee_{t\in\mathbb{Z}} h^y_t & h^y_- &:= \bigvee_{t\leqslant 0} h^y_t \\
h^y_+ &:= \bigvee_{t>0} h^y_t & h^y_{+,k} &:= \bigvee_{t>k} h^y_t \\
h^y_{-,k} &:= \bigvee_{t\leqslant k} h^y_t & H^y &:= \bigoplus_{i=1}^p h^y \\
\mathcal{P}^y &:= \bigoplus_{i=1}^p h^y_- & \mathcal{F}^y &:= \bigoplus_{i=1}^p h^y_+ \\
\mathcal{P}^y_k &:= \bigoplus_{i=1}^p h^y_{-,k} & \mathcal{F}^y_k &:= \bigoplus_{i=1}^p h^y_{+,k} \\
\mathcal{P}^y_{-\infty} &:= \bigcap_{k<0} \mathcal{P}^y_k & \mathcal{F}^y_\infty &:= \bigcap_{k>0} \mathcal{F}^y_k
\end{aligned}
$$

where all closures are taken in $\mathcal{L}^2\left((\Omega,\mathcal{F},\mathbb{P}),\mathbb{R}\right)$. $\bigoplus$ denotes an external direct sum, which will be represented by column vectors.

Here, H^y represents the complete history of the process, $\mathcal{P}^y$ its past (and present) , $\mathcal{F}^y$ its future, and $\mathcal{F}^y_k$ the future from time k onwards. Another way to look at this structure is as the $\mathbb{R}^{p\times p}$-module generated by the elements of the process y, and closed with respect to the norm

$$\|x\|_{H^y} := \operatorname{tr}\, \mathbb{E}(xx^T) \tag{2.39}$$

which is induced from the inner product

$$\langle x, y\rangle_{H^y} := \operatorname{tr}\, \mathbb{E}(xy^T) = \mathbb{E}(y^T x). \tag{2.40}$$

It is clear that the space H^y is a Hilbert space when equipped with the inner product (2.40). This contains all of the information available about the process, and is taken as the space of primary interest when studying wide sense stationary processes.

The process itself provides a set of spanning vectors (over $\mathbb{R}^{p\times p}$), but these do not form an orthogonal collection. Such a basis can be generated using a Gram-Schmidt orthogonalisation method.

Definition 2.22 (Innovations) *Let the orthogonal projection of $y \in H^y$ onto a closed subspace $\mathcal{M} \leqslant H^y$ be denoted by $(f|\mathcal{M})$. Then define the* forward innovations process $(e(t))_{t\in\mathbb{Z}}$ *by*

$$e(t) := y(t) - (y(t)|\mathcal{P}_{t-1})$$

and the backward innovations process $(\bar{e}(t))_{t\in\mathbb{Z}}$ *by*

$$\bar{e}(t) := y(t) - (y(t)|\mathcal{F}_t).$$

■

Theorem 2.23 (Wold Decomposition [125, 126]) *If $(x(n))_{n\in\mathbb{Z}}$ is a wide-sense stationary process with $(e(t))_{t\in\mathbb{Z}}$ its forward innovation process and $(\bar{e}(t))_{t\in\mathbb{Z}}$ its backward innovation process. Then:*

1) $x(n) = u(n) + v(n)$, where $u(n) := (x(n)|\mathcal{P}^e_n)$, $v(n) := (x(n)|\mathcal{P}^x_{-\infty})$, where $u(n) \perp v(n)$ for all $n \in \mathbb{Z}$.

2) $x(n) = \bar{u}(n) + \bar{v}(n)$, where $\bar{u}(n) := (x(n)|\mathcal{F}^{\bar{e}}_{n-1})$, $\bar{v}(n) := (x(n)|\mathcal{F}^x_\infty)$, where $\bar{u}(n) \perp \bar{v}(n)$ for all $n \in \mathbb{Z}$.

3) The processes $(u(t))_{t\in\mathbb{Z}}$ and $(\bar{u}(t))_{t\in\mathbb{Z}}$ are wide-sense stationary and have the representations

$$\begin{aligned} u(n) &= \sum_{k=0}^{\infty} A_k e(n-k)\ \forall n \in \mathbb{Z}, \\ \bar{u}(n) &= \sum_{k=0}^{\infty} B_k \bar{e}(n+k)\ \forall n \in \mathbb{Z}, \end{aligned}$$

where $\mathbb{E}e(j)e^T(k) = \Sigma\delta_{jk} = \mathbb{E}\bar{e}_j\bar{e}_k^T$. Furthermore,

$$0 \leqslant \sum_{k=0}^{\infty} A_k \Sigma A_k^T < \infty,$$

$$\mathbb{E}x_0 e_{-k}^T = \mathbb{E}u_0 e_{-k}^T = A_k\Sigma,\ \forall k > 0,$$

and

$$A_0\Sigma = \Sigma = \Sigma A_0^T.$$

Analogous results hold for the coefficients of the backwards innovation expansion.

4) v and $\bar{v}$ are linearly deterministic processes, that is $v_k \in \mathcal{P}_{k-1}^v$ and $\bar{v}_k \in \mathcal{F}_k^{\bar{v}}$. Also, $\mathcal{P}_k^v = \mathcal{P}_{-\infty}^x$ and $\mathcal{F}_k^{\bar{v}} = \mathcal{F}_{\infty}^x$ for all $k \in \mathbb{Z}$.

■

The rank of the matrix Σ is called the *rank of the process.* If the process has full rank then the sequence $(A_i)_{i=0}^{\infty}$ is uniquely defined. This sequence and Σ together are called the *Wold sequence of* x. The *normalised innovations processes* f and $\bar{f}$ are given by $f_t := \Sigma^{-1/2}e(t)$ and $\bar{f}_t := \Sigma^{-1/2}\bar{e}(t)$.

To show the importance of the innovations define the following

$$H_{(t)}^y := \bigoplus_{i=1}^p h_t^y, \quad h_{(s,t)}^y := \bigvee_{i=s}^t h_i^y$$

and

$$H_{(s,t)}^y := \bigoplus_{i=1}^p h_{(s,t)}^y.$$

Lemma 2.24 *1) $\mathcal{P}_k^x = \mathcal{P}_{k-p}^x \textcircled{\perp} H_{(k-p+1,k)}^e = \mathcal{P}_{k-p}^x \textcircled{\perp} H_{(k-p+1,k)}^f$ where $p > 0$.*

2) $\mathcal{F}_k^x = \mathcal{F}_{k+p}^x \textcircled{\perp} H_{(k+1,k+p)}^{\bar{e}} = \mathcal{F}_{k-p}^x \textcircled{\perp} H_{(k+1,k+p)}^{\bar{f}}$ where $p > 0$.

3) $\mathcal{P}_k^x = \mathcal{P}_{-\infty}^x \textcircled{\perp} \mathcal{P}_k^e$, $\mathcal{P}_{-\infty}^x \perp H^e$ and $\mathcal{P}_{-\infty}^e = 0$. Similar results hold for the backwards innovations relationship to x.

■

The following class of wide sense stationary stochastic processes will be used most commonly, because they are completely described by their Wold decomposition.

Definition 2.25 (Pure Linear Non-Determinism) *A stochastic process $(y(t))_{t\in\mathbb{Z}}$ is* purely linearly non-deterministic *if* $\mathcal{P}_{-\infty}^y = 0$.

■

2.3.2 Frequency-Domain Representation

Following similar motivation to that in the deterministic case, the following frequency domain representation relates the process to a Hilbert space of square integrable functions.

As the stochastic process has no input-output structure, there can be no immediate notions of an associated system. However, an analogue to this will be introduced in §2.3.3.

There are two complications to the frequency domain representation which were not paralleled in the deterministic case. Firstly, the random variables defining the process at different instants are not independent: this leads to the use of weighted Lebesgue spaces. Secondly, the definition of H^y leads to the use of matrix frequency domain representations. This will be discussed at length in Note 2.29.

The following object will be used to summarise the covariance structure of the stochastic process.

Definition 2.26 (Matrix Distribution Function) *A* matrix distribution function *is a function*

$$F : \mathbb{T} \to \mathbb{C}^{p \times p}$$

which is bounded, increasing, right-continuous (in the sense of increasing $\arg(z)$*), has* $F(1) = 0_{p\times p}$*, and has positive increments, i.e.*

$$F(e^{i\theta_2}) - F(e^{i\theta_1}) \geqslant 0, \quad 0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 2\pi.$$

■

Such a function may be used to define a matrix measure. The associated measure has been widely studied, and Chapters 10 & 11 of Champeney [17] give an excellent overview of the results relating this measure to positive-definite sequences.

Definition 2.27 (Positive-Definite Sequences) *A matrix sequence* $(R_t)_{t=-\infty}^{\infty}$, $R_t \in \mathbb{C}^{p\times p}$ *is said to be* positive *if all of the matrices*

$$\hat{R}_n := \begin{bmatrix} R_0 & R_1 & R_2 & \cdots & R_n \\ R_{-1} & R_0 & R_1 & \cdots & R_{n-1} \\ R_{-2} & R_{-1} & R_0 & \cdots & R_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{-n} & R_{-n+1} & R_{-n+2} & \cdots & R_0 \end{bmatrix}$$

are non-negative definite. A matrix sequence is said to be strictly positive *if* $\hat{R}_n$ *is positive definite for each* $n \in \mathbb{N}$.

■

The relationship to stationary stochastic processes is easily noted if R_t is taken to be $\Gamma_y(t)$ and then

$$\hat{R}_n = \mathbb{E} \begin{bmatrix} y(0) \\ \vdots \\ y(n) \end{bmatrix} \begin{bmatrix} y(0)^T & \cdots & y(n)^T \end{bmatrix}.$$

The following result is the key relationship between the time and frequency domain descriptions of wide sense stationary stochastic processes. It shows the existence of a measure which completely describes the covariance structure of a process, and hence summarises its second order structure.

Theorem 2.28 (Riesz-Herglotz Theorem) *$(R_t)_{t=-\infty}^{\infty}$ is a positive sequence of matrices if and only if there exists a matrix distribution function F on $\mathbb{T}$ such that*

$$R_n = \frac{1}{2\pi}\int_{\mathbb{T}} e^{-in\lambda}\,dF(\lambda),\ \ \forall n \in \mathbb{Z}$$

where the integral is interpreted as a Lebesgue-Stieltjes integral.

Proof : See for example Theorem 11.30 (c) Rudin [113] for the univariate case, and Foias and Frahzo [32] for the matrix case. ■

In the case where $R_t = \Gamma_y(t)$ then the associated matrix distribution function is known as the *spectral distribution function* of $(y(t))_{t\in\mathbb{Z}}$.

The Riesz-Herglotz Theorem suffices to allow the construction of the frequency domain representation. The process will be represented by a space of matrix functions, thus let F be the spectral distribution function of $(y(t))_{t\in\mathbb{Z}}$ and

$$\mathcal{L}^2_{p\times p}(F) := \left\{ f : \mathbb{T} \to \mathbb{C}^{p\times p} \,\middle|\, f \text{ is measurable, } \|f\|_{\mathcal{L}^2_{p\times p}(F)} < \infty \text{ and } \overline{f(e^{i\theta})} = f(e^{-i\theta}) \right\} \tag{2.41}$$

where $\|f\|_{\mathcal{L}^2_{p\times p}(F)}$ is the norm induced by the inner-product

$$\langle f, g\rangle_{\mathcal{L}^2_{p\times p}(F)} := \frac{1}{2\pi}\mathrm{tr}\int_0^{2\pi} f^*(e^{i\theta})\,dF(\theta)\,g(e^{i\theta}) \tag{2.42}$$

where this is interpreted as a Lebesgue-Stieltjes integral.

The following linear mapping, which is known as the *Cramer* or *Kolmogorov isomorphism*, is defined

$$\begin{aligned} L : H^y &\to \mathcal{L}^2_{p\times p}(F) \\ L : Ay(t) &\mapsto A^T e^{-it\theta} \end{aligned} \tag{2.43}$$

where the mapping is extended by linearity (over $\mathbb{R}^{p\times p}$) and continuity (since bounded) to all of H^y. The similarity, in both form and function, to the Hilbert isomorphism is obvious.

Noting that

$$\begin{aligned} \langle Ay(s), By(t)\rangle_{H^y} &= \mathbb{E}\mathrm{tr}\ Ay(s)y(t)^T B^T \\ &= \mathrm{tr}\ A\Gamma_y(t-s)B^T \\ &\text{and} \\ \left\langle A^T e^{-is\theta}, B^T e^{-it\theta}\right\rangle_{\mathcal{L}^2_{p\times p}(F)} &= \frac{1}{2\pi}\mathrm{tr}\int_0^{2\pi} Ae^{is\theta}\,dF(\theta)\,B^T e^{-it\theta} \\ &= \frac{1}{2\pi}\mathrm{tr}\ A\left(\int_0^{2\pi} e^{i(s-t)\theta}\,dF(\theta)\right)B^T \\ &= \mathrm{tr}\ A\Gamma_y(t-s)B^T \end{aligned}$$

and hence it is clear that L is an isometric mapping. Since the exponential functions provide a basis (over $\mathbb{R}^{p\times p}$) for the latter space, and the process provides a basis (over $\mathbb{R}^{p\times p}$) for the former this shows that L is an isometric isomorphism.

Note 2.29 (Matrix Functions) *If $x \in H^y$ then x_i is composed of elements from each of the histories of the components $y_j(t)$ and the (i,j)th element of the representation describes this contribution.*

■

The Cramer Isomorphism has reduced the study of the second order properties (least squares prediction etc.) to that of the study of positive matrix-valued Borel measures, and the properties of linear maps on such weighted spaces. Important papers and books in the development of this point of view are [55, 63, 125, 126, 106, 56, 62, 54, 79, 29]. Many more modern papers have continued this effort, such as [76, 110, 111, 12, 66, 67, 131, 24, 118, 91, 23]. Thus characterizations of indeterminacy, predictability, minimality, the angle between the future and past, the remote future and past etc. have all been obtained in terms of properties of the Borel measure. This has proved a rich and complex area which has motivated several important developments in functional analysis and operator theory.

The inverse relationship to the Cramer Isomorphism is expressed in the Spectral Representation Theorem, again due to Cramer, which states that each random variable in the stochastic process $(x(n))_{n\in\mathbb{Z}}$ may be represented in the following stochastic integral form:

$$x_n = \int_0^{2\pi} e^{-in\theta} I_p \, d\zeta_x(\theta) \tag{2.44}$$

where ζ_x is a continuous-time orthogonal increment process which for each $\theta \in [0, 2\pi]$ is a random variable in H^x. See Caines pp.46–57 [16] for a discussion of this result. Such integrals have a simple construction avoiding any of the complications which occur when the integrand is also a random variable e.g. the Itô or Stratonovich integrals, Jazwinski [64].

The following result summarises some of the most important results from the papers of Wiener and Masani [125] and [126] which established this framework for stochastic processes. The result connects together the notion of spectral distribution, Wold decomposition and pure linear non-determinism.

Theorem 2.30 (Kolmogorov-Wiener-Masani (e.g. p36 [16])) *Let $(x(n))_{n\in\mathbb{Z}}$ be a p-component wide-sense stationary stochastic process, with forward and backward Wold decomposition as in Theorem 2.23. Let F, F_u, and F_v be the spectral distributions of the processes described in Theorem 2.23. Then*

(1) x is full rank if and only if

$$\frac{1}{2\pi}\int_0^{2\pi} \log\det\left(\frac{dF_u(\theta)}{dm}\right) d\theta > -\infty. \tag{2.45}$$

In this case

$$\begin{aligned} \det(\Sigma) &= \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \log\det\left(\frac{dF_a(\theta)}{dm}\right) d\theta\right] \\ &= \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \log\det\left(V(e^{i\theta})V^T(e^{-i\theta})\right) d\theta\right] \end{aligned} \tag{2.46}$$

which is known as the Szegö formula. V is defined by

$$V(e^{i\theta}) := \sum_{k=0}^{\infty} A_k \Sigma^{1/2} e^{ik\theta} := \lim_{\rho \nearrow 1} \sum_{k=0}^{\infty} A_k \Sigma^{1/2} \rho^k e^{ik\theta}$$

where the radial limit function exists a.e. and lies in $\mathcal{L}^2_{p\times p}(\mathbb{T})$.

(2) If x is full rank, the Lebesgue decomposition of F is given by

$$F = F_u + F_v$$

with F_u absolutely continuous and $\frac{dF_v}{dm} = 0$ a.e. (the Radon-Nikodym derivative with respect to the Lebesgue measure m on $\mathbb{T}$ e.g. see Rudin [113]). Also

$$F_u(e^{i\theta}) = \int_0^\theta \frac{dF_u}{dm}\, d\theta = \int_0^\theta V(e^{i\theta})V^T(e^{-i\theta})\, d\theta.$$

■

From this result it is clear that, for full rank processes, the spectral distribution is absolutely continuous (see Rudin [113] for definition) if and only if the underlying process is purely linearly non-deterministic.

Definition 2.31 (Spectral-Density) *If $(x(n))_{n\in\mathbb{Z}}$ is a full rank, wide sense stationary, purely linearly non-deterministic process then*

$$\frac{dF(\theta)}{dm} = \Phi(e^{i\theta}) \in \mathcal{L}^1_{p\times p}(\mathbb{T}) \tag{2.47}$$

where $\Phi(e^{i\theta})$ is the spectral-density function *of the process. Moreover*

$$\begin{aligned}\Gamma_y(t) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-it\theta}\, dF(\theta)\\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-it\theta}\Phi(e^{i\theta})\, d\theta\end{aligned}$$

and hence Φ has a Fourier series representation

$$\Phi(e^{i\theta}) = \sum_{k=-\infty}^{\infty} \Gamma_y(k)e^{ik\theta}. \tag{2.48}$$

The spectral-density is Hermitian and positive semi-definite.

■

Definition 2.32 (Finite Dimensional Process) *A stochastic process is said to be* finite-dimensional *of* order n *if its spectral density $\Phi(e^{i\theta})$ is rational of degree $2n$.*

■

The spectral-density of a full rank, wide sense stationary, purely linearly non-deterministic process may also be factorised as

$$\Phi(e^{i\theta}) = V(e^{i\theta})V(e^{i\theta})^* \text{ a.e.}$$

as given in the Kolmogorov-Wiener-Masani Theorem. $V(e^{i\theta})$ is called the *right spectral factor* of $\Phi(e^{i\theta})$ and may be extended to a function analytic in the disc. A similar factorisation exists in terms of the backwards Wold decomposition. Hence

$$\Phi(e^{i\theta}) = W(e^{i\theta})^*W(e^{i\theta}) \text{ a.e.}$$

where

$$W(e^{i\theta}) := \sum_{k=0}^{\infty} B_k\Sigma^{1/2}e^{ik\theta} := \lim_{\rho\nearrow 1}\sum_{k=0}^{\infty} B_k\Sigma^{1/2}\rho^k e^{ik\theta}.$$

Such relationships provide equalities of the associated Fourier series.

In the case where the spectral-density function is positive definite the uniqueness of the spectral factors is described by the following result. A more general version of this result is found in Masani Corollary 4.6 [87]. Refer to Wiener & Masani [125], Clancey & Gohberg [19] and Gohberg & Kaashoek [38] for further details and discussion of the general rôle of factorisation in various fields of applied mathematics.

Lemma 2.33 (Conditions for Existence of Spectral Factors e.g. [39]) *If $\Phi(e^{i\theta}) \in \mathcal{L}^{\infty}_{p\times p}(\mathbb{T})$ and is positive-definite on $\mathbb{T}$ then there exist right and left spectral factors V, W units in $\mathcal{H}^2_{p\times p}(\mathbb{D})$ such that*

$$V(e^{i\theta})V^T(e^{-i\theta}) = \Phi(e^{i\theta}) = W^T(e^{-i\theta})W(e^{i\theta})a.e.$$

and

$$\det(V(z)) \neq 0 \neq \det(W(z)),\ \forall |z| \leqslant 1.$$

The left (right) spectral factor is unique up to left (right) multiplication by a unitary matrix.

Proof : See Gohberg & Krein [39] §8. ∎

A similar development might be given for continuous-time stochastic processes. The major difference is that Bochner's Theorem is used instead of the Riesz-Herglotz Theorem 2.28, and functions are defined on the imaginary axis and left and right half-planes. However, let it suffice to say that a similar notion of spectral-density appears in the theory of continuous-time processes and in a similar manner to above this factorises as

$$\begin{aligned} \Phi_c(i\omega) &= V_c(i\omega)V_c^T(-i\omega) \\ &= W_c^T(-i\omega)W_c(i\omega) \end{aligned}$$

where the left and right spectral factors may be extended to analytic functions in the left and right half-planes.

2.3.3 Introduction to Canonical Correlation

The following well known construction, as developed by Hotelling [59, 60], will serve as a motivation for the construction described in the remainder of this subsection.

Given two sets of univariate random variables $(x_i)_{i=1}^l$ and $(y_j)_{j=1}^m$ (for definiteness let $l < m$) then two orthonormal bases $(u_i)_{i=1}^l$ of $\bigvee_{i=1}^l x_i$ and $(v_i)_{i=1}^m$ of $\bigvee_{j=1}^m y_j$ such that

$$\mathbb{E}(u_i v_j) = \sigma_i \delta_{ij},\ \sigma_i \in (0,1]$$

are sought. The σ_i are to be as large as possible, and in decreasing order.

To obtain these bases, stack the above sequences into column vectors

$$X^T := [x_1, \ldots, x_l]\,,\quad Y^T := [y_1, \ldots, y_m]$$

which for convenience are assumed to have non-singular covariance matrices N_x and N_y. The *normalised covariance matrix* is then

$$N = \mathbb{E} N_x^{-\frac{1}{2}} X Y^T N_y^{-\frac{1}{2}}.$$

This has singular value decomposition

$$N = UCV^T$$

where $U \in \mathrm{O}_l(\mathbb{R}),\ V \in \mathrm{O}_m(\mathbb{R})$ (groups of real orthogonal $l \times l$ and $m \times m$ matrices) and

$$C = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_l & 0 & \cdots & 0 \end{bmatrix}.$$

From the properties of the singular value decomposition it is clear that the random variables given by $U^T N_x^{-\frac{1}{2}} X$ and $V^T N_y^{-\frac{1}{2}} Y$ satisfy the requirements above and are hence called the *canonical variables* and the constants $\sigma_i \in (0,1]$ are the *canonical correlation coefficients.*

For another method of solution of this problem based on Lagrange' multipliers, and discussion of the statistical properties of the canonical correlation coefficients an excellent reference is Anderson [6] Chapter 12.

In the context of stochastic processes the following *correlation operator* will be the focus of study. It is related directly to the normalised covariance matrix above.

Throughout this section all processes will be taken to be full rank, wide sense stationary and purely linearly non-deterministic.

Definition 2.34 (Correlation Operator) *Given $(y(t))_{t\in\mathbb{Z}}$ a full rank, wide sense stationary and purely linearly non-deterministic, p-component stochastic process with forward normalised innovations $(f_t)_{t\in\mathbb{Z}}$ and backwards normalised innovations $\left(\bar{f}_t\right)_{t\in\mathbb{Z}}$ then the following is a weak definition of the* correlation operator *of the process:*

$$\begin{aligned} E_y : H^y &\to H^y \\ \left\langle E_y f_i, \bar{f}_j \right\rangle_{H^y} &= \begin{cases} \left\langle f_i, \bar{f}_j \right\rangle_{H^y} & i \leqslant 0 \ \&\ j > 0 \\ 0 & otherwise. \end{cases} \end{aligned} \tag{2.49}$$

This may be extended by linearity and continuity to all of H^y.

■

Lemma 2.35

$$E_y = \mathbf{P}_{\mathcal{F}^y} \mathbf{P}_{\mathcal{P}^y},$$

$$E_y^* = \mathbf{P}_{\mathcal{P}^y} \mathbf{P}_{\mathcal{F}^y}.$$

Proof : From Lemma 2.24

$$\mathcal{P}^y = \mathcal{P}^f \text{ and } \mathcal{F}^y = \mathcal{F}^{\bar{f}}$$

and

$$\left\langle E_y f_i, \bar{f}_j \right\rangle_{H^y} = \left\langle \mathbf{P}_{\mathcal{P}^y} f_i, \mathbf{P}_{\mathcal{F}^y} \bar{f}_i \right\rangle_{H^y}.$$

The second statement follows from the fact that the projections are orthogonal. ■

To analyse this structure further the frequency domain equivalent is now considered.

$$L(\mathcal{P}^y) = \bigvee_{\substack{k\geqslant 0 \\ A\in\mathbb{R}^{p\times p}}} Ae^{ik\theta} \leqslant \mathcal{L}^2_{p\times p}(F)$$

which is closed with respect to the norm $\|.\|_{\mathcal{L}^2_{p\times p}(F)}$, and which shall be denoted $\mathcal{H}^2_{p\times p}(F)$.

$$L(\mathcal{F}^y) = \bigvee_{\substack{k< 0 \\ A\in\mathbb{R}^{p\times p}}} Ae^{ik\theta} \leqslant \mathcal{L}^2_{p\times p}(F)$$

which shall be denoted $\mathcal{H}^2_{p\times p}(F)_-$. Due to the assumptions on $(y(t))_{t\in\mathbb{Z}}$ it is clear that

$$\mathcal{L}^2_{p\times p}(F) = \mathcal{H}^2_{p\times p}(F) \vee \mathcal{H}^2_{p\times p}(F)_-$$

but that these are not orthogonal e.g.

$$\Gamma_y(s+t) = \left\langle I_p e^{is\theta}, I_p e^{-it\theta} \right\rangle_{\mathcal{L}^2_{p\times p}(F)}, \forall s,t \in \mathbb{N}.$$

The following lemma describes the spaces $\mathcal{H}^2_{p\times p}(F)$ and $\mathcal{H}^2_{p\times p}(F)_-$ more fully.

Lemma 2.36 *If Φ is given by (2.47) and (2.48) and has left and right spectral factors V and W, then the following set equality holds between equivalence classes of almost everywhere equal matrix functions defined on $\mathbb{T}$ with analytic extensions into the interiors of $\mathbb{D}$:*

$$\begin{aligned} \mathcal{H}^2_{p\times p}(F) &= W^{-1}\mathcal{H}^2_{p\times p}(\mathbb{D}) := \left\{W^{-1}f \mid f \in \mathcal{H}^2_{p\times p}(\mathbb{D})\right\} \\ \mathcal{H}^2_{p\times p}(F)_- &= V^{-*}\mathcal{H}^2_{p\times p}(\bar{\mathbb{D}}^{\complement}) \end{aligned} \tag{2.50}$$

Proof : Let $f \in \mathcal{H}^2_{p\times p}(F)$, and $(p_k)_{k=1}^{\infty}$ be a sequence of matrix polynomials such that

$$\begin{aligned} \|p_n - f\|_{\mathcal{L}^2_{p\times p}(F)} &= \frac{1}{2\pi}\mathrm{tr}\int_0^{2\pi}(p_n-f)^*\Phi(e^{i\theta})(p_n-f)\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left\|(Wp_n)(e^{i\theta})-(Wf)(e^{i\theta})\right\|_2^2 d\theta \to 0 \text{ as } n\to\infty. \end{aligned}$$

That is, Wf is the $\mathcal{L}^2_{p\times p}(\mathbb{T})$ limit of Wp_n.

W and p_n have radius of convergence at least 1 and so Wp_n is analytic in the disc. Also

$$\sup_{r\leqslant 1}\int_0^{2\pi}\left\|W(re^{i\theta})p_n(re^{i\theta})\right\|_2^2 d\theta \leqslant \|W\|_2^2\sum_{k=1}^{d_n}\left\|p_n^{(k)}\right\|_2^2 < \infty$$

so $Wp_n \in \mathcal{H}^2_{p\times p}(\mathbb{D})$ and since this space is closed in $\mathcal{L}^2_{p\times p}(\mathbb{T})$ then $Wf \in \mathcal{H}^2_{p\times p}(\mathbb{D})$. Hence

$$Wf = g,\ g\in\mathcal{H}^2_{p\times p}(\mathbb{D}) \Rightarrow\ f = W^{-1}g$$

which is analytic inside the disc since W has no zeros inside $\mathbb{D}$. Hence, $\mathcal{H}^2_{p\times p}(F) \subseteq W^{-1}\mathcal{H}^2_{p\times p}(\mathbb{D})$.

Letting $g \in \mathcal{H}^2_{p\times p}(\mathbb{D})$ and noting from the Closure Theorem, Masani [87], that the closure of $W\mathcal{H}^2_{p\times p}(\mathbb{D})$ is just $\mathcal{H}^2_{p\times p}(\mathbb{D})$, then there exists a sequence of polynomials $(p_k)_{k=1}^{\infty}$ in $e^{i\theta}$ approximating g. That is

$$\frac{1}{2\pi}\int_0^{2\pi}\left\|g(e^{i\theta})-(Wp_n)(e^{i\theta})\right\|_2^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left\|(W^{-1}g)(e^{i\theta})-p_n(e^{i\theta})\right\|_2^2 d\theta \to 0 \text{ as } n\to\infty.$$

Hence $W^{-1}g \in \mathcal{H}^2_{p\times p}(F)$ by the definition above.

A dual argument gives the second result. ■

This result obviously justifies the notation used for $L(\mathcal{P}^y)$ and $L(\mathcal{F}^y)$. The following result extend this set equality.

Lemma 2.37 (Isometry Between Weighted and Unweighted Spaces) *With V and W defined as in Lemma 2.36, the mappings*

$$\begin{aligned} \mathbf{M}_W : \mathcal{L}^2_{p\times p}(F) &\to \mathcal{L}^2_{p\times p}(\mathbb{T}) \\ \mathbf{M}_W : f &\mapsto Wf \\ &\textit{and} \\ \mathbf{M}_{V^*} : \mathcal{L}^2_{p\times p}(F) &\to \mathcal{L}^2_{p\times p}(\mathbb{T}) \\ \mathbf{M}_{V^*} : f &\mapsto V^*f \end{aligned}$$

are isometric isomorphisms. Furthermore

$$\begin{aligned} \mathbf{M}_W\mathcal{H}^2_{p\times p}(F) &= \mathcal{H}^2_{p\times p}(\mathbb{D}) \\ &\textit{and} \\ \mathbf{M}_{V^*}\mathcal{H}^2_{p\times p}(F)_- &= \mathcal{H}^2_{p\times p}(\bar{\mathbb{D}}^{\complement}). \end{aligned}$$

Note: double-headed arrows running vertically represent isomorphisms
the other double-headed morphisms are surjections.

$$\begin{tikzcd}
\mathcal{L}^2_{p\times p}(F) \arrow[r, two heads, "\mathbf{P}_{\mathcal{H}^2_{p\times p}(F)}"] \arrow[d, two heads, "\mathbf{M}_W"'] & \mathcal{H}^2_{p\times p}(F) \arrow[d, two heads, "\mathbf{M}_W"] \\
\mathcal{L}^2_{p\times p}(\mathbb{T}) \arrow[r, two heads, "\mathbf{P}_+"'] & \mathcal{H}^2_{p\times p}(\mathbb{D})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{L}^2_{p\times p}(F) \arrow[r, two heads, "\mathbf{P}_{\mathcal{H}^2_{p\times p}(F)_-}"] \arrow[d, two heads, "\mathbf{M}_{V^*}"'] & \mathcal{H}^2_{p\times p}(F)_- \arrow[d, two heads, "\mathbf{M}_{V^*}"] \\
\mathcal{L}^2_{p\times p}(\mathbb{T}) \arrow[r, two heads, "\mathbf{P}_-"'] & \mathcal{H}^2_{p\times p}(\bar{\mathbb{D}}^{\complement})
\end{tikzcd}$$

Figure 2.1: Commuting Diagram Connecting Projections

Proof : Lemma 2.36 establishes the existence of a bijection between these sets. It remains to be checked that these are isometries.

For all $f, g \in \mathcal{L}^2_{p\times p}(F)$ then

$$\begin{aligned}
\langle f, g\rangle_{\mathcal{L}^2_{p\times p}(F)} &= \frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\, f^*\Phi g \, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\, (Wf)^*(Wg)\, d\theta \\
&= \langle Wf, Wg\rangle_{\mathcal{L}^2_{p\times p}(\mathbb{T})} \\
&= \langle \mathbf{M}_W f, \mathbf{M}_W g\rangle_{\mathcal{L}^2_{p\times p}(\mathbb{T})}
\end{aligned}$$

hence $\mathbf{M}_W$ is an isometry.

The final part is readily deduced from the first, and Lemmas 2.36 and 2.37. ∎

Lemma 2.38 (Projection Lemma)

$$\begin{aligned}
\mathbf{P}_{\mathcal{H}^2_{p\times p}(F)} &= \mathbf{M}_W^*\mathbf{P}_+\mathbf{M}_W, \\
\mathbf{P}_{\mathcal{H}^2_{p\times p}(F)_-} &= \mathbf{M}_{V^*}^*\mathbf{P}_-\mathbf{M}_{V^*}.
\end{aligned}$$

Proof : This follows from the commuting diagrams in Figure 2.1 and Lemma 2.37. ∎

Before demonstrating the structure of the correlation operator it is necessary to introduce a slightly modified definition of a Hankel operator to deal with the action upon spaces of matrix valued functions.

Definition 2.39 (Matrix Hankel Operators) *A matrix Hankel operator with symbol* $\phi \in \mathcal{L}^\infty_{p\times p}(\mathbb{T})$ *is a linear mapping*

$$\begin{aligned}
\mathbf{H}_\phi : \mathcal{H}^2_{p\times p}(\mathbb{D}) &\rightarrow \mathcal{H}^2_{p\times p}(\bar{\mathbb{D}}^{\complement}) \\
\mathbf{H}_\phi : f &\mapsto \mathbf{P}_-\phi f.
\end{aligned}$$

∎

Note 2.40 (Matrix Hankel Operators) *There are several important implications of the change in definition of a Hankel operator.*

The matrix space $\mathcal{L}^2_{p\times p}(\mathbb{T})$ *is isomorphic to*

$$\bigoplus_{i=1}^{p} \mathcal{L}^2_{p\times 1}(\mathbb{T})$$

and it is clear that the equivalent operator on this space is

$$\bigoplus_{i=1}^{p} \mathbf{H}_\phi$$

where

$$\mathbf{H}_\phi : \mathcal{L}^2_{p\times 1}(\mathbb{T}) \to \mathcal{L}^2_{p\times 1}(\mathbb{T}).$$

Thus the matrix Hankel operator has rank p times that of the same vector operator. Likewise, its spectrum will be identical to that of the vector operator, but with each eigenvalue occurring with p times greater multiplicity.

■

Analogues of Lemma 2.13 and Proposition 2.19 hold with the only change that $F_i \in \mathbb{R}^{n\times p}$ with the ith row arbitrary.

Lemma 2.41 (Canonical Correlation Operator)

$$E_y = L^*\mathbf{M}^*_{V^*}\mathbf{H}_{V^{-1}W^*}\mathbf{P}_+\mathbf{M}_W L$$

and hence $E_y\,|_{\mathcal{P}^y}$ is unitarily equivalent to $\mathbf{H}_{V^{-1}W^*}$.

Proof :

$$\begin{aligned}
\mathbf{P}_{\mathcal{F}^y}\mathbf{P}_{\mathcal{P}^y} &= L^*\mathbf{P}_{\mathcal{H}^2_{p\times p}(F)_-}LL^*\mathbf{P}_{\mathcal{H}^2_{p\times p}(F)}L \\
&= L^*\mathbf{M}^*_{V^*}\mathbf{P}_-\mathbf{M}_{V^*}\mathbf{M}^*_W\mathbf{P}_+\mathbf{M}_W L \\
&= (\mathbf{M}_W L)^*\,\mathbf{H}_{V^*W^{-1}}\mathbf{P}_+\,(\mathbf{M}_{V^*}L)\,.
\end{aligned}$$

Hence $E_y\,|_{\mathcal{P}^y}$ is unitarily equivalent to $\mathbf{H}_{V^*W^{-1}}$ and

$$V^{-1}W^* = V^{-1}(VV^*)(W^*W)^{-1}W^* = V^*W^{-1}.$$

■

Definition 2.42 (Phase Factor) *If $\Phi(e^{i\theta}) > 0_{p\times p}$ is the spectral-density of a stochastic process with spectral factors V and W then a* phase factor *of the process is*

$$\phi := V^{-1}W^* = V^*W^{-1}.$$

■

Note 2.43 (Uniqueness of Phase Factors) *There is inherent non-uniqueness in this notion as any choice of spectral factors will lead to a phase factor. Hence, from the uniqueness result in Lemma 2.33*

$$TV^{-1}W^*U$$

is a phase factor for the same process for any $T, U \in \mathrm{O}_p(\mathbb{R})$.

■

Definition 2.44 (Toeplitz Operators) *A* matrix Toeplitz Operator *is a linear map*

$$\begin{aligned} \mathbf{T}_\phi : \mathcal{H}^2_{p\times p}(\mathbb{D}) &\rightarrow \mathcal{H}^2_{p\times p}(\mathbb{D}) \\ \mathbf{T}_\phi : f &\mapsto \mathbf{P}_+ \phi f \end{aligned}$$

where ϕ is an element of $\mathcal{L}^\infty_{p\times p}(\mathbb{T})$ and is called the symbol *of the operator.*

A dual matrix Toeplitz Operator *is a linear map*

$$\begin{aligned} \tilde{\mathbf{T}}_\phi : \mathcal{H}^2_{p\times p}(\bar{\mathbb{D}}^{\complement}) &\rightarrow \mathcal{H}^2_{p\times p}(\bar{\mathbb{D}}^{\complement}) \\ \tilde{\mathbf{T}}_\phi : f &\mapsto \mathbf{P}_- \phi f \end{aligned}$$

where ϕ is an element of $\mathcal{L}^\infty_{p\times p}(\mathbb{T})$ and is called the symbol *of the operator.*

■

The following is a simplified criterion for the existence of bounded inverses to Toeplitz operators. For a more comprehensive picture of the available results on this area see Böttcher & Silbermann [13] and Nikol'skiĭ [94].

Lemma 2.45 *If A, B are units in $\mathcal{H}^\infty_{p\times p}(\mathbb{D})$ and $\phi = A^*B$ then $\mathbf{T}_\phi$ is boundedly invertible, and $\mathbf{T}_\phi^{-1} = \mathbf{T}_{\phi^{-1}}$.*

Proof :

$$\begin{aligned} \mathbf{T}_\phi &= \mathbf{T}_{A^*B} = \mathbf{P}_+\mathbf{M}_{A^*B}\mathbf{P}_+ \\ &= \mathbf{P}_+\mathbf{M}_{A^*}\mathbf{M}_B\mathbf{P}_+ \\ &= (\mathbf{P}_+\mathbf{M}_{A^*}\mathbf{P}_+)(\mathbf{P}_+\mathbf{M}_B\mathbf{P}_+) \text{ since } B\mathcal{H}^2_{p\times p}(\mathbb{D}) \subseteq \mathcal{H}^2_{p\times p}(\mathbb{D}) \\ &= \mathbf{T}_{A^*}\mathbf{T}_B \\ &= \mathbf{T}_A^*\mathbf{T}_B \end{aligned}$$

As $\mathbf{T}_A^*$ is invertible if and only if $\mathbf{T}_A$ is invertible, and has the same norm, then it suffices to show that $\mathbf{T}_A$ and $\mathbf{T}_B$ are boundedly invertible, as

$$\mathbf{T}_\phi^{-1} = (\mathbf{T}_A^*\mathbf{T}_B)^{-1} = \mathbf{T}_B^{-1}\mathbf{T}_A^{-*}.$$

For any unit C in $\mathcal{H}^\infty_{p\times p}(\mathbb{D})$ then $\mathbf{T}_{C^{-1}}$ is bounded as $\|\mathbf{T}_{C^{-1}}\| = \|C^{-1}\|_\infty < \infty$ and for all $f \in \mathcal{H}^2_{p\times p}(\mathbb{D})$

$$\mathbf{T}_{C^{-1}}\mathbf{T}_C f = \mathbf{P}_+\mathbf{M}_{C^{-1}}\mathbf{P}_+ C f = \mathbf{P}_+ f = f = \mathbf{T}_C\mathbf{T}_{C^{-1}} f$$

since $Cf \in \mathcal{H}^2_{p\times p}(\mathbb{D})$. Hence $\mathbf{T}_C$ is boundedly invertible. ■

Lemma 2.46 *If the spectral factors of a stochastic process are such that $W^{-1}, V^{-1} \in \mathcal{H}^\infty_{p\times p}(\mathbb{D})$ then*

$$\mathbf{H}_{V^{-1}W^*} = \tilde{\mathbf{T}}_{V^{-1}}\mathbf{H}_\Phi\mathbf{T}_{W^{-1}} = \tilde{\mathbf{T}}_V^{-1}\mathbf{H}_\Phi\mathbf{T}_W^{-1}$$

and

$$R\mathbf{H}_{V^{-1}W^*} = \mathbf{T}_{V^{-1}(e^{-i\theta})}(R\mathbf{H}_\Phi)\mathbf{T}_{W^{-1}} = \mathbf{T}_{V(e^{-i\theta})}^{-1}(R\mathbf{H}_\Phi)\mathbf{T}_W^{-1}$$

Proof : Using the definition of the various operators, and the natural embedding of $\mathcal{H}^2(\mathbb{D})$ into $\mathcal{L}^2(\mathbb{T})$:

$$\begin{aligned}\tilde{\mathbf{T}}_{V^{-1}}\mathbf{H}_\Phi\mathbf{T}_{W^{-1}} &= (\mathbf{P}_-\mathbf{M}_{V^{-1}}\mathbf{P}_-)(\mathbf{P}_-\mathbf{M}_\Phi\mathbf{P}_+)(\mathbf{P}_+\mathbf{M}_{W^{-1}}\mathbf{P}_+)\\ &= \mathbf{P}_-\mathbf{M}_{V^{-1}}\mathbf{P}_-\mathbf{M}_\Phi\mathbf{P}_+\mathbf{M}_{W^{-1}}\mathbf{P}_+\\ &= \mathbf{P}_-\mathbf{M}_{V^{-1}}\mathbf{P}_-\mathbf{M}_\Phi\mathbf{M}_{W^{-1}}\mathbf{P}_+\end{aligned}$$

using the idempotence of orthogonal projections, and the observation that $W^{-1} \in \mathcal{H}^\infty_{p\times p}(\mathbb{D})$ and hence $W^{-1}\mathcal{H}^2_{p\times p}(\mathbb{D}) \subseteq \mathcal{H}^2_{p\times p}(\mathbb{D})$. Also

$$\begin{aligned}\tilde{\mathbf{T}}_{V^{-1}}\mathbf{H}_\Phi\mathbf{T}_{W^{-1}} &= \mathbf{P}_-\mathbf{M}_{V^{-1}}(I_{\mathcal{L}^2_{p\times p}(F)} - \mathbf{P}_+)\mathbf{M}_{\Phi W^{-1}}\mathbf{P}_+\\ &= \mathbf{P}_-\mathbf{M}_{V^{-1}}\mathbf{M}_{\Phi W^{-1}}\mathbf{P}_+ - \mathbf{P}_-\mathbf{M}_{V^{-1}}\mathbf{P}_+\mathbf{M}_{\Phi W^{-1}}\mathbf{P}_+\\ &= \mathbf{P}_-\mathbf{M}_{V^{-1}\Phi W^{-1}}\mathbf{P}_+\\ &= \mathbf{P}_-\mathbf{M}_{V^{-1}W^*}\mathbf{P}_+\\ &= \mathbf{H}_{V^{-1}W^*},\end{aligned}$$

where $V^{-1}\mathcal{H}^2_{p\times p}(\mathbb{D}) \subseteq \mathcal{H}^2_{p\times p}(\mathbb{D})$ implies $\mathbf{P}_-\mathbf{M}_{V^{-1}}$ is identically zero on $\mathcal{H}^2_{p\times p}(\mathbb{D})$.

The second equality is a consequence of the assumption that $V^{-1} \in \mathcal{H}^\infty_{p\times p}(\mathbb{D})$, in that

$$\begin{aligned}(\mathbf{P}_+\mathbf{M}_{V^{-1}}\mathbf{P}_+)(\mathbf{P}_+\mathbf{M}_V\mathbf{P}_+) &= \mathbf{P}_+\mathbf{M}_{V^{-1}}\mathbf{P}_+\mathbf{M}_V\mathbf{P}_+\\ &= \mathbf{P}_+\mathbf{M}_{V^{-1}}\mathbf{M}_V\mathbf{P}_+\\ &= I_{\mathcal{H}^2_{p\times p}(\mathbb{D})}\end{aligned}$$

where the second line is obtained from the assumption mentioned. Hence $\mathbf{T}_{V^{-1}}$ is the inverse of $\mathbf{T}_V$ on $\mathcal{H}^2_{p\times p}(\mathbb{D})$.

The second result uses the properties of R given in Lemma 2.16

$$\begin{aligned}R\tilde{\mathbf{T}}_{V^{-1}} &= R\mathbf{P}_-\mathbf{M}_{V^{-1}}\mathbf{P}_-\\ &= \mathbf{P}_+R\mathbf{M}_{V^{-1}}\mathbf{P}_-\\ &= \mathbf{P}_+\mathbf{M}_{V^{-1}(e^{-i\theta})}\mathbf{P}_+R.\end{aligned}$$

■

Lemma 2.47

$$\tilde{\mathbf{H}}_{V^{-1}W^*} = \mathbf{T}_{V^*}\tilde{\mathbf{H}}_{\Phi^{-1}}\tilde{\mathbf{T}}_{W^*}$$

and

$$\tilde{\mathbf{H}}_{V^{-1}W^*}R = \mathbf{T}_{V^*}\left(R\mathbf{H}_{\Phi^{-1}}\right)\mathbf{T}_{W^T}.$$

Proof : This follows the pattern of the previous result upon noting that

$$V^{-1}W^* = V^*\left(V^{-*}V^{-1}\right)W^* = V^*\Phi^{-1}W^*.$$

■

Definition 2.48 (Spectral Summand) *If*

$$\Phi(e^{i\theta}) = \sum_{k=-\infty}^{\infty} \Gamma_y(k)e^{ik\theta} > 0, \ \forall\theta \in (0, 2\pi]$$

is the spectral-density of a stochastic process then the spectral summand *of the process is given by*

$$Z(z) := \frac{\Gamma_y(0)}{2} + \sum_{k=1}^{\infty}\Gamma_y(k)z^k$$

which is analytic in $\mathbb{D}$*. Obviously, this is closely connected to* $\mathbf{P}_+\Phi$*.*

■

2.3.4 Relationship to Mutual Information

The information contained in one random variable about another has been the subject of much study and has given rise to the subject of Information Theory. The short summary presented here is taken from the paper by Gel'fand and Yaglom [34] in which they extend the work of Shannon [115] on this subject.

Definition 2.49 (Information Between Random Variables) *If ξ and η are random variables (possibly vectors) on $(\Omega, \mathcal{F}, \mathbb{P})$ with probability densities $p_\xi(x)$, $p_\eta(y)$ and joint density $p_{\xi\eta}(x, y)$ then the* mutual information *is*

$$J(\xi, \eta) := \int\int p_{\xi\eta}(x, y) \log\left(\frac{p_{\xi\eta}(x, y)}{p_\xi(x) p_\eta(y)}\right) dx\, dy,$$

assuming that the density is never zero (see [34] for the fully general case).

The information between two infinite sets of random variables $x = (x(n))_{n\in\mathbb{Z}}$ and $y = (y(t))_{t\in\mathbb{Z}}$ is defined as the supremum of the information over all finite subsets of the space generated by the elements i.e.

$$J(x, y) := \sup\left\{J\left((u_1, \ldots, u_k), (v_1, \ldots, v_l)\right) \mid u_i \in H^x,\ i = 1, \ldots, k,\ v_j \in H^y,\ j = 1, \ldots, l,\ \forall k, l \in \mathbb{N}\right\}.$$

∎

The notion of information depends upon the complete probability distribution of the random variables involved and so goes beyond the second-order properties discussed up to this point. However, Gaussian processes are specified only by their first two moments, so in that case the two points of view coincide. Hence, for the remainder of this subsection it will be assumed that the processes and random variables under consideration are Gaussian (see Ibragimov & Rozanov [63] for an encyclopaedic treatment of such processes, including this material).

Theorem 2.50 (Information Between Sets of Gaussian Random Variables [34]) *If*

$$\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_k \end{bmatrix}, \eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_l \end{bmatrix} \quad \text{and } \zeta = \begin{bmatrix} \xi^T & \eta^T \end{bmatrix}^T$$

are jointly normally distributed random vectors with zero mean and

$$A = \mathbb{E}(\xi\xi^T),\ B = \mathbb{E}(\eta\eta^T) \text{ and } C = \mathbb{E}(\zeta\zeta^T)$$

are non-singular then

$$J(\xi, \eta) = \frac{1}{2} \log \frac{\det A \det B}{\det C}.$$

∎

This can be extended in the case of stochastic processes.

Theorem 2.51 (Information Between Past and Future [34] Theorem 2.3) *If $(y(t))_{t\in\mathbb{Z}}$ is a finite-dimensional Gaussian stochastic process of order n, then using the notation of the previous subsections,*

$$\begin{aligned} J\left((y_i)_{i\leqslant 0}, (y_j)_{j>0}\right) &= -\frac{1}{2} \log \det\left(I_{H^y} - E_y^* E_y\right) \\ &= -\frac{1}{2} \sum_{i=1}^{n} \log(1 - \sigma_i(E_y)^2). \end{aligned}$$

■

Desai and Pal [21, 22] have related this measure to the solution of two matrix quadratic equations, as will be discussed in §4.5.1. This has been suggested as a basis for modelling stochastic processes by those of smaller order. This may also be exploited to provide a method of stochastic realisation, as described in [21].

While this is an attractive method for constructing simple models of stochastic processes, this method is not universally applicable. In Arun's thesis [74] it is argued that this model reduction scheme is not suitable for models whose purpose is prediction. This is because of the renormalisation that is inherent in canonical correlation analysis. From Lemma 2.46 it is clear that the canonical correlation operator E_y is unitarily equivalent to $\tilde{\mathbf{T}}_{V^{-1}}\mathbf{H}_\Phi\mathbf{T}_{W^{-1}}$. Thus, this operator is equivalent to the covariance Gramian between the past and future of the process weighted by the inverse of right and left spectral factors respectively. This corresponds to the inner product Gramian between the forward innovations of the past and backward innovations of the future. Thus implicitly the method renormalises the future and the model reduction procedure selects elements of the future which are important after this normalization. The result may then be substantially different from the important unweighted components of the future used for prediction.

Chapter 3

Modelling and Identification Problems

In this brief Chapter the first step in the identification schema of §1.2 is carried out without orientation to one particular type of application or problem. This involves determining the general classes of models to be used in the subsequent sections.

This Chapter's contribution, other than a brief summary of the relevant theory, is to formulate the general model classes such that they will be readily parameterised in subsequent Chapters.

3.1 Modelling

3.1.1 Time-Series

Consideration of equations (2.20) and (2.21) in conjunction with the Wold decomposition of a p-component, wide sense stationary, purely linearly non-deterministic stochastic process $(y(t))_{t\in\mathbb{Z}}$ with a rational spectrum suggests that the stochastic process might be represented as

$$\left.\begin{array}{rcl} \xi(k+1) & = & A\xi(k)+Fw(k) \\ y(k) & = & C\xi(k)+Gw(k) \end{array}\right\} \tag{3.1}$$

for all $k \in \mathbb{Z}$, where

$$\lambda(A) \subseteq \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$
$$(A, F, C, G) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m} \text{ is minimal and}$$

$(w(k))_{k\in\mathbb{Z}}$ is an m-component wide sense stationary orthogonal process ($p \leqslant m$). Such a representation is termed a *strong stochastic state-space realisation* and a discussion of the existence of such a representation may be found in Chapter 4 of Caines [16].

If the process is also full-rank, then it may be shown that there exist n and p-component processes $(x(n))_{n\in\mathbb{Z}}$ and $(\nu(k))_{k\in\mathbb{Z}}$ such that

$$\left.\begin{array}{rcl} x(k+1) & = & Ax(k)+B\nu(k) \\ y(k) & = & Cx(k)+\nu(k) \end{array}\right\} \tag{3.2}$$

with $\nu(k) = Df(k)$, $f(k)$ the normalised forward innovations sequence of the process, $D \in \mathbb{R}^{p\times p}$ positive-definite, $B \in \mathbb{R}^{n\times p}$. Such a representation is termed an *innovations representation.*

Under the assumption of stationarity of the predictor i.e. that it has been running from the distant past, the optimal one-step-ahead predictor of a system in the form of (3.2) is given by the process with the state-space realisation

$$\left.\begin{array}{rcl} \hat{x}(k+1) & = & A\hat{x}(k)+\Delta(y(k)-C\hat{x}(k)) \\ & = & (A-\Delta C)\hat{x}(k)+\Delta y(k) \\ \hat{y}(k) & = & C\hat{x}(k), \end{array}\right\} \tag{3.3}$$

where

$$\Delta = \Delta(A,B,C,D) = (AV_\infty C^T + BDD^T)(CV_\infty C^T + DD^T)^{-1}, \tag{3.4}$$

and V_∞ is a positive semi-definite solution of

$$V_\infty = (A-\Delta C)V_\infty(A-\Delta C)^T + BDD^TB^T + \Delta DD^T\Delta^T - \Delta DD^TB^T - BDD^T\Delta^T. \tag{3.5}$$

For a process of the form (3.2) $V_\infty = 0$, $\Delta = B$ are solutions to (3.4) and (3.5). Since $\lambda(A) \subseteq \mathbb{D}$ then, by standard results on solutions of the discrete filter Riccati equation (e.g. [16] Chapter 3), this is the unique non-negative definite solution to the Riccati equation (3.5). Thus

$$\left.\begin{array}{rcl} \hat{x}_{k+1} & = & A\hat{x}_k + B(y_k - C\hat{x}_k) \\ & = & (A-BC)\hat{x}_k + By_k \\ \hat{y}_k & = & C\hat{x}_k, \end{array}\right\} \tag{3.6}$$

is the one-step-ahead predictor for the system (3.2).

It is well known (e.g. [16] Chapter 4) that this system is stable, that is $\lambda(A - BC) \subseteq \mathbb{D}$.

Two particular classes of systems will be of importance in this context.

Definition 3.1 (Minimum-Phase Realisations) *If* $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times p} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times p}$ *is minimal with* $D \in \mathrm{GL}_p(\mathbb{R})$ *then* (A, B, C, D) *is said to be a* discrete-time minimum-phase *realisation if both* $\lambda(A) \subseteq \mathbb{D}$ *and* $\lambda(A - BD^{-1}C) \subseteq \mathbb{D}$. *The set of all such realisations will be denoted* $\mathrm{DM}_\mathrm{n}^\mathrm{m}$, *and the set of associated transfer functions will be denoted* $\mathrm{TDM}_\mathrm{n}^\mathrm{m}$.

Such a realisation is said to be a continuous-time minimum-phase *realisation if both* $\lambda(A) \subseteq \mathbb{C}_-$ *and* $\lambda(A - BD^{-1}C) \subseteq \mathbb{C}_-$. *The set of such realisations will be denoted* $\mathrm{M}_\mathrm{n}^\mathrm{m}$ *with transfer functions* $\mathrm{TM}_\mathrm{n}^\mathrm{m}$.

■

Using the notation of this definition, and noting the stability of A and $A - BC$, it is clear that the class of models of systems in the form given by (3.2) is given by

$$\mathrm{Innov}_\mathrm{n}^\mathrm{p} = \{(A, B, C, I) \in \mathrm{DM}_\mathrm{n}^\mathrm{p}\} \times \{D \in \mathrm{GL}_p(\mathbb{R}) \mid D > 0\}$$

and that the class of predictors of such models is given by

$$\mathrm{Pred}_\mathrm{n}^\mathrm{p} = \{(A, B, C) \mid (A, B, C, I) \in \mathrm{DM}_\mathrm{n}^\mathrm{p}\}.$$

Note 3.2 *No parameters relating to the covariance of the innovations appear in the parametrisation of the predictors. This is due to the fact that the optimal predictor for such systems is independent of the noise structure.*

■

3.1.2 ARMAX Systems

ARMAX models combine both deterministic and stochastic components in one model. However, this is only done in a simple additive fashion which does not allow for feedback between the components.

An ARMAX process is a stochastic process $(y(t))_{t\in\mathbb{Z}}$ which satisfies a vector difference equation

$$\sum_{i=0}^{n} A(i)y(t-i) = \sum_{j=1}^{n} B(j)u(t-j) + \sum_{k=1}^{n} C(k)e(t-k)$$

for some deterministic p-dimensional sequence $(u(t))_{t\in\mathbb{Z}}$ and $(e(t))_{t\in\mathbb{Z}}$ an identity covariance white-noise process. Their existence, and several other representations are discussed in Chapter 7 of [16]. For such processes to be stationary it is sufficient that $\det \sum_{i=1}^{n} A(i)z^i$ has no zeros in $\bar{\mathbb{D}}$. Also, in an analogous manner to the previous section, the representation of the noise component may be chosen so that $\det \sum_{k=1}^{n} C(k)z^i$ only has zeros outside $\mathbb{D}$. To avoid complications this may be further restricted to having zeros in $\bar{\mathbb{D}}^\mathsf{C}$ (see Hannan & Deistler [46] p25 and §2.8).

Such a system may be rewritten in terms of the delay operator S (definition in §2.2.1) as

$$y(t) = G(S)u(t) + H(S)e(t) \tag{3.7}$$

where $G(S) = A^{-1}(S)B(S)$ and $H(S) = A^{-1}(S)C(S)$ (these operators being defined using a suitable functional calculus). By the assumptions made in the previous paragraph $G(z)$, $H(z)$

and $H^{-1}(z)$ are analytic in $\mathbb{D}$. From this representation the steady-state one-step-ahead predictor may be derived e.g. p195 Söderström & Stoica [116]

$$\hat{y}(t|t-1) = H^{-1}(S)G(S)u(t) + \left(I - H^{-1}(S)\right) y(t). \tag{3.8}$$

It is also possible to write the system as a stochastic state-space system

$$\left.\begin{array}{rcl} x(t+1) & = & Ax(t) + Gu(t) + B\nu(t) \\ y(t) & = & Cx(t) + \nu(t) \end{array}\right\} \tag{3.9}$$

where $(A, [G, B], C, I)$ is minimal. Using the equations (3.4) and (3.5), a one-step-ahead predictor

$$\left.\begin{array}{rcl} \hat{x}(t+1|t) & = & (A - BC)\,\hat{x}(t|t-1) + Gu(t) + By(t) \\ \hat{y}(t|t-1) & = & C\hat{x}(t|t-1) \end{array}\right\} \tag{3.10}$$

may be constructed (see p.429 [16]).

Both the vector-difference and the stochastic state-space representations assume that the noise process and the deterministic process share the same dynamics given by the denominator polynomial $A(S)$. Any differences are accounted for by pole-zero cancellations or by non-minimality of the realisations (A, G, C) and (A, B, C, I).

It will be expedient, for the purposes of parametrisation, to assume that the process is described by (3.7) with $G \in \mathrm{TDC}_{\mathrm{n}}^{\mathrm{p,m}}$ and $H \in \mathrm{Innov}_{\mathrm{n}}^{\mathrm{p}}$ and that each has a state-space realisation as described in the following Chapter. This is algebraically equivalent to both of the other representations.

Hence, the set of models used in Chapter 7 will be

$$\mathrm{ARMAX}_{\mathrm{n}}^{\mathrm{p,m}} := \{(A, B, C, 0) \in \mathrm{DC}_{\mathrm{n}}^{\mathrm{p,m}}\} \times \mathrm{Innov}_{\mathrm{n}}^{\mathrm{p}}. \tag{3.11}$$

3.2 Conclusions

In this Chapter two important classes of models and various representations for each have been discussed. The state-space representations of these models will be exploited in the remainder of the thesis. While all of the available alternative model types have advantages of their own, the state-space representation is preferred because of its relationship to associated Hankel operators (see Lemma 2.13 and Proposition 2.19). In addition, such models are now the most commonly used in control system design, which is an important application of any model. In addition many of the deeper relationships between these models and other objects of interest can most readily (some necessarily) be expresses in terms of the state-space representation.

Chapter 4

Parametrisation Problems

This Chapter sets out an approach to the parametrisation problems proposed as the second part of the schema in §1.2. Parametrisation results are presented for the classes of minimum-phase systems $\mathrm{DM}_{\mathrm{n}}^{\mathrm{p}}$ (hence $\mathrm{Innov}_{\mathrm{n}}^{\mathrm{p}}$ and $\mathrm{Pred}_{\mathrm{n}}^{\mathrm{p}}$) as well as for classes of stable and positive-real transfer functions (see §4.2 for definition).

The parameters considered in this Chapter are meaningful in that they are associated with some important property of the system under consideration. The complete elaboration of this will not be completed until the end of the subsequent Chapter, and until that point some heuristic arguments (e.g. Remark 4.5) will have to suffice to indicate the significance of the chosen parameters.

The approach taken is to develop state-space canonical forms for the associated continuous-time systems. It is possible to exploit the simple structure of the algebraic Riccati equations associated with continuous-time systems so as to derive state-space canonical forms with simple parametrisations.

In §4.5 the structure of the canonical form is exploited to produce error bounds for a specific approximation scheme based on balanced truncation. The error bounds are obtained by relating a minimum-phase balanced realisation to a stochastically balanced realisation thus allowing the application of the error bounds known for that class.

Many of the results of this Chapter are highly derivative although most are new in particulars. Hence the presentation will be relatively brief with references to more complete treatments.

4.1 Introduction

Much of the analysis below concerns the choice of bases in which to represent various linear operators. This leads to a relatively large amount of detailed and specific manipulation. However, despite their elementary nature, the results of this Chapter provide insights which are used in the later Chapters.

As stated in the conclusion of the previous Chapter, minimal state-space representations of linear time-invariant systems will be used in the remainder of this thesis. This choice has the serious drawback that any two elements of the same $\sim$-equivalence class (see §2.2.1 p15 for the definition of this equivalence relation) represent the same linear system, and hence such models are indistinguishable from their input-output behaviour. This problem is one of *identifiability* i.e. the map from the parameterising set (here the set of minimal state-space realisations) to the set of systems is not injective. However, as indicated in §2.2.1, $\mathrm{L}_{\mathrm{n}}^{\mathrm{p,m}}/\sim$ is isomorphic to $\mathrm{TL}_{\mathrm{n}}^{\mathrm{p,m}}$. To overcome this problem of identifiability there are two common approaches: the overlapping parametrisation method involving the construction of a C^1-atlas of local co-ordinate functions covering $\mathrm{TL}_{\mathrm{n}}^{\mathrm{p,m}}$ (see Hu [61] for the relevant definitions); or the construction a single *canonical form* for the relation $\sim$.

Definition 4.1 (Canonical Form) *If $M \subseteq \mathrm{L}_{\mathrm{n}}^{\mathrm{p,m}}$, then*

$$\Gamma : M \to M$$

is a $\sim$-canonical form if

$$\forall (A,B,C,D) \in M, \Gamma\left((A,B,C,D)\right) \sim (A,B,C,D)$$

and

$$\forall (A,B,C,D), (\hat{A},\hat{B},\hat{C},\hat{D}) \in M,$$
$$(A,B,C,D) \sim (\hat{A},\hat{B},\hat{C},\hat{D}) \Leftrightarrow \Gamma\left((A,B,C,D)\right) = \Gamma\left((\hat{A},\hat{B},\hat{C},\hat{D})\right).$$

■

The overlapping parametrisation method has been strongly recommended in Glover & Willems [37] and Clark [20]. The atlas is constructed from a collection of locally identifiable open sets which together cover the manifold of systems. This approach has the advantage that any model has at least one co-ordinate function in which its realisation is well conditioned, so avoiding certain numerical difficulties. There are difficulties moving from one such set of co-ordinates to another, but several algorithms have been proposed to deal with these e.g. [20].

While it has been shown by Hazewinkel and Kalman [50] that there is no global continuous canonical forms for $\mathrm{L}_{\mathrm{n}}^{\mathrm{p,m}}$ if $\min(m, p) > 1$, it is possible to construct continuous canonical forms which divide the underlying manifold up into several pieces. In Helmke & Hinrichsen [52] many topological properties of such canonical forms are discussed. These representations allow study of the whole structure of the manifold via the connections between the cells. This is an alternative to the co-ordinate transformation maps of the atlas although subject to numerical problems [37]. For example, Ober [101] uses such cell decompositions to study the connectivity properties of various manifolds of systems.

The difference between the two approaches should not be overstated, bacause the atlas method requires a sufficiently rich class of local canonical forms as its coordinate charts which are often dense in the underlying manifold. The largest cell of a cell decomposition is often dense in the underlying manifold. Hence, this cell, in combination with a rich class of automorphisms of the manifold may provide an atlas. A simple example of automorphisms of the manifold is left multiplication of the transfer function by a non-singular matrix which, for the purposes of identification, is equivalent to transforming the data by the inverse of that matrix.

The approach taken in this thesis will be to construct canonical forms for the state-space realisations of the classes $\mathrm{TM}_{\mathrm{n}}^{\mathrm{p}}$, $\mathrm{TC}_{\mathrm{n}}^{\mathrm{p}}$ and $\mathrm{TP}_{\mathrm{n}}^{\mathrm{p}}$ (defined in the next section). Balanced realisations of several types will be used as a starting point for the canonical forms both because of their connections with Hankel operators and for their model reduction properties.

The following attempt to overcome the problem of identifiability of state-space realisations immediately leads to a solution to the problem of parametrising the chosen model classes. This is a partial reply to the objection to state-space realisations in terms of identifiability. The results shown in the later sections of this thesis may also be seen as a *posteriori* justification for the approach presented here. The approach to the derivation of canonical forms presented in this Chapter is readily adapted to other contexts which suggests that the technique might be applied elsewhere. A further advantage of the similarity of the canonical forms derived below is that it is then possible to derive connections between the parameters of each. These relationships are explored in Chapter 5.

4.2 Balanced Realisations

4.2.1 Non-Symmetric Case

In Chapter 2 the idea of considering linear systems as mappings from sequences of inputs to sequences of outputs was discussed. It is in this context that the notion of balancing state-space realisations of linear systems, as introduced by Moore [92], is seen as a natural choice for a canonical realisation. It can be shown, e.g. §2.3 Glover [35], that the solution to the Discrete-time Control Lyapunov Equation (2.23) is related to the energy transferred to the state at time 0 from the input up to time 0 of a system with realisation (A, B, C, D). The solution to (2.24) is related to the energy transferred to the output from time 0 onwards from the state at time 0 of a system with realisation (A, B, C, D). Hence the balanced realisation keeps these two quantities matched, and so the state is "balanced" with respect to the inputs and the outputs.

The principal reason for interest in balanced realisations is because of their model reduction properties i.e. that certain subsystems of balanced systems are also minimal, stable and balanced, and that the $\mathcal{L}^\infty$-error induced by this truncation is simply expressed in terms of the Hankel singular values (see Pernebo & Silverman [107], Enns [30], and Glover [35] for these results).

Balanced realisations are also intimately related to stability theory via the two Lyapunov equations which characterise them.

Balanced realisations also have convenient uniqueness properties (see Proposition 4.11) which shall be exploited in this Chapter. Also, they have an important relationship to Hankel operators, as shown in Lemma 2.13, and are often used in other analytical techniques.

In the following subsections two other forms of balancing the state-space realisations of linear systems are considered, one of which is novel. The basic properties of these realisations are studied, and error bounds derived for a model reduction scheme based upon the second form. Further structural results will be demonstrated for the subclass of systems which are symmetric.

The first concern of control is for the stability of the systems under consideration. It is therefore natural for control theory to study the class of stable systems and Lyapunov balancing. As discussed in Chapter 2 there is no relevant concept of stability for stochastic processes. However, it was demonstrated that the spectral-density, spectral factors and phase factor were all important in the development of representations of such systems. This provides motivation for the development of balancing schemes appropriate to these functions.

The following property characterises a rational spectral summand.

Definition 4.2 (Positive-Real Realisation) *A transfer function* $G \in \mathrm{TC}^{\mathrm{p,p}}_{\mathrm{n}}$ *is said to be* continuous-time positive-real *if*

$$G(i\omega) + G(-i\omega)^T > 0, \;\; \omega \in \mathbb{R} \cup \{\infty\}.$$

Let $\mathrm{TP}^{\mathrm{p}}_{\mathrm{n}}$ *denote the set of all such positive-real transfer functions, and let* $\mathrm{P}^{\mathrm{p}}_{\mathrm{n}}$ *be their set of realisations.*

$G(z) \in \mathrm{TDC}^{\mathrm{p,p}}_{\mathrm{n}}$ *is called* discrete-time positive-real *if*

$$G(e^{i\theta}) + G^T(e^{-i\theta}) > 0, \;\; \theta \in [0, 2\pi).$$

The subset of $\mathrm{TDC}^{\mathrm{p,p}}_{\mathrm{n}}$ *containing all discrete-time positive-real systems will be denoted by* $\mathrm{TDP}^{\mathrm{p}}_{\mathrm{n}}$, *with realisations* $\mathrm{DP}^{\mathrm{p}}_{\mathrm{n}}$.

■

For this class of transfer functions Desai and Pal [21] (see §4.5.1 for more details) introduced a balancing scheme based upon the continuous-time positive-real Riccati equations. Thus, given $\mathcal{Q} := (A, B, C, D) \in \mathrm{P}^{\mathrm{p}}_{\mathrm{n}}$

$$\begin{aligned} R_P(\mathcal{Q}; X) &:= (A - BN^{-1}C)X + X(A - BN^{-1}C)^T + XC^TN^{-1}CX + BN^{-1}B^T = 0 \quad (4.1) \\ R_P^D(\mathcal{Q}; X) &:= (A - BN^{-1}C)^TX + X(A - BN^{-1}C) + XBN^{-1}B^TX + C^TN^{-1}C = 0. \quad (4.2) \end{aligned}$$

where $N = (D + D^T)$. These may be rewritten in the following way, which is preferable for certain manipulations:

$$R_P(\mathcal{Q}; X) := AX + XA^T + \left(B - XC^T\right) N^{-1} \left(B - XC^T\right)^T = 0 \tag{4.3}$$

$$R_P^D(\mathcal{Q}; X) := A^TX + XA + \left(C^T - PB\right) N^{-1} \left(C^T - PB\right)^T = 0 \tag{4.4}$$

The following pair of Riccati equations give rise to a discrete-time version of positive-real balancing. Given $\mathcal{Q} := (A, B, C, D) \in \mathrm{DP}^{\mathrm{p}}_{\mathrm{n}}$

$$R_{DP}(\mathcal{Q}; X) := AXA^T - X + (B - AXC^T)M^{-1}(B - AXC^T)^T = 0 \tag{4.5}$$

$$R_{DP}^D(\mathcal{Q}; X) := A^TXA - X + (C^T - A^TXB)N^{-1}(C^T - A^TXB)^T = 0. \tag{4.6}$$

where $M = (D + D^T - CXC^T)$ and $N = (D + D^T - B^T XB)$.

The associated form of balancing is as follows.

Definition 4.3 (Positive-Real Balancing) *Given* $(A, B, C, D) \in \mathrm{P}^{\mathrm{p}}_{\mathrm{n}}$ *with minimal solutions (under the normal ordering on Hermitian matrices)* P *and* Q *to*

$$\begin{aligned} R_P((A, B, C, D); P) &= 0 \\ R_P^D((A, B, C, D); Q) &= 0 \end{aligned}$$

then (A, B, C, D) *is said to be* continuous-time positive-real balanced *with* positive-real Gramian P *if* $P = Q = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_k I_{n_k}) = \mathrm{diag}(\mu_1, \ldots, \mu_n)$ *where* $\sigma_1 > \ldots > \sigma_k$ *and* $\mu_1 \geqslant \ldots \geqslant \mu_n$.

Any $(A, B, C, D) \in \mathrm{DP}^{\mathrm{p}}_{\mathrm{n}}$ *such that the minimal solutions* P *and* Q *to*

$$\begin{aligned} R_{DP}((A, B, C, D); P) &= 0 \\ R_{DP}^D((A, B, C, D); Q) &= 0 \end{aligned}$$

are similarly equal, diagonal, and have diagonal elements in decreasing order is discrete-time positive-real balanced *with analogous Gramians.*

■

These equations also lie behind the method of stochastic balancing analysed by Green and by Wang and Safonov [44, 41, 43]. This method is discussed in §4.5.2.

As has been shown in §3.1.1, in the context of prediction, the class of minimum-phase systems is of interest. These transfer functions are distinguished by their stability simultaneous with that of their algebraic inverse. As in the case of Lyapunov balancing, it is known that stability of a minimal realisation is equivalent to the existence of a positive-definite solution to an associated Lyapunov equation. It is this thought that motivates the following definition.

Definition 4.4 (Right Minimum-Phase Balanced) *Given* $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times p} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times p}$ *with solutions* P *and* Q *to*

$$\begin{aligned} R_M((A, B, C, D); P) &:= AP + PA^T + BB^T = 0 && (4.7) \\ R_M^D((A, B, C, D); Q) &:= (A - BD^{-1}C)^T Q + Q(A - BD^{-1}C) + C^T D^{-T} D^{-1} C = 0 && (4.8) \end{aligned}$$

then (A, B, C, D) *is said to be* continuous-time right minimum-phase balanced *with* right minimum-phase Gramian Σ *if* $\Sigma = P = Q = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_k I_{n_k}) = \mathrm{diag}(\mu_1, \ldots, \mu_n)$ *where* $\sigma_1 > \ldots > \sigma_k$ *and* $\mu_1 \geqslant \ldots \geqslant \mu_n$.

Any $(A, B, C, D) \in \mathrm{DM}^{\mathrm{p}}_{\mathrm{n}}$ *such that the solutions* P *and* Q *to*

$$\begin{aligned} R_{DM}((A, B, C, D); P) &:= APA^T - P + BB^T = 0 && (4.9) \\ R_{DM}^D((A, B, C, D); Q) &:= (A - BD^{-1}C)^T Q(A - BD^{-1}C) - Q + C^T D^{-T} D^{-1} C = 0 && (4.10) \end{aligned}$$

are similarly equal, diagonal, and have diagonal elements in decreasing order is discrete-time left minimum-phase balanced *with analogous Gramians and characteristic values.*

■

Without further qualification "minimum-phase balanced" will be taken to mean "right minimum-phase balanced". Definition 5.1 makes the purpose of the terminology clear.

Remark 4.5 (Mutual Information) *§2.3.4 discusses the mutual information interpretation of canonical correlation coefficients. Minimum-phase balancing equates the controllability Gramian of the innovations model of a stochastic process to the observability Gramian of the inverse. By analogy with the energy interpretation of Lyapunov balancing this symmetric relationship between the past and the future might be thought to be related to the mutual information. Theorem 4.51 shows that the Gramians are closely related to the canonical correlation coefficients.*

■

The important fact concerning this balancing scheme is that it characterises minimum-phase functions, which the next lemma demonstrates for the continuous case.

Lemma 4.6 (Characterisation of Minimal Minimum-Phase Systems) *If (A, B, C, D) is minimal then:*

$$(A, B, C, D) \in \mathrm{M}_{\mathrm{n}}^{\mathrm{p}} \Leftrightarrow \exists P, Q \textit{ solutions to } (4.1) \textit{ and } (4.2) \textit{ which are positive-definite.}$$

Proof : From Glover [35] Theorem 3.3 (4) it is known that with (A, B) controllable

$$\pi(A) = \nu(P) = 0$$

and

$$\nu(A) = \pi(P) = n$$

hence $\lambda(A) \subset \mathbb{C}_- \Leftrightarrow P > 0$.

Following familiar reasoning e.g. Wonham [132],

$$\begin{gathered}(A, C) \text{ is observable iff} \\ (A, -D^{-1}C) \text{ is observable iff} \\ (A - FD^{-1}C, -D^{-1}C) \text{ is observable } \forall F \in \mathbb{R}^{n \times p}\end{gathered}$$

hence $\lambda(A - BD^{-1}C) \subset \mathbb{C}_- \Leftrightarrow Q > 0$. ■

Note 4.7 *From the spectral mapping theorem it is known that*

$$\lambda(A) \subseteq \mathbb{D} \Leftrightarrow \lambda(M(A)) \subseteq M(\mathbb{D}) = \mathbb{C}_-.$$

Hence it is obvious that the bilinear transform maps rational discrete-time stable transfer functions to continuous-time stable transfer functions. Since the inverse of a minimum-phase functions is also stable (and minimum-phase) the bilinear transform maps minimum-phase discrete-time transfer function to minimum-phase continuous-time transfer functions.

■

It is not obvious that the bilinear transformation and the operation of inversion commute. The next result demonstrates this relationship.

Proposition 4.8 (Inversion Commutes with the Bilinear Transform) *Given a continuous time transfer function $G_c(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}$ with discrete-time equivalent $G_d(z) = C(z^{-1}I_n - A)^{-1}B + D$ then*

$$\begin{aligned}
(I - \tilde{A} + \tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}(I + \tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C}) &= (I - \tilde{A})^{-1}(I + \tilde{A}) \\
&\quad - 2(I - \tilde{A})^{-1}\tilde{B}(\tilde{D} + \tilde{C}(I - \tilde{A})^{-1}\tilde{B})^{-1}\tilde{C}(I - \tilde{A})^{-1} \\
-\sqrt{2}\tilde{D}^{-1}\tilde{C}(I - \tilde{A} + \tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}\tilde{B}\tilde{D}^{-1} &= -\sqrt{2}(I - \tilde{A})^{-1}\tilde{B}(\tilde{D} + \tilde{C}(I - \tilde{A})^{-1}\tilde{B})^{-1} \\
\sqrt{2}\tilde{D}^{-1}\tilde{C}(I - \tilde{A} + \tilde{B}\tilde{D}^{-1}\tilde{C})^{-1} &= \sqrt{2}(\tilde{D} + \tilde{C}(I - \tilde{A})^{-1}\tilde{B})^{-1}\tilde{C}(I - \tilde{A})^{-1} \\
\tilde{D}^{-1} - \tilde{D}^{-1}\tilde{C}(I - \tilde{A} + \tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}\tilde{B}\tilde{D}^{-1} &= (\tilde{D} + \tilde{C}(I - \tilde{A})^{-1}\tilde{B})^{-1}
\end{aligned}$$

Proof : The equivalence of the $\tilde{D}$ terms is merely the Matrix Inversion Lemma

$$(\tilde{D}+\tilde{C}(I-\tilde{A})^{-1}\tilde{B})^{-1}=\tilde{D}^{-1}-\tilde{D}^{-1}\tilde{C}(I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}\tilde{B}\tilde{D}^{-1}.$$

$$\begin{aligned}
&(\tilde{D}+\tilde{C}(I-\tilde{A})^{-1}\tilde{B})^{-1}\tilde{C}(I-\tilde{A})^{-1}\\
&= \left[\tilde{D}^{-1}-\tilde{D}^{-1}\tilde{C}(I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}\tilde{B}\tilde{D}^{-1}\right]\tilde{C}(I-\tilde{A})^{-1}\\
&= \tilde{D}^{-1}\tilde{C}(I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}\left\{(I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})-\tilde{B}\tilde{D}^{-1}\tilde{C}\right\}(I-\tilde{A})^{-1}\\
&= \tilde{D}^{-1}\tilde{C}(I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}
\end{aligned}$$

The $\tilde{B}$ term is checked in much the same way.

$$\begin{aligned}
&(I-\tilde{A})^{-1}(I+\tilde{A})-2(I-\tilde{A})^{-1}\tilde{B}(\tilde{D}+\tilde{C}(I-\tilde{A})^{-1}\tilde{B})^{-1}\tilde{C}(I-\tilde{A})^{-1}\\
&= (I-\tilde{A})^{-1}\left\{I+\tilde{A}-2\tilde{B}\tilde{D}^{-1}\tilde{C}(I-\tilde{A})^{-1}+2\tilde{B}\tilde{D}^{-1}\tilde{C}(I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}\tilde{B}\tilde{D}^{-1}\tilde{C}(I-\tilde{A})^{-1}\right\}\\
&= (I-\tilde{A})^{-1}\left\{I+\tilde{A}-2\left[I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C}-\tilde{B}\tilde{D}^{-1}\tilde{C}\right](I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}\tilde{B}\tilde{D}^{-1}\tilde{C}(I-\tilde{A})^{-1}\right\}\\
&= (I-\tilde{A})^{-1}\left\{I+\tilde{A}-2(I-\tilde{A})(I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}\tilde{B}\tilde{D}^{-1}\tilde{C}(I-\tilde{A})^{-1}\right\}\\
&= (I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}\left\{(I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})(I+\tilde{A})-2\tilde{B}\tilde{D}^{-1}\tilde{C}\right\}(I-\tilde{A})^{-1}\\
&= (I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}\left\{I+\tilde{A}-\tilde{A}-\tilde{A}^2+\tilde{B}\tilde{D}^{-1}\tilde{C}+\tilde{B}\tilde{D}^{-1}\tilde{C}\tilde{A}-2\tilde{B}\tilde{D}^{-1}\tilde{C}\right\}(I-\tilde{A})^{-1}\\
&= (I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}\left\{I-\tilde{A}^2-\tilde{B}\tilde{D}^{-1}\tilde{C}+\tilde{B}\tilde{D}^{-1}\tilde{C}\tilde{A}\right\}(I-\tilde{A})^{-1}\\
&= (I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}\left\{(I+\tilde{A}-\tilde{B}\tilde{D}^{-1}\tilde{C})(I-\tilde{A})\right\}(I-\tilde{A})^{-1}\\
&= (I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}(I+\tilde{A}-\tilde{B}\tilde{D}^{-1}\tilde{C})
\end{aligned}$$

■

Corollary 4.8.1 (Realisation for $G_d^{-1}(z)$) *Using the notation of Proposition 4.8, $G_d^{-1}(z)$ has a realisation given by*

$$\begin{pmatrix}
(I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}(I+\tilde{A}-\tilde{B}\tilde{D}^{-1}\tilde{C}),\\
-\sqrt{2}(I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}\tilde{B}\tilde{D}^{-1},\\
\sqrt{2}\tilde{D}^{-1}\tilde{C}(I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1},\\
(\tilde{D}+\tilde{C}(I-\tilde{A})^{-1}\tilde{B})^{-1}
\end{pmatrix}$$

■

The relationships between these realisations are of significance because they lead to an analysis of the relationship between the solutions to the Lyapunov equations (4.7), (4.8), (4.9) and (4.10). That the solution to discrete-time Lyapunov equations may be related to that of continuous-time Lyapunov equations by the use of the bilinear transform is found in Glover [35]. The following result makes explicit the relationship between the Lyapunov equations considered above, and this is subsequently used to characterise discrete-time minimum-phase transfer functions.

Lemma 4.9 (Equivalence of Continuous-Time and Discrete-Time Lyapunov Equations) *Let $G_d(z)$ have realisation (A,B,C,D) and $G_c(s)=G_d(M(s))$ has realisation $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ given by Proposition 2.6. In this case, where $P,Q>0$*

$$R_{DM}((A,B,C,D);P)=0\Leftrightarrow R_M((\tilde{A},\tilde{B},\tilde{C},\tilde{D});P)=0$$

and

$$R^D_{DM}((A,B,C,D);Q)=0\Leftrightarrow R^D_M((\tilde{A},\tilde{B},\tilde{C},\tilde{D});Q)=0.$$

Proof : The argument to show that the bilinear transform preserves the solution to Lyapunov equations is found in Glover [35]. It may be applied to the controllability Lyapunov equation:

$$\begin{aligned}
&(I+A)\left\{\tilde{A}\Sigma + \Sigma\tilde{A}^T + \tilde{B}\tilde{B}^T\right\}(I+A^T)\\
&= (A-I)\Sigma(I+A^T) + (I+A)\Sigma(A^T-I) + 2BB^T\\
&= A\Sigma A^T - \Sigma A^T + A\Sigma - \Sigma + A\Sigma A^T - A\Sigma + \Sigma A^T - \Sigma + 2BB^T\\
&= 2\left\{A\Sigma A^T - \Sigma + BB^T\right\}
\end{aligned}$$

(4.10) shows that the bracketed terms in this equation correspond to the A-matrix in Corollary 4.8.1. Hence from (4.10)

$$\begin{aligned}
&(A^T - C^TD^{-T}B^T)\Sigma(A - BD^{-1}C) - \Sigma + C^TD^{-T}D^{-1}C\\
&= (I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^{-T}\begin{bmatrix} (I+\tilde{A}^T - \tilde{C}^T\tilde{D}^{-T}\tilde{B}^T)\Sigma(I+\tilde{A}-\tilde{B}\tilde{D}^{-1}\tilde{C})\\ -(I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^T\Sigma(I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})\\ +2\tilde{C}^T\tilde{D}^{-T}\tilde{D}^{-1}\tilde{C}\end{bmatrix}(I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}\\
&= 2(I-\tilde{A}^T+\tilde{C}^T\tilde{D}^{-T}\tilde{B}^T)^{-1}\begin{bmatrix}\tilde{A}^T\Sigma + \Sigma\tilde{A} - \Sigma\tilde{B}\tilde{D}^{-1}\tilde{C}\\ -\tilde{C}^T\tilde{D}^{-T}\tilde{B}^T\Sigma + \tilde{C}^T\tilde{D}^{-T}\tilde{D}^{-1}\tilde{C}\end{bmatrix}(I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}\\
&= 2(I-\tilde{A}^T+\tilde{C}^T\tilde{D}^{-T}\tilde{B}^T)^{-1}\begin{bmatrix}(\tilde{A}^T - \tilde{C}^T\tilde{D}^{-T}\tilde{B}^T)\Sigma + \Sigma(\tilde{A}-\tilde{B}\tilde{D}^{-1}\tilde{C})\\ +\tilde{C}^T\tilde{D}^{-T}\tilde{D}^{-1}\tilde{C}\end{bmatrix}(I-\tilde{A}+\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}
\end{aligned}$$

Hence, since this last bracketed term is (4.8) and the outer terms are always invertible, then (4.10) and (4.8) hold simultaneously, as required. ■

Lemma 4.10 (Characterisation of Minimal Discrete-Time Minimum-Phase Systems) *If (A, B, C, D) is minimal then:*

$$(A,B,C,D) \in \mathrm{DM}^{\mathrm{p}}_{\mathrm{n}} \Leftrightarrow \exists P, Q \textit{ solutions to } (4.9) \textit{ and } (4.10) \textit{ which are positive-definite.}$$

Proof : This follows immediately from Lemma 4.9, Lemma 4.6 and the observation that the bilinear transformation provides a bijection between $\mathrm{TM}^{\mathrm{p}}_{\mathrm{n}}$ and $\mathrm{TDM}^{\mathrm{p}}_{\mathrm{n}}$. ■

Having established the relationships between the classes of transfer functions of interest and the allied Riccati equations and Lyapunov equations, it is now possible to demonstrate the existence and uniqueness of the appropriate balanced realisations.

Proposition 4.11 (Existence of Minimum-Phase Balanced Realisations)

1) If P solves

$$\begin{aligned}
R_M((A,B,C,D);P) &= 0\\
R_{DM}((\tilde{A},\tilde{B},\tilde{C},\tilde{D});P) &= 0
\end{aligned}$$

and Q solves

$$\begin{aligned}
R^D_M((A,B,C,D);Q) &= 0\\
R^D_{DM}((\tilde{A},\tilde{B},\tilde{C},\tilde{D});Q) &= 0
\end{aligned}$$

then, given $T \in \mathrm{GL}_n(\mathbb{R})$, the equations

$$\begin{aligned} R_M(T(A,B,C,D)T^{-1};X) &= 0 \\ R_{DM}(T(\tilde{A},\tilde{B},\tilde{C},\tilde{D})T^{-1};X) &= 0 \end{aligned}$$

and

$$\begin{aligned} R_M^D(T(A,B,C,D)T^{-1};Y) &= 0 \\ R_{DM}^D(T(\tilde{A},\tilde{B},\tilde{C},\tilde{D})T^{-1};Y) &= 0 \end{aligned}$$

have solutions $X = TPT^T$ and $Y = T^{-T}QT^{-1}$.

2) Each transfer function $G \in \mathrm{TM}_\mathrm{n}^\mathrm{p}$ has a continuous-time minimum-phase balanced realisation and each transfer function $G \in \mathrm{TDM}_\mathrm{n}^\mathrm{p}$ has a discrete-time minimum-phase balanced realisation.

3) All balanced realisations of $G \in \mathrm{TM}_\mathrm{n}^\mathrm{p}$ have the same Gramian (similarly for $G \in \mathrm{TDM}_\mathrm{n}^\mathrm{p}$).

4) Let $\mathcal{Q}$ be a balanced realisation of $G \in \mathrm{TM}_\mathrm{n}^\mathrm{p}$ or of $G \in \mathrm{TDM}_\mathrm{n}^\mathrm{p}$ with minimum-phase Gramian $\Sigma = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_k I_{n_k})$ where the σ_i are distinct. All balanced realisations of G are given by $Q\mathcal{Q}Q^T$ where $Q = \mathrm{diag}(Q_1, \ldots, Q_k)$ with $Q_i \in \mathrm{O}_{n_i}(\mathbb{R})$, $i = 1, \ldots, k$.

Proof : The existence and uniqueness properties of balanced realisations for discrete-time systems may be deduced from the continuous-time results and Lemma 4.9.

1)

$$\begin{aligned} 0 &= R_M\left(T(A,B,C,D)T^{-1};X\right) \\ &= (TAT^{-1})X + X(T^{-T}A^TT^T) + TBB^TT^T \\ &= TR_M\left((A,B,C,D);T^{-1}XT^{-T}\right)T^T \end{aligned}$$

Since T is invertible

$$0 = R_M\left((A,B,C,D);T^{-1}XT^{-T}\right)$$

and hence, since the solution to such a Lyapunov equation is unique (see Glover [35] for conditions) then

$$P = T^{-1}XT^{-T}.$$

The other equations behave in an analogous fashion.

2) Let $\mathcal{Q} = (A,B,C,D) \in \mathrm{M}_\mathrm{n}^\mathrm{p}$ be a realisation of G. Let $\lambda_i(A)$ be the ith eigenvalue of A (see §C.4 for details). Also let $P((A,B,C,D))$ be the unique solution to

$$R_M((A,B,C,D);P) = 0$$

and $Q((A,B,C,D))$ be the unique solution to

$$R_M^D((A,B,C,D);Q) = 0.$$

Observe that $\lambda_i(P(\mathcal{Q})Q(\mathcal{Q}))$ is invariant under state-space transformation of $\mathcal{Q}$, since from 1) above

$$\begin{aligned} P\left(T\mathcal{Q}T^{-1}\right) Q\left(T\mathcal{Q}T^{-1}\right) &= TP\left(\mathcal{Q}\right)T^{T}T^{-T}Q\left(\mathcal{Q}\right)T^{-1} \\ &= TP\left(\mathcal{Q}\right)Q\left(\mathcal{Q}\right)T^{-1}. \end{aligned}$$

From $P(\mathcal{Q})$ and $Q(\mathcal{Q})$ two state-space transformations are derived which transform the realisation to a balanced one. Let $\delta_i := \lambda_i(P(\mathcal{Q})Q(\mathcal{Q}))$, $i = 1, \ldots, n$ and $\Delta := \operatorname{diag}(\sqrt{\delta_1}, \ldots, \sqrt{\delta_n})$. $P(\mathcal{Q})$ is positive-definite and so

$$P(\mathcal{Q}) = UDU^T, \; D = \operatorname{diag}(d_1, \ldots, d_n), \; d_1 \geqslant \ldots \geqslant d_n > 0, \; U \in \mathrm{O}_n(\mathbb{R}).$$

So, let

$$\bar{\mathcal{Q}} := (D^{-1/2}U^T)\mathcal{Q}(D^{-1/2}U^T)^{-1}$$

and thus

$$P\left(\bar{\mathcal{Q}}\right) = I_n \text{ and } Q\left(\bar{\mathcal{Q}}\right) = V\Delta^2V^T, \text{ for some } \; V \in \mathrm{O}_n(\mathbb{R}).$$

Let

$$\hat{\mathcal{Q}} = (\Delta^{1/2}V^T)\bar{\mathcal{Q}}(\Delta^{1/2}V^T)^{-1},$$

which gives

$$P(\hat{\mathcal{Q}}) = \Delta = Q(\hat{\mathcal{Q}}),$$

as required.

3) The above construction, and the State-Space Isomorphism Theorem (Kailath [72]) guarantee that every realisation has the same set of characteristic values. The conditions that a balanced realisation $\mathcal{Q}$ has $P(\mathcal{Q}) = Q(\mathcal{Q})$ diagonal and positive-definite with an ordering on the diagonal elements in combination with the invariance of the δ_i, implies that Δ as defined above is the Gramian for all balanced realisations of G.

4) If $\mathcal{P}$ and $\mathcal{Q}$ are balanced realisation of G then, by the State-Space Isomorphism Theorem,

$$\mathcal{P} = T\mathcal{Q}T^{-1} \text{ for some } T \in \mathrm{GL}_n(\mathbb{R})$$

and so from 1)

$$\Delta = P(\mathcal{Q}) = P(\mathcal{P}) = T\Delta T^T;$$

and from the ordering on the diagonal elements of Δ it is observed that T is of the form given in the theorem.

By similar considerations it is seen that if $\mathcal{Q}$ is a balanced realisation of G then $Q\mathcal{Q}Q^{-1}$ is also balanced if Q is conformably block diagonal.

■

Remark 4.12 *The above proof shows that the characteristic values are independent of the realisation of G, and so the Gramian may be referred to as the Gramian of G.*

■

4.2.2 Symmetric Balanced Realisation

In this section a further set of important parameters are examined. These arise from consideration of the form of the Riccati equations appearing in the definitions of balancing above.

Examining the pairs of Riccati equations (2.28), (2.29) and (4.1), (4.2) it is evident that the dual equation is merely a primary equation applied to the transposed realisation i.e.

$$\begin{aligned} R_L^D\left((A,B,C,D);X\right) &= R_L\left((A^T,C^T,B^T,D^T);X\right) \\ R_P^D\left((A,B,C,D);X\right) &= R_P\left((A^T,C^T,B^T,D^T);X\right). \end{aligned}$$

This leads to the consideration of the set of systems for which the realisations (A,B,C,D) and (A^T,C^T,B^T,D^T) realise the same transfer function and, as is shown below, to consideration of symmetric transfer functions and to sign-symmetric realisations. As was mentioned in §2.2.1, a sign-symmetric realisation (A,B,C,D) is one for which there exists a sign matrix S (that is a diagonal matrix with ± 1 entries on the diagonal) such that

$$S(A,B,C,D)S = (A^T,C^T,B^T,D^T).$$

Such realisations have been considered in Fuhrmann [33] and Willems [127].

The next theorem summarises the well-known properties of such realisations.

Theorem 4.13 ([128, 33]) *If $G \in \mathrm{TL}_{\mathrm{n}}^{\mathrm{p,m}}$ is symmetric and $\mathcal{Q} = (A,B,C,D) \in \mathrm{L}_{\mathrm{n}}^{\mathrm{p,m}}$ is a realisation of G then*

1. *$\mathcal{Q}^T$ is also a realisation of G.*
2. *The state-space transformation T from $\mathcal{Q}$ to $\mathcal{Q}^T$ is symmetric.*
3. *There exists a sign-symmetric realisation of G.*

Proof :

1. This is a simple consequence of the definitions involved.
2. (Proposition 4 [128]) Observe that $A^T = TAT^{-1}, B^T = CT^{-1}, C^T = TB$, and thus $A^T = T^TAT^{-T}, B^T = CT^{-T}, C^T = T^TB$; so by the State-Space Isomorphism Theorem, $T = T^T$.
3. T is symmetric, and so has decomposition

$$T = USU^T$$

where $UU^T = \mathrm{diag}(\mu_1,\ldots,\mu_n)$, μ_i is the ith singular value of T, and S is a sign matrix. Define

$$\mathcal{P} = U^T\mathcal{Q}U$$

which is sign-symmetric, since

$$\begin{aligned} \mathcal{P}^T &= U^T\mathcal{Q}^TU \\ &= U^TT^{-1}\mathcal{Q}TU \\ &= SU^T\mathcal{Q}US \\ &= S\mathcal{P}S. \end{aligned}$$

∎

Corollary 4.13.1 (Characterisation of Symmetric Transfer Functions) *G is symmetric if and only if it has a sign-symmetric realisation, since*

$$\begin{aligned} G^T(s) &= B^T(sI_n - A^T)^{-1}C^T + D^T \\ &= CS(sI_n - SAS)^{-1}SB + D \\ &= C(sI_n - A)^{-1}B + D \\ &= G(s) \end{aligned}$$

and also 3) of Theorem 4.13.

■

The significance of such realisations derives not merely from the relationship with Hankel operators (Proposition 2.19) and the subsequently derived canonical forms, but also from their relationship with an important input-output invariant - the Cauchy-Index.

Definition 4.14 (Cauchy-Index [11]) *A function $f : \mathbb{R} \to \mathbb{R}$ is said to have a* jump *of size k at x_0 if*

$$k = \lim_{x \nearrow x_0} f(x) - \lim_{x \searrow x_0} f(x)$$

where each one of these quantities exists and is finite.

If $f(x) \to -\infty$ as $x \nearrow x_0$ and $f(x) \to \infty$ as $x \searrow x_0$ then the jump at x_0 is said to be of size ∞, and f is said to have jump of size $-\infty$ at x_0 if $(-f)$ has jump of ∞ at x_0. The jump is undefined in other cases.

Let $p(x)$ and $q(x)$ be relatively prime polynomials with real coefficients. The Cauchy-index $\mathrm{I}_{-\infty}^{\infty} g(x)$ of $g(x) = p(x)/q(x)$ is defined as the number of jumps of size ∞ minus the number of jumps of size $-\infty$ of $g(x)$ for $x \in \mathbb{R}$.

A matrix Cauchy-Index may also be defined for symmetric matrices $S(x)$ with rational entries. This is the sum of the Cauchy-Indices of the eigenvalues of $S(x)$ for $x \in \mathbb{R}$.

■

That the Cauchy-Index is related to sign-symmetry is the upshot of the following theorem.

Theorem 4.15 (Bitmead and Anderson [11]) *Let $G(s)$ be an $m \times m$ real rational symmetric matrix with proper part $\hat{G}(s)$. Let $\hat{G}(s) = C(sI_n - A)^{-1}B + D$ with A of dimension $\delta[\hat{G}(s)]$ (the M^cMillan degree). Then there exists a unique P such that*

$$PA = A^T P \ , \ \ PB = C^T$$

and

$$\mathrm{sig}(P) = \mathrm{I}_{-\infty}^{\infty} G(s).$$

■

Corollary 4.15.1 *If $G \in \mathrm{TL}_{\mathrm{n}}^{\mathrm{m,m}}$ has a realisation $\mathcal{Q} \in \mathrm{L}_{\mathrm{n}}^{\mathrm{m,m}}$ which is sign symmetric with respect to S, then*

$$\mathrm{I}_{-\infty}^{\infty} G(s) = \mathrm{sig}(S) = \mathrm{tr}\,(S) = \sum_{i=1}^{n} s_i.$$

∎

This result has previously been exploited to distinguish connected components of the space of symmetric transfer functions by Brockett [14] for $p = m = 1$ and, for the stable case without p and m constraints, by Helmke [53]. It was also utilised by Ober [101] for similar purposes.

Of course, sign-symmetric realisations will be of little importance if they cannot be reconciled with the structures considered above, namely balanced realisations. That this is possible depends on the symmetry between the Riccati equations mentioned above, and the fact that the solutions are chosen in a compatible way e.g. the minimal solution to both equations (4.1) and (4.2) are chosen.

Proposition 4.16 (Sign-Symmetric Lyapunov Balanced Realisations) *Every symmetric $G \in \mathrm{TC}_{\mathrm{n}}^{\mathrm{p,p}}$ has a Lyapunov balanced sign-symmetric realisation $\mathcal{Q}$.*

Proof : Let $\mathcal{Q}$ be a balanced realisation of G with Gramian $\Sigma = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_k I_{n_k})$. As G is symmetric, from Theorem 4.13, there exists a unique, symmetric $T \in \mathrm{GL}_n(\mathbb{R})$ such that $T\mathcal{Q}T^{-1} = \mathcal{Q}^T$. $\mathcal{Q}^T$ is also a balanced realisation of G as transposition merely interchanges the Lyapunov equations (2.28) and (2.29) and therefore, by Lemma 4.1 Glover [35],

$$T = \mathrm{diag}(Q_1, \ldots, Q_k), \quad Q_i \in \mathrm{O}_{n_i}(\mathbb{R}).$$

As T is symmetric and orthogonal, $T^2 = I_n$, and so T has all its eigenvalues equal to either 1 or -1. T thus has an eigenvalue-eigenvector decomposition,

$$T = USU^T, \; U = \mathrm{diag}(U_1, \ldots, U_k), \; U_i \in \mathrm{O}_{n_i}(\mathbb{R})$$

and

$$S = \mathrm{diag}(s_1, \ldots, s_n), \; s_i \in \{\pm 1\}.$$

A state-space transformation effected by U^T gives $U^T\mathcal{Q}U$ a balanced realisation since U has the appropriate block structure to preserve balancing. $U^T\mathcal{Q}U$ is then sign-symmetric with respect to S, since

$$(U^T\mathcal{Q}U)^T = U^T\mathcal{Q}^TU = U^TT\mathcal{Q}T^{-1}U = S(U^T\mathcal{Q}U)S.$$

∎

Proposition 4.17 (Existence of Sign-Symmetric Positive-Real Balanced Realisations) *Every symmetric $G \in \mathrm{TP}_{\mathrm{n}}^{\mathrm{p,m}}$ has a positive-real balanced sign-symmetric realisation $\mathcal{Q}$.*

Proof : The same argument as in Proposition 4.16 applies in this context since transposition interchanges the Riccati equations (4.1) and (4.2). ∎

Proposition 4.18 (Uniqueness of Sign-Symmetric Balanced Realisations) *Any two input-output equivalent realisations $\mathcal{P}, \mathcal{Q} \in \mathrm{C}_{\mathrm{n}}^{\mathrm{p,p}}$ ($\mathrm{P}_{\mathrm{n}}^{\mathrm{p}}$) which are Lyapunov (Positive-Real) balanced with Gramian $\Sigma = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_k I_{n_k})$ and sign-symmetric with respect to S are related by an orthogonal state-space transformation $Q = \mathrm{diag}(Q_1, \ldots, Q_k)$, $Q_i \in \mathrm{O}_{n_i}(\mathbb{R})$ such that $QSQ^T = S$.*

Proof : If $\mathcal{P}$ and $\mathcal{Q}$ are balanced realisation of $G \in \mathcal{T}$ sign symmetric with respect to S, then by Lemma 4.1 Glover [35] there exists a unique orthogonal $Q = \mathrm{diag}(Q_1, \ldots, Q_k)$ such that

$$\mathcal{P} = Q\mathcal{Q}Q^T$$

and so

$$\begin{aligned}
\mathcal{P} &= S\mathcal{P}^T S \\
&= SQ\mathcal{Q}^T Q^T S \\
&= (SQS)\mathcal{Q}(SQS)^T \\
&= (SQSQ^T)\mathcal{P}(SQSQ^T)^T
\end{aligned}$$

thus from the State-Space Isomorphism Theorem

$$SQSQ^T = I_n$$

which is equivalent to the required result

$$S = QSQ^T.$$

Also, if Q satisfies the above equation and if $\mathcal{Q}$ is a balanced realisation of G sign-symmetric with respect to S then so is $Q\mathcal{Q}Q^T$.

The positive-real case follows the same logic, and relies upon the uniqueness result in Desai & Pal [22]. ■

This is not the only matter to be dealt with in considering the development of canonical forms. It must also be made clear what internal relationships are imposed upon a realisation which is both sign-symmetric and balanced.

Theorem 4.19 (Internal B-Orthogonality) *Let $(A, B, C, D) \in \mathrm{C}_{\mathrm{n}}^{\mathrm{p,p}}$ be Lyapunov balanced with Gramian σI_n and sign-symmetric with respect to $S = \mathrm{diag}(s_1, \ldots, s_n)$ then if $s_i s_j = -1$, the ith row of B is orthogonal to the jth row of B.*

Proof : From (2.28) and (2.29)

$$\begin{aligned}
\sigma A + \sigma A^T - BB^T &= 0 \\
\sigma A^T + \sigma A - C^T C &= 0 \\
\Rightarrow \quad -BB^T = -C^T C &= -SBB^T S.
\end{aligned}$$

Hence

$$\begin{aligned}
\left[BB^T\right]_{ij} &= \left[SBB^T S\right]_{ij} \\
&= s_i s_j \left[BB^T\right]_{ij}
\end{aligned}$$

and so if $s_i s_j = -1$ then $\left[BB^T\right]_{ij} = 0$ i.e. the ith row of B is orthogonal to the jth row of B. ■

Theorem 4.20 (Internal B-Orthogonality) *Let $(A, B, C, D) \in \mathrm{P}_{\mathrm{n}}^{\mathrm{p}}$ be positive-real balanced with Gramian σI_n and sign-symmetric with respect to $S = \mathrm{diag}(s_1, \ldots, s_n)$ then if $s_i s_j = -1$*

$$b_i \left(D + D^T\right)^{-1} b_j^T = 0$$

where b_i is the ith row of B.

Proof : From (4.1) and (4.2), where $N = (D + D^T)$,

$$\begin{aligned} \sigma\left(A - BN^{-1}C\right) + \sigma\left(A - BN^{-1}C\right)^T + \sigma^2 C^T N^{-1} C + BN^{-1}B^T &= 0 \\ \sigma\left(A - BN^{-1}C\right)^T + \sigma\left(A - BN^{-1}C\right) + \sigma^2 B^T N^{-1} B + CN^{-1}C^T &= 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow\ (1-\sigma^2)BN^{-1}B^T &= (1-\sigma^2)C^T N^{-1} C \\ \Rightarrow\ BN^{-1}B^T &= SBN^{-1}B^T S \end{aligned}$$

and hence the result follows as previously. ■

Finally, it would seem appropriate to consider whether there might be an analogous case of interest related to minimum-phase balancing. In considering the symmetric systems it was the relationship between the Riccati equations which motivated the discussion. In the case of minimum-phase balancing it is obvious that

$$R_M^D\left((A,B,C,D);X\right) = R_M\left((A^T,C^T,B^T,D^T)^{-1};X\right)$$

i.e. the dual is the observability Lyapunov equation for the realisation of the inverse. However, there is no class of systems for which

$$G(s) = G^{-T}(s),\ \forall s \in \mathbb{C}$$

since this would imply that G was orthogonal at every point in $\mathbb{C}$ and hence of bounded norm on $\mathbb{C}$. Therefore the transfer function would have no poles, and hence because of rationality would be a constant.

4.3 Canonical Forms

This section deals with the details of the programme for deriving a canonical form for the class $\mathrm{M}_{\mathrm{n}}^{\mathrm{p}}$. The details presented in the remainder of this Chapter are based upon the foundations of balanced minimal realisations described in §4.2. This section may be viewed as the continuation of the project initiated by Maciejowski and Ober [81, 96, 99, 82, 101, 95, 98, 100] to utilise balanced realisations as a basis for canonical forms.

In addition, a canonical form for $\mathrm{P}_{\mathrm{n}}^{\mathrm{p}}$ is described in Appendix A, which is different in detail from previously published versions, Ober [96]. Here the positive-real parametrisation is derived directly from the Positive Real Ricatti Equation (4.1) and (4.2), rather than via the Bounded Real Ricatti Equations e.g. used in Ober [96]. The structure of the decomposition of the A matrix into symmetric and antisymmetric parts is also different, as is the factorisation of the BB^T terms.

Obviously, the main purpose of the derivation of a canonical form is to produce a parametrisation of the same class. However, the two constructions differ in character and in method of proof, and hence to distinguish these two tasks, the concepts of reducibility and augmentability are introduced. These two concepts relate to the way in which balanced models may be decomposed into balanced subsystems (reducibility), or how balanced models may be joined together to form a larger balanced model (augmentability).

One other aspect of this Chapter is the relationship between balancing and symmetry. Lemma 4.1 Glover [35] and Propositions 4.11 and 4.18 above describe the remaining freedom for minimal state-space realisations which are balanced and sign-symmetric. The procedure followed in this section is to select those realisations with given structural properties so as to reduce the degrees of freedom and eventually to ensure that only one such realisation lies in any input-output equivalence class.

The structure of the canonical forms will be introduced in the following four sections:

§4.3.1, in which the block structure induced by balancing is described. This naturally leads to examination of the diagonal subsystems.

§4.3.2, in which the structure imposed upon the B-matrix is considered, and §4.3.3 in which the structure imposed upon the A-matrix is considered.

Finally in §4.3.4 the actual canonical form and demonstration that this provides a parametrisation are stated.

In §4.4 this structure is mimicked for symmetric systems.

Throughout this section the use of the unqualified term "balanced" will either take its meaning from the context, or will mean "any one of Lyapunov balanced, positive-real balanced or minimum-phase balanced". The terms "Gramian" and "characteristic value" will be used in a similar manner.

4.3.1 Natural Induced Structure

If $\mathcal{Q} = (A, B, C, D)$ is either a Lyapunov balanced realisation of $G \in \mathrm{TC}_{\mathrm{n}}^{\mathrm{p,m}}$, or a positive-real balanced realisation of $G \in \mathrm{TP}_{\mathrm{n}}^{\mathrm{p}}$, or a minimum-phase balanced realisation of $G \in \mathrm{TM}_{\mathrm{n}}^{\mathrm{p}}$ with Gramian $\Sigma = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_k I_{n_k})$, then the matrices A, B and C have a natural subpartitioning which corresponds to the multiplicity of the distinct characteristic values. Let (A, B, C, D) be said to be *partitioned according to* $(n_i)_{i=1}^k$, where $\sum_{i=1}^k n_i = n$, if

$$\begin{aligned} A &= (A_{ij})_{1\leqslant i,j\leqslant k}, & A_{ij} &\in \mathbb{R}^{n_i \times n_j}, \\ B^T &= (B_1^T, \ldots, B_k^T), & B_j &\in \mathbb{R}^{n_j \times m}, \\ C &= (C_1, \ldots, C_k), & C_j &\in \mathbb{R}^{p \times n_j}. \end{aligned}$$

The A_{ij}, B_i and C_j will be referred to as the *partition of* (A, B, C, D).

Given that (A, B, C, D) is partitioned according to $(n_i)_{i=1}^k$ and $Q = \mathrm{diag}(Q_1, \ldots, Q_k)$, $Q_i \in \mathrm{O}_{n_i}(\mathbb{R})$ then

$$\mathcal{P} := (\bar{A}, \bar{B}, \bar{C}, \bar{D}) = Q\mathcal{Q}Q^T$$

is also partitioned according to $(n_i)_{i=1}^k$ with

$$\begin{aligned} \bar{A}_{ij} &= Q_i A_{ij} Q_j^T, \\ \bar{B}_j &= Q_j B_j, \\ \bar{C}_i &= C_i Q_i^T. \end{aligned}$$

Since all balanced realisations of G are given by some such $Q\mathcal{Q}Q^T$, there is a natural block structure arising for balanced realisations. From this simple observation it is clear that any canonical form imposed on all of the diagonal blocks of the A-matrix of a balanced realisation also fixes the form of the off-diagonal blocks.

4.3.2 B-matrix Structure

If $\mathcal{Q} = (A, B, C, D)$ is a balanced minimal realisation of G with Gramian σI_n, then all possible B-matrices corresponding to balanced realisations are given by QB, $Q \in \mathrm{O}_n(\mathbb{R})$. A natural structure with appropriate uniqueness properties is given by the following generalisation of the familiar concept of row reduced echelon form (Birkhoff & MacLane [10] p162).

Definition 4.21 (Positive-Upper-Triangular Matrix) *A matrix* $M = [m_{ij}]_{1 \leqslant j \leqslant l}^{1 \leqslant i \leqslant n} \in \mathbb{R}^{n \times l}$ *which has rank r, is said to be a* positive-upper-triangular matrix *if there exist indices* $(t_i)_{i=1}^{r}$ *with* $1 \leqslant t_1 < \ldots < t_r \leqslant l$ *such that*

$$\begin{array}{ll} m_{i,j} = 0 & \text{if either } 1 \leqslant j < t_i \text{ and } 1 \leqslant i \leqslant r, \text{ or } i > r \\ m_{i,t_i} > 0 & \text{for } 1 \leqslant i \leqslant r \\ m_{i,j} \in \mathbb{R} & \text{otherwise,} \end{array}$$

That is, the non-zero entries occur in the first r rows, and the first non-zero entry in each row is positive, and these leading entries are strictly to the right of those above them.

If M^T *is positive-upper-triangular then* M *is said to be positive-lower-triangular.*

■

Remark 4.22 *If* $M \in \mathbb{R}^{n \times n}$ *and* $\operatorname{rank} M = n$ *then* M *positive-upper triangular implies that* M *is upper-triangular with positive entries on the diagonal.*

■

The following result shows the type of uniqueness imposed by the preservation of such a structure, and constructs an orthogonal transformation which brings the B-matrix to this form. The basis for the proof may be found in any linear algebra text, or for the full details [89].

Lemma 4.23 (QP **Factorisation)** *If* $M \in \mathbb{R}^{n \times l}$ *has rank r, then there exists* $Q \in \mathrm{O}_n(\mathbb{R})$ *such that* QM *is positive-upper-triangular.* Q *is unique up to left-multiplication by an orthogonal matrix of the form* $\operatorname{diag}(I_r, \bar{Q})$, $\bar{Q} \in \mathrm{O}_{n-r}(\mathbb{R})$.

■

Remark 4.24

i. If $\operatorname{rank} M = n$ *then* Q *is unique.*

ii. A QP*-factorisation of* M *may be derived from the above result. If* $P = QM$ *is positive-upper-triangular then* $M = Q^T P$ *is such a factorisation.*

iii. This construction is better known as a QR *factorisation.*

■

4.3.3 A-matrix Structure

Following the assumptions regarding (A, B, C, D) in the previous subsection, and that the B-matrix is positive-upper-triangular then this B-matrix canonical form is preserved by orthogonal state space transformations of the form

$$Q = \operatorname{diag}(I_r, \bar{Q})$$

where $r = \operatorname{rank}(B)$.

The following definition gives the structure of the canonical form for the A-matrix.

Definition 4.25 (Balanced Matrix) *A matrix* $A \in \mathbb{R}^{n \times n}$ *is said to be in* r-balanced *form,* $1 \leqslant \tau \leqslant r \leqslant n$, *if the matrix* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $A_{11} \in \mathbb{R}^{r \times r}$, *is such that:*

1) A_{11} *is skew symmetric.*

2) A_{12} *and* A_{22} *are given by the set of indices*

$$\begin{array}{ccccccccc} 1 & = h_1 & < \ldots < & h_i & < & h_{i+1} & < \ldots < & h_q & \leqslant & n-r, \\ 1 & \leqslant g_q & < \ldots < & g_{i+1} & < & g_i & < \ldots < & g_1 & \leqslant & r, \end{array}$$

in the following way:

a) for $A_{12} = [a_{st}]_{1 \leqslant t \leqslant n-r}^{1 \leqslant s \leqslant r}$*:*

$$\begin{array}{lll} a_{g_i,h_i} & > 0 & \text{for} \quad 1 \leqslant i \leqslant q \\ a_{g_i,t} & = 0 & \text{for} \quad t > h_i, \text{ where } 1 \leqslant i \leqslant q \\ a_{s,t} & = 0 & \text{for} \quad t \geqslant h_i \text{ and } s > g_i, \text{ where } 1 \leqslant i \leqslant q, \end{array}$$

i.e.

$$A_{12} = \begin{pmatrix} \vdots & \vdots & & \vdots & \vdots & \vdots & \\ \cdot & \cdot & \ldots & a_{g_2-1,h_2-1} & a_{g_2-1,h_2} & a_{g_2-1,h_2+1} & \ldots \\ \cdot & \cdot & \ldots & a_{g_2,h_2-1} & a_{g_2,h_2} & 0 & \ldots \\ \cdot & \cdot & \ldots & a_{g_2+1,h_2-1} & 0 & 0 & \ldots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ a_{g_1-1,h_1} & a_{g_1-1,h_1+1} & \ldots & a_{g_1-1,h_2-1} & 0 & 0 & \ldots \\ a_{g_1,h_1} & 0 & \ldots & 0 & 0 & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \end{pmatrix}.$$

b) A_{22} *is given by*

$$A_{22} = \begin{pmatrix} 0 & \alpha_2 & & & & & \\ -\alpha_2 & 0 & \alpha_3 & & & & \\ & -\alpha_3 & 0 & \cdot & & & 0 \\ & & \ddots & \ddots & & \ddots & \\ & 0 & & \cdot & & 0 & \alpha_{(n-r)} \\ & & & & & -\alpha_{(n-r)} & 0 \end{pmatrix},$$

where for $2 \leqslant i \leqslant (n-r)$

$$\alpha_i \begin{cases} = 0 & \text{if } i = h_s \text{ for some } 1 \leqslant s \leqslant q \\ > 0 & \text{otherwise} \end{cases}.$$

3) $A_{21} = -A_{12}^T$.

■

The main feature of this structure is the A_{22} block which corresponds to the Schwarz canonical form for matrices modulo similarity (see Schwarz [114]). If $r = 1$ then A is in precisely this form. The A_{12} block is related to the A_{22}-block in that a "step up" occurs wherever $\alpha_j = 0$.

That the A-matrix may be brought to such a form is the content of Proposition 5.8 Ober [96].

Proposition 4.26 (Canonical Form for A-matrix [96]) *Let (A, B, C, D) be a minimal balanced realisation with Gramian $\Sigma = \sigma I_n$. If B is positive-upper-triangular of rank r then among the class*

$$\left\{QAQ^T \mid Q = \mathrm{diag}(I_r, \bar{Q}),\ \bar{Q} \in \mathrm{O}_{n-r}(\mathbb{R})\right\}$$

there is exactly one element in r-balanced form.

■

It is worthy of note that this complicated structure is only present when there are repeated characteristic values. In the simpler non-repeated case Kabamba [71] derived a less complex canonical form.

4.3.4 Canonical Form and Parametrisation

The previous sections have described a structure for the diagonal blocks of a balanced realisation which, upon consideration of the partitioning mentioned in §4.3.1, would seem to be the basis for a canonical form. Furthermore, the parameters appearing in the descriptions of those structures might also provide a parametrisation of the class of systems under consideration.

However, before such an argument can be pressed to conclusion several major considerations must be taken into account. Firstly, if it is sought to fix the canonical form for (A, B, C, D) by applying the above structures to each (A_{ii}, B_i, C_i, D) then it must be known that each such diagonal subsystem is minimal otherwise Proposition 4.26 does not apply. Such a property will be termed *reducibility*. This will be established for minimum-phase balancing in Proposition 4.28.

In showing that all realisations in the the following canonical form are indeed minimal realisations of minimum-phase systems it will be necessary first to show that each diagonal sub-block is minimal (Proposition 4.31) and that minimality is preserved by stacking blocks together. This final property of being able to build up larger minimum-phase systems from smaller building blocks will be termed *augmentability* and is dealt with in Proposition 4.27.

In the next two propositions let $(\bar{A}, \bar{B}, \bar{C}, \bar{D}) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times p} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times p}$ be minimum-phase balanced with Gramian $\Sigma > 0$, and let $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ be partitioned according to (n_1, n_2).

Proposition 4.27 (Augmentability of Minimum-Phase Balanced Realisations)
If $\bar{A}_{11}$, $\bar{A}_{22}$, $(\bar{A}_{11} - \bar{B}_1\bar{D}^{-1}\bar{C}_1)$ and $(\bar{A}_{22} - \bar{B}_2\bar{D}^{-1}\bar{C}_2)$ are asymptotically stable and $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$, $\Sigma_1 \in \mathbb{R}^{n_1\times n_1}$ with $\lambda(\Sigma_1) \cap \lambda(\Sigma_2) = \emptyset$ then $\bar{A}$ and $(\bar{A} - \bar{B}\bar{D}^{-1}\bar{C})$ are asymptotically stable and $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is minimal.

Proof : Consideration of $\Sigma \times (4.7) \times \Sigma + (4.8)$ (where $P = Q = \Sigma$) gives

$$\begin{aligned} 0 &= \left(I + \Sigma^2\right)\bar{A}^T\Sigma + \Sigma\bar{A}\left(I + \Sigma^2\right) + \bar{C}^T\bar{D}^{-T}\bar{D}^{-1}\bar{C} - \bar{C}^T\bar{D}^{-T}\bar{B}^T\Sigma - \Sigma\bar{B}\bar{D}^{-1}\bar{C} + \Sigma\bar{B}\bar{B}^T\Sigma \\ &= \left(I + \Sigma^2\right)\bar{A}^T\Sigma + \Sigma\bar{A}\left(I + \Sigma^2\right) + \left(\bar{C}^T\bar{D}^{-T} - \Sigma\bar{B}\right)\left(\bar{D}^{-1}\bar{C} - \bar{B}^T\Sigma\right) \end{aligned}$$

hence

$$\begin{aligned} 0 &= \bar{A}^T\Sigma\left(I + \Sigma^2\right)^{-1} + \Sigma\left(I + \Sigma^2\right)^{-1}\bar{A} \\ &\quad + \left(I + \Sigma^2\right)^{-1}\left(\bar{C}^T\bar{D}^{-T} - \Sigma\bar{B}\right)\left(\bar{D}^{-1}\bar{C} - \bar{B}^T\Sigma\right)\left(I + \Sigma^2\right)^{-1} = 0 \end{aligned} \tag{4.11}$$

Applying a state-space transformation $(A, B, C, D) = (I + \Sigma^2)^{1/4}(\bar{A}, \bar{B}, \bar{C}, \bar{D})(I + \Sigma^2)^{-1/4}$ to equations (4.7) and (4.11) followed by pre- and post-multiplying by $(I + \Sigma^2)^{-1/4}$ for the first equation and $(I+\Sigma^2)^{1/4}$ for the second, the following are obtained:

$$\begin{aligned} 0 &= A\Sigma\left(I+\Sigma^2\right)^{-\frac{1}{2}} + \Sigma\left(I+\Sigma^2\right)^{-\frac{1}{2}} A^T + BB^T \\ 0 &= A^T\Sigma\left(I+\Sigma^2\right)^{-\frac{1}{2}} + \Sigma\left(I+\Sigma^2\right)^{-\frac{1}{2}} A \\ &\quad + \left(I+\Sigma^2\right)^{-1}\left(C^TD^{-T} - \Sigma\left(I+\Sigma^2\right)^{-\frac{1}{2}} B\right)\left(D^{-1}C - B^T\Sigma\left(I+\Sigma^2\right)^{-\frac{1}{2}}\right)\left(I+\Sigma^2\right)^{-1} \end{aligned}$$

which upon the substitutions $X = \Sigma\,(I+\Sigma^2)^{-\frac{1}{2}}$ and $Y = \left(D^{-1}C - B^T\Sigma\,(I+\Sigma^2)^{-\frac{1}{2}}\right)(I+\Sigma^2)^{-1}$ appear as

$$\begin{aligned} AX + XA^T + BB^T &= 0 \\ A^TX + XA + Y^TY &= 0. \end{aligned}$$

Note that the state-space transformation applied respects the partitioning, and hence the (i,i), $i=1,2$ blocks of A have the same spectrum as those of $\bar{A}$. Applying Proposition 4 from Kabamba [71] leads to the conclusion that $\bar{A}$ is stable and that $(\bar{A},\bar{B})$ is controllable.

A similar sequence of manipulations leads to the conclusion that $(\bar{A}-\bar{B}\bar{D}^{-1}\bar{C})$ is stable and $(\bar{A}-\bar{B}\bar{D}^{-1}\bar{C}, -\bar{D}\bar{C})$ is observable. These manipulations involve rewriting (4.7) as

$$\left(\bar{A}-\bar{B}\bar{D}^{-1}\bar{C}\right)\Sigma + \Sigma\left(\bar{A}^T - \bar{C}^T\bar{D}^{-T}\bar{B}^T\right) + \bar{B}\bar{B}^T + \bar{B}\bar{D}^{-1}\bar{C}\Sigma + \Sigma\bar{C}^T\bar{D}^{-T}\bar{B}^T = 0 \tag{4.12}$$

and then considering $(4.12) + \Sigma \times (4.8) \times \Sigma$ to which is applied a state space transformation equal to the inverse of the one above. This leads to a pair of Lyapunov equations from which the desired conclusion may be obtained.

Finally, the argument in Lemma 4.6 shows that $(\bar{A},\bar{C})$ is observable. ■

Proposition 4.28 (Reducibility Property of Minimum-Phase Balanced Realisations) *If $\bar{A}$ and $\bar{A}-\bar{B}\bar{D}^{-1}\bar{C}$ are asymptotically stable then $\bar{A}_{ii}$ and $\bar{A}_{ii}-\bar{B}_i\bar{D}^{-1}\bar{C}_i$, $i=1,2$ are all stable.*

Proof : Following the proof of Proposition 4.27 it is seen that (A,B,Y) is Lyapunov balanced. Hence from Theorem 3.2 Pernebo & Silverman [107] $\bar{A}_{ii}$, $i=1,2$ are asymptotically stable.

Stability of $\bar{A}_{ii}-\bar{B}_i\bar{D}^{-1}\bar{C}_i$, $i=1,2$ is derived in a similar manner from the later part of the proof of Proposition 4.27. ■

Having the basic results in place, the form of the parametrisation is as follows.

Definition 4.29 (Standard Form for Parameters) *A* set of standard parameters *is a set*

$$\left\{\sigma_j, n_j, r_j, \check{A}_j, \check{B}_j, \check{U}_j, \check{D} \,|\, j = 1,\ldots,k\right\}$$

where:

$$\begin{array}{lllll} \sigma_1 > \ldots > \sigma_k > 0 & , & \textit{where} & k \in \mathbb{N}; \sigma_i \in \mathbb{R}, \\ n_1\ ,\ \ldots,\ n_k & , & \text{''} & n_j \in \mathbb{N}\ ;\ \sum_{j=1}^k n_j = n, \\ r_1\ ,\ \ldots,\ r_k & , & \text{''} & r_j \in \mathbb{N};\ 1 \leqslant r_j \leqslant \min(n_j, m), \\ \check{A}_1\ ,\ \ldots,\ \check{A}_k & , & \text{''} & \check{A}_j \in \mathbb{R}^{n_j\times n_j}\ r_j\textit{-balanced}, \\ \check{B}_1\ ,\ \ldots,\ \check{B}_k & , & \text{''} & \check{B}_j \in \mathbb{R}^{r_j\times m}\ \textit{full rank positive-upper-triangular}, \\ \check{U}_1,\ \ldots,\ \check{U}_k & , & \text{''} & \check{U}_j \in \mathrm{O}_{p\times r_j}(\mathbb{R}), \\ \check{D}\ , & & \text{''} & \check{D} \in \mathbb{R}^{m\times m}. \end{array}$$

A realisation (A, B, C, D) is in standard form *if it is parametrised by a set of standard parameters in the following way: if (A, B, C, D) is partitioned according to $(n_i)_{i=1}^k$, then*

(1) $A_{ij} = \begin{bmatrix} \bar{A}_{ij} & 0 \\ 0 & 0 \end{bmatrix}$, $i \neq j$, *where* $\bar{A}_{ij} \in \mathbb{R}^{r_i \times r_j}$ *is a function of* $\sigma_i, \sigma_j, \check{B}_i, \check{B}_j, \check{U}_i, \check{U}_j, r_i$, *and* $\check{D}$.

(2) $A_{jj} = \check{A}_j + \begin{bmatrix} \bar{A}_{jj} & 0 \\ 0 & 0 \end{bmatrix}$, *where* $\bar{A}_{jj} \in \mathbb{R}^{r_j \times r_j}$ *is a function of* $\sigma_j, \check{B}_j, \check{U}_j, r_i$, *and* $\check{D}$.

(3) $B_j = \begin{bmatrix} \bar{B}_j \\ 0 \end{bmatrix}$, *where* $\bar{B}_j$ *is a function of* $\check{D}$ *and* $\check{B}_j$.

(4) C_j^T *is a function of* $\check{U}_j$, $\check{B}_j$ *and* $\check{D}$.

(5) D *is a function of* $\check{D}, \check{U}_j$ *and* r_j.

■

Remark 4.30 *In the case where $n_i = 1$ then the associated parameters τ_i^j, $\check{A}_i$ and $\check{U}_i^j$ become degenerate and they may all be replaced by a single sign $s_i \in \{1, -1\}$.*

■

In the one block case any such realisation has the important property of being minimal. This was first demonstrated in Ober [96] in a more specialised form.

Lemma 4.31 (Minimality of Simple Balanced Case) *If (A, B, C, D) is in standard form with $k = 1$ then (A, B, C, D) is minimal.*

Proof : See Theorem 7.1 [96]. ■

This leads to the main result of this section:

Theorem 4.32 (Minimum-Phase Canonical Form and Parametrisation) *$G \in \mathrm{TM}_{\mathrm{n}}^{\mathrm{m}}$ if and only if there exists an unique minimal realisation (A, B, C, D) given by the standard set of parameters where:*

$$\begin{aligned}
B_i &= \begin{bmatrix} \check{B}_i \\ 0 \end{bmatrix} \\
A_{ii} &= \check{A}_i - \frac{1}{2\sigma_i} B_i B_i^T \\
A_{ij} &= \frac{1}{\sigma_i^2 - \sigma_j^2} \left\{ \sigma_j (1 + \sigma_i^2) B_i B_j^T - \sigma_i \sqrt{1 + \sigma_i^2} \sqrt{1 + \sigma_j^2} \sqrt{B_i B_i^T} U_i U_j^T \sqrt{B_j B_j^T} \right\} \\
C_i &= D \left(\sigma_i B_i^T + \sqrt{1 + \sigma_i^2} U_i^T \sqrt{B_i B_i^T} \right) \\
D &= \check{D} \\
U_i &= \begin{bmatrix} \check{U}_i \\ 0 \end{bmatrix} \in \mathbb{R}^{n \times m}
\end{aligned}$$

Furthermore, (A, B, C, D) is minimum-phase balanced with Gramian $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_k)$.

Proof : "$\Rightarrow$"Given any minimal realisation of G, the state-space transformation described in Proposition 4.11 transforms it to a minimum-phase balanced realisation $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ which may be partitioned according to $(n_i)_{i=1}^k$, the multiplicities of the distinct minimum-phase characteristic values.

From Proposition 4.23 there exist $Q_i \in \mathrm{O}_{n_i}(\mathbb{R})$ for each B_i such that $Q_i B_i$ is positive-upper-triangular. Hence the whole realisation may be transformed via the state-space transformation $T = \mathrm{diag}(Q_1, \ldots, Q_k)$ and results in a balanced realisation with the B_i's positive-upper-triangular.

The next step involves performing a sequence of transformations to impose r_j-balanced structure on the A-matrix. It is noted first that Proposition 4.28 implies that each diagonal block is indeed minimal. Hence each block may be considered in isolation and Proposition 4.26 applied to the diagonal realisations. Here r_j is the rank of B_j and this process makes the realisation unique, i.e. the only transformation which preserves the balancedness and the structure is the identity. This results in a canonical form (A, B, C, D).

It only remains to check the form of the parametrisation. Consider the (i, i)th block of the two defining Lyapunov Equations

$$\begin{aligned} \sigma_i A_{ii} + \sigma_i A_{ii}^T &= -B_i B_i^T \\ \sigma_i A_{ii}^T + \sigma_i A_{ii} - \sigma_i C_i^T D^{-T} B_i^T - \sigma_i B_i D^{-1} C_i &= -C_i^T D^{-T} D^{-1} C_i \end{aligned}$$

which upon substitution gives

$$A_{ii} = \hat{A}_i - \frac{1}{2\sigma_i} B_i B_i^T$$

where $\hat{A}_i$ is antisymmetric, and so

$$\begin{aligned} -B_i B_i^T - \sigma_i C_i^T D^{-T} B_i - \sigma_i B_i D^{-1} C_i + C_i^T D^{-T} D^{-1} C_i &= 0 \\ \Rightarrow \left(C_i^T D^{-T} - \sigma_i B_i\right)\left(D^{-1} C_i - \sigma_i B_i^T\right) &= (1 + \sigma_i^2) B_i B_i^T \qquad (4.13) \\ \Rightarrow C_i^T D^{-T} - \sigma_i B_i &= \sqrt{1 + \sigma_i^2}\sqrt{B_i B_i^T}\, U_i, \qquad (4.14) \end{aligned}$$

where

$$U_i = \begin{bmatrix} \check{U}_i \\ 0 \end{bmatrix}$$

following the convention that

$$\sqrt{\begin{bmatrix} \check{B}_i \check{B}_i^T & 0 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} \sqrt{\check{B}_i \check{B}_i^T} & 0 \\ 0 & 0 \end{bmatrix}$$

where the square root is taken to be the unique positive-definite square root of a positive-definite matrix. And thus,

$$\begin{aligned} C_i^T &= \left(\sigma_i B_i + \sqrt{1 + \sigma_i^2}\sqrt{B_i B_i^T}\, U_i\right) D^T, \\ \Rightarrow C_i &= D\left(\sigma_i B_i^T + \sqrt{1 + \sigma_i^2}\, U_i^T \sqrt{B_i B_i^T}\right) \end{aligned}$$

Now consider the (i, j)th term

$$\sigma_j A_{ij} + \sigma_i A_{ji}^T = -B_i B_j^T \qquad (4.15)$$

$$\sigma_j A_{ji}^T + \sigma_i A_{ij} - \sigma_j C_i^T D^{-T} B_j^T - \sigma_i B_i D^{-1} C_j = -C_i^T D^{-T} D^{-1} C_j \qquad (4.16)$$

so taking $\sigma_j \times(4.15) - \sigma_i \times(4.16)$ gives

$$(\sigma_j^2 - \sigma_i^2)A_{ij} + \sigma_i\sigma_j C_i^T D^{-T} B_j^T + \sigma_i^2 B_i D^{-1} C_j = -\sigma_j B_i B_j^T + \sigma_i C_i^T D^{-T} D^{-1} C_j$$

$$\begin{aligned}
A_{ij} &= -\frac{1}{\sigma_j^2 - \sigma_i^2}\left\{\sigma_j B_i B_j^T + \sigma_i\sigma_j C_i^T D^{-T} B_j^T + \sigma_i^2 B_i D^{-1} C_j - \sigma_i C_i^T D^{-T} D^{-1} C_j\right\} \qquad (4.17)\\
&= \frac{1}{\sigma_i^2 - \sigma_j^2}\left\{\begin{array}{l} \sigma_i\sigma_j\left(\sigma_i B_i + \sqrt{1+\sigma_i^2}\sqrt{B_i B_i^T} U_i\right) B_j^T + \sigma_i^2 B_i\left(\sigma_j B_j^T + \sqrt{1+\sigma_j^2} U_j^T \sqrt{B_j B_j^T}\right) \\ -\sigma_i\left(\sigma_i B_i + \sqrt{1+\sigma_i^2}\sqrt{B_i B_i^T} U_i\right)\left(\sigma_j B_j^T + \sqrt{1+\sigma_j^2} U_j^T \sqrt{B_j B_j^T}\right) + \sigma_j B_i B_j^T \end{array}\right\}\\
&= \frac{1}{\sigma_i^2 - \sigma_j^2}\left\{\sigma_j(1+\sigma_i^2) B_i B_j^T - \sigma_i\sqrt{1+\sigma_i^2}\sqrt{1+\sigma_j^2}\sqrt{B_i B_i^T} U_i U_j^T \sqrt{B_j B_j^T}\right\}
\end{aligned}$$

"$\Leftarrow$"The next step is to demonstrate that any such realisation is a minimal realisation, and also that it is minimum-phase.

Firstly, minimality will be dealt with. For the diagonal blocks $k = 1$ so Lemma 4.31 implies minimality and hence stability of A_{ii} and $(A_{ii} - B_i D^{-1} C_i)$. Then, from the construction of the realisation it is clear that the whole satisfies the necessary Lyapunov equations and hence, applying Proposition 4.27 one block at a time, implies both that it is minimal and also minimum-phase. ∎

In the single-input single-output case the canonical form takes on a much simplified, and more elegant form as the following shows.

Corollary 4.32.1 (SISO Parametrisation) *$G \in \mathrm{TM}_{\mathrm{n}}^{1}$ if and only if there exists a unique minimal realisation (A, B, C, D) given by the standard set of parameters where:*

$$\begin{aligned}
B_i &= \begin{bmatrix} \check{b}_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}\\
C_i &= d(\sigma_i + s_i\sqrt{1+\sigma_i^2}) B_i\\
A_{ii} &= \check{A}_i - \frac{1}{2\sigma_i} B_i B_i^T\\
A_{ij} &= \frac{B_i B_j^T}{\sigma_i^2 - \sigma_j^2}\left\{\sigma_j(1+\sigma_i^2) - s_i s_j \sigma_i\sqrt{1+\sigma_i^2}\sqrt{1+\sigma_j^2}\right\}\\
&= \frac{-B_i B_j^T\sqrt{1+\sigma_i^2}}{\sigma_j^2(1+\sigma_i^2) - (s_i s_j)^2\sigma_i^2(1+\sigma_j^2)}\left\{\sigma_j\sqrt{1+\sigma_i^2} - s_i s_j \sigma_i\sqrt{1+\sigma_j^2}\right\}\\
&= \frac{-\sqrt{1+\sigma_i^2} B_i B_j^T}{\sigma_j\sqrt{1+\sigma_i^2} + s_i s_j \sigma_i\sqrt{1+\sigma_j^2}}
\end{aligned}$$

where $s_i \in \{\pm 1\}$ takes the place of U_i, D is a scalar d, and the A_{ii} term is unaffected.

∎

Definition 4.33 (Minimum-Phase Canonical Form) *Define the family of canonical forms $(\Gamma_{M,p,n})_{p,n\in\mathbb{N}}$ to be*

$$\Gamma_{M,p,n} : \mathrm{M}_{\mathrm{n}}^{\mathrm{p}} \to \mathrm{M}_{\mathrm{n}}^{\mathrm{p}}$$

where $\Gamma_{M,p,n}((A, B, C, D))$ is the unique minimal input-output equivalent realisation as given in Theorem 4.32.

∎

Note 4.34 (Canonical Form and Parametrisation for $\mathrm{Innov}_{\mathrm{n}}^{\mathrm{p}}$ and $\mathrm{Pred}_{\mathrm{n}}^{\mathrm{p}}$) *From the descriptions of* $\mathrm{Innov}_{\mathrm{n}}^{\mathrm{p}}$ *and* $\mathrm{Pred}_{\mathrm{n}}^{\mathrm{p}}$ *given in §3.1 it is immediately clear that these results also give canonical forms for these classes.*

∎

4.4 Symmetric Canonical Forms

The special problems which arise in the symmetric cases of positive-real balancing and Lyapunov balancing are now considered.

Throughout this section G will be assumed to be symmetric, and the unqualified term "balanced" will mean either Lyapunov balanced or positive-real balanced.

4.4.1 B-matrix Structure

In Ober [101] the structure imposed upon the B-matrix sub-blocks is that of positive-upper-triangularity. However, in the symmetric case the B-matrix has an internal orthogonality, as shown in Theorem 4.19. This orthogonality is problematic from the point of view of parametrising such matrices, as clearly the entries of the matrix itself are no longer independent. The following result will be used to make this internal orthogonality explicit.

Lemma 4.35 (PQ Factorisation) *If $M \in \mathbb{R}^{n\times l}$ is of rank n then there exists a unique positive-upper-triangular matrix of rank n, $R \in \mathbb{R}^{n\times n}$, and a unique $U \in \mathbb{R}^{n\times l}$ with $UU^T = I_n$, such that $M = RU$.*

∎

The following definition gives the form of the B-matrix in the canonical form developed in the next section, and the next proposition shows that it is indeed achievable.

Definition 4.36 (τ-Positive-Upper-Triangular Matrix) *A matrix $M = [m_{ij}]_{1\leqslant j\leqslant l}^{1\leqslant i\leqslant n} \in \mathbb{R}^{n\times l}$ with $\operatorname{rank} M = r$, is said to be* τ-positive-upper-triangular *if*

$$M = \begin{bmatrix} M_1 \\ M_2 \\ 0 \end{bmatrix}, \quad \textit{where } M_1 \in \mathbb{R}^{\tau\times n},\ M_2 \in \mathbb{R}^{(r-\tau)\times n},$$

and M_1, M_2 are positive-upper-triangular.

Proposition 4.37 *Let $\mathcal{Q} = (A, B, C, D)$ be a balanced realisation of G which is sign-symmetric with respect to $S = \operatorname{diag}(s_1, \ldots, s_n)$ and has Gramian σI_n. If*

$$\begin{aligned} \tau_1 &:= \operatorname{rank}(B + SB), \\ \tau_2 &:= \operatorname{rank}(B - SB) \;=\; \operatorname{rank}(B) - \tau_1 \end{aligned}$$

then

1) *There exists a balanced realisation $\mathcal{P} = Q\mathcal{Q}Q^T$, $Q \in \mathrm{O}_n(\mathbb{R})$ which is sign-symmetric with respect to sign matrix $S = \operatorname{diag}(I_{\tau_1}, -I_{\tau_2}, S_1)$, and has QB τ_1-positive-upper-triangular.*

2) Let $\mathcal{P} = (\bar{A}, \bar{B}, \bar{C}, \bar{D})$ be a balanced realisation of G sign-symmetric with respect to the sign matrix $\bar{S} = \mathrm{diag}(I_{\tau_1}, -I_{\tau_2}, S_2)$ and with $\bar{B}$ τ_1-positive-upper-triangular. All other balanced realisations which are sign-symmetric with respect to $\hat{S} = \mathrm{diag}(I_{\tau_1}, -I_{\tau_2}, S_3)$ with S_3 a sign matrix and with $\bar{B}$ τ_1-positive-upper-triangular are given by $\bar{Q}\mathcal{P}\bar{Q}^T$ where $\bar{Q} = \mathrm{diag}(I_{\tau_1+\tau_2}, \hat{Q})$, $\hat{Q} \in \mathrm{O}_{n-\tau_1-\tau_2}(\mathbb{R})$ and $\hat{Q}S_3\hat{Q}^T$ is a sign matrix of the same inertia.

Proof : It is required that τ_1 and τ_2 be shown to be well defined. Define the indices $(i_t)_{t=1}^{l}$ and $(j_t)_{t=1}^{n-l}$ so that $s_{i_1}, \ldots, s_{i_l}$ are each equal to 1, and $s_{j_1}, \ldots, s_{j_{n-l}}$ are each equal to -1. From the orthogonality conditions Theorems 4.19 and 4.20 it is seen that the spans of $\bar{B}_1 = \{b_{i_t} | \, t = 1, \ldots, l\}$ and $\bar{B}_2 = \{b_{j_t} | \, t = 1, \ldots, n-l\}$ are orthogonal and hence independent (as the terms I_p and $(D+D^T)^{-1}$ may be seen as non-degenerate inner products). These are also the non-zero rows of $\frac{1}{2}(B+SB)$ and $\frac{1}{2}(B-SB)$ respectively, and hence the result holds.

1) Consider the matrices

$$B_1 = \begin{bmatrix} b_{i_1} \\ \vdots \\ b_{i_l} \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} b_{j_1} \\ \vdots \\ b_{j_{n-l}} \end{bmatrix}.$$

B_1 and B_2 are of rank τ_1 and τ_2 respectively. Now apply Lemma 4.23 to B_i to obtain Q_i such that Q_iB_i is positive-upper-triangular, $i = 1, 2$. If

$$Q_1 = \left[q_{ij}^{(1)}\right]_{1\leqslant i,j\leqslant l}, \; Q_2 = \left[q_{ij}^{(2)}\right]_{1\leqslant i,j\leqslant n-l},$$

then define

$$Q = [q_{ij}]_{1\leqslant i,j\leqslant n} \text{ where } q_{ij} = \begin{cases} 0 & s_i s_j = -1, \\ q_{kl}^{(1)} & i = i_k, \; j = i_l, \\ q_{kl}^{(2)} & i = j_k, \; j = j_l. \end{cases}$$

Q is orthogonal, as it is similar, via a permutation to $\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}$, and permutation matrices are orthogonal. QB is such that each non-zero row has its first non-zero entry positive, and all its non-zero rows are linearly independent. Also, rows $i_1, \ldots, i_l$ and $j_1, \ldots, j_{n-l}$ form positive-upper-triangular matrices.

QB is brought to the desired form by premultiplying by a permutation matrix representing the permutation

$$\begin{bmatrix} i_1 & \cdots & i_{\tau_1} & j_1 & \cdots & j_{\tau_2} & i_{\tau_1+1} & \cdots & i_l & j_{\tau_2+1} & \cdots & j_{n-l} \\ 1 & \cdots & \tau_1 & \tau_1+1 & \cdots & \tau_1+\tau_2 & \tau_1+\tau_2+1 & \cdots & l+\tau_2 & l+\tau_2+1 & \cdots & n \end{bmatrix}.$$

2) From Lemma 4.1 Glover [35] or Desai & Pal [22] any other balanced realisation is of the form $Q\mathcal{P}Q^T$, with Q orthogonal. Consider a realisation of G, represented as $\mathcal{R} = Q\mathcal{P}Q^T$ to be a balanced realisation which is sign-symmetric with respect to $\hat{S} = \mathrm{diag}(I_{\tau_1}, -I_{\tau_2}, S_3)$ with S_3 a sign matrix. Also let $Q\bar{B}$ be τ_1-positive-upper-triangular. By a similar argument to that of Proposition 4.17, it may be shown that

$$Q = \bar{S}Q\hat{S}, \tag{4.18}$$

which implies that if Q is partitioned according to $(\tau_1, \tau_2, n-\tau_1-\tau_2)$ then

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} = \begin{bmatrix} Q_{11} & -Q_{12} & Q_{13}S_3 \\ -Q_{21} & Q_{22} & -Q_{23}S_3 \\ S_2Q_{31} & -S_2Q_{32} & S_2Q_{33}S_3 \end{bmatrix} \tag{4.19}$$

and hence $Q_{12} = Q_{21}^T = 0$. Also, since

$$\begin{bmatrix} Q_{11} & 0 & Q_{13} \\ 0 & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \\ 0 \end{bmatrix} = \begin{bmatrix} Q_{11}\bar{B}_1 \\ Q_{22}\bar{B}_2 \\ Q_{31}\bar{B}_1 + Q_{32}\bar{B}_2 \end{bmatrix} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \\ 0 \end{bmatrix}$$

and $\bar{B}_1$ and $\bar{B}_2$ are positive-upper-triangular, then the last block of this equation and the relevant blocks of (4.19) imply that

$$\begin{aligned} 0 &= Q_{31}\bar{B}_1 + Q_{32}\bar{B}_2 \\ &= S_2\left(Q_{31}\bar{B}_1 - Q_{32}\bar{B}_2\right) \end{aligned}$$

and hence

$$0_{(n-\tau_1-\tau_2)\times p} = Q_{31}\bar{B}_1 \tag{4.20}$$
$$= Q_{32}\bar{B}_2. \tag{4.21}$$

If it were known that Q_{11} were orthogonal then the uniqueness property of Lemma 4.23 would give $Q_{11} = I_{\tau_1}$. This is indeed so, because $\bar{B}_1$ is full rank, and hence has image $\mathbb{R}^{\tau_1\times 1}$, to which the rows of Q_{31} are orthogonal from (4.20). Hence $Q_{31} = 0_{(n-\tau_1-\tau_2)\times\tau_1}$ and so from the orthogonality of Q i.e. $Q^TQ = I_n$, it follows that $Q_{11}^TQ_{11} = I_{\tau_1}$. The same follows for Q_{22} using (4.21), and this implies that the off-diagonal blocks are all zero.

The final part of the statement follows from equation (4.18).

■

Imposition of this structure upon the B-matrix of a balanced realisation immediately leads to the following simple observation regarding the A-matrix.

Corollary 4.37.1 *If $\mathcal{P} = (A, B, C, D)$ is partitioned according to $(r, n-r)$, $r = \operatorname{rank}(B)$, as in the previous proposition, and so A has the form $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ then the matrix*

$$\begin{bmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

is skew-symmetric.

Proof : This will be demonstrated for the positive-real case. The Lyapunov balanced case is similar though simpler.

Consider the equations (4.1) and (4.2) in the case $P = pI_n$

$$\begin{aligned} p\left(A + A^T\right) &= p^2C^TN^{-1}(D)C - p\left(BN^{-1}(D)C + C^TN^{-1}(D)B^T\right) - BN^{-1}B^T \\ &= p^2SBN^{-1}(D)B^TS - p\left(BN^{-1}(D)B^TS + SBN^{-1}(D)B^T\right) - BN^{-1}B^T \\ &= \begin{bmatrix} Z & 0 \\ 0 & 0 \end{bmatrix}, \text{ for some } Z \in \mathbb{R}^{r\times r} \end{aligned}$$

■

The previous proposition gives the canonical form for the B-matrix. The following result gives a parametrisation of these matrices.

Proposition 4.38 *Given a balanced sign-symmetric realisation as in the previous section, there exists a factorisation*

$$B = \begin{bmatrix} \check{B}^1\check{U}^1 \\ \check{B}^2\check{U}^2 \\ 0 \end{bmatrix} \mathcal{M}^{-1/2}(D)$$

where

$$\begin{array}{rcl} \mathcal{M}(D) & = & I_p \text{ in Lyapunov balanced case, or } (D+D^T)^{-1} \text{ in the Positive-real case,} \\ \check{B}^i & \in & \mathbb{R}^{\tau_i\times\tau_i} \text{ full rank positive upper-triangular,} \\ \check{U}^i & \in & \mathrm{O}_{\tau_i\times m}(\mathbb{R}) := \left\{U\in\mathbb{R}^{\tau_i\times m} : UU^T = I_{\tau_i} \text{ if } \tau_i\leqslant m, \text{ or } U^TU = I_m \text{ if } m\leqslant\tau_i\right\} \\ & & \text{positive-upper-triangular, } i=1,2. \end{array}$$

furthermore

$$\begin{bmatrix} \check{U}^1 \\ \check{U}^2 \end{bmatrix} \in \mathrm{O}_{(\tau_1+\tau_2)\times m}(\mathbb{R}).$$

Proof : This follows immediately from Lemma 4.35, Proposition 4.37 and Theorems 4.19 and 4.20. ■

This proposition shows clearly the structure of the B-matrix of a sign-symmetric realisation. The internal orthogonality of the B-matrix is expressed in the $\check{U}$-matrix, which is orthogonal. This matrix

$$\check{U} = \begin{bmatrix} \check{U}_1 \\ \check{U}_2 \end{bmatrix}$$

is τ_1-positive-upper-triangular. This structure is problematic from the point of view of parametrisation, as the author is unaware of any simple parametrisation for all such matrices, unlike the case of the class $\mathrm{O}_n(\mathbb{R})$. However, this is an inherent problem induced by sign symmetry and the structure of the Lyapunov/Riccati equations involved.

Define the set of τ_1-positive-upper triangular $(\tau_1+\tau_2)\times m$ orthogonal matrices as follows:

$$\mathrm{P}_m(\tau_1,\tau_2) = \left\{ U\in\mathrm{O}_{(\tau_1+\tau_2)\times m}(\mathbb{R}) \;\middle|\; \begin{array}{c} U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix},\ U_i\in\mathrm{O}_{\tau_i\times m}(\mathbb{R}), \\ U_i \text{ positive-upper-triangular },\ i=1,2 \end{array} \right\}.$$

Note that $\tau_1+\tau_2\leqslant m$. This set will be used to parametrise the B matrices.

4.4.2 A-matrix Structure

In the case of a symmetric transfer function there are several more structural constraints on the canonical form for the A-matrix. These are described in the following definition:

Definition 4.39 Symmetric-Balanced Matrix *A matrix $A\in\mathbb{R}^{n\times n}$ is said to be in (r,τ)-*symmetric-balanced form, *$1\leqslant\tau\leqslant r\leqslant n$, if the matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $A_{11}\in\mathbb{R}^{r\times r}$, is such that:*

1) $A_{11} = \begin{bmatrix} 0 & \check{A}^{(12)} \\ -\check{A}^{(12)T} & 0 \end{bmatrix}$, $\check{A}^{(12)}\in\mathbb{R}^{\tau\times(r-\tau)}$,

2) A_{12} *and* A_{22} *are given by the set of indices*

$$\begin{array}{ccccccccc} 1 & = h_1 & <\ldots< & h_i & < & h_{i+1} & <\ldots< & h_q & \leqslant & n-r, \\ 1 & \leqslant g_q & <\ldots< & g_{i+1} & < & g_i & <\ldots< & g_1 & \leqslant & r, \end{array}$$

in the following way:

a) for $A_{12} = [a_{st}]_{1\leqslant t\leqslant n-r}^{1\leqslant s\leqslant r}$*:*

$$\begin{array}{llll} a_{g_i,h_i} & > 0 & for & 1 \leqslant i \leqslant q \\ a_{g_i,t} & = 0 & for & t > h_i, \text{ where } 1 \leqslant i \leqslant q \\ a_{s,t} & = 0 & for & t \geqslant h_i \text{ and } s > g_i, \text{ where } 1 \leqslant i \leqslant q, \end{array}$$

i.e.

$$A_{12} = \begin{pmatrix} \vdots & \vdots & & \vdots & \vdots & \vdots & \\ \cdot & \cdot & \dots & a_{g_2-1,h_2-1} & a_{g_2-1,h_2} & a_{g_2-1,h_2+1} & \dots \\ \cdot & \cdot & \dots & a_{g_2,h_2-1} & a_{g_2,h_2} & 0 & \dots \\ \cdot & \cdot & \dots & a_{g_2+1,h_2-1} & 0 & 0 & \dots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ a_{g_1-1,h_1} & a_{g_1-1,h_1+1} & \dots & a_{g_1-1,h_2-1} & 0 & 0 & \dots \\ a_{g_1,h_1} & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots \end{pmatrix}.$$

and, in addition to the above, the entries are such that

$$a_{ij} = 0 \text{ if } s_i s_{r+j} = 1$$

where $S = \operatorname{diag}(s_1, \dots . s_n)$*, is the sign symmetry matrix of* A*, derived inductively from the parameters* τ*,* $(g_i)_{i=1}^q$*, and* $(h_i)_{i=1}^q$ *as follows:*

$$S = \operatorname{diag}(I_\tau, -I_{r-\tau}, \tilde{S}) = \operatorname{diag}(s_1, \dots, s_n)$$

with

$$\begin{array}{lll} s_{r+1} & = & -s_{g_1}, \text{ and} \\ s_{r+p} & = & \begin{cases} -s_{g_\eta}, & \exists \eta \leqslant q \text{ such that } p = h_\eta, \\ -s_{r+p-1}, & \text{otherwise,} \end{cases} \quad \forall p \in \mathbb{N} \text{ such that } 2 \leqslant p \leqslant n-r. \end{array}$$

b) A_{22} *is given by*

$$A_{22} = \begin{pmatrix} 0 & \alpha_2 & & & & \\ -\alpha_2 & 0 & \alpha_3 & & & \\ & -\alpha_3 & 0 & \cdot & & 0 \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \cdot & 0 & \alpha_{(n-r)} \\ & & & & -\alpha_{(n-r)} & 0 \end{pmatrix},$$

where for $2 \leqslant i \leqslant (n-r)$

$$\alpha_i \begin{cases} = 0 & \text{if } i = h_s \text{ for some } 1 \leqslant s \leqslant q \\ > 0 & \text{otherwise} \end{cases}.$$

3) $A_{21} = -A_{12}^T$.

A is, by construction, sign-symmetric with respect to S.

■

The differences between r-balanced form and (r,τ)-symmetric balanced form are that there is a substructure to the A_{11}-block and that many of the A_{12} entries are zero.

Note that the A-matrix defines the S-matrix via the recursive definition above.

Note 4.40 *If A is (r,τ)-symmetric-balanced then A is r-balanced.*

■

Proposition 4.41 (Existence of Canonical Form for A-matrix (Symmetric Case))

If (A,B,C,D) is a balanced realisation of a symmetric transfer function with Gramian $\Sigma = \sigma I_n$, B τ-positive-upper-triangular ($\operatorname{rank}(B) = r$), and which is sign-symmetric with respect to $S = \operatorname{diag}(I_\tau, -I_{r-\tau}, S_3)$, then among the set

$$\left\{ QA^TQ^T \mid Q = \operatorname{diag}(I_r, \bar{Q}),\ \bar{Q} \in \mathrm{O}_{n-r}(\mathbb{R}) \text{ and } \bar{Q}S_3\bar{Q}^T \text{ is a sign matrix} \right\}$$

there exists exactly one element in (r,τ)-symmetric balanced form.

Proof : Refer to Ober [97] or Theorem 5.40 [89]. ■

4.4.3 Symmetric Canonical Form and Parametrisation

Definition 4.42 (Standard Symmetric Form for Parameters) *A* set of standard symmetric parameters *is a set*

$$\left\{ \sigma_j, n_j, \tau_j^i, \check{A}_j, \check{B}_j^i, \check{U}_j^i, \check{D} \mid j = 1,\ldots,k,\ i = 1,2 \right\}$$

where:

$$\begin{array}{llll}
\sigma_1 > \ldots > \sigma_k > 0 & , & \text{where} & k \in \mathbb{N}; \sigma_i \in \mathbb{R}, \\
n_1\ ,\ \ldots,\ n_k & , & " & n_j \in \mathbb{N}\ ;\ \sum_{j=1}^k n_j = n, \\
\tau_1^i\ ,\ \ldots,\ \tau_k^i & , & " & \tau_j^i \in \mathbb{N}\ ;\ i = 1,2\ ;\ 1 \leqslant r_j^1 \leqslant \min(n_j, m), \\
\check{A}_1,\ \ldots,\ \check{A}_k & , & " & \check{A}_j \in \mathbb{R}^{n_j \times n_j}\ (r_j^1, \tau_j^1)\text{-symmetric-balanced}, \\
\check{B}_1^i,\ \ldots,\ \check{B}_k^i & , & " & \check{B}_j^i \in \mathbb{R}^{\tau_j^i \times \tau_j^i} \text{ full rank positive-upper-triangular};\ i = 1,2, \\
\check{U}_1^i,\ \ldots,\ \check{U}_k^i & , & " & \begin{bmatrix} \check{U}_j^1 \\ \check{U}_j^2 \end{bmatrix} \in \mathrm{P}_m(\tau_1, \tau_2), \\
\check{D} = \check{D}^T\ , & & " & \check{D} \in \mathbb{R}^{m \times m},
\end{array}$$

with $r_j^1 = \tau_j^1 + \tau_j^2$ and $r_j^2 = n - r_j^1$.

A realisation (A,B,C,D) is in standard symmetric form *if it is parametrised by a set of standard symmetric parameters in the following way: if (A,B,C,D) is partitioned according to $(n_i)_{i=1}^k$, then*

(1) $A_{ij} = \begin{bmatrix} \bar{A}_{ij} & 0 \\ 0 & 0 \end{bmatrix}$, *$i \neq j$, where $\bar{A}_{ij} \in \mathbb{R}^{r_i^1 \times r_j^1}$ is a function of $\sigma_i, \sigma_j, \check{B}_i^k, \check{B}_j^k, \check{U}_i^k, \check{U}_j^k, \tau_i^k$, and $\check{D}$.*

(2) $A_{jj} = \check{A}_j + \begin{bmatrix} \bar{A}_{jj} & 0 \\ 0 & 0 \end{bmatrix}$, *where $\bar{A}_{jj} \in \mathbb{R}^{r_j^1 \times r_j^1}$ is a function of $\sigma_j, \check{B}_j^k, \check{U}_j^k, \tau_i^k$, and $\check{D}$.*

(3) $B_j = \begin{bmatrix} \bar{B}_j^1 \\ \bar{B}_j^2 \\ 0 \end{bmatrix}$, *where* $\bar{B}_j^k = \breve{B}_j^k \breve{U}_j^k g(\breve{D}) \in \mathbb{R}^{\tau_j^k \times m}$ *where* $g(\breve{D})$ *is a non-singular matrix function of* $\breve{D}$.

(4) $C_j^T = S_j B_j$ *where* S_j *is the sign matrix associated with* $\breve{A}_j$.

(5) D *is a function of* $\breve{D}, \breve{U}_j^k$ *and* τ_j^k,

with $k = 1, 2$ *in all of the above.*

■

Lemma 4.43 (Minimality of Simple Balanced Case) *If* (A, B, C, D) *is in standard form with* $k = 1$ *then* (A, B, C, D) *is minimal.*

Proof : This result is proven by contradiction, using the Popov-Belevitch-Hautus eigenvector test (Kailath [72] p.366).

Consider (A, B, C, D), and let

$$x^T = (x_1, \ldots, x_n) \in \mathbb{R}^{1 \times n}$$

such that

$$x^T B = 0 \tag{4.22}$$

and

$$x^T A = \lambda x^T. \tag{4.23}$$

As B is τ-positive-upper-triangular, (4.22) implies that $x_1, \ldots, x_r$ are all zero. This is seen by restricting (4.22) to the first r coordinates

$$(x_1, \ldots, x_r) \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = 0.$$

From the definition of B it is seen that $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ is full rank. Hence it is injective as a linear map, and so the pre-image of 0 is also $\{0\}$. The remaining x_i may be shown to be zero by the following inductive argument:

1) If $x_{r+1} \neq 0$ then since A is (r, τ)-symmetry balanced, $x^T A$ has g_1th element $(-a_{g_1 h_1} x_{r+1})$. $g_1 \leqslant r$ by definition, and so equation (4.23) gives

$$-a_{g_1 h_1} x_{r+1} = \lambda x_{g_1}.$$

Since $x_{g_1} = 0$ and $a_{g_1 h_1} > 0$ then $x_{r+1} = 0$.

2) Assume $x_1, \ldots, x_k$ are equal to zero, $k \geqslant r + 1$. If $x_{k+1} \neq 0$ then either

   a) $\alpha_{k+1-r} \neq 0$, so $x^T A$ has kth element $(-\alpha_{k+1-r} x_{k+1})$ contradicting (4.23) by the argument in 1), or,

   b) $\alpha_{k+1-r} = 0$, so that $k + 1 - r = h_l$ for some $l \in \mathbb{N}$, and the g_lth element of $x^T A$ is $(-a_{g_l h_l} x_{k+1})$ contradicting (4.23) since $g_l \leqslant r$.

Thus $x = 0$ and the Popov-Belevitch-Hautus Lemma implies that (A, B) is controllable. (C, A) is observable from the sign symmetry of the realisation i.e. $C = B^T S$. ■

Theorem 4.44 (Symmetric Asymptotically Stable) *$G \in \mathrm{TC}_{\mathrm{n}}^{\mathrm{m,m}}$ is symmetric if and only if there exists a unique standard symmetric realisation (A, B, C, D) having a standard symmetric set of parameters where:*

$$\begin{aligned} B_j &= \begin{bmatrix} \check{B}_j^1 \check{U}_j^1 \\ \check{B}_j^2 \check{U}_j^2 \\ 0 \end{bmatrix}, \\ A_{jj} &= \check{A}_j - \frac{1}{2\sigma_j} B_j B_j^T, \\ A_{ij} &= \frac{1}{\sigma_i^2 - \sigma_j^2} \left(\sigma_j B_i B_j^T - \sigma_i S_i B_i B_j^T S_j \right), \ i \neq j, \\ C_j^T &= S_j B_j, \\ D &= \check{D}. \end{aligned}$$

where S_i is the sign symmetry matrix of $\check{A}_i$.

Furthermore:

1. *(A, B, C, D) is Lyapunov balanced with Gramian $\Sigma = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_k I_{n_k})$.*
2. *(A, B, C, D) is sign-symmetric with respect to $S = \mathrm{diag}(S_1, \ldots, S_k)$.*
3. *$\mathrm{I}_{-\infty}^{\infty} G(s) = \mathrm{tr}\,(S)$.*

Proof : The argument for the canonical form follows that of Theorem 4.32 except in invoking Propositions 4.37 and 4.41 instead of Propositions 4.23 and 4.26 to establish the existence of the canonical form for the B_i and A_{ii}-matrices respectively.

Let i be fixed for the following discussion. After the B_i structure has been imposed, B_i is of the form

$$B_i = \begin{bmatrix} B_i^1 \\ B_i^2 \\ 0 \end{bmatrix}$$

and hence applying Lemma 4.35 to each of B_i^1 and B_i^2 gives the parameters required. Their properties may be deduced from Propositions 4.37 and 4.38.

It remains to check the details of this canonical form.

Considering the (i, i) blocks of equations (2.28) and (2.29) it is seen that

$$\begin{aligned} \sigma_i \left(A_{ii} + A_{ii}^T \right) + B_i B_i^T &= 0 \\ \sigma_i \left(A_{ii} + A_{ii}^T \right) + S_i B_i B_i^T S_i &= 0 \end{aligned}$$

and noting that due to the orthogonality relations in Proposition 4.38

$$B_i B_i^T = \begin{bmatrix} B_i^1 B_i^{1^T} & 0 & 0 \\ 0 & B_i^2 B_i^{2^T} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.24}$$

hence $S_i B_i B_i^T S_i = B_i B_i^T$.

Therefore,

$$A_{ii} + A_{ii}^T = -\frac{1}{\sigma_i} B_i B_i^T$$

and so

$$A_{ii} = \hat{A}_i - \frac{1}{2\sigma_i} B_i B_i^T \tag{4.25}$$

for some anti-symmetric $\hat{A}_i$.

Letting (A_{ii}, B_i, C_i, D) be denoted by (F, G, H, J) and $\hat{A}_i$ by X let these be partitioned according to $(\tau_i^1, \tau_i^2, n_i - r_i)$. From the sign symmetry property of the canonical form

$$A_{ii}^T = S_i A_{ii} S_i = S_i \hat{A}_i S_i - \frac{1}{2\sigma_i} B_i B_i^T \tag{4.26}$$

then

$$\begin{aligned} F_{11}^T &= F_{11} \\ F_{22}^T &= F_{22} \\ \text{and } F_{12}^T &= -F_{21}. \end{aligned}$$

This implies, since $\hat{A}_i$ is anti-symmetric, that X_{11} and X_{22} are zero, and that $X_{12}^T = -X_{21}$. This shows that $\hat{A}_i$ is indeed (r, τ_i^1)-symmetric balanced.

(4.25) is a decomposition of A_{ii} into symmetric and antisymmetric parts.

Consider now the (i, j)-block of equations (2.28) and (2.29).

$$\sigma_j A_{ij} + \sigma_i A_{ji}^T + B_i B_j^T = 0 \tag{4.27}$$

$$\sigma_j A_{ji}^T + \sigma_i A_{ij} + S_i B_i B_j^T S_j = 0 \tag{4.28}$$

so taking $\sigma_j \times (4.27) - \sigma_i \times (4.28)$ gives

$$\left(\sigma_j^2 - \sigma_i^2\right) A_{ij} = -\sigma_j B_i B_j^T + \sigma_i S_i B_i B_j^T S_j$$

and hence

$$A_{ij} = \frac{1}{\sigma_i^2 - \sigma_j^2} \left(\sigma_j B_i B_j^T - \sigma_i S_i B_i B_j^T S_j\right)$$

as required.

The parametrisation result holds by Proposition 4 from Kabamba [71].

Finally, parts 1) and 2) are implicit in the construction, and 3) follows from Proposition 4.15 regarding the Cauchy-Index. ■

Definition 4.45 (Symmetric Stable Canonical Form) *Let* $\mathrm{TC}_{\mathrm{n}}^{\mathrm{m,m}}[sym]$ *be the set of all symmetric elements of* $\mathrm{TC}_{\mathrm{n}}^{\mathrm{m,m}}$ *and* $\mathrm{C}_{\mathrm{n}}^{\mathrm{m,m}}[sym]$ *the set of corresponding realisations.*

Define the family of canonical forms $(\Gamma_{SC,m,n})_{m,n\in\mathbb{N}}$ *by*

$$\Gamma_{SC,m,n} : \mathrm{C}_{\mathrm{n}}^{\mathrm{m,m}}[sym] \to \mathrm{C}_{\mathrm{n}}^{\mathrm{m,m}}[sym]$$

where $\Gamma_{SC,m,n}((A, B, C, D))$ *is the unique minimal realisation input-output equivalent to* (A, B, C, D) *given by Theorem 4.44.*

■

Theorem 4.46 (Canonical Form for Symmetric Positive-Real Transfer Functions) *$G \in \mathrm{TP}^{\mathrm{m}}_{\mathrm{n}}$ is symmetric if and only if there exists an unique realisation (A, B, C, D) of G given by a standard set of parameters with the following restrictions:*

$$\begin{array}{c} p_i = \sigma_i \in (0,1), \\ D > 0 \end{array}$$

The matrices (A, B, C, D) are partitioned according to $(n_i)_{i=1}^{k}$ and the individual blocks are given by

$$\begin{array}{rcl} B_i &=& \begin{bmatrix} \check{B}_i^1 \check{U}_i^1 \\ \check{B}_i^2 \check{U}_i^2 \\ 0 \end{bmatrix} \sqrt{\check{D} + \check{D}^T} \\ A_{ii} &=& \check{A}_i + B_i (D + D^T)^{-1} B_i^T S_i - \dfrac{1 + p_i^2}{2p_i} B_i (D + D^T)^{-1} B_i^T \\ A_{ij} &=& B_i (D + D^T)^{-1} B_j^T S_j \\ && -\dfrac{1}{p_j^2 - p_i^2} \left\{ p_j (1 - p_i^2) B_i (D + D^T)^{-1} B_j^T - p_i (1 - p_j^2) S_i B_i^T (D + D^T)^{-1} B_j^T S_j \right\} \\ C_i &=& B_i^T S_i \\ D &=& \check{D} \end{array}$$

where S_i is the sign symmetry matrix of $\check{A}_i$.

Furthermore:

1. *(A, B, C, D) is positive-real balanced with Gramian $P = \mathrm{diag}(p_1 I_{n_1}, \ldots, p_k I_{n_k})$.*

2. *(A, B, C, D) is sign-symmetric with respect to $S = \mathrm{diag}(S_1, \ldots, S_k)$.*

3. *$\mathrm{I}_{-\infty}^{\infty} G(s) = \mathrm{tr}\,(S)$.*

Proof : See Appendix A Theorem A.1 for reducibility and augmentability references. The details of the canonical form follow from this theorem and the properties of sign-symmetric realisations in a similar manner to the previous theorem. ■

Definition 4.47 (Symmetric Positive-Real Canonical Form) *Let $\mathrm{TP}^{\mathrm{p}}_{\mathrm{n}}[sym]$ be the set of all symmetric elements of $\mathrm{TP}^{\mathrm{p}}_{\mathrm{n}}$ and $\mathrm{P}^{\mathrm{p}}_{\mathrm{n}}[sym]$ the set of corresponding realisations.*

Define the family of canonical forms $(\Gamma_{SP,m,n})_{m,n \in \mathbb{N}}$ by

$$\Gamma_{SP,m,n} : \mathrm{P}^{\mathrm{p}}_{\mathrm{n}}[sym] \to \mathrm{P}^{\mathrm{p}}_{\mathrm{n}}[sym]$$

where $\Gamma_{SP,m,n}((A, B, C, D))$ is the unique minimal realisation input-output equivalent to (A, B, C, D) given by Theorem 4.46.

■

4.4.4 Minimum-Phase Canonical Form For Symmetric Φ_c

The previous section showed that the class of symmetric positive-real functions has a much simplified parametrisation (c.f. Theorem A.1) which exhibits sign symmetry. While it is clear that the associated spectral factors could not share the same symmetry properties, some associated canonical form may exist which incorporates the sign symmetry matrix.

To demonstrate such a canonical form for $\mathrm{M}_{\mathrm{n}}^{\mathrm{p}}$ it is first necessary to explore in detail the relationship between spectral factors and summands. These relationships are well known, and are essentially described in the Kalman-Yakubovich-Popov Lemma e.g. [31, 3, 108].

Theorem 4.48 (The Kalman-Yakubovich-Popov Lemma [31, 3, 108]) *Let Z_c be a rational $p \times p$-matrix function with real coefficients, where*

$$Z_c(s) = \tilde{C}_z(sI_n - \tilde{A}_z)^{-1}\tilde{B}_z + \tilde{D}_z$$

with $(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)$ a minimal realisation of state dimension n.

Then Z_c is positive-real with $\tilde{A}_z$ asymptotically stable if and only if there exist real matrices $0 < P \in \mathbb{R}^{n\times n}$, $\tilde{L}_o \in \mathbb{R}^{p\times n}$, and $\tilde{W} \in \mathbb{R}^{p\times p}$ such that $(\tilde{L}_o, \tilde{A}_z)$ is observable and

$$\begin{aligned} \tilde{A}_z^T P + P\tilde{A}_z &= -\tilde{L}_o^T\tilde{L}_o && (4.29a)\\ P\tilde{B}_z &= \tilde{C}_z^T - \tilde{L}_o^T\tilde{W} && (4.29b)\\ \tilde{W}^T\tilde{W} &= \tilde{D}_z + \tilde{D}_z^T. && (4.29c) \end{aligned}$$

Furthermore, $\tilde{W}^T\tilde{W} > 0$ and $\tilde{A}_z - \tilde{B}_z\tilde{W}^{-1}\tilde{L}_o$ is asymptotically stable.

Also, since Z_c^T is also positive-real iff Z_c is positive-real then Z_c is positive-real with $\tilde{A}_z$ asymptotically stable if and only if there exist real matrices $0 < Q \in \mathbb{R}^{n\times n}$, $\tilde{K}_o \in \mathbb{R}^{n\times p}$, and $\tilde{V} \in \mathbb{R}^{p\times p}$ such that $(\tilde{A}_z, \tilde{K}_o)$ is controllable and

$$\begin{aligned} \tilde{A}_z Q + Q\tilde{A}_z^T &= -\tilde{K}_o\tilde{K}_o^T && (4.30a)\\ Q\tilde{C}_z^T &= \tilde{B}_z - \tilde{K}_o\tilde{V}^T && (4.30b)\\ \tilde{V}\tilde{V}^T &= \tilde{D}_z + \tilde{D}_z^T. && (4.30c) \end{aligned}$$

Furthermore, $\tilde{V}\tilde{V}^T > 0$ and $\tilde{A}_z - \tilde{K}_o\tilde{V}^{-1}\tilde{C}_z$ is asymptotically stable.

■

Corollary 4.48.1 ([31, 3, 108]) *The transfer functions*

$$\begin{aligned} V_c(s) &= \tilde{C}_z(sI_n - \tilde{A}_z)^{-1}\tilde{K}_o + \tilde{V}\\ W_c(s) &= \tilde{L}_o(sI_n - \tilde{A}_z)^{-1}\tilde{B}_z + \tilde{W} \end{aligned}$$

where

$$\begin{aligned} \tilde{K}_o &= (\tilde{B}_z - Q\tilde{C}_z^T)\tilde{V}^{-T} && (4.31a)\\ \tilde{L}_o &= \tilde{W}^{-T}(\tilde{C}_z - \tilde{B}_z^T P) && (4.31b)\\ \tilde{V} &= \sqrt{\tilde{D}_z + \tilde{D}_z^T} && (4.31c)\\ \tilde{W} &= \sqrt{\tilde{D}_z + \tilde{D}_z^T} && (4.31d) \end{aligned}$$

are minimum-phase iff Z_c above is positive-real.

Furthermore the realisations are minimal if and only if $\left(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z\right)$ is minimal.

■

Note 4.49 *The non-uniqueness of these spectral factors, as described in Lemma 2.33, is exactly that of choosing a square root of* $(\tilde{D}_z + \tilde{D}_z^T)$. *In the next Chapter normalising assumptions will be made on all the spectral factors which will impose a particular choice for the sake of definiteness.*

■

The converse relationship is described as follows:

Lemma 4.50 *If*

$$V_c(s) = \tilde{V} + \tilde{C}_z(sI_n - \tilde{A}_z)^{-1}\tilde{K}_o$$

then

$$Z_c(s) = \frac{1}{2}\tilde{V}\tilde{V}^T + \tilde{C}_z(sI_n - \tilde{A}_z)^{-1}(\tilde{K}_o\tilde{V}^T + \Sigma\tilde{C}_z^T),$$

where

$$\tilde{A}_z\Sigma + \Sigma\tilde{A}_z^T + \tilde{K}_o\tilde{K}_o^T = 0.$$

■

Proof :

$$\begin{aligned}
V_c(s)V_c^T(-s) &= [\tilde{C}_z(sI_n - \tilde{A}_z)^{-1}\tilde{K}_o + \tilde{V}][\tilde{K}_o^T(-sI_n - \tilde{A}_z^T)^{-1}\tilde{C}_z^T + \tilde{V}^T] \\
&= \tilde{C}_z(sI_n - \tilde{A}_z)^{-1}\tilde{K}_o\tilde{K}_o^T(-sI_n - \tilde{A}_z^T)^{-1}\tilde{C}_z^T + \tilde{V}\tilde{K}_o^T(-sI_n - \tilde{A}_z^T)^{-1}\tilde{C}_z^T \\
&\quad + \tilde{C}_z(sI_n - \tilde{A}_z)^{-1}\tilde{K}_o\tilde{V}^T + \tilde{V}\tilde{V}^T \\
&= \tilde{C}_z(sI_n - \tilde{A}_z)^{-1}\left[-\tilde{A}_z\Sigma - \Sigma\tilde{A}_z^T\right](-sI_n - \tilde{A}_z^T)^{-1}\tilde{C}_z^T \\
&\quad + \tilde{V}\tilde{K}_o^T(-sI_n - \tilde{A}_z^T)^{-1}\tilde{C}_z^T + \tilde{C}_z(sI_n - \tilde{A}_z)^{-1}\tilde{K}_o\tilde{V}^T + \tilde{V}\tilde{V}^T \\
&= \tilde{C}_z(sI_n - \tilde{A}_z)^{-1}\left[(sI_n - \tilde{A}_z)\Sigma + \Sigma(-sI_n - \tilde{A}_z^T)\right](-sI_n - \tilde{A}_z^T)^{-1}\tilde{C}_z^T \\
&\quad + \tilde{V}\tilde{K}_o^T(-sI_n - \tilde{A}_z^T)^{-1}\tilde{C}_z^T + \tilde{C}_z(sI_n - \tilde{A}_z)^{-1}\tilde{K}_o\tilde{V}^T + \tilde{V}\tilde{V}^T \\
&= \tilde{C}_z\Sigma(-sI_n - \tilde{A}_z^T)^{-1}\tilde{C}_z^T + \tilde{C}_z(sI_n - \tilde{A}_z)^{-1}\Sigma\tilde{C}_z^T \\
&\quad + \tilde{V}\tilde{K}_o^T(-sI_n - \tilde{A}_z^T)^{-1}\tilde{C}_z^T + \tilde{C}_z(sI_n - \tilde{A}_z)^{-1}\tilde{K}_o\tilde{V}^T + \tilde{V}\tilde{V}^T \\
&= (\tilde{V}\tilde{K}_o^T + \tilde{C}_z\Sigma)(-sI_n - \tilde{A}_z^T)^{-1}\tilde{C}_z^T + \tilde{C}_z(sI_n - \tilde{A}_z)^{-1}(\tilde{K}_o\tilde{V}^T + \Sigma\tilde{C}_z^T) + \tilde{V}\tilde{V}^T \\
&= Z_c^T(-s) + Z_c(s)
\end{aligned}$$

■

The following result shows that a realisation closely related to that given in Corollary 4.48.1 is minimum-phase balanced.

Theorem 4.51 (Transformation of $\mathrm{P}_\mathrm{n}^\mathrm{m}$ **to** $\mathrm{M}_\mathrm{n}^\mathrm{m}$**)** *If* $\left(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z\right)$ *is a realisation of a positive-real function* $Z_c(s)$ *which is positive-real balanced with respect to the positive-real Gramian* $P = \mathrm{diag}(p_1 I_{n_1}, \ldots, p_k I_{n_k})$ *then the transfer function* V_c *with realisation* $(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V})$ *where*

$$\begin{aligned}
\tilde{A}_v &= (I_n - P^2)^{-\frac{1}{4}}\tilde{A}_z(I_n - P^2)^{\frac{1}{4}} \\
\tilde{K} &= (I_n - P^2)^{-\frac{1}{4}}(\tilde{B}_z - P\tilde{C}_z^T)\sqrt{\tilde{D}_z + \tilde{D}_z^T}^{\,-T} \\
\tilde{C} &= \tilde{C}_z(I_n - P^2)^{\frac{1}{4}} \\
\tilde{V} &= \sqrt{\tilde{D}_z + \tilde{D}_z^T}
\end{aligned}$$

where the positive-lower-triangular root is used, is minimum-phase balanced with respect to the Gramian

$$\Sigma = \operatorname{diag}\left(\frac{p_1}{\sqrt{1-p_1^2}} I_{n_1}, \ldots, \frac{p_k}{\sqrt{1-p_k^2}} I_{n_k}\right).$$

Moreover, $(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V})$ *is a minimal realisation of the left spectral factor of the spectral-density* $\Phi_c(i\omega) = Z_c(i\omega) + Z_c^T(-i\omega)$.

Proof : To check that the given realisation is minimum-phase balanced, it is substituted into the Lyapunov equations (4.7) and (4.8).

Assuming that $\left(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z\right)$ and $(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V})$ are of the above form:

$$\begin{aligned} X &= \tilde{A}_v\Sigma + \Sigma\tilde{A}_v^T + \tilde{K}\tilde{K}^T \\ &= (I_n - P^2)^{-\frac{1}{4}}\tilde{A}_z(I_n - P^2)^{\frac{1}{4}}\Sigma + \Sigma(I_n - P^2)^{\frac{1}{4}}\tilde{A}_z^T(I_n - P^2)^{-\frac{1}{4}} \\ &\quad + (I_n - P^2)^{-\frac{1}{4}}(\tilde{B}_z - P\tilde{C}_z^T)(\tilde{D}_z + \tilde{D}_z^T)^{-1}(\tilde{B}_z - P\tilde{C}_z^T)^T(I_n - P^2)^{-\frac{1}{4}}. \end{aligned}$$

Hence, by pre- and post-multiplying by $(I_n - P^2)^{\frac{1}{4}}$, and substituting for Σ

$$\hat{X} = \tilde{A}_z P + P\tilde{A}_z^T + (\tilde{B}_z - P\tilde{C}_z^T)(\tilde{D}_z + \tilde{D}_z^T)^{-1}(\tilde{B}_z - P\tilde{C}_z^T)^T$$

is obtained, which is recognised to be (4.4). Hence $\hat{X} = 0$ and so $X = 0$.

Let

$$(\hat{A}, \hat{B}, \hat{C}, \hat{D}) = (I_n - P^2)^{\frac{1}{4}}(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V})(I_n - P^2)^{-\frac{1}{4}}.$$

Now considering the Lyapunov equation (4.8) it is seen that

$$\begin{aligned} Y &= (\tilde{A}_v - \tilde{K}\tilde{V}^{-1}\tilde{C})^T\Sigma + \Sigma(\tilde{A}_v - \tilde{K}\tilde{V}^{-1}\tilde{C}) + \tilde{C}^T\tilde{V}^{-T}\tilde{V}^{-1}\tilde{C} \\ &= (I_n - P^2)^{\frac{1}{4}}(\hat{A} - \hat{B}\hat{D}^{-1}\hat{C})^T(I_n - P^2)^{-\frac{1}{4}}\Sigma + \Sigma(I_n - P^2)^{-\frac{1}{4}}(\hat{A} - \hat{B}\hat{D}^{-1}\hat{C})(I_n - P^2)^{\frac{1}{4}} \\ &\quad + (I_n - P^2)^{\frac{1}{4}}\hat{C}^T\hat{D}^{-T}\hat{D}^{-1}\hat{C}(I_n - P^2)^{\frac{1}{4}} \end{aligned}$$

hence, by pre- and post-multiplying by $(I_n - P^2)^{-\frac{1}{4}}$, and substituting for Σ

$$\begin{aligned} \hat{Y} &= (\hat{A} - \hat{B}\hat{D}^{-1}\hat{C})^T P(I_n - P^2)^{-1} + P(I_n - P^2)^{-1}(\hat{A} - \hat{B}\hat{D}^{-1}\hat{C}) + \hat{C}^T\hat{D}^{-T}\hat{D}^{-1}\hat{C} \\ &= (\tilde{A}_z - (\tilde{B}_z - P\tilde{C}_z^T)(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{C}_z)^T P(I_n - P^2)^{-1} \\ &\quad + P(I_n - P^2)^{-1}(\tilde{A}_z - (\tilde{B}_z - P\tilde{C}_z^T)(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{C}_z) + \tilde{C}_z^T(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{C}_z. \end{aligned}$$

is obtained. Let $\check{A} = (\tilde{A}_z - B(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{C}_z)$, so that

$$\begin{aligned} \hat{Y} &= \check{A}^T P(I_n - P^2)^{-1} + P(I_n - P^2)^{-1}\check{A} + \tilde{C}_z^T(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{C}_z \\ &\quad + \tilde{C}_z^T(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{C}_z P^2(I_n - P^2)^{-1} + P^2(I_n - P^2)^{-1}\tilde{C}_z^T(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{C}_z \\ &= \check{A}^T P(I_n - P^2)^{-1} + P(I_n - P^2)^{-1}\check{A} + \tilde{C}_z^T(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{C}_z\left(I_n + P^2(I_n - P^2)^{-1}\right) \\ &\quad + P^2(I_n - P^2)^{-1}\tilde{C}_z^T(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{C}_z \\ &= \check{A}^T P(I_n - P^2)^{-1} + P(I_n - P^2)^{-1}\check{A} + \tilde{C}_z^T(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{C}_z(I_n - P^2)^{-1} \\ &\quad + P^2(I_n - P^2)^{-1}\tilde{C}_z^T(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{C}_z. \end{aligned}$$

Now pre- and post-multiply by $(I_n - P^2)$ to obtain

$$\begin{aligned}
\bar{Y} &= (I_n - P^2)\breve{A}^T P + P\breve{A}(I_n - P^2) + (I_n - P^2)\tilde{C}_z^T(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{C}_z \\
&\quad + P^2\tilde{C}_z^T(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{C}_z(I_n - P^2) \\
&= \breve{A}^T P + P\breve{A} - P\left(P\breve{A}^T + \breve{A}P\right)P + \tilde{C}_z^T(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{C}_z \\
&\quad - P^2\tilde{C}_z^T(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{C}_z + P^2\tilde{C}_z^T(\tilde{D}_z + \tilde{D}_z^T)^{-1}C - P^2\tilde{C}_z^T(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{C}_z P^2 \\
&= \breve{A}^T P + P\breve{A} - P\left(P\breve{A}^T + \breve{A}P + P\tilde{C}_z^T(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{C}_z P\right)P + \tilde{C}_z^T(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{C}_z.
\end{aligned}$$

The bracketed term is seen from (4.1) to be equal to $\tilde{B}_z(D + \tilde{D}_z^T)^{-1}\tilde{B}_z^T$, and thus:

$$\bar{Y} = \breve{A}^T P + P\breve{A} + P\tilde{B}_z(D + \tilde{D}_z^T)^{-1}\tilde{B}_z^T P + \tilde{C}_z^T(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{C}_z$$

which is (4.2). Hence $\bar{Y} = 0$ and thus $Y = 0$ showing that the realisation $(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V})$ is minimum-phase balanced.

To complete the proof of this theorem observe that the proposed realisation is a non-singular state-space transformation of the realisation given in Corollary 4.48.1, and hence the minimality of this realisation derives from the minimality of spectral factors in Corollary 4.48.1. The fact that this realisation is minimum-phase derives from the fact that it solves the two Lyapunov equations (4.7) and (4.8). ■

Note 4.52

$$\Sigma = P\sqrt{I_n - P^2}^{-1} \Leftrightarrow P = \Sigma\sqrt{I_n + \Sigma^2}^{-1}$$

$$(I_n - P^2) = (I_n + \Sigma^2)^{-1}$$

■

Theorem 4.53 (Inverse Relationship) *If $(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V})$ is a minimal realisation of a minimum-phase function $V_c(s)$ which is minimum-phase balanced with respect to the minimum-phase Gramian $\Sigma = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_k I_{n_k})$ then the transfer function Z_c with realisation $(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)$ where*

$$\begin{aligned}
\tilde{A}_z &= (I_n + \Sigma^2)^{-\frac{1}{4}}\tilde{A}_v(I_n + \Sigma^2)^{\frac{1}{4}} \\
\tilde{B}_z &= (I_n + \Sigma^2)^{-\frac{1}{4}}(\tilde{K}\tilde{V}^T + \Sigma\tilde{C}^T) \\
\tilde{C}_z &= \tilde{C}(I_n + \Sigma^2)^{\frac{1}{4}} \\
\tilde{D}_z &= \frac{1}{2}\tilde{V}\tilde{V}^T
\end{aligned}$$

is positive-real balanced with respect to the Gramian $P = \Sigma\sqrt{I_n + \Sigma^2}^{-1}$.

Moreover, $(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)$ is a minimal realisation of the spectral summand as given above.

Proof : The realisation $(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)$ is a state-space transformation of the realisation in Lemma 4.50. Hence given a realisation of V_c as above, $(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)$ is certainly a minimal realisation of the associated positive real spectral summand. The relationship between the Gramians may then be established by the argument of Theorem 4.51 when it is noted that this transformation is the algebraic inverse of that in Theorem 4.51. ■

Note 4.54 (Relationship to Left Spectral Factor) *The above two results deal with the relationship between the spectral summand and the right spectral factor. A dual result exists for the relationship between the spectral summand and the left spectral factor. This involves a dual notion of minimum-phase balancing and will be discussed in Chapter 5.*

■

The following result shows how the canonical form in Theorem 4.51 may simplify if the spectral-density is symmetric.

Theorem 4.55 (Transformation of $\mathrm{P}_{\mathrm{n}}^{\mathrm{m}}$ to $\mathrm{M}_{\mathrm{n}}^{\mathrm{m}}$: Symmetric Case) *Let Φ_c be symmetric and the spectral summand Z_c have realisation $(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)$ given by a set of standard symmetric parameters*

$$\left\{p_j, n_j^p, \tau_j^i, \check{A}_j^p, \check{B}_j^{ip}, \check{U}_j^{ip}, \check{D}^p \mid \min(\tau_j^1, \tau_j^2) = 0,\ j = 1, \ldots, k^p,\ i = 1, 2\right\}$$

as given in Theorem 4.46. The realisation $(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V})$ of V_c given in Theorem 4.51 may be described by a standard set of parameters

$$\left\{\sigma_j, n_j^m, r_j, \check{A}_j^m, \check{B}_j^m, \check{U}_j^m, \check{D}^m \mid j = 1, \ldots, k^m\right\}$$

given by

$$\begin{aligned}
k^m &= k^p \\
\sigma_i &= \frac{p_i}{\sqrt{1 - p_i^2}} \\
n_i^m &= n_i^p \\
r_i &= \tau_i^1 + \tau_i^2 \\
\check{A}_i^m &= \check{A}_i^p \\
\check{B}_i^m &= \frac{1}{(1 - p_i^2)^{\frac{1}{4}}}(I_{n_i} - p_i \hat{S}_i) \begin{bmatrix} \check{B}_i^{1p} \check{U}_i^{1p} \\ \check{B}_i^{2p} \check{U}_i^{2p} \end{bmatrix} \\
\check{U}_i^m &= \sqrt{\check{B}_i^m \check{B}_i^{mT}}^{-1} \hat{S}_i \check{B}_i^m \, \textit{(positive-definite)} \\
\check{D}^m &= \sqrt{2\check{D}^p} \, \textit{(positive-lower-triangular)}
\end{aligned}$$

where $\hat{S}_i = I_{r_i}$ if $\tau_i^2 = 0$ and $\hat{S}_i = -I_{r_i}$ otherwise.

Furthermore, the realisations are related as follows:

$$\begin{aligned}
\tilde{A}_v &= (I - P^2)^{-\frac{1}{4}} \tilde{A}_z (I - P^2)^{\frac{1}{4}} \\
\tilde{K} &= (1 - P^2)^{-\frac{1}{4}} (I - PS) \tilde{B}_z \tilde{V}^{-T} \\
\tilde{C} &= \tilde{C}(1 - P^2)^{\frac{1}{4}} \\
\tilde{V} &= \sqrt{2\tilde{D}_z} \, \textit{(positive-lower-triangular)}
\end{aligned}$$

with $S = \operatorname{diag}(S_1, \ldots, S_k)$ the sign matrix derived from $\check{A}_i^m$. The following holds internally for the spectral factor's realisation:

$$\tilde{C} = \tilde{V}\tilde{K}^T \left(\Sigma + \sqrt{1 + \Sigma^2} S\right).$$

Proof : The relationship between the Gramians Σ and P in Theorem 4.51 demonstrates the equality of k^m with k^p and n_i^m with n_i^p and the relationship between p_i and σ_i.

Again from Theorem 4.51

$$\tilde{K}_i = (I_{n_i} - P_i^2)^{-\frac{1}{4}}(\tilde{B}_{z_i} - P_i\tilde{C}_{z_i}^T)\sqrt{\tilde{D}_z + \tilde{D}_z^T}^{-1}$$

and hence from the sign symmetry of the positive-real realisation

$$\tilde{K}_i = \frac{1}{(1-p_i^2)^{\frac{1}{4}}}(I_{n_i} - p_iS_i)\tilde{B}_{z_i}\sqrt{2\tilde{D}_z}^{-1}$$

where S_i is the sign matrix associated with $\breve{A}_j^p$. From this it is obvious that $\tilde{K}_i$ has the same rank as $\tilde{B}_{z_i}$ and that $r_i = \tau_i^1 + \tau_i^2$. Also, because one of the two blocks of $\tilde{B}_{z_i}$ is of zero dimension; $\frac{I_{n_i} - p_iS_i}{(1-p_i^2)^{\frac{1}{4}}} > 0$; and because of the the orthogonality result Theorem 4.20 then $\tilde{K}_i$ is positive-upper-triangular.

Now consider the expression

$$\begin{aligned}
\tilde{C}_i^T\tilde{V}^{-T} - \sigma_i\tilde{K}_i &= (1-p_i^2)^{\frac{1}{4}}\tilde{C}_{z_i}^T\sqrt{2\tilde{D}_z}^{-1} - \frac{\sigma_i}{(1-p_i^2)^{\frac{1}{4}}}(I_{n_i} - p_iS_i)\tilde{B}_{z_i}\sqrt{2\tilde{D}_z}^{-1} \\
&= \left((1-p_i^2)^{\frac{1}{4}}S_i - \frac{\sigma_i}{(1-p_i^2)^{\frac{1}{4}}}(I_{n_i} - p_iS_i)\right)\tilde{B}_{z_i}\sqrt{2\tilde{D}_z}^{-1} \\
&= \left(-\frac{p_i}{(1-p_i^2)^{\frac{3}{4}}}I_{n_i} + (1-p_i^2)^{\frac{1}{4}}S_i + \frac{p_i^2}{(1-p_i^2)^{\frac{3}{4}}}S_i\right)\tilde{B}_{z_i}\sqrt{2\tilde{D}_z}^{-1} \\
&= \left(-\frac{p_i}{(1-p_i^2)^{\frac{3}{4}}}I_{n_i} + \frac{1-p_i^2+p_i^2}{(1-p_i^2)^{\frac{3}{4}}}S_i\right)\tilde{B}_{z_i}\sqrt{2\tilde{D}_z}^{-1} \\
&= \frac{1}{(1-p_i^2)^{\frac{3}{4}}}(S_i - p_iI_{n_i})\tilde{B}_{z_i}\sqrt{2\tilde{D}_z}^{-1} \\
&= \frac{1}{(1-p_i^2)^{\frac{3}{4}}}S_i(I_{n_i} - p_iS_i)\tilde{B}_{z_i}\sqrt{2\tilde{D}_z}^{-1} \\
&= \frac{1}{\sqrt{1-p_i^2}}S_i\tilde{K}_i \\
&= \sqrt{1+\sigma_i^2}S_i\tilde{K}_i.
\end{aligned}$$

Hence,

$$\tilde{C}_i = \tilde{V}\left(\sigma_i\tilde{K}_i^T + \sqrt{1+\sigma_i^2}\tilde{K}_i^TS_i\right).$$

Comparing these last two results with (4.13) gives

$$S_i\tilde{K}_i = \sqrt{\tilde{K}_i\tilde{K}_i^T}\breve{U}_i^m.$$

Upon consideration of the block structure of $\tilde{K}_i$, $\breve{U}_i^m$ and S_i

$$\breve{U}_i^m = \sqrt{\breve{B}_i^m\breve{B}_i^{mT}}^{-1}\hat{S}_i\breve{B}_i^m.$$

Finally, it only remains to check the form of the parameters corresponding to the A-matrices. Since the diagonal blocks of $\tilde{A}_z$ are preserved under the mapping from positive-real to minimum-phase

$$A_{ii}^m = A_{ii}^p$$

$$= \breve{A}_i^p + \tilde{B}_{z_i}(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{B}_{z_i}^T S_i - \frac{1+p_i^2}{2p_i}\tilde{B}_{z_i}(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{B}_{z_i}^T$$

and, since $\tilde{B}_{z_i}(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{B}_{z_i}^T$ is block diagonal

$$\begin{aligned}
&= \breve{A}_i^p + \frac{1}{2}\left(\tilde{B}_{z_i}(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{B}_{z_i}^T S_i + S_i\tilde{B}_{z_i}(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{B}_{z_i}^T\right) \\
&\quad - \frac{(1+p_i^2)}{2\sigma_i\sqrt{1-p^2}}\tilde{B}_{z_i}(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{B}_{z_i}^T \\
&= \breve{A}_i^p + \frac{1}{2\sigma_i}\left(\frac{p_i}{\sqrt{1-p_i^2}}\tilde{B}_{z_i}(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{B}_{z_i}^T S_i + \frac{p_i}{\sqrt{1-p_i^2}}S_i\tilde{B}_{z_i}(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{B}_{z_i}^T\right) \\
&\quad - \frac{(1+p_i^2)}{2\sigma_i\sqrt{1-p^2}}\tilde{B}_{z_i}(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{B}_{z_i}^T \\
&= \breve{A}_i^p + \frac{1}{2\sigma_i}\frac{1}{\sqrt{1-p_i^2}}(I_{n_i} - p_iS_i)\tilde{B}_{z_i}(\tilde{D}_z + \tilde{D}_z^T)^{-1}\tilde{B}_{z_i}^T(I_{n_i} - p_iS_i) \\
&= \breve{A}_i^p + \frac{1}{2\sigma_i}\tilde{B}_i\tilde{B}_i^T
\end{aligned}$$

Hence equality of $\breve{A}_i^p$ and $\breve{A}_i^m$ is established. ■

Remark 4.56 *This result considers only symmetric systems with one sign appearing in each block. This arises because the mapping of the B-terms considered above maps τ-positive-upper-triangular matrices onto τ-positive-upper-triangular matrices. This is a problem in the case of the minimum-phase canonical form where the B-term is positive-upper-triangular.*

Only systems with distinct singular values will be considered at the estimation stage.

This problem could of course be obviated if the canonical form for symmetric positive-real systems used only upper-triangular B-terms. However, this would result in a B-term with complicated internal orthogonality conditions.

■

For SISO systems the situation is simpler, as the following Theorem shows.

Corollary 4.56.1 (Transformation of $\mathrm{P}_\mathrm{n}^\mathrm{m}$ to $\mathrm{M}_\mathrm{n}^\mathrm{m}$: SISO Case) *Assume that the transfer functions and realisations are as in Theorem 4.55 and further that the transfer functions are single-input single-output. Then if $(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)$ is defined by Theorem 4.46 where $p = 1$, with parameters*

$$\left\{b_i^p,\, p_i,\, \alpha_{i,j}^p,\, d^p,\, k^p,\, n_i^p,\, s_i^p \mid i = 1, \dots k^p, j = 2, \dots, n_i^p\right\}$$

and $(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V})$ is given by Theorem 4.51 then this realisation may be described by Corollary 4.32.1 with parameters

$$\left\{b_i^m,\, \sigma_i,\, \alpha_{i,j}^m,\, d^m,\, k^m,\, n_i^m,\, s_i^m \mid i = 1, \dots k^m, j = 2, \dots, n_i^m\right\}$$

where

$$\begin{aligned}
\sigma_i &= \frac{p_i}{\sqrt{1-p_i^2}} \\
b_i^m &= \frac{b_i^p(1 - p_is_i^p)}{\sqrt{2d^p}(1-p_i^2)^{\frac{1}{4}}} \\
\alpha_{i,j}^p &= \alpha_{i,j}^m
\end{aligned}$$

$$\begin{aligned} d^m &= \sqrt{2d^p} \\ s_i^p &= s_i^m \\ n_i^m &= n_i^p \\ k^m &= k^p. \end{aligned}$$

Proof : All of the expressions follow immediately from Theorem 4.55 except the equality of the sign matrices.

Consider

$$\begin{aligned} \tilde{C}_i &= s_i^p \tilde{B}_{z_i}(1-p_i^2)^{1/4} \\ & \quad \text{and, by Theorem 4.53} \\ \tilde{C}_i &= \tilde{V}(\sigma_i + s_i^m\sqrt{1+\sigma_i^2})\tilde{K}_i \\ & \quad \text{from the parametrisation of SISO minimum-phase systems} \\ &= \sqrt{2\tilde{D}_z}(\sigma_i + s_i^m\sqrt{1+\sigma_i^2})\frac{\tilde{B}_{z_i}(1-p_i s_i^p)}{\sqrt{2\tilde{D}_z}(1-p_i^2)^{1/4}} \\ &= \frac{s_i^m \tilde{B}_{z_i}(1-p_i s_i^p)}{(1-p_i^2)^{1/4}}\left(\frac{s_i^m p_i}{\sqrt{1-p_i^2}} + \frac{1}{\sqrt{1-p_i^2}}\right) \\ &= \frac{s_i^m \tilde{B}_{z_i}(1-p_i s_i^p)}{(1-p_i^2)^{1/4}}\frac{1+s_i^m p_i}{\sqrt{1-p_i^2}} \\ &= \frac{s_i^m \tilde{B}_{z_i}(1-p_i s_i^p)(1+s_i^m p_i)}{(1-p_i^2)^{3/4}} \\ &= \frac{s_i^m \tilde{B}_{z_i}(1+(s_i^m - s_i^p)p_i - s_i^p s_i^m p_i^2)}{(1-p_i^2)^{3/4}} \end{aligned}$$

and hence by the equality of the first and last lines

$$\frac{s_i^p}{s_i^m}(1-p_i^2) = 1 + (s_i^m - s_i^p)p_i - s_i^p s_i^m p_i^2$$

and hence by comparing coefficients

$$s_i^m = s_i^p.$$

∎

Definition 4.57 *Define* $\mathrm{P}_\mathrm{n}^\mathrm{p}[\alpha] \subseteq \mathrm{P}_\mathrm{n}^\mathrm{p}[sym]$ *to be all those positive-real realisations* (A,B,C,D) *such that* $\Gamma_{SP,p,n}((A,B,C,D))$ *is given by a standard set of symmetric parameters such that either* $\tau_i^1 = 0$ *or* $\tau_i^2 = 0$ *for each* $i = 1,\ldots,k$.

Define $\mathrm{P}_\mathrm{n}^\mathrm{p}[\omega] \subseteq \mathrm{P}_\mathrm{n}^\mathrm{p}[\alpha]$ *to be all those realisations* (A,B,C,D) *such that* $\Gamma_{SP,p,n}((A,B,C,D))$ *is given by a standard set of symmetric parameters such that* $k = n$ *i.e. that there are* n *distinct characteristic values.*

∎

Definition 4.58 (Mapping from $\mathrm{P}_\mathrm{n}^\mathrm{p}$ to $\mathrm{M}_\mathrm{n}^\mathrm{p}$) *Define a family of maps* $(\Lambda_{p,n})_{p,n\in\mathbb{N}}$ *by*

$$\Lambda_{p,n} : \Gamma_{SP,p,n}\left(\mathrm{P}_\mathrm{n}^\mathrm{p}[\alpha]\right) \to \mathrm{M}_\mathrm{n}^\mathrm{p}$$

by

$$\Lambda_{p,n}((\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)) = (\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V})$$

as defined in Theorem 4.55.

Also define $\mathrm{M}_{\mathrm{n}}^{\mathrm{p}}[\alpha]$ *to be the set of all realisations in* $\mathrm{M}_{\mathrm{n}}^{\mathrm{p}}$ *which are input-output equivalent to some* $(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V}) \in \mathrm{im}\Lambda_{p,n}$ *and* $\mathrm{M}_{\mathrm{n}}^{\mathrm{p}}[\omega]$ *to be all those realisations that are input-output equivalent to some realisation in* $\Lambda_{p,n}(\Gamma_{SP,p,n}(\mathrm{P}_{\mathrm{n}}^{\mathrm{p}}[\omega]))$.

The sets $\mathrm{TM}_{\mathrm{n}}^{\mathrm{p}}[\alpha]$ *and* $\mathrm{TM}_{\mathrm{n}}^{\mathrm{p}}[\omega]$ *are the sets of transfer function with realisations in the above classes.*

■

Note 4.59 *Part of the result in Theorem 4.55 may be expressed as*

$$\Lambda_{p,n}(\Gamma_{SP,p,n}(\mathrm{P}_{\mathrm{n}}^{\mathrm{p}}[\alpha])) \subseteq \Gamma_{M,p,n}(\mathrm{M}_{\mathrm{n}}^{\mathrm{p}})$$

■

Theorem 4.60 (Symmetric Form of Canonical Form $\mathrm{M}_{\mathrm{n}}^{\mathrm{p}}[\alpha]$) $G \in \mathrm{TM}_{\mathrm{n}}^{\mathrm{p}}[\alpha]$ *if and only if there exists a unique minimal realisation* (A, B, C, D) *given by a set of parameters*

$$\left\{\sigma_j, n_j, r_j, \check{A}_j, \check{B}_j, \check{D} \mid j = 1, \ldots, k\right\}$$

where:

$$\begin{array}{lllll}
\sigma_1 > \ldots > \sigma_k > 0 & , & \text{where} & k \in \mathbb{N};\ \sigma_i \in \mathbb{R}, \\
n_1\ ,\ \ldots,\ n_k & , & " & n_j \in \mathbb{N}\ ;\ \sum_{j=1}^{k} n_j = n, \\
r_1\ ,\ \ldots,\ r_k & , & " & r_j \in \mathbb{N};\ 1 \leqslant r_j \leqslant \min(n_j, p), \\
\check{A}_1\ ,\ \ldots,\ \check{A}_k & , & " & \check{A}_j \in \mathbb{R}^{n_j \times n_j}\ (r_j, r_j)\text{-symmetric-balanced}, \\
\check{B}_1\ ,\ \ldots,\ \check{B}_k & , & " & \check{B}_j \in \mathbb{R}^{r_j \times p}\ \text{full rank positive-upper-triangular}; \\
\check{D} = \check{D}^T & , & " & \check{D} \in \mathbb{R}^{p \times p},
\end{array}$$

with realisations

$$\begin{aligned}
B_i &= \begin{bmatrix} \check{B}_i \\ 0 \end{bmatrix} \\
A_{ii} &= \check{A}_i - \frac{1}{2\sigma_i} B_i B_i^T \\
A_{ij} &= \frac{1}{\sigma_i^2 - \sigma_j^2}\left\{\sigma_j(1+\sigma_i^2) B_i B_j^T - \sigma_i\sqrt{1+\sigma_i^2}\sqrt{1+\sigma_j^2} S_i B_i B_j^T S_j\right\} \\
C_i &= D B_i^T \left(\sigma_i I_{n_i} + \sqrt{1+\sigma_i^2} S_i\right) \\
D &= \check{D}
\end{aligned}$$

where $S = \mathrm{diag}(S_1, \ldots, S_k)$, S_i *the sign matrix of* $\check{A}_i$.
Furthermore, (A, B, C, D) *is minimum-phase balanced with Gramian* $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_k)$.

Proof : Since $\mathrm{M}_{\mathrm{n}}^{\mathrm{p}}[\alpha]$ is the image of $\mathrm{P}_{\mathrm{n}}^{\mathrm{p}}[\alpha]$ under $\Lambda_{p,n}$ the above follows immediately from Theorems 4.55 and 4.32.

The A_{ij} expression comes from simplification of (4.17). ■

Definition 4.61 (Symmetric Minimum-Phase Canonical Form) *Define the family of canonical forms* $(\Gamma_{SM,p,n})_{p,n\in\mathbb{N}}$ *by*

$$\Gamma_{SM,p,n} : \mathrm{M}_{\mathrm{n}}^{\mathrm{p}}[\alpha] \to \mathrm{M}_{\mathrm{n}}^{\mathrm{p}}[\alpha]$$

where $\Gamma_{SM,p,n}((A,B,C,D))$ *is the unique minimal realisation input-output equivalent to* (A,B,C,D) *given by Theorem 4.60.*

■

Note 4.62 *This canonical form is a restriction of* $\Gamma_{M,p,n}$ *but has a different parametrisation.*

■

Corollary 4.62.1 (Canonical Form for $\mathrm{M}_{\mathrm{n}}^{\mathrm{p}}[\omega]$**)** $G \in \mathrm{TM}_{\mathrm{n}}^{\mathrm{p}}[\omega]$ *if and only if there exists a unique minimal realisation* (A,B,C,D) *given by a set of parameters*

$$\left\{\sigma_j, s_i, \check{B}_j, \check{D} \,|\, j = 1,\ldots,k\right\}$$

as in Theorem 4.58, but where

$$\begin{aligned} n_i &= 1 \\ r_i &= 1 \\ \check{A}_i &= 0_{1\times 1} \end{aligned}$$

■

Theorem 4.63 (Inverse to Theorem 4.55) *If* $(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V}) \in \mathrm{M}_{\mathrm{n}}^{\mathrm{p}}[\alpha]$ *is a minimum-phase balanced realisation with Gramian* Σ*, then*

$$\begin{aligned} \tilde{A}_z &= (I+\Sigma^2)^{-\frac{1}{4}}\tilde{A}_v(I+\Sigma^2)^{\frac{1}{4}} \\ \tilde{B}_z &= (I+\Sigma^2)^{-\frac{1}{4}}(I-PS)^{-1}\tilde{K}\tilde{V}^T \\ \tilde{C}_z &= \tilde{C}(I+\Sigma^2)^{\frac{1}{4}} \\ \tilde{D}_z &= \frac{1}{2}\tilde{V}\tilde{V}^T \end{aligned}$$

where $P = \Sigma(I+\Sigma^2)^{-\frac{1}{2}}$, *is a realisation in* $\Gamma_{P,p,n}(\mathrm{P}_{\mathrm{n}}^{\mathrm{p}}[\alpha])$ *of the associated spectral summand which is sign-symmetric with respect to* S*, the sign matrix of* $\tilde{A}_v$*.*

Proof : This is a special case of Theorem 4.53, and is clear from that and previous results. ■

4.5 Model Reduction Properties

It is an especially attractive feature of many balancing schemes that they possess a conceptually simple model reduction scheme. This is shown in the following section for minimum-phase balancing.

4.5.1 Balanced Stochastic Truncation

A model reduction scheme based upon the equations (4.1) and (4.2) was introduced by Desai & Pal [21] and extended to discrete-time systems in Desai & Pal [22]. The stability and minimality properties have been studied in Harshavardhana *et al* [49]. Balanced stochastic truncation is a closely related method (as demonstrated in Lemma 4.65) which instead of concentrating on the positive-real factor approximated the right spectral factor. The method allows for the treatment of non-square, non-minimum-phase spectral factors. Several authors have discussed this method e.g. [7, 42, 41, 70, 102, 122]. Green [41] originally obtained a relative error bound for this scheme and Wang and Safonov [122, 121, 123] have extended this by producing a suite of error-bounds improving on Green's result.

The following method depend crucially upon the relationships between spectral factors and spectral summands as detailed in §4.4.4.

Definition 4.64 (Stochastic Balancing) *Given Z_c, V_c and W_c spectral summand and factors as in Theorem 4.48 and Corollary 4.48.1, so that*

$$\begin{array}{lcl} Z_c(s) & \textit{has realisation} & (\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z) \\ V_c(s) & " & (\tilde{A}_z, \tilde{K}_o, \tilde{C}_z, \tilde{V}) \\ W_c(s) & " & (\tilde{A}_z, \tilde{B}_z, \tilde{L}_o, \tilde{W}) \end{array}$$

where $\tilde{K}_o$, $\tilde{L}_o$, $\tilde{V}$ and $\tilde{W}$ are given by (4.31a)–(4.31d), then define $F(s) = \tilde{L}_o(sI - \tilde{A}_z)^{-1}\tilde{K}_o$.

The realisations of Z_c, V_c and W_c are said to be stochastically balanced with Gramian *P if $(\tilde{A}_z, \tilde{K}_o, \tilde{L}_o)$ is Lyapunov balanced with the same Gramian.*

■

Lemma 4.65 (Relationship Between Stochastic and Positive-Real Balancing) *If $Z_c \stackrel{s}{=} (\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)$ and $\tilde{K}_o$, $\tilde{L}_o$, $\tilde{V}$ and $\tilde{W}$ are given by (4.31a)–(4.31d) then $(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)$ satisfies (4.1) and (4.2) iff $(\tilde{A}_z, \tilde{K}_o, \tilde{L}_o)$ satisfies (2.28) and (2.29).*

Proof : This is obvious from consideration of the alternative form of the positive-real Riccati equations (4.3) and (4.4). ■

If the realisations of Z_c, V_c and W_c are stochastically balanced with Gramian

$$P = \operatorname{diag}(p_1 I_{n_1}, \ldots, p_k I_{n_k}) = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

such that $P_i \in \mathbb{R}^{m_i \times m_i}$, $i = 1, 2$, and $\lambda(P_1) \cap \lambda(P_2) = \emptyset$ then partitioning all of the realisations according to (m_1, m_2) gives the following:

$$\tilde{A}_z = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \quad \tilde{B}_z = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} \quad \tilde{C}_z = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}$$
$$\tilde{K}_o = \begin{bmatrix} \tilde{K}_1 \\ \tilde{K}_2 \end{bmatrix} \quad \tilde{L}_o = \begin{bmatrix} \tilde{L}_1 & \tilde{L}_2 \end{bmatrix}$$

Define

$$\begin{aligned} \hat{V}_S(s) &= \tilde{C}_1 \left(sI_{m_1} - \tilde{A}_{11}\right)^{-1} \tilde{K}_1 + \tilde{V} \\ \hat{W}_S(s) &= \tilde{L}_1 \left(sI_{m_1} - \tilde{A}_{11}\right)^{-1} \tilde{B}_1 + \tilde{W} \\ \hat{Z}_S(s) &= \tilde{C}_1 \left(sI_{m_1} - \tilde{A}_{11}\right)^{-1} \tilde{B}_1 + \tilde{D}_z \end{aligned}$$

which are the *Balanced Stochastic Approximants of* V, W and Z_c. Of course, this is merely positive-real model reduction of Z_c, but it is the model reduction of V and W that is of main interest. Not only are the realisations of these approximands minimal, the following relative error-bounds also hold.

Lemma 4.66 (Lemma 1.2 and Theorem 3.1 Wang & Safonov [121]) *Given the above definitions then*

$$\left\| V_c^{-1}(s)\left(V_c(s) - \hat{V}_S(s)\right)\right\|_\infty \leqslant 2 \sum_{i=r+1}^{k} \frac{p_i}{1-p_i} \tag{4.32}$$

$$\left\| \left(W_c(s) - \hat{W}_S(s)\right) W_c^{-1}(s)\right\|_\infty \leqslant 2 \sum_{i=r+1}^{k} \frac{p_i}{1-p_i} \tag{4.33}$$

$$\left\| \hat{V}_S^{-1}(s)\left(V_c(s) - \hat{V}_S(s)\right)\right\|_\infty \leqslant 2 \sum_{i=r+1}^{k} \frac{p_i}{1-p_i} \tag{4.34}$$

$$\left\| \left(W_c(s) - \hat{W}_S(s)\right) \hat{W}_S^{-1}(s)\right\|_\infty \leqslant 2 \sum_{i=r+1}^{k} \frac{p_i}{1-p_i} \tag{4.35}$$

■

4.5.2 Minimum-Phase Transfer Functions

An analogous model reduction scheme, based upon the minimum-phase balanced realisation, suggests itself. Hence if $V_c \in \mathrm{TM}_\mathrm{n}^\mathrm{p}$ has a realisation $(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V}) \in \mathrm{M}_\mathrm{n}^\mathrm{p}$ which is minimum-phase balanced with respect to Gramian

$$\Sigma = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_k I_{n_k}) = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

such that $\Sigma_i \in \mathbb{R}^{m_i \times m_i}$, $i = 1, 2$, where $m_1 = n_1 + \cdots + n_l$ $(1 \leqslant l < k)$ and $m_2 = n - m_1$ then it may be partitioned according to (m_1, m_2).

If

$$\hat{V}_{MP}(s) = \tilde{C}_1(sI_{m_1} - \tilde{A}_{11})^{-1}\tilde{K}_1 + \tilde{V}$$

then Theorem 4.51 may be used to relate the result from the previous section to the case considered here.

Theorem 4.67 (Error-Bounds for Minimum-Phase Balanced Truncation) *Given* $V_c(s)$ *and* $\hat{V}_{MP}(s)$ *as defined above then*

$$\left\| V_c^{-1}(s)\left(V_c(s) - \hat{V}_{MP}(s)\right)\right\|_\infty \leqslant 2 \sum_{i=l+1}^{k} \sigma_i \left(\sigma_i + \sqrt{1+\sigma_i^2}\right) \tag{4.36}$$

$$\left\| \hat{V}_{MP}^{-1}(s)\left(V_c(s) - \hat{V}_{MP}(s)\right)\right\|_\infty \leqslant 2 \sum_{i=l+1}^{k} \sigma_i \left(\sigma_i + \sqrt{1+\sigma_i^2}\right) \tag{4.37}$$

Proof : If it may be shown that

$$\hat{V}_{MP}(s) = \hat{V}_S(s)$$

then application of Lemma 4.66 and some simple algebra will establish the desired result.

Firstly, assume that Z_c, V_c and W_c have the realisations given in Definition 4.64, and that these are stochastically balanced with respect to $P = \operatorname{diag}(p_1 I_{n_1}, \ldots, p_k I_{n_k})$. Partition all of these realisations according to $(n_i)_{i=1}^k$.

Theorem 4.51 gives a realisation $(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V})$ which is minimum-phase balanced with minimum-phase Gramian $\Sigma = P\sqrt{I-P^2}^{-1}$ which may also be partitioned according to $(n_i)_{i=1}^k$ since it inherits the same block structure as the above realisations.

From comparisons of Theorem 4.51 and Corollary 4.48.1 it is clear that

$$(\tilde{A}_z, \tilde{K}_o, \tilde{C}_z, \tilde{V}) = (I-P^2)^{\frac{1}{4}}(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V})(I-P^2)^{-\frac{1}{4}}$$

(given appropriate choice of constant term - see Chapter 5) and hence each block is merely scaled by a scalar factor.

Taking m_1 and m_2 as at the head of the subsection and partitioning all of the above realisations according to (m_1, m_2) and also partitioning

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \; P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \text{ where } \Sigma_i, P_i \in \mathbb{R}^{m_i \times m_i}, \; i = 1, 2$$

then, from the above block equalities

$$(\tilde{A}_{z_{11}}, \tilde{K}_{o_1}, \tilde{C}_{z_1}, \tilde{V}) = (I_{m_1} - P_1^2)^{\frac{1}{4}}(\tilde{A}_{v_{11}}, \tilde{K}_1, \tilde{C}_1, \tilde{V})(I_{m_1} - P_1^2)^{-\frac{1}{4}}$$

and hence

$$\hat{V}_{MP}(s) = \hat{V}_S(s).$$

The final form of the result is obtained by considering the relationship between p_i and σ_i:

$$\begin{aligned} p_i &= \frac{\sigma_i}{\sqrt{1+\sigma_i^2}} \\ \frac{p_i}{1-p_i} &= p_i\left(\frac{1}{1-p_i^2} + \frac{p_i}{1-p_i^2}\right) \end{aligned}$$

hence since

$$\frac{1}{1-p_i^2} = 1 + \sigma_i^2$$

and

$$\frac{p_i}{1-p_i^2} = \sigma_i\sqrt{1+\sigma_i^2}$$

then

$$\begin{aligned} \frac{p_i}{1-p_i} &= p_i\left(1+\sigma_i^2 + \sigma_i\sqrt{1+\sigma_i^2}\right) \\ &= \sigma_i\left(\sigma_i + \sqrt{1+\sigma_i^2}\right) \end{aligned}$$

■

Chapter 5

Interpretation of Balanced Parameters

In the previous Chapters a variety of inter-related transfer functions have been suggested as vehicles for the study of stochastic processes. In Chapter 2 it was shown that the canonical correlation structure of certain stochastic processes was related to properties of the phase-factor $V^{-1}W^*$ associated with the process' spectral density; in Chapter 3 the use of the discrete-time minimum-phase spectral factor $V(z)$ and its inverse were used to model the process and its predictors; and in Chapter 4 the continuous-time equivalent spectral factor was used as a basis for a parametrisation of the model class $\mathrm{M}_{\mathrm{n}}^{\mathrm{p}}$.

It has also been shown that a form of balancing corresponds to each transfer function e.g. minimum-phase balancing of spectral factors and positive-real balancing of spectral summands. In Theorem 4.51 a relationship between continuous-time spectral factors and summands was demonstrated which relates the balancedness of the realisations.

The primary purpose of this Chapter is the demonstration of the relationship between the properties of the Hankel operator $R\mathbf{H}_{V^{-1}W^*}$ and the parameters appearing in the parametrisations in Theorems 4.32 and 4.44. In the process of obtaining these results many other connections will be examined.

5.1 Connections Between Realisations

5.1.1 Realisations and Balancing

Before proceeding to analyse the relationships between the various factors mentioned in §2.3 it is necessary to describe their state-space realisations and to consider forms of balancing suitable to each.

Balancing schemes have been introduced for spectral summands and right spectral factors. It will be shown below that the stable and anti-stable parts of the phase factors may be Lyapunov balanced separately. The following definition modifies Definition 4.4 to apply to left spectral factors.

Definition 5.1 (Left Minimum-Phase Balanced) *Given* $(A,B,C,D) \in \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times p}\times\mathbb{R}^{p\times n}\times\mathbb{R}^{p\times p}$ *with solutions P and Q to*

$$\begin{aligned} R_{LM}((A,B,C,D);P) &:= (A-BD^{-1}C)P+P(A-BD^{-1}C)^T+BD^{-1}D^{-T}B^T=0 && (5.1)\\ R^D_{LM}((A,B,C,D);Q) &:= AQ+QA+C^TC=0 && (5.2)\end{aligned}$$

then (A,B,C,D) is said to be continuous-time left minimum-phase balanced *with* left minimum-phase Gramian Σ *if* $\Sigma=P=Q=\mathrm{diag}(\sigma_1 I_{n_1},\ldots,\sigma_k I_{n_k})=\mathrm{diag}(\mu_1,\ldots,\mu_n)$ *where* $\sigma_1>\ldots>\sigma_k$ *and* $\mu_1\geqslant\ldots\geqslant\mu_n$.

Any $(A,B,C,D)\in \mathrm{DM}_{\mathrm{n}}^{\mathrm{p}}$ such that the solutions P and Q to

$$\begin{aligned} R_{DLM}((A,B,C,D);P) &:= (A-BD^{-1}C)P(A-BD^{-1}C)^T-P+BD^{-1}D^{-T}B^T=0 && (5.3)\\ R^D_{DLM}((A,B,C,D);Q) &:= A^TQA-Q+C^TC=0 && (5.4)\end{aligned}$$

are similarly equal, diagonal, and have diagonal elements in decreasing order is discrete-time left minimum-phase balanced *with analogous Gramians and characteristic values.*

■

In order to distinguish the realisations of the eight transfer functions under consideration the following notations are introduced. For each of the following realisations the appropriate balancing scheme is mentioned and its Gramian named.

Right Spectral Factor $V_d(z) = C(z^{-1}I - A_v)^{-1}K + V \stackrel{s}{=} (A_v, K, C, V)$.

This realisation will be taken to be Discrete Minimum-Phase Balanced with Gramian Σ.

Right Spectral Factor $V_c(s) = \tilde{C}(sI - \tilde{A}_v)^{-1}\tilde{K} + \tilde{V} \stackrel{s}{=} (\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V})$.

This realisation will be taken to be Minimum-Phase Balanced with Gramian Σ.

Spectral Summand $Z_d(z) = C_z(z^{-1}I - A_z)^{-1}B_z + D_z \stackrel{s}{=} (A_z, B_z, C_z, D_z)$.

This realisation will be taken to be Discrete Positive-Real Balanced with Gramian P.

Spectral Summand $Z_c(s) = \tilde{C}_z(sI - \tilde{A}_z)^{-1}\tilde{B}_z + \tilde{D}_z \stackrel{s}{=} (\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)$.

This realisation will be taken to be Positive-Real Balanced with Gramian P.

Left Spectral Factor $W_d(z) = L(z^{-1}I - A_w)^{-1}B + W \stackrel{s}{=} (A_w, B, L, W)$.

This realisation will be taken to be Discrete Left Minimum-Phase Balanced with Gramian Σ.

Left Spectral Factor $W_c(s) = \tilde{L}(sI - \tilde{A}_w)^{-1}\tilde{B} + \tilde{W} \stackrel{s}{=} (\tilde{A}_w, \tilde{B}, \tilde{L}, \tilde{W})$.

This realisation will be taken to be Left Minimum-Phase Balanced with Gramian Σ.

The stable and anti-stable parts of the Phase functions are handled separately, although both may be Lyapunov balanced with the same Gramian. The definition of the various realisations will be given in §5.1.3 below.

Discrete-time Phase Factor - Anti-stable part

$\left[V_d^{-1}(z)W_d^*(z)\right]_- = K_o^T(zI - A_z^T)^{-1}L_o^T \stackrel{s}{=} (A_z^T, L_o^T, K_o^T)$

This realisation will be taken to be Discrete Lyapunov Balanced with Gramian P.

Discrete-time Phase Factor - Stable part

$\left[V_d^{-1}(z)W_d^*(z)\right]_+ = \hat{C}_d(z^{-1}I - \hat{A}_d)^{-1}\hat{B}_d \stackrel{s}{=} (\hat{A}_d, \hat{B}_d, \hat{C}_d)$

This realisation will be taken to be Discrete Lyapunov Balanced with Gramian P.

Continuous-time Phase Factor - Anti-stable part

$[V_c^{-1}(s)W_c^*(s)]_- = \tilde{K}_o^T(-sI - \tilde{A}_z^T)^{-1}\tilde{L}_o^T \stackrel{s}{=} (\tilde{A}_z^T, \tilde{L}_o^T, \tilde{K}_o^T)$.

This realisation will be taken to be Lyapunov Balanced with Gramian P.

Continuous-time Phase Factor - Stable part

$[V_c^{-1}(s)W_c^*(s)]_+ = \hat{C}_c(sI - \hat{A}_c)^{-1}\hat{B}_c \stackrel{s}{=} (\hat{A}_c, \hat{B}_c, \hat{C}_c)$

This realisation will be taken to be Lyapunov Balanced with Gramian P.

5.1.2 Conventions and Normalisations

For the sake of definiteness it is necessary to choose one particular representative spectral factor from the class of minimal, square factors

$$\left\{V_dU \mid U \in \mathbb{C}^{p\times p} \text{ unitary}\right\}$$

for some spectral factor V_d.

U may be restricted to the class of orthogonal matrices because only real spectral factors are being considered.

A unique element from this set may be chosen by specifying that it is positive-upper-triangular at one particular point.

For the remainder of this work it will be assumed that all spectral factors are normalised as in Table 5.1.

	Normalise at	Positive-Lower-Triangular	Positive-Upper-Triangular
Discrete Time	$z=-1$	$W_d(-1) = W - L(I+A_w)^{-1}B$	$V_d(-1) = V - C(I+A_v)^{-1}K$
Continuous Time	$s=\infty$	$W_c(\infty) = \tilde{W}$	$V_c(\infty) = \tilde{V}$

Table 5.1: Summary of Normalisation Conventions

Remark 5.2 *The discrete-time spectral factor is normalised at $z=-1$ because this is the image of $s=\infty$ under the bilinear map M. Thus the two normalisations are equivalent, and this can be checked by reference to Proposition 2.7.*

■

The spectral summands will be normalised by taking the summand to have constant term equal to half of that of the spectral-density i.e. to be symmetric.

Lemma 5.3 *If Φ_c and Φ_d are symmetric and V_c, W_c, V_d and W_d are normalised as in Table 5.1 then*

$$\begin{aligned} V_c &= W_c^T \\ V_d &= W_d^T \end{aligned}$$

Proof :

$$\begin{aligned} \Phi_d &= \Phi_d^T \\ \Leftrightarrow V_d V_d^* &= (W_d^* W_d)^T \\ &= W_d^T \bar{W}_d \end{aligned}$$

and since $W_d^T \in \mathrm{TM}_{\mathrm{n}}^{\mathrm{p}}$ then W_d^T is a right spectral factor of Φ_d. Hence, from Lemma 2.33

$$W_d^T = V_d U, \;\; U \in \mathrm{O}_p(\mathbb{R})$$

but W_d is positive-lower-triangular at the point -1, and V_d is positive-upper-triangular there. Hence $U = I_p$, which gives the required result.

The continuous-time case follows from the same argument considered at $s=\infty$. ■

Note 5.4 *If $\Phi_d = \Phi_d^T$ and V_d, W_d are as in Lemma 5.3 then $R\mathbf{H}_{V_d^{-1}W_d^*}$ is self-adjoint.*

Proof :

$$\begin{aligned} \left(V_d^{-1} W_d^*\right)^T &= \bar{W}_d V_d^{-T} \\ &= V_d^* W_d^{-1} \\ &= V_d^{-1} W_d^* \end{aligned}$$

■

$$
\begin{tikzcd}
 & V_d(z) \overset{s}{=} (A_v, K, C, V) & V_c(s) \overset{s}{=} (\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V}) \arrow[l, "V_2\ (a)"'] & \\
V_d^{-1}(z)W_d^*(z) & Z_d(z) \overset{s}{=} (A_z, B_z, C_z, D_z) \arrow[l, "(i)"] \arrow[u, "(d)"] \arrow[d, "(f)"'] & Z_c(s) \overset{s}{=} (\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z) \arrow[l, "V_2\ (c)"] \arrow[u, "(b)"'] \arrow[d, "(e)"] \arrow[r, "(h)"'] & V_c^{-1}(s)W_c^*(s) \\
 & W_d(z) \overset{s}{=} (A_w, B, L, W) & W_c(s) \overset{s}{=} (\tilde{A}_w, \tilde{B}, \tilde{L}, \tilde{W}) \arrow[l, "V_2\ (g)"] &
\end{tikzcd}
$$

Figure 5.1: Diagram of Relationships Between Functions Involved in Identification

5.1.3 The Interrelationships Between Spectral Factors, Summands and Phase Factors

Figure 5.1 shows all of the relevant transfer functions and the relationships described in this Chapter. As will be demonstrated, there is a simple duality between the left and right spectral factors which is suggested by the symmetry of the diagram.

In the following analyses two related aims are pursued. Firstly, it is necessary to show that there exist relationships between the state-space realisations of the different transfer functions enumerated in §5.1.1 such that the balancing schemes may also be related. As will be shown in §5.2, this will be used to relate the σ parameters appearing in Theorem 4.32 to the canonical correlation coefficients. Secondly, it will be possible to relate the canonical forms derived in Chapter 4 to each other. This will show the relationship between the sign parameters which appear in the various realisations, and hence connect these and other parameters to properties of the canonical correlation Hankel operator.

The various relationships are now described in detail.

(a) [Continuous-time Right Spectral Factor $\leftrightarrow$ Discrete-time Right Spectral Factor]

Propositions 2.6 and 2.7 give the details of this mapping. In Note 4.7 it is shown that this is a bijective mapping from $\mathrm{M}_\mathrm{n}^\mathrm{p}$ to $\mathrm{DM}_\mathrm{n}^\mathrm{p}$, and Lemma 4.9 shows that V_c's realisation is continuous minimum-phase balanced iff V_d's realisation is discrete minimum-phase balanced with the same Gramian.

(b) [Spectral Summand $\leftrightarrow$ Right Spectral Factor]

Theorems 4.51 describes this relationship and shows that $(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)$ is positive-real balanced if and only if $(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V})$ is minimum-phase balanced.

Further note that the choice of constant term corresponds to the normalisation chosen in 5.1.2.

It is possible to say considerably more about this relationship because of the canonical forms available for such realisations, as was shown in §4.4.4.

(c) [Continuous-Time Spectral Summand $\leftrightarrow$ Discrete-Time Spectral Summand]

This relationship is described in the paper Anderson *et al* [5]. The following summarises this relationship:

Theorem 5.5 ([5]) *If* $Z_c(s) \stackrel{s}{=} (\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)$ *and* $Z_d(z) := Z_c(M(z)) \stackrel{s}{=} (A_z, B_z, C_z, D_z)$ *as given in Proposition 2.6 then* $(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)$ *is continuous-time Positive-Real balanced with Gramian* P *iff* (A_z, B_z, C_z, D_z) *is discrete-time Positive-Real balanced with Gramian* P.

Proof : See Anderson *et al* [5] Theorem 1. This is also proved in Popov [108] from the Kalman-Yakubovich-Popov equations (see Theorem 5.7). ■

Note 5.6 (Sign Symmetry) *From Proposition 2.6 1)* $(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)$ *is sign-symmetric with respect to* S *iff* (A_z, B_z, C_z, D_z) *is sign-symmetric with respect to* S.

■

(d) [Spectral Summand $\leftrightarrow$ Right Spectral Factor]

This relationship is established from the discrete-time version of the Kalman-Yakubovich-Popov Lemma.

Theorem 5.7 (The Kalman-Yakubovich-Popov Lemma [31, 3, 108]) *Let* Z_d *be a rational* $p \times p$*-matrix function with real coefficients, where*

$$Z_d(z^{-1}) = C_z(z^{-1}I_n - A_z)^{-1}B_z + D_z$$

with (A_z, B_z, C_z, D_z) *a minimal realisation of state dimension* n.

Then Z_d *is discrete positive-real with* $\lambda(A_z) \subseteq \mathbb{D}$ *if and only if there exist real matrices* $0 < P \in \mathbb{R}^{n \times n}$, $L_o \in \mathbb{R}^{p \times n}$, *and* $W \in \mathbb{R}^{p \times p}$ *such that* (L_o, A_z) *is observable and*

$$\begin{aligned} A_z^T P A_z - P &= -L_o^T L_o \\ A_z^T P B_z &= C_z^T - L_o^T W \\ W^T W &= D_z + D_z^T - B_z^T P B_z. \end{aligned}$$

Furthermore, $W^T W > 0$ *and* $A_z - B_z W^{-1} L_o$ *is asymptotically stable.*

Also, since Z_d^T *is also positive-real iff* Z_d *is positive-real then* Z_d *is positive-real with* $\lambda(A_z) \subseteq \mathbb{D}$ *i.e. asymptotically stable, if and only if there exist real matrices* $0 < Q \in \mathbb{R}^{n \times n}$, $K_o \in \mathbb{R}^{n \times p}$, *and* $V \in \mathbb{R}^{p \times p}$ *such that* (A_z, K_o) *is controllable and*

$$\begin{aligned} A_z Q A_z^T - Q &= -K_o K_o^T \\ A_z Q C_z^T &= B_z - K_o V^T \\ V V^T &= D_z + D_z^T - C_z Q C_z^T. \end{aligned}$$

Furthermore, $V V^T > 0$ *and* $A_z - K_o V^{-1} C_z$ *is asymptotically stable.*

■

Corollary 5.7.1 ([31, 3, 108]) *The transfer functions*

$$\begin{aligned} V_d(z) &= C_z(z^{-1}I_n - A_z)^{-1}K_o + V \\ W_d(z) &= L_o(z^{-1}I_n - A_z)^{-1}B_z + W \end{aligned}$$

where

$$K_o = (B_z - A_z Q C_z^T)V^{-T} \tag{5.5a}$$
$$L_o = W^{-T}(C_z - B_z^T P A_z) \tag{5.5b}$$
$$V = \sqrt{D_z + D_z^T - C_z Q C_z^T} \tag{5.5c}$$
$$W = \sqrt{D_z + D_z^T - B_z^T P B_z} \tag{5.5d}$$

are minimum-phase iff Z_d above is positive-real.

■

The converse relationship is described as follows:

Lemma 5.8 *If*

$$V_d(z) = V + C_z(z^{-1}I_n - A_z)^{-1}K_o$$

then

$$Z_d(z) = \frac{1}{2}VV^T + C_z(z^{-1}I_n - A_z)^{-1}(K_oV^T + A_z\Sigma C_z{}^T),$$

where

$$A_z\Sigma A_z{}^T + \Sigma + K_oK_o{}^T = 0.$$

■

Theorem 5.9 *If (A_z, B_z, C_z, D_z) is a realisation of a discrete-time positive-real function $Z_d(z)$ which is discrete positive-real balanced with respect to the positive-real Gramian $P = \mathrm{diag}(p_1 I_{n_1}, \ldots, p_k I_{n_k})$ then the transfer function V_d with realisation (A_v, K, C, V) where*

$$\begin{aligned} A_v &= (I - P^2)^{-\frac{1}{4}}A_z(I - P^2)^{\frac{1}{4}} \\ K &= (I - P^2)^{-\frac{1}{4}}(B_z - A_z P C_z^T)V^{-T} \\ C &= C_z(I - P^2)^{\frac{1}{4}} \\ V &= \sqrt{D_z + D_z^T - C_z P C_z^T} \end{aligned}$$

is minimum-phase balanced with respect to the Gramian

$$\Sigma = \mathrm{diag}\left(\frac{p_1}{\sqrt{1-p_1^2}}I_{n_1}, \ldots, \frac{p_k}{\sqrt{1-p_k^2}}I_{n_k}\right).$$

Moreover this is a realisation of the right spectral factor of the spectral-density $\Phi_d = Z_d + Z_d^*$.

Proof : It immediate follows from the Kalman-Yakubovich-Popov Lemma 5.7 and Corollary 5.7.1 that $V_d(z)$ is indeed a right-spectral factor of $Z_d + Z_d^*$.

To conclude the proof it suffices to show that the relationships described in sections (a), (b), (c) and (d) in Figure 5.1 are commutative, and hence

$$\begin{array}{c} (A_v, K, C, V) \text{ satisfies (4.9) and (4.10)} \\ \text{iff (see section (a))} \\ (\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V}) \text{ satisfies (4.7) and (4.8)} \\ \text{iff (see section (b))} \\ (\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z) \text{ satisfies (4.1) and (4.2)} \\ \text{iff (see section (c))} \\ (A_z, B_z, C_z, D_z) \text{ satisfies (4.5) and (4.6)} \end{array}$$

To establish commutativity consider the mapping from $(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)$ to (A_v, K, C, L) via $(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V})$ as compared with the route via (A_z, B_z, C_z, D_z).

$$\begin{aligned} A_v &= (I - \tilde{A}_v)^{-1}(I + A_v) \\ &= (I - P^2)^{-\frac{1}{4}}(I - \tilde{A}_z)^{-1}(I + \tilde{A}_z)(I - P^2)^{\frac{1}{4}} \end{aligned}$$

and on the other hand

$$\begin{aligned} A_v &= (I - P^2)^{-\frac{1}{4}} A_z (I - P^2)^{\frac{1}{4}} \\ &= (I - P^2)^{-\frac{1}{4}}(I - \tilde{A}_z)^{-1}(I + \tilde{A}_z)(I - P^2)^{\frac{1}{4}}. \end{aligned}$$

$$\begin{aligned} C &= \sqrt{2}\tilde{C}(I - \tilde{A}_v)^{-1} \\ &= \sqrt{2}\tilde{C}_z(I - \tilde{A}_z)^{-1}(I - P^2)^{\frac{1}{4}} \end{aligned}$$

and on the other hand

$$\begin{aligned} C &= C_z(I - P^2)^{\frac{1}{4}} \\ &= \sqrt{2}\tilde{C}_z(I - \tilde{A}_z)^{-1}(I - P^2)^{\frac{1}{4}}. \end{aligned}$$

$$\begin{aligned} VV^T &= \left(\tilde{V} + \tilde{C}(I - \tilde{A}_v)^{-1}\tilde{K}\right)\left(\tilde{V} + \tilde{C}(I - \tilde{A}_v)^{-1}\tilde{K}\right)^T \\ &= \tilde{V}\tilde{V}^T + \tilde{V}\tilde{K}^T(I - \tilde{A}_v^T)^{-1}\tilde{C}^T + \tilde{C}(I - \tilde{A}_v)^{-1}\tilde{K}\tilde{V}^T \\ &\quad + \tilde{C}(I - \tilde{A}_v)^{-1}\tilde{K}\tilde{K}^T(I - \tilde{A}_v^T)^{-1}\tilde{C}^T \\ &= \tilde{D}_z + \tilde{D}_z^T + (\tilde{B}_z^T - \tilde{C}_z P)(I - \tilde{A}_z^T)^{-1}\tilde{C}_z^T + \tilde{C}_z(I - \tilde{A}_z)^{-1}(\tilde{B}_z - P\tilde{C}_z^T) \\ &\quad - \tilde{C}_z(I - \tilde{A}_z)^{-1}\left(\tilde{A}_z P + P\tilde{A}_z^T\right)(I - \tilde{A}_z^T)^{-1}\tilde{C}_z^T \\ &= (\tilde{D}_z + \tilde{C}_z(I - \tilde{A}_z)^{-1}\tilde{B}_z) + (\tilde{D}_z^T + \tilde{B}_z^T(I - \tilde{A}_z^T)^{-1}\tilde{C}_z^T) \\ &\quad - \tilde{C}_z(I - \tilde{A}_z)^{-1}\left(\tilde{A}_z P + P\tilde{A}_z^T + (I - \tilde{A}_z)P + P(I - \tilde{A}_z^T)\right)(I - \tilde{A}_z^T)^{-1}\tilde{C}_z^T \\ &= (\tilde{D}_z + \tilde{C}_z(I - \tilde{A}_z)^{-1}\tilde{B}_z) + (\tilde{D}_z^T + \tilde{B}_z^T(I - \tilde{A}_z^T)^{-1}\tilde{C}_z^T) \\ &\quad - 2\tilde{C}_z(I - \tilde{A}_z)^{-1}P(I - \tilde{A}_z^T)^{-1}\tilde{C}_z^T \end{aligned}$$

and, following the other path gives

$$\begin{aligned} VV^T &= D_z + D_z^T - C_z P C_z^T \\ &= (\tilde{D}_z + \tilde{C}_z(I - \tilde{A}_z)^{-1}\tilde{B}_z) + (\tilde{D}_z^T + \tilde{B}_z^T(I - \tilde{A}_z^T)^{-1}\tilde{C}_z^T) \\ &\quad - 2\tilde{C}_z(I - \tilde{A}_z)^{-1}P(I - \tilde{A}_z^T)^{-1}\tilde{C}_z^T \end{aligned}$$

as required.

Also, comparing (d) to (a), (b) and (c) composed,

$$KV^T = (I - P^2)^{-\frac{1}{4}}(B_z - A_z P C_z^T)$$

and

$$\begin{aligned}
KV^T &= \sqrt{2}(I-\tilde{A}_v)^{-1}\tilde{K}\tilde{V}^T + \sqrt{2}(I-\tilde{A}_v)^{-1}\tilde{K}\tilde{K}^T(I-\tilde{A}_v^T)^{-1}\tilde{C}^T \\
&= \sqrt{2}(I-\tilde{A}_v)^{-1}(I-P^2)^{-\frac{1}{4}}(\tilde{B}_z - P\tilde{C}_z^T) \\
&\quad + \sqrt{2}(I-P^2)^{-\frac{1}{4}}(I-\tilde{A}_z)^{-1}\tilde{K}_o\tilde{K}_o^T(I-\tilde{A}_z^T)^{-1}\tilde{C}_z^T \\
&= (I-P^2)^{-\frac{1}{4}}B_z - \frac{1}{\sqrt{2}}(I-P^2)^{-\frac{1}{4}}(I+A_z)P\tilde{C}_z^T \\
&\quad + \frac{1}{2\sqrt{2}}(I-P^2)^{-\frac{1}{4}}(I+A_z)\left(\tilde{A}_z P + P\tilde{A}_z^T\right)(I+A_z^T)\tilde{C}_z^T \\
&= (I-P^2)^{-\frac{1}{4}}B_z - \frac{1}{2\sqrt{2}}(I-P^2)^{-\frac{1}{4}}(I+A_z)\left(\tilde{A}_z P + P\tilde{A}_z^T + P(I-\tilde{A}_z^T)\right)(I+A_z^T)\tilde{C}_z^T \\
&= (I-P^2)^{-\frac{1}{4}}B_z + \frac{1}{\sqrt{2}}(I-P^2)^{-\frac{1}{4}}(I+A_z)(I+\tilde{A}_z)P(I+A_z^T)\tilde{C}_z^T \\
&= (I-P^2)^{-\frac{1}{4}}\left(B_z - A_z P C_z^T\right)
\end{aligned}$$

Finally, it is necessary to check that the two methods of deriving V are equivalent, since the above calculation showed only that they were equivalent up to right multiplication by an orthogonal matrix. This final point is settled by the chosen normalisations. ■

(e) [Spectral Summand ↔ Left Spectral Factor]

Theorem 5.10 *If $(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)$ is a realisation of a positive-real function $Z_c(s)$ which is positive-real balanced with respect to the Gramian $P = \mathrm{diag}(p_1 I_{n_1}, \ldots, p_k I_{n_k})$ then the transfer function W_c with realisation $(\tilde{A}_w, \tilde{B}, \tilde{L}, \tilde{W})$ where*

$$\begin{aligned}
\tilde{A}_w &= (I-P^2)^{\frac{1}{4}}\tilde{A}_z(I-P^2)^{-\frac{1}{4}} \\
\tilde{B} &= (I-P^2)^{\frac{1}{4}}\tilde{B}_z \\
\tilde{L} &= \tilde{W}^{-T}\left(\tilde{C}_z - \tilde{B}_z^T P\right)(I-P^2)^{-\frac{1}{4}} \\
\tilde{W} &= \sqrt{\tilde{D}_z + \tilde{D}_z^T}
\end{aligned}$$

and $\tilde{W}$ is normalised so that $\tilde{W} - \tilde{L}(I+\tilde{A}_w)^{-1}\tilde{B}$ is positive-lower-triangular, is left minimum-phase balanced with respect to the Gramian

$$\Sigma = \mathrm{diag}\left(\frac{p_1}{\sqrt{1-p_1^2}}I_{n_1}, \ldots, \frac{p_k}{\sqrt{1-p_k^2}}I_{n_k}\right).$$

Moreover, $(\tilde{A}_w, \tilde{B}, \tilde{L}, \tilde{W})$ is a realisation of the left spectral factor of the spectral-density $Z_c + Z_c^*$.

Proof : This is the dual of Theorem 4.51. ■

Theorem 5.11 *If* $(\tilde{A}_w, \tilde{B}, \tilde{L}, \tilde{W})$ *is a minimal realisation of a minimum-phase function* $W_c(s)$ *which is minimum-phase balanced with respect to the minimum-phase Gramian* $\Sigma = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_k I_{n_k})$ *then the transfer function* Z_c *with realisation* $(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)$ *is positive-real balanced with respect to the positive-real Gramian* $P = \Sigma\sqrt{I_n + \Sigma^2}^{-1}$*, where*

$$\begin{aligned}
\tilde{A}_z &= (I+\Sigma^2)^{\frac{1}{4}}\tilde{A}_w(I+\Sigma^2)^{-\frac{1}{4}} \\
\tilde{B}_z &= (I+\Sigma^2)^{\frac{1}{4}}\tilde{B} \\
\tilde{C}_z &= \left(\tilde{B}^T\Sigma + \tilde{W}^T\tilde{L}\right)(I+\Sigma^2)^{-\frac{1}{4}} \\
\tilde{D}_z &= \frac{1}{2}\tilde{W}^T\tilde{W}.
\end{aligned}$$

Moreover, $(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)$ *is a realisation of the spectral summand as above.*

■

(f) [Spectral Summand $\leftrightarrow$ Left Spectral Factor]

Theorem 5.12 *If* (A_z, B_z, C_z, D_z) *is a realisation of a discrete-time positive-real function* $Z_d(z)$ *which is discrete positive-real balanced with respect to the positive-real Gramian* $P = \mathrm{diag}(p_1 I_{n_1}, \ldots, p_k I_{n_k})$ *then the transfer function* V_d *with realisation* (A_w, B, L, W) *where*

$$\begin{aligned}
A_w &= (I-P^2)^{\frac{1}{4}}A_z(I-P^2)^{-\frac{1}{4}} \\
B &= (I-P^2)^{\frac{1}{4}}B_z \\
L &= W^{-T}\left(C_z - B_z^T P A_z\right)(I-P^2)^{-\frac{1}{4}} \\
W &= \sqrt{D_z + D_z^T - B_z^T P B_z}
\end{aligned}$$

and W *is normalised so that* $W - L(I + A_w)^{-1}B$ *is positive-lower-triangular,*

is minimum-phase balanced with respect to the Gramian

$$\Sigma = \mathrm{diag}\left(\frac{p_1}{\sqrt{1-p_1^2}}I_{n_1}, \ldots, \frac{p_k}{\sqrt{1-p_k^2}}I_{n_k}\right).$$

Moreover this is a realisation of the left spectral factor of the spectral-density $\Phi_d = Z_d + Z_d^*$*.*

Proof : This is the dual of Theorem 5.9. ■

Theorem 5.13 *If* (A_w, B, L, W) *is a minimal realisation of a minimum-phase function* $W_c(s)$ *which is minimum-phase balanced with respect to the Gramian* $\Sigma = \mathrm{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_k I_{n_k})$ *then the transfer function* Z_c *with realisation* $(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z)$ *where*

$$\begin{aligned}
A_z &= (I+\Sigma^2)^{\frac{1}{4}}A_w(I+\Sigma^2)^{-\frac{1}{4}} \\
B_z &= (I+\Sigma^2)^{\frac{1}{4}}B \\
C_z &= \left(B^T\Sigma A_w + W^T L\right)(I+\Sigma^2)^{-\frac{1}{4}} \\
D_z &= \frac{1}{2}\left\{W^T W + B^T\Sigma B\right\}
\end{aligned}$$

is positive-real balanced with respect to the Gramian $P = \Sigma\sqrt{I_n + \Sigma^2}^{-1}$.

Moreover, (A_z, B_z, C_z, D_z) *is a realisation of the spectral summand as above.*

■

(g) [Continuous Dual M.P. Balanced $\leftrightarrow$ Discrete Dual M.P. Balanced]

The following relationships are clearly dual to those in (a) above.

$$\begin{aligned}
\tilde{A}_w &= (I + A_w)^{-1}(A_w - I) \\
\tilde{B} &= \sqrt{2}(I + A_w)^{-1}B \\
\tilde{L} &= \sqrt{2}L(I + A_w)^{-1} \\
\tilde{W} &= W - L(I + A_w)^{-1}B
\end{aligned}$$

$$\begin{aligned}
A_w &= (I - \tilde{A}_w)^{-1}(I + \tilde{A}_w) \\
B &= \sqrt{2}(I - \tilde{A}_w)^{-1}\tilde{B} \\
L &= \sqrt{2}\tilde{L}(I - \tilde{A}_w)^{-1} \\
W &= \tilde{W} + \tilde{L}(I - \tilde{A}_w)^{-1}\tilde{B}
\end{aligned}$$

(h) [Spectral Summand $\rightarrow$ Phase Factor]

Previous sections have established the relationship between the realisations of $Z_c(s), V_c(s)$ and $W_c(s)$ which are respectively positive-real balanced, right minimum-phase balanced, and left minimum-phase balanced. The balancing schemes involve four Lyapunov equations, as Theorems 4.51 and 5.10 demonstrate that the Positive Real Riccati Equations (4.1) and (4.2) are equivalent to the Lyapunov equations (4.7), (4.8), (5.1) and (5.2) under the conditions described there.

Using the notation of the sections above, then

$$\begin{aligned}
Z_c(s) &\overset{s}{=} \left(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z\right) \\
V_c(s) &\overset{s}{=} \left(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V}\right) = (I - P^2)^{-\frac{1}{4}}\left(\tilde{A}_z, \tilde{K}_o, \tilde{C}_z, \tilde{V}\right)(I - P^2)^{\frac{1}{4}} \\
W_c(s) &\overset{s}{=} \left(\tilde{A}_w, \tilde{B}, \tilde{L}, \tilde{W}\right) = (I - P^2)^{\frac{1}{4}}\left(\tilde{A}_z, \tilde{B}_z, \tilde{L}_o, \tilde{W}\right)(I - P^2)^{-\frac{1}{4}}
\end{aligned}$$

then from the Kalman-Yacubovich-Popov Lemma it is clear that

$$\begin{aligned}
\tilde{A}_z^T P + P\tilde{A}_z &= -\tilde{L}_o^T\tilde{L}_o \\
\tilde{A}_z P + P\tilde{A}_z^T &= -\tilde{K}_o\tilde{K}_o^T
\end{aligned}$$

showing that $(\tilde{A}_z^T, \tilde{L}_o^T, \tilde{K}_o^T)$ is a Lyapunov balanced realisation with Lyapunov Gramian P. This utilises two of the four Lyapunov equations mentioned in the preceding paragraph.

Since $\left(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V}\right)$ is right minimum-phase balanced with respect to $P(I - P^2)^{-\frac{1}{2}}$ it is therefore clear that if $X = (I - P^2)^{\frac{1}{4}}$ then

$$\left[X\left(\tilde{A}_v - \tilde{K}\tilde{V}^{-1}\tilde{C}\right)^T X^{-1}\right]P + P\left[X^{-1}\left(\tilde{A}_v - \tilde{K}\tilde{V}^{-1}\tilde{C}\right)X\right] + X\tilde{C}^T\tilde{V}^{-T}\tilde{V}^{-1}\tilde{C}X = 0 \quad (5.6)$$

(a state-space transformation of (4.8)).

Also, since $\left(\tilde{A}_w, \tilde{B}, \tilde{L}, \tilde{W}\right)$ being left minimum-phase balanced with respect to $P(I - P^2)^{-\frac{1}{2}}$ it is clear that

$$\left[X\left(\tilde{A}_w - \tilde{B}\tilde{W}^{-1}\tilde{L}\right)X^{-1}\right]P + P\left[X^{-1}\left(\tilde{A}_w - \tilde{B}\tilde{W}^{-1}\tilde{L}\right)X\right]^T + X\tilde{B}\tilde{W}^{-1}\tilde{W}^{-T}\tilde{B}^T X = 0. \quad (5.7)$$

That these two equations are connected is demonstrated by the following Lemma adapted from a similar result in Green [42], but presented here with an elementary proof.

Lemma 5.14 ([42])

$$\begin{aligned}\tilde{K}_o\tilde{V}^T + P\tilde{L}_o^T\tilde{W} &= (I-P^2)\tilde{B}_z \quad (5.8)\\ \tilde{V}\tilde{K}_o^T P + \tilde{W}^T\tilde{L}_o &= \tilde{C}_z(I-P^2) \quad (5.9)\end{aligned}$$

and

$$\left(\tilde{A}_z - \tilde{K}_o\tilde{V}^{-1}\tilde{C}_z\right)(I-P^2) = (I-P^2)\left(\tilde{A}_z - \tilde{B}_z\tilde{W}^{-1}\tilde{L}_o\right). \quad (5.10)$$

Proof : Substituting $\tilde{C}_z^T$ from (4.29b) in (4.30b) gives (5.8), and substituting $\tilde{B}_z$ from (4.30b) in (4.29b) gives (5.9).

$$\begin{aligned}\left(\tilde{A}_z - \tilde{K}_o\tilde{V}^{-1}\tilde{C}_z\right)(I-P^2) &= \tilde{A}_z(I-P^2) - \tilde{K}_o\tilde{V}^{-1}\tilde{C}_z(I-P^2)\\ &= \tilde{A}_z(I-P^2) - \tilde{K}_o\tilde{K}_o^T P - \tilde{K}_o\tilde{V}^{-1}\tilde{W}\tilde{L}_o \text{ from (5.8)}\\ &= \tilde{A}_z(I-P^2) + \tilde{A}_z P^2 + P\tilde{A}_z P - \tilde{K}_o\tilde{V}^{-1}\tilde{W}\tilde{L}_o \text{ from (4.30a)}\\ &= \tilde{A}_z + P\left(-P\tilde{A}_z - \tilde{L}_o^T\tilde{L}_o\right) - \tilde{K}_o\tilde{V}^{-1}\tilde{W}\tilde{L}_o \text{ from (4.29a)}\\ &= (I-P^2)\tilde{A}_z - \left(P\tilde{L}_o^T\tilde{W} + \tilde{K}_o\tilde{V}^{-1}\tilde{W}^T\tilde{W}\right)\tilde{W}^{-1}\tilde{L}_o\\ &\quad \text{since } \tilde{W}^T\tilde{W} = \tilde{V}\tilde{V}^T\\ &= (I-P^2)\tilde{A}_z - \left(P\tilde{L}_o^T\tilde{W} + \tilde{K}_o\tilde{V}^T\right)\tilde{W}^{-1}\tilde{L}_o\\ &= (I-P^2)\left(\tilde{A}_z - \tilde{B}_z\tilde{W}^{-1}\tilde{L}_o\right) \text{ from (5.8).}\end{aligned}$$

■

Hence, it is clear that

$$\begin{aligned}X^{-1}\left(\tilde{A}_v - \tilde{K}\tilde{V}^{-1}\tilde{C}\right)X &= X^{-2}\left(\tilde{A}_z - \tilde{K}_o\tilde{V}^{-1}\tilde{C}_z\right)X^2\\ &= X^2\left(\tilde{A}_z - \tilde{B}_z\tilde{W}^{-1}\tilde{L}_o\right)X^{-2}\\ &= X\left(\tilde{A}_w - \tilde{B}\tilde{W}^{-1}\tilde{L}\right)X^{-1} \quad (5.11)\end{aligned}$$

and therefore equations (5.6) and (5.7) show that the realisation

$$\left(\hat{A}_c, \hat{B}_c, \hat{C}_c, \hat{D}_c\right) = \left(X(\tilde{A}_w - \tilde{B}\tilde{W}^{-1}\tilde{L})X^{-1}, -X\tilde{B}\tilde{W}^{-1}, \tilde{V}^{-1}\tilde{C}X, \tilde{V}^{-1}\tilde{W}^T\right)$$

is Lyapunov balanced with Gramian P.

Having discussed the properties of these two realisations, the following result shows what transfer functions they realise.

Proposition 5.15 ([42]) *If V_c and W_c are defined as in §5.1.1 then*

$$E_c(s) = \left(V_c^{-1}W_c^*\right)(s) = \tilde{K}_o^T\left(-sI - \tilde{A}_z^T\right)^{-1}\tilde{L}_o^T + \tilde{V}^{-1}\tilde{W}^T + \hat{C}_c\left(sI - \hat{A}_c\right)^{-1}\hat{B}_c$$

where $\tilde{K}_o^T\left(-sI - \tilde{A}_z^T\right)^{-1}\tilde{L}_o^T$ is strictly anti-stable, and $\hat{C}_c\left(sI - \hat{A}_c\right)^{-1}\hat{B}_c$ is strictly stable. Furthermore, both realisations are Lyapunov balanced with Gramian P.

Proof : $\lambda(\tilde{A}_z) \subseteq \mathbb{C}_-$ by assumption, hence $\tilde{K}_o^T\left(-sI - \tilde{A}_z^T\right)^{-1}\tilde{L}_o^T$ is anti-stable.

$\hat{A}_c$ is similar to $\tilde{A}_z - \tilde{K}_o\tilde{V}^{-1}\tilde{C}_z$, and hence by Theorem 4.48 it is clear that $\lambda(\hat{A}_c) \subseteq \mathbb{C}_-$ and hence $\hat{C}_c\left(sI - \hat{A}_c\right)^{-1}\hat{B}_c$ is stable.

The balancedness of both realisations was established immediately preceding this Proposition, thus leaving the details of the realisations as the only remaining part of the proposition to be proven.

Consider

$$\begin{aligned}
&V_c(s)E_c(s)\\
&= \left(\tilde{V} + \tilde{C}_z(sI - \tilde{A}_z)^{-1}\tilde{K}_o\right)\left(\tilde{K}_o^T\left(-sI - \tilde{A}_z^T\right)^{-1}\tilde{L}_o^T + \tilde{V}^{-1}\tilde{W}^T + \hat{C}_c\left(sI - \hat{A}_c\right)^{-1}\hat{B}_c\right)\\
&= \tilde{V}\tilde{K}_o^T\left(-sI - \tilde{A}_z^T\right)^{-1}\tilde{L}_o^T + \tilde{W}^T + \tilde{V}\hat{C}_c\left(sI - \hat{A}_c\right)^{-1}\hat{B}_c + \tilde{C}_z(sI - \tilde{A}_z)^{-1}\tilde{K}_o\tilde{V}^{-1}\tilde{W}^T\\
&\quad - \tilde{C}_z(sI - \tilde{A}_z)^{-1}\tilde{K}_o\tilde{K}_o^T\left(sI + \tilde{A}_z\right)^{-T}\tilde{L}_o^T\\
&\quad + \tilde{C}_z(sI - \tilde{A}_z)^{-1}\tilde{K}_o\hat{C}_c\left(sI - \hat{A}_c\right)^{-1}\hat{B}_c
\end{aligned} \tag{5.12}$$

From (4.30a)

$$\begin{aligned}
\tilde{K}_o\tilde{K}_o^T &= -\tilde{A}_zP - P\tilde{A}_z^T\\
&= \left(sI - \tilde{A}_z\right)P + P\left(-sI - \tilde{A}_z^T\right).
\end{aligned}$$

Also, from the above definition of $\hat{C}_c$

$$\tilde{K}_o\hat{C}_c = \tilde{K}_o\tilde{V}^{-1}\tilde{C}_zX^2$$

and hence from (5.11)

$$\begin{aligned}
\tilde{K}_o\tilde{V}^{-1}\tilde{C}_zX^2 &= \tilde{A}_zX^2 - X^2\hat{A}_c\\
&= X^2\left(sI - \hat{A}_c\right) - \left(sI - \tilde{A}_z\right)X^2.
\end{aligned}$$

Substituting these relationships in equation (5.12), cancelling obvious terms, and collecting common factors gives

$$\begin{aligned}
&V_c(s)E_c(s)\\
&= \left(\tilde{V}\tilde{K}_o^T + \tilde{C}_zP\right)\left(-sI - \tilde{A}_z^T\right)\tilde{L}_o^T + \tilde{W}^T + \left(\tilde{V}\hat{C}_c - \tilde{C}_zX^2\right)\left(sI - \hat{A}_c\right)^{-1}\hat{B}_c\\
&= \tilde{C}_z\left(sI - \tilde{A}_z\right)^{-1}\left(X^2\hat{B}_c + P\tilde{L}_o^T + \tilde{K}_o\tilde{V}^{-1}\tilde{W}^T\right)\\
&= W(-s)^T
\end{aligned}$$

since

$$\tilde{V}\hat{C}_c - \tilde{C}_zX^2 = \tilde{V}\tilde{V}^{-1}\tilde{C}_zX^2 - \tilde{C}_zX^2 = 0$$

and

$$\begin{aligned}
X^2\hat{B}_c &= -X^4\tilde{B}_z\tilde{W}^{-1}\\
&= -\left(\tilde{K}_o\tilde{V}^T + P\tilde{L}_o\tilde{W}\right)\tilde{W}^{-1} \text{ using (5.8)}\\
&= -\left(\tilde{K}_o\tilde{V}^{-1}\tilde{W}^T + P\tilde{L}_o\right)
\end{aligned}$$

and hence the second term is also zero. ■

Corollary 5.15.1 *If* $\left(\tilde{A}_z, \tilde{B}_z, \tilde{C}_z, \tilde{D}_z\right)$ *is sign-symmetric with respect to* S *then both* $(\tilde{A}_z^T, \tilde{L}_o^T, \tilde{K}_o^T)$ *and* $(\hat{A}_c, \hat{B}_c, \hat{C}_c)$ *are also.*

■

Note 5.16 (Phase Factor Constant Term) *Due to the normalisations assumed in §5.1.2* $\tilde{V}$ *is positive-upper-triangular and* $\tilde{W}$ *positive-lower-triangular and hence, from Lemma 4.23,* $U_o = I_p$ *if the spectral-density is symmetric.*

■

(i) [Spectral Summand → Phase Factor]

The realisation of the discrete-time phase factor may be almost entirely inferred from the previous section and the application of the results in §2.2.3 and §4.2.1.

Hence, given

$$\begin{aligned}
Z_d(z) &\stackrel{s}{=} (A_z, B_z, C_z, D_z) \\
V_d(z) &\stackrel{s}{=} (A_v, K, C, V) = (I - P^2)^{\frac{1}{4}}(A_z, K_o, C_z, V)(I - P^2)^{-\frac{1}{4}} \\
W_d(z) &\stackrel{s}{=} (A_w, B, L, W) = (I - P^2)^{-\frac{1}{4}}(A_z, B_z, L_o, W)(I - P^2)^{\frac{1}{4}}.
\end{aligned}$$

It has also been established that all of these realisations are the bilinear transforms of their continuous-time counterparts.

From the Kalman-Yakubovich-Popov Lemma (Theorem 5.7) it is known that

$$\begin{aligned}
A_z^T P A_z - P &= -L_o^T L_o \\
A_z P A_z^T - P &= -K_o K_o^T
\end{aligned}$$

It may also be derived from (5.4) and (4.10), or from the bilinear transform of the results in the previous section that

$$\begin{aligned}
\left[X\left(A_w - BW^{-1}L\right)X^{-1}\right] P \left[X^{-1}\left(A_w - BW^{-1}L\right)^T X\right] & \\
- P + XBW^{-1}W^{-T}B^T X &= 0 \qquad (5.13) \\
\left[X\left(A_v - KV^{-1}C\right)^T X^{-1}\right] P \left[X^{-1}\left(A_v - KV^{-1}C\right) X\right] & \\
- P + XC^T V^{-T} V^{-1} C X &= 0 \qquad (5.14)
\end{aligned}$$

where $X = (I - P^2)^{\frac{1}{4}}$.

Hence, define the following realisation

$$\left(\hat{A}_d, \hat{B}_d, \hat{C}_d, \hat{D}_d\right) = \left(X(A_w - BW^{-1}L)X^{-1}, -XBW^{-1}, V^{-1}CX, V^{-1}W^T\right)$$

which by (5.13) and (5.14) is Lyapunov balanced with Gramian P.

Proposition 5.17 (Realisation of the Phase Factor) *If* V_d *and* W_d *are as defined in §5.1.1 then*

$$E_d(z) = \left(V_d^{-1}W_d^*\right)(z) = K_o^T(zI - A_z^T)^{-1}L_o^T + V^{-1}W^T + \hat{C}_d(z^{-1}I - \hat{A}_d)^{-1}\hat{B}_d$$

where $K_o^T(zI - A_z^T)^{-1}L_o^T$ *is strictly anti-stable, and* $\hat{C}_d(z^{-1}I - \hat{A}_d)^{-1}\hat{B}_d$ *is strictly stable. Furthermore, both realisations are Lyapunov balanced with Gramian* P.

Proof : That (A_z^T, K_o^T, L_z^T) and $(\hat{A}_d, \hat{B}_d, \hat{C}_d)$ are the images of $(\tilde{A}_z^T, \tilde{K}_o^T, \tilde{L}_o^T)$ and $(\hat{A}_c, \hat{B}_c, \hat{C}_c)$ under the bilinear transform is established in (a) and (g) above. Proposition 2.7 shows that both realisations are balanced with respect to the Lyapunov equations (5.13) and (5.14).

Finally, $V_d^{-1}(z)W_d^*(z) = V_c^{-1}(M(z))W_c^*(M(z))$, hence the above realisations do represent the phase factor. ■

Corollary 5.17.1 *If (A_z, B_z, C_z, D_z) is sign-symmetric with respect to S then both (A_z^T, L_o^T, K_o^T) and $(\hat{A}_d, \hat{B}_d, \hat{C}_d)$ are also.*

■

Note 5.18 (Phase Factor Constant Term) *Due to the normalisations assumed in §5.1.2 $\tilde{V}$ is positive-upper-triangular and $\tilde{W}$ positive-lower-triangular and hence, from Lemma 4.23, $U_o = I_p$ if the spectral-density is symmetric.*

■

5.1.4 Relationships for Symmetric Spectral Densities

When Φ_c is symmetric several of the relationships described in §5.1.3 specialise. Consider the case where $V_c \in \mathrm{M}_{\mathrm{n}}^{\mathrm{p}}[\alpha]$ and has realisation $(\tilde{A}_z, \tilde{K}, \tilde{C}, \tilde{V})$ given by Theorem 4.60. This is right minimum-phase balanced with Gramian Σ, and a sign matrix S may be derived from the parameters of $\tilde{A}_v$ as in Definition 4.39.

Following the sequence of relationships given by Theorems 4.63 and 5.5 it is clear that

$$\begin{aligned} A_z &= (I - \tilde{A}_z)^{-1}(I + \tilde{A}_z) \\ &= (I + \Sigma^2)^{-\frac{1}{4}}(I - \tilde{A}_v)^{-1}(I + \tilde{A}_v)(I + \Sigma^2)^{\frac{1}{4}} \end{aligned}$$

and

$$K_o = \sqrt{2}(I + \Sigma^2)^{-\frac{1}{4}}(I - \tilde{A}_v)^{-1}\tilde{K}$$

from Proposition 2.6 and Theorem 4.51.

Also,

$$\begin{aligned} L_o &= K_o^T S \\ &= \sqrt{2}\tilde{K}^T(I - \tilde{A}_v^T)^{-1}(I + \Sigma^2)^{-\frac{1}{4}} S \end{aligned}$$

from the discussion of sign symmetry in (h) above. That this relationship is equivalent to the more complex equality

$$L_o = \sqrt{2}\left(\tilde{V}^{-1}\tilde{C} - \tilde{K}^T\Sigma\right)(I + \Sigma^2)^{-1}(I - \tilde{A}_v)^{-1}(I + \Sigma^2)^{\frac{3}{4}}$$

is an extension of the argument that $\tilde{B}_z^T = \tilde{C}_z S$ given in Theorem 4.63.

5.2 Interpretation of Parameters

5.2.1 Spectral Densities and Related Functions

In Chapter 4, a continuous-time parametrisation was used to obtain a parametrisation of the discrete-time models for the underlying time-series process considered in §3.1.1.

The relationships described in this Chapter may now be exploited to show that the parameters arising in the continuous-time parametrisation may be related to the properties of objects which involve only discrete-time transfer functions. In Chapters 6 and 7 it is shown that such objects may indeed be derived from observations of the underlying process.

As in §5.1.4 assume that Φ_c is symmetric, $V_c \in \mathrm{M}_{\mathrm{n}}^{\mathrm{p}}[\omega]$, and that the relationships between the realisations are as described in §5.1.3.

Proposition 2.19 shows:

Theorem 5.19 (Spectrum Properties of Canonical Correlation Hankel Operator) *The canonical correlation operator $R\mathbf{H}_{V_d^{-1}W_d^*}$ has rank $p \times n$ with eigenvectors*

$$\begin{aligned} u_i(z) &= L_o(z^{-1}I_n - A_z)^{-1}A_zF_i + L_oF_i \\ &\stackrel{s}{=} (A_z, A_zF_i, L_o, L_oF_i) \end{aligned}$$

with

$$\left(R\mathbf{H}_{V_d^{-1}W_d^*}\right) u_i(z) = \frac{s_i\sigma_i}{\sqrt{1+\sigma^2}} u_i(z).$$

where (A_z^T, L_o^T, K_o^T) is sign-symmetric with respect to S, and F_i is zero except for arbitrary entries in the ith row.

Proof : This follows immediately from the realisation of $V_d^{-1}W_d^*$, Proposition 5.17 and from Lemma 2.13. ■

Corollary 5.19.1 *The eigenvectors of $R\mathbf{H}_{V_d^{-1}W_d^*}$ may be expressed in terms of the realisation of V_c as*

$$\begin{aligned} u_i(z) &= \sqrt{2}s_i\tilde{K}^T\left((1-z)I - (1+z)\tilde{A}_v^T\right)^{-1}(I+\Sigma^2)^{-\frac{1}{4}}F_i \\ &= \frac{\sqrt{2}s_i}{(1+\sigma_i^2)^{\frac{1}{4}}}\tilde{K}^T\left((1-z)I - (1+z)\tilde{A}_v^T\right)^{-1}F_i \end{aligned}$$

■

Theorem 5.20 (Norm of Schmidt Vectors)

$$\|u_i\|_2^2 = \frac{\sigma_i}{\sqrt{1+\sigma_i^2}}\|F_i\|_2^2$$

Proof :

$$\begin{aligned} \|u_i\|_2^2 &= \left\|\sum_{k=0}^{\infty} z^k K_o^T A_z^{k^T} F_i\right\|_2^2 \\ &= \operatorname{tr}\sum_{k=0}^{\infty} F_i A_z^k K_o K_o^T A_z^{k^T} F_i^T \end{aligned}$$

$$
\begin{aligned}
&= \operatorname{tr} F_i \left(\sum_{k=0}^{\infty} A_z^k K_o K_o^T A_z^{k^T} \right) F_i^T \\
&= \operatorname{tr} F_i P F_i^T \\
&= p_i \operatorname{tr} F_i F_i^T \\
&= \frac{\sigma_i}{\sqrt{1+\sigma_i^2}} \|F_i\|_2^2
\end{aligned}
$$

■

The following Theorem describes the properties of a closely related operator.

Theorem 5.21 *The canonical correlation operator $\tilde{\mathbf{H}}_{V_d^{-1}W_d^*}R$ has rank $p \times n$ with eigenvectors*

$$
\begin{aligned}
u_i(z) &= \hat{B}_d^T(z^{-1}I_n - \hat{A}_d^T)^{-1}\hat{A}_d^T F_i + \hat{B}_d^T F_i \\
&\stackrel{s}{=} (\hat{A}_d^T, \hat{A}_d^T F_i, \hat{B}_d^T, \hat{B}_d^T F_i)
\end{aligned}
$$

with

$$
\left(\tilde{\mathbf{H}}_{V_d^{-1}W_d^*}R\right) u_i(z) = \frac{s_i\sigma_i}{\sqrt{1+\sigma^2}} u_i(z).
$$

where $(\hat{A}_d, \hat{B}_d, \hat{C}_d)$ is sign-symmetric with respect to S, and F_i is zero except for arbitrary entries in the ith row.

■

It is also clear that the norm of these eigenvectors are the same as those of Theorem 5.20.

Corollary 5.21.1 *The normalised eigenvectors of $\tilde{\mathbf{H}}_{V_d^{-1}W_d^*}R$ may be expressed in terms of the realisation of V_c as*

$$
\begin{aligned}
u_i(z) &= \sqrt{2}\tilde{V}^{-1}\tilde{C}\left((1-z)I - (1+z)\left(\tilde{A}_v - \tilde{K}\tilde{V}^{-1}\tilde{C}\right)\right)^{-1}\Sigma^{-\frac{1}{2}}SF_i \\
&= \sqrt{\frac{2}{\sigma_i}}\tilde{V}^{-1}\tilde{C}\left((1-z)I - (1+z)\left(\tilde{A}_v - \tilde{K}\tilde{V}^{-1}\tilde{C}\right)\right)^{-1} F_i.
\end{aligned}
$$

Furthermore,

$$
u_i(-1) = \frac{s_i\sigma_i + \sqrt{1+\sigma_i^2}}{\sqrt{2\sigma_i}} \tilde{K}_i^T.
$$

Proof : The main result follows by standard manipulations based on the relationships described above. The final part derives from the relationship between the $\tilde{C}$ and $\tilde{K}$ described in Theorem 4.55. ■

5.2.2 Stable Transfer Functions

Similar results to those of the preceding section will be of use when estimating the parameters describing the deterministic part of an ARMAX model.

Let $G(s)_c$ be a symmetric stable continuous-time transfer function with balanced realisation $(\tilde{E}, \tilde{F}, \tilde{G}, \tilde{H})$ which is sign symmetric with respect to the sign matrix S, and have Gramian $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$. Let its discrete-time equivalent $G_d(z)$ have realisation (E, F, G, H) which is related to $(\tilde{E}, \tilde{F}, \tilde{G}, \tilde{H})$ by Proposition 2.7.

Theorem 5.22 (Hankel Operator with Stable Symbol) *The anti-Hankel operator* $\tilde{\mathbf{H}}_G R$ *has eigenvectors*

$$\begin{aligned} u_i(z) &= F^T(z^{-1}I - E^T)^{-1}E^T F_i + F^T F_i \\ &= z^{-1}F^T(z^{-1}I - E^T)^{-1}F_i \\ &= \sqrt{2}\tilde{F}^T\left((1-z)I - (1+z)\tilde{E}\right)^{-1} F_i \end{aligned}$$

with eigenvalue $s_i\sigma_i$*, where* F_i *is zero except for arbitrary entries in the* i*th row.*

■

Theorem 5.23 (Norm of Eigenvectors)

$$\|u_i(z)\|_2^2 = \sigma_i \|F_i\|_2^2$$

■

5.3 Conclusions Relevant to System Identification

From §5.2.1 it is clear that in the case where $V_c \in \mathrm{M}_{\mathrm{n}}^{\mathrm{p}}[\omega]$, the parameters appearing in the parametrisation of V_c (see Theorem 4.60) namely s_i, σ_i and $\check{B}_i$, $i = 1, \ldots, n$, are all related to properties of the operator $R\mathbf{H}_{V_d^{-1}W_d^*}$.

This observation is the basis for the estimation scheme proposed in Chapter 6 which seeks to approximate $R\mathbf{H}_{V_d^{-1}W_d^*}$ and hence to obtain estimates for s_i, σ_i and $\check{B}_i$.

Chapter 6

Estimation of Parameters: Time-Series

Models and parametrisations have been provided in Chapters 3 and 4 for the identification of stationary time-series. These Chapters completed the first two stages of the identification schema proposed in §1.2.

In Chapter 5 the parameters in the canonical forms for minimum-phase and stable systems (Theorems 4.60 and 4.44) have been associated with properties of the spectrum and eigenvectors of the canonical correlation Hankel operator associated with the underlying stochastic process.

This Chapter attempts to exploit these relationships to provide an estimation scheme for the parameters of the model. This provides an approach to the solution of the third step of the identification schema. The main idea behind this Chapter is that by approximating the canonical correlation Hankel operator (where observations provide information), estimates of the parameters may be obtained using the relationships explored in Chapter 5.

A secondary aim of the Chapter is to describe and analyse estimation algorithms. As will be shown in §6.4 this has not been completely successful. §6.5 contains some suggested methods for the continuation of the project.

The first section, §6.1, discusses several sorts of approximation appropriate to operators and random variables. The limiting properties of such forms of approximation and sufficient conditions for such approximation are also addressed. The results of this section establish which properties will be sought of the estimation schemes proposed in the later sections.

The second section §6.2 presents the first stage of the description and analysis of the proposed approximation scheme. This includes several assumptions which make the algorithm proposed unrealisable in practice. It is shown that under certain conditions this idealised scheme produces consistent estimates of the desired parameters.

In §6.3 the proposed algorithm is altered to remove the assumption that infinite-dimensional Hilbert space operators may be used directly in the algorithm. The method used to implement this is a form of finite-section method. As in the previous section consistency of the resulting parameter estimates is obtained.

Section §6.4 considers the problems with the construction of the spectral factors associated with the estimated spectral densities appearing in the previous algorithms. The properties of available spectral factorisation algorithms are examined and a variety of approaches to proving global and local covergence of spectral factorisation algorithms are discussed. Consistent estimators are only obtained for some of the parameters sought.

In §6.5 some of the unresolved problems of §6.4 are discussed and partial results indicating the possible direction of solutions are obtained.

Throughout this Chapter many of the details of the scheme are proposed for the non-symmetric spectral density case, where the calculations are essentially the same. This may in future lead to results for the general non-symmetric case.

These three methods use the realisation of the stochastic process only to construct the covariance estimates and to derive their convergence rates to the true covariances. Direct measurement of these lagged covariances with certain convergence properties ensured would permit the similar analysis. Arun's thesis [74] describes several such situations which arise in practice.

Note 6.1 *Since all work in this Chapter is done using discrete-time transfer functions, the subscript d will be dropped from the spectral densities and factors. The notation $\hat{V}_d$ will be used to denote a dth order approximant to V.*

6.1 Approximation Results

The two types of object used in this Chapter are random variables and operators. The following subsections describe several ways in which these objects may be said to be close to others of the same type.

6.1.1 Spectral Approximation

This subsection describes the conditions necessary to guarantee that if

$$\lim_{n\to\infty} T_n = T$$

in some appropriate sense then

$$\lim_{n\to\infty} \lambda(T_n) = \lambda(T)$$

again in some sense as yet to be specified.

The connection between the norm convergence of operator sequences and spectral convergence has been studied in many places in the mathematical literature. An excellent summary of this area is presented in Chatelin's work [18] and the references therein.

The first problem is whether, in the relevant sense to be specified, the spectrum is continuous with respect to norm perturbations. This problem has solutions reaching back at least to Hellinger and Toeplitz [51].

The following definitions allow the notion of convergence of the spectrum to be made precise.

Definition 6.2 (Set Convergence) *If E, $E_n \subseteq \mathbb{C}$, $n \in \mathbb{N}$, then:*

1) $E := \limsup E_n$ *iff, for each $x \in E$ there exists an infinite subset $N_1 \subseteq \mathbb{N}$ such that $x \in E_n$, $\forall n \in N_1$.*

2) $E := \liminf E_n$ *iff, for each $x \in E$, there exists a sequence $x_n \in E_n$ such that* $\lim_{n\to\infty} x_n = x$.

3) $E := \lim_{n\to\infty} E_n$ *iff* $\limsup E_n = \liminf E_n =: \lim E_n$.

■

Definition 6.3 (Hausdorff Distance) *If $F, G \subseteq \mathbb{C}$ are closed then*

$$c(F,G) := \sup_{f\in F} \inf_{g\in G} |f-g|$$

and

$$d(F,G) := \max\left(c(F,G), c(G,F)\right).$$

d is called the Hausdorff Distance *of F from G. Let $K(\mathbb{C})$ be the set of all compact subsets of* $\mathbb{C}$. *Then $(K(\mathbb{C}), d)$ is a metric space known as the* Hausdorff metric space.

■

Using this notation, Newburgh [93] demonstrated the following solution to the above continuity problem.

Theorem 6.4 (Newburgh [93]) *If $\lambda(T)$ is totally disconnected, and $\|T - T_n\|_{op} \to 0$, then the mapping between metric spaces*

$$\lambda : (\mathrm{End}(H), e) \to (K(\mathbb{C}), d)$$

(the spectrum of an operator) where $e(A,B) = \|A - B\|_{op}$ for A, $B \in$ End(H) *is continuous at T, and*

$$\lim_{n\to\infty} \lambda(T_n) = \lambda(T).$$

■

6.1.2 Approximation Preserving Multiplicity

Despite the definitive answer to the first problem provided by Theorem 6.4, a second problem arises concerning whether the multiplicities of the approximate eigenvalues are the same as those of the approximated eigenvalues. The scheme presented in §6.2.2 shows that having several eigenvalues of the approximants converge to the same eigenvalue of the approximand would be problematic. This problem is related to the convergence of the spectral projections as is shown in Anselone's result, Proposition 6.10. This gives necessary and sufficient conditions for the preservation of the multiplicity of eigenvalues by norm approximations of operators.

All of the following results and notation are drawn from Chatelin [18], where a more complete consideration of the concepts and their interrelations is presented.

Assume that H is a Hilbert space and that T and T_n, $n \in \mathbb{N}$ are bounded linear maps on H.

Definition 6.5 (Resolvent Set) *Given $T \in \mathrm{End}(H)$, the* resolvent set *$\rho(T)$ is defined as*

$$\rho(T) := \left\{ z \in \mathbb{C} \mid (T - zI_H)^{-1} \in \mathrm{End}(H) \right\}$$

and the resolvent operator function *by*

$$\begin{aligned} R(T,.) &: \rho(T) \to \mathrm{End}(H) \\ R(T,z) &:= (T - zI_H)^{-1} \end{aligned}$$

■

Definition 6.6 (Stable Convergence) *A sequence of bounded operators $(T_n)_{n=1}^{\infty}$ on H is said to* converge stably *to $T \in \mathrm{End}(H)$, written $T_n \xrightarrow{s} T$, if*

1) T_n is pointwise convergent *to T, written $T_n \xrightarrow{p} T$, if*

$$T_n x \to Tx, \ \forall x \in H,$$

2) $\exists M > 0, N \in \mathbb{N}$ such that $\forall n > N$

$$T_n^{-1} \in \mathrm{End}(H) \ \text{and} \ \left\| T_n^{-1} \right\|_{op} \leqslant M.$$

■

Throughout the remainder of this section let $\lambda \in \lambda(T)$ be an eigenvalue of T of algebraic multiplicity $m < \infty$, and let $\lambda \in \Delta \subseteq \mathbb{C}$, where Δ is a simply connected closed subset of $\mathbb{C}$ containing no other elements of $\lambda(T)$. Let Γ be the boundary of Δ with positive orientation.

Definition 6.7 (Spectral Projection) *The* spectral projection *of T associated with eigenvalue λ is defined as*

$$\mathbf{P}(T;\lambda) := -\frac{1}{2\pi i}\int_{\Gamma} R(T,z)\,dz \ \in \mathrm{End}(H).$$

This is an orthogonal projection.

■

Note 6.8 (Independence from Γ) *$\mathbf{P}(T;\lambda)$ is independent of the particular Γ. Any boundary of a closed, simply connected region intersecting the spectrum only at this eigenvalue would give the same projection.*

■

In the case where λ is an eigenvalue of finite multiplicity then $\mathbf{P}(T;\lambda)$ has properties similar to those arising in the case of finite-dimensional operators. Hence $\mathrm{im}\mathbf{P}(T;\lambda)$ is of dimension m and contains the eigenspace $\ker(T-\lambda I_H)$. Also,

$$(T-\lambda I_H)^m \mathbf{P}(T;\lambda) = 0.$$

Hence, in the finite-dimensional case, $\mathrm{im}\mathbf{P}(T;\lambda)$ would be the span of all the eigenvectors and principal vectors associated with λ.

Preservation of multiplicity may be expressed as

$$\lim_{m\to\infty} \dim \mathrm{im}\mathbf{P}(T_n;\lambda) = \dim \mathrm{im}\mathbf{P}(T;\lambda)$$

This observation shows the importance of the following result of Anselone's.

Definition 6.9 (Collectively Compact Convergence) *The sequence $(T_n)_{n=1}^{\infty}$ is said to exhibit* collectively compact convergence *to T, written $T_n \xrightarrow{cc} T$, if*

1) $T_n \xrightarrow{p} T$,

2) *the set*
$$X := \cup_{n=1}^{\infty}(T-T_n)\{h \in H \mid \|h\|_H \leqslant 1\}$$
is relatively compact in H.

■

Proposition 6.10 (Anselone, see p128 [18]) *For projections P and $(P_n)_{n=1}^{\infty}$ with P compact, the following are equivalent:*

1) $P_n \xrightarrow{p} P$ *and* $\dim \mathrm{im}P_n = \dim \mathrm{im}P < \infty$ *for n large enough; and*

2) $P_n \xrightarrow{cc} P$.

■

The following notion of convergence of $(T_n)_{n=1}^{\infty}$ captures this requirement, and will be sufficient to establish the result required in this subsection.

Definition 6.11 (Strongly Stable Convergence) *With λ and Δ as above, $(T_n)_{n=1}^{\infty}$ is said to be* strongly stably convergent *to T in Δ, written $T_n \xrightarrow{ss} T$ in Δ, if*

1) $(T_n - zI_H) \xrightarrow{s} (T - zI_H)$, $\forall z \in \Delta \setminus \{\lambda\}$.

2) $\dim \mathrm{im}\mathbf{P}(T_n;\lambda) = m$ *for n large enough.*

■

Proposition 6.12 (Theorem 5.6 [18]) *If $T_n \xrightarrow{ss} T$ in Δ, then $\lambda(T_n) \cap \Delta$ consists, for n large enough, of exactly m eigenvalues, counting their multiplicities.*

■

It will therefore suffice to show that any approximation scheme is strongly stable. However, a simpler condition may be obtained which is more easily applied.

Definition 6.13 (Spectral Radius) *If $T \in \mathrm{End}(H)$ then*

$$\lambda_\sigma(T) := \lim_{n\to\infty} \|T^n\|_{op}^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|T^n\|_{op}^{\frac{1}{n}}$$

exists and is called the spectral radius *of T.*

■

Definition 6.14 (Radial Convergence) *$(T_n)_{n=1}^\infty$ is said to be* radially convergent *to T, written $T_n \xrightarrow{\sigma} T$, if*

1) $0 \in \rho(T)$.

2) $T_n \xrightarrow{s} T$.

3) $\forall \epsilon > 0$, $\exists N$ such that $\forall n > N$

$$\lambda_\sigma\left[(T - T_n)T^{-1}\right] < \epsilon.$$

■

Definition 6.15 (Uniform Radial Convergence) *$(T_n)_{n=1}^\infty$ is said to be* uniformly radially convergent *to T in $\rho(T)$, written $T_n - zI_H \xrightarrow{u\sigma} T - zI_H$ for z in $\rho(T)$, if*

1) $(T_n - zI_H) \xrightarrow{\sigma} (T - zI_H)$ $\forall z \in \rho(T)$.

2) $\forall \epsilon > 0, \forall K \subseteq \rho(T)$ compact, $\exists N(K)$ such that $\forall n > N(K)$

$$\sup_{z\in K} \lambda_\sigma\left[(T - T_n)R(T, z)\right] < \epsilon.$$

■

Lemma 6.16 (p. 246 [18]) *Given $T, T_n \in \mathrm{End}(H)$, $n \in \mathbb{N}$*

$$\|T - T_n\|_{op} \to 0 \;\Rightarrow (T_n - zI_H) \xrightarrow{u\sigma} (T - zI_H) \text{ in } \rho(T)$$

■

Lemma 6.17 (Theorem 5.26 [18] Lemordant) *If T is compact then*

$$(T_n - zI_H) \xrightarrow{u\sigma} (T - zI_H) \text{ in } \rho(T) \Leftrightarrow (T_n - zI_H) \xrightarrow{ss} (T - zI_H)\ \forall z \in \mathbb{C} \setminus \{0\}.$$

■

Theorem 6.18 (Spectral Convergence) *If T and $(T_n)_{n=1}^\infty$ are such that $\|T - T_n\|_{op} \to 0$ and T is compact then*

1) $\lambda(T_n) \to \lambda(T)$ in $(K(\mathbb{C}), d)$, and

2) $\forall \lambda \in \lambda(T)\setminus\{0\}$ of multiplicity $m < \infty$, and $\Delta \in K(\mathbb{C})$ simply connected with $\Delta \cap \lambda(T) = \{\lambda\}$ then $\exists N_2 > 0$ such that $\forall n > N_2$, $\lambda(T_n) \cap \Delta$ consists of exactly m eigenvalues, counting multiplicity.

Proof : From Lemma 6.16

$$(T_n - zI_H) \xrightarrow{u\sigma} (T - zI_H) \text{ in } \rho(T).$$

Hence, from Lemma 6.17

$$(T_n - zI_H) \xrightarrow{ss} (T - zI_H)\ \forall z \in \mathbb{C} \setminus \{0\}.$$

Since T is compact its spectrum is totally disconnected with all non-zero elements in the point spectrum with finite multiplicity. For any $\lambda \in \lambda(T) \setminus \{0\}$, define $\Delta_\lambda \in K(\mathbb{C})$ which is simply connected and is such that $\Delta_\lambda \cap \lambda(T) = \{\lambda\}$. Then from Proposition 6.12 the conclusion required is reached. ■

6.1.3 Schmidt Vector Approximation

The other significant connections made in §5.2 related the Schmidt vectors of the canonical correlation Hankel operator to the B-parameters of the underlying process. It is therefore necessary to understand the manner in which the Schmidt vectors of operator approximants are related to the Schmidt vectors of the limit operator.

Theorem 6.19 (Convergence of Compact Operators e.g. Glover *et al* **[36])** *If $(T_m)_{m=1}^{\infty}$ and T are compact operators on Hilbert space with $\left\|m^l(T-T_m)\right\|_{op} \to 0$, for some $l \in \mathbb{N}$, as $m \to \infty$ and T with each singular value of multiplicity 1 then*

$$m^l|\sigma_k(T) - \sigma_k(T_m)| \to 0$$

and

$$\left\|m^l(v_k - v_k^m)\right\|_2 \to 0, \ \left\|m^l(w_k - w_k^m)\right\|_2 \to 0$$

where $\{v_k, w_k\}$ and $\{v_k^m, w_k^m\}$ are Schmidt pairs of T and T_m respectively corresponding to singular values $\sigma_k(T)$ and $\sigma_k(T_m)$ respectively.

Proof : See [36] Appendix 2. A simple extension of the proof gives this result. ∎

Proposition 6.20 (Pointwise Convergence of Schmidt Vectors in the Disc) *A convergent sequence $(F_n)_{n=1}^{\infty}$ of functions in $\mathcal{H}^2_{p\times p}$ with limit F is such that*

$$F_n(z) \to F(z) \quad \forall z \in \mathbb{D}$$

in any matrix norm.

Proof : Given $z \in \mathbb{D}$ let $r = |z|$ and $r' = (1+r)/2$. Hence, noting that

$$|r'e^{i\theta} - z| \;\geqslant\; \frac{1-r}{2}$$

then

$$\begin{aligned}
\|F(z) - F_n(z)\|_2^2 &= \sum_{i=1}^{p}\sum_{j=1}^{p} |F(z)_{ij} - F_n(z)_{ij}|^2 \\
\text{which by Cauchy's Formula} &= \sum_{i=1}^{p}\sum_{j=1}^{p} \left| \frac{1}{2\pi i}\int_0^{2\pi} \frac{F(r'e^{i\theta})_{ij} - F_n(r'e^{i\theta})_{ij}}{r'e^{i\theta} - z} r'e^{i\theta}\, d\theta \right|^2 \\
\text{and from Jensen's Inequality} &\leqslant \frac{(1+r)^2}{4\pi^2(1-r)^2} \sum_{i=1}^{p}\sum_{j=1}^{p} \int_0^{2\pi} \left| F(r'e^{i\theta})_{ij} - F_n(r'e^{i\theta})_{ij} \right|^2 d\theta \\
&\leqslant \frac{(1+r)^2}{4\pi^2(1-r)^2} \sum_{i=1}^{p}\sum_{j=1}^{p} \|F_{ij} - (F_n)_{ij}\|_2^2 \\
&= \frac{(1+r)^2}{4\pi^2(1-r)^2} \|F - F_n\|_2^2
\end{aligned}$$

Finally note the equivalence of all norms on finite matrices. ∎

The previous result does not deal with convergence of the Schmidt vectors on $\mathbb{T}$. The following lemma provides sufficient conditions for such convergence.

Lemma 6.21 (Pointwise Convergence on the Unit Circle) *If $(F_n)_{n=1}^{\infty}$ is a sequence of functions in $\mathcal{L}^2_{p\times p}(\mathbb{T})$ which are continuously differentiable and for which*

$$\lim_{n\to\infty} \|F_n\|_2 = 0 \text{ and} \tag{6.1a}$$

$$\lim_{n\to\infty} \left\| \frac{dF_n}{d\theta} \right\|_2 = 0 \tag{6.1b}$$

then

$$F_n(e^{i\theta}) \to 0 \ \forall \theta \in [0, 2\pi).$$

Proof : From (6.1a) it is clear that for almost all $\theta \in [0, 2\pi)$ the desired result holds. Take θ' to be one such point.

Given $\theta \in [0, 2\pi) \setminus \{\theta'\}$

$$\begin{aligned}
\left\| F_n(e^{i\theta}) \right\|_1 &= \left\| \int_{\theta'}^{\theta} \left(\frac{dF_n}{d\theta}(e^{it}) \right) dt + F_n(e^{i\theta'}) \right\|_1 \\
&\leqslant \int_{\theta'}^{\theta} \left\| \frac{dF_n}{d\theta}(e^{it}) \right\|_1 dt + \left\| F_n(e^{i\theta'}) \right\|_1 \\
&\quad \text{which by the Cauchy-Schwarz inequality} \\
&\leqslant |\theta - \theta'| \left(\int_{\theta'}^{\theta} \left\| \frac{dF_n}{d\theta} \right\|_2^2 dt \right)^{\frac{1}{2}} + \left\| F_n(e^{i\theta'}) \right\|_1 \\
&\quad \text{and since } [\theta', \theta) \subseteq [0, 2\pi) \text{ then} \\
&\leqslant |\theta - \theta'| \left\| \frac{dF_n}{d\theta} \right\|_2 + \left\| F_n(e^{i\theta'}) \right\|_1 \\
&\to 0 \text{ as } n \to \infty \text{ by assumption (6.1b).}
\end{aligned}$$

■

6.1.4 Probabilistic Approximations

Definition 6.22 (Probability One Convergence) *A sequence $(y_t)_{t=1}^{\infty}$ of p-component random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to* converge with probability one (with respect to the probability measure $\mathbb{P}$) *if there exists another random variable y on the same probability space such that*

$$\lim_{t\to\infty} \|y_t(\omega) - y(\omega)\| = 0, \ \forall \omega \in \Omega \setminus N$$

for any vector norm $\|.\|$ on $\mathbb{R}^{p\times 1}$ and for some $\mathbb{P}$-null set N. y is the limit *of the sequence and the convergence if denoted*

$$y_t \to y \text{ w.p. } 1 \ [\mathbb{P}] \text{ as } t \to \infty,$$

■

Lemma 6.23 *If $(y_t)_{t=1}^{\infty}$ and y are are p-component random variables such that $y_t \to y$ w.p. 1 $[\mathbb{P}]$ as $t \to \infty$ and $f : \mathbb{R}^{p\times 1} \to \mathbb{R}^{q\times 1}$ is a continuous function then*

$$f(y_t) \to f(y) \text{ w.p. } 1 \ [\mathbb{P}] \text{ as } t \to \infty.$$

■

6.2 Approximation: Stage 1

6.2.1 Assumptions

Before proceeding with the details of the proposed estimation scheme, the main assumptions necessary for subsequent results are collected here. It is assumed below that κ and l are non-negative integers.

A Assumptions on $(y(t))_{t\in\mathbb{Z}}$.

- A.1 $(y(t))_{t\in\mathbb{Z}}$ is purely linearly non-deterministic.
- A.2 $(y(t))_{t\in\mathbb{Z}}$ is wide-sense stationary.
- A.3 $(y(t))_{t\in\mathbb{Z}}$ is zero-mean.
- A.4 $(y(t))_{t\in\mathbb{Z}}$ is full rank.

B Assumptions on the spectrum $\Phi(e^{i\theta})$ (as defined by (2.48)) of $(y(t))_{t\in\mathbb{Z}}$.

- B.1 $0 < \alpha I_p \leqslant \Phi(e^{i\theta}) \leqslant \beta I_p \ \forall\theta \in [0, 2\pi]$.
- B.2 $\lim_{d\to\infty} d^{\kappa} \sum_{k=d+1}^{\infty} \|\Gamma_y(k)\|_{\sigma} = 0$[1]
- B.3 $\Phi(e^{i\theta})$ is rational of degree $2n$.
- B.4 $\frac{d\Phi}{d\theta}$ belongs to $\mathcal{L}^2_{p\times p}(\mathbb{T})$.
- B.5 The Hankel singular values of the phase factor $V^{-1}W^*$ of $\Phi(e^{i\theta})$ are each of multiplicity one.

C Assumptions on $R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}}$.

- C.1 $\operatorname{card}\left(\lambda\left(R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}}\right) \cap \mathbb{T}\right) = 1$.

D Assumptions on the Innovations Sequence $(e(t))_{t\in\mathbb{Z}}$.

- D.1 $\mathbb{E}e_i^4(0) < \infty$.
- D.2 $\mathbb{E}e_i(0)e_j(0)e_k(t)e_l(t) = \mathbb{E}e_i(0)e_j(0)\mathbb{E}e_k(t)e_l(t) = \gamma_{ij}^{(e)}(0)\gamma_{kl}^{(e)}(0)$.

E Assumptions on the Function $N(d)$.

- E.1 $\displaystyle\sum_{d=1}^{\infty} \frac{d^{2(\kappa+1)}}{N(d)} < \infty$.
- E.2 $\displaystyle\sum_{d=1}^{\infty} \frac{d^3}{N(d)} < \infty$.

F Collections of Assumptions.

- F.1 Assume that A.1, A.2, A.3, A.4, D.1 and D.2 hold.
- F.2 Assume that A.1, A.2, A.3, A.4, B.1, D.1 and D.2 hold.
- F.3 Assume that F.2, B.2, B.4, E.1 with $\kappa = l = 0$ and E.2 all hold.
- F.4 Assume that F.2, B.2, B.4, E.1 with $\kappa = 2l$, $l \geqslant 1$.

[1] For $\kappa = 0$ this is merely the Cauchy condition equivalent to $\Phi \in A_{p\times p}(\mathbb{T})$, the Wiener algebra. For $\kappa \geqslant 1$ it is weaker than the κth derivative being an element of Wiener algebra.

6.2.2 Details of Stage 1

This section suggests an estimation scheme for the parameters appearing in the parametrisation of $\mathrm{M}_{\mathrm{n}}^{\mathrm{p}}[\omega]$. The following describes a method of constructing a sequence of mappings $\hat{\theta}^N$

$$\hat{\theta}^N : (y_t)_{t=1}^N \mapsto \left\{\hat{\sigma}_i, \hat{s}_i, \hat{B}_i \,|\, i = 1, \ldots, n\right\}$$

where N is the number of data observations available.

The following property of the estimators $\hat{\theta}^N$ will be sought:

Definition 6.24 (Strongly Consistent Estimator) *An estimator* $\hat{\theta}^N$ *of* θ *(a constant) is* strongly consistent for θ *if*

$$\hat{\theta}^N \to \theta \quad \text{w.p. } 1 \ [\mathbb{P}] \ \textit{as } N \to \infty.$$

■

It will be assumed that the parameters are non-stochastic quantities i.e. assuming a non-Bayesian framework.

It is also assumed throughout that n, the order of the system is known, and that at step d of the estimation procedure there are $N(d)+d$ data measurements available, $(y_t)_{t=1}^{N(d)+d}$.

The following steps are taken to estimate the appropriate parameters:

1) An estimate of the spectral-density is formed from estimated covariance lags:

$$\hat{\Phi}_d(e^{i\theta}) := \sum_{k=-d}^{d} \hat{\Gamma}^{(N)}(k) e^{-ik\theta}. \tag{6.2}$$

where

$$\hat{\Gamma}^{(N)}(k) := \begin{cases} \frac{1}{N(d)} \sum_{i=1}^{N(d)} y_i y_{i+k}^T & k = 0, 1, \ldots, d \\ \left(\hat{\Gamma}^{(N)}(-k)\right)^T & k = -d, \ldots, -1 \end{cases}. \tag{6.3}$$

2) Given that $\hat{\Phi}_d(e^{i\theta})$ is a spectral-density function (see Proposition 6.35), then the following are defined:

$$\begin{aligned} \hat{\Phi}_d(e^{i\theta}) &= \hat{V}_d(e^{i\theta}) \hat{V}_d(e^{i\theta})^* \\ &= \hat{W}_d(e^{i\theta})^* \hat{W}_d(e^{i\theta}) \end{aligned}$$

$$\hat{V}_d(e^{i\theta}) = \sum_{k=0}^{d} \hat{A}_{d,k} e^{ik\theta} \tag{6.4a}$$

$$\hat{W}_d(e^{i\theta}) = \sum_{k=0}^{d} \hat{B}_{d,k} e^{ik\theta} \tag{6.4b}$$

where the standard normalisations assumptions are made (see §5.1.2).

3) From the coefficients of $\hat{\Phi}_d(e^{i\theta})$, $\hat{V}_d(e^{i\theta})$ and $\hat{W}_d(e^{i\theta})$ the following operators are formed:

$$X_d = R\mathbf{H}_{\hat{\Phi}_d} = \begin{bmatrix} \hat{\Gamma}^{(N)}(1) & \hat{\Gamma}^{(N)}(2) & \hat{\Gamma}^{(N)}(3) & \hat{\Gamma}^{(N)}(4) & \cdots \\ \hat{\Gamma}^{(N)}(2) & \hat{\Gamma}^{(N)}(3) & \hat{\Gamma}^{(N)}(4) & \ddots & \ddots \\ \hat{\Gamma}^{(N)}(3) & \hat{\Gamma}^{(N)}(4) & \ddots & \hat{\Gamma}^{(N)}(d) & \ddots \\ \hat{\Gamma}^{(N)}(4) & \ddots & \hat{\Gamma}^{(N)}(d) & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \in \mathrm{End}(\mathcal{H}^2_{p\times p})$$

$$Y_d = \mathbf{T}_{\bar{\hat{V}}_d} = \begin{bmatrix} \hat{A}_{d,0} & \hat{A}_{d,1} & \hat{A}_{d,2} & \hat{A}_{d,3} & \cdots \\ 0 & \hat{A}_{d,0} & \hat{A}_{d,1} & \hat{A}_{d,2} & \ddots \\ 0 & 0 & \hat{A}_{d,0} & \hat{A}_{d,1} & \ddots \\ 0 & 0 & 0 & \hat{A}_{d,0} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \end{bmatrix} \in \mathrm{End}(\mathcal{H}^2_{p\times p})$$

$$Z_d = \mathbf{T}_{\hat{W}_d} = \begin{bmatrix} \hat{B}_{d,0} & 0 & 0 & 0 & \cdots \\ \hat{B}_{d,1} & \hat{B}_{d,0} & 0 & 0 & \ddots \\ \hat{B}_{d,2} & \hat{B}_{d,1} & \hat{B}_{d,0} & 0 & \ddots \\ \hat{B}_{d,3} & \hat{B}_{d,2} & \hat{B}_{d,1} & \hat{B}_{d,0} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \end{bmatrix} \in \mathrm{End}(\mathcal{H}^2_{p\times p})$$

Note 6.25 *By the assumptions in §5.1.2 these operator matrices are upper and lower triangular respectively.*

■

4) The approximate correlation operator

$$\mathcal{C}_d = Y_d^{-1} X_d Z_d^{-1} \tag{6.5}$$

is formed. The first n eigenvalues of $\mathcal{C}_d$, ranked by magnitude, are

$$\left\{\lambda_i^{(d)} \,|\, i = 1, \ldots, n\right\} \subseteq \lambda\left(\mathcal{C}_d\right)$$

and the associated normalised eigenvectors are

$$\left\{\hat{u}_i^{(d)} \,|\, i = 1, \ldots, n\right\}$$

where

$$\mathcal{C}_d \hat{u}_i^{(d)} = \lambda_i^{(d)} \hat{u}_i^{(d)}. \tag{6.6}$$

5) The estimates

$$\hat{s}_i^{(d)} = \mathrm{signum}(\lambda_i^{(d)})$$
$$\hat{\sigma}_i^{(d)} = \frac{|\lambda_i^{(d)}|}{\sqrt{1 - \lambda_i^{(d)2}}}$$

and

$$\hat{B}_i^{(d)} = \sqrt{2\hat{\sigma}_i^{(d)}}\, \hat{s}_i^{(d)} \hat{u}_i^{(d)}(-1)$$

$i = 1, \ldots n$ are calculated.

The estimator is then given by

$$\hat{\theta}^{N(d)+d} = \left\{ \hat{s}_i^{(d)}, \hat{\sigma}_i^{(d)}, \hat{B}_i^{(d)} \mid i = 1, \ldots, n \right\}.$$

Note 6.26 *No estimator for the noise covariance D will be described in this Chapter.*

■

6.2.3 Analysis of Stage 1

The proof of consistency of the various estimators is based upon the proof that the Hankel operator $R\mathbf{H}_{V_d^{-1}W_d^*}$ may be approximated in the operator norm at a sufficiently fast rate. It is then clear, from §6.1.1 that the spectrum of the approximating operators converges to that of the true operator, and that the eigenvectors converge in a suitably strong sense.

The following inequalities will be required in subsequent results.

Lemma 6.27 (Chebychev's Inequality) *Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\epsilon > 0$ then*

$$\mathbb{P}\left(|X| > \epsilon\right) \leqslant \frac{\mathbb{E}(X^2)}{\epsilon^2}$$

Proof : See e.g. §7.3 Grimmett & Stirzaker [45]. ■

The following standard result relates the Euclidean norm of a n-vector to its l^1 norm. See for example Horn & Johnson [58].

Lemma 6.28 (Inequality) *Let $a_i \in \mathbb{C},\ i = 1, \ldots, n$ then*

$$\left| \sum_{i=1}^{n} a_i \right|^2 \leqslant n \sum_{i=1}^{n} |a_i|^2$$

■

Lemma 6.29 (Inequality) *Let $a_i \in \mathbb{C},\ i \in \mathbb{Z}$ then*

$$\left| \sum_{i=-\infty}^{\infty} a_i \bar{a}_{i+k} \right| \leqslant \sum_{i=-\infty}^{\infty} |a_i|^2, \ \forall k \in \mathbb{Z}$$

Proof : This is merely the Cauchy-Schwartz inequality applied to $(a_i)_{i=-\infty}^{\infty}$ and $(a_{i+k})_{i+k=-\infty}^{\infty}$. ■

Lemma 6.30 (Relationship between norms) *Let $B = [b_{ij}]_{i,j=1}^{p} \in \mathbb{C}^{p \times p}$ then*

$$\|B\|_1 = \sum_{i,j=1}^{p} |b_{ij}| \leqslant p \left(\sum_{i,j=1}^{p} |b_{ij}|^2 \right)^{\frac{1}{2}} = p \, \|B\|_2$$

Proof : In Lemma 6.28 take $n = p^2$ and $a_i = |b_{jk}|$ where $p(j-1) + k = i$; $j, k \in \{1, \ldots, p\}$. Then take the square root of both sides. ■

Lemma 6.31 (Lower bound on Weighted Matrix Norm) *Let* $B = [b_{ij}]_{i,j=1}^{n} \in \mathbb{C}^{n\times n}$ *and* $\Gamma \in \mathbb{C}^{n\times n}$, $\Gamma > 0$ *then*

$$\left\| B\Gamma^{\frac{1}{2}} \right\|_2^2 \geqslant \lambda_{\min}(\Gamma) \left\| B \right\|_2^2$$

Proof : Firstly, recall (e.g. from 4.2.2 Horn & Johnson [58], due to Rayleigh and Ritz) that

$$\lambda_{\min}(\Gamma) = \min_{x\in\mathbb{C}^n\setminus\{0\}} \frac{x^*\Gamma x}{x^* x}.$$

Hence, letting b_i be the ith row of B, gives

$$\begin{aligned}
\left\| B\Gamma^{\frac{1}{2}} \right\|_2^2 &= \operatorname{tr} B\Gamma B^* \\
= \sum_{i=1}^{p} [B\Gamma B^*]_{ii} &= \sum_{i=1}^{p} b_i \Gamma b_i^* \\
\text{which by above} &\geqslant \sum_{i=1}^{p} \lambda_{\min}(\Gamma) b_i b_i^* \\
= \lambda_{\min}(\Gamma) \operatorname{tr} BB^* &= \lambda_{\min}(\Gamma) \left\| B \right\|_2^2
\end{aligned}$$

■

The following Proposition contains the main theoretical result of this Chapter.

Proposition 6.32 *Under conditions F.2 and E.1 for* $\kappa \geqslant 0$ *then*

$$\lim_{d\to\infty} d^{\kappa} \sum_{k=0}^{d} \left\| \Gamma(k) - \hat{\Gamma}^{(N(d))}(k) \right\|_{\sigma} = 0 \quad \text{w.p. 1 } [\mathbb{P}].$$

Under conditions F.2 and E.2 then

$$\lim_{d\to\infty} \sum_{k=0}^{d} k^2 \left\| \Gamma(k) - \hat{\Gamma}^{(N(d))}(k) \right\|_2^2 = 0 \quad \text{w.p. 1 } [\mathbb{P}].$$

Proof : This result is a generalisation of a result in Caines & Baykal-Gürsoy [15] where a similar proof method is used. First, consider the following quantity:

$$\begin{aligned}
\mathbb{P}\left(d^{\kappa} \sum_{k=0}^{d} \left\| \Gamma(k) - \hat{\Gamma}^{(N)}(k) \right\|_{\sigma} \geqslant \epsilon \right) &\leqslant \frac{d^{2\kappa}}{\epsilon^2} \mathbb{E}\left(\sum_{k=0}^{d} \left\| \Gamma(k) - \hat{\Gamma}^{(N)}(k) \right\|_{\sigma} \right)^2 \text{ by Chebychev's inequality} \\
\text{and from Lemma 6.28} \quad & \\
&\leqslant \frac{d^{2\kappa}(d+1)}{\epsilon^2} \sum_{k=0}^{d} \mathbb{E} \left\| \Gamma(k) - \hat{\Gamma}^{(N)}(k) \right\|_{\sigma}^2 \\
&\leqslant \frac{d^{2\kappa}(d+1)}{\epsilon^2} \sum_{k=0}^{d} \mathbb{E} \left\| \Gamma(k) - \hat{\Gamma}^{(N)}(k) \right\|_2^2 (\text{ as } \left\| A \right\|_{\sigma} \leqslant \left\| A \right\|_2) \quad (6.7) \\
&= \frac{d^{2\kappa}(d+1)}{\epsilon^2} \sum_{k=0}^{d} \sum_{i,j=1}^{p} \mathbb{E} \left(\gamma_{ij}(k) - \hat{\gamma}_{ij}^{(N)}(k) \right)^2
\end{aligned}$$

Appendix B derives the following expression for the variance of $\hat{\gamma}_{ij}^{(N)}(k)$.

$$\begin{aligned}
\operatorname{Var}\left(\hat{\gamma}_{ij}^{(N)}(k) \right) &= \mathbb{E}\left(\gamma_{ij}(k) - \hat{\gamma}_{ij}^{(N)}(k) \right)^2 \\
&= \left(\frac{1}{N(d)^2} \sum_{m,n=1}^{N(d)} \gamma_{ijij}(k, m-n, m-n+k) \right) - \gamma_{ij}(k)^2 \\
&= \left(\frac{1}{N(d)} \sum_{\nu=-(N(d)-1)}^{(N(d)-1)} \frac{N(d)-|\nu|}{N(d)} \gamma_{ijij}(k, \nu, \nu+k) \right) - \gamma_{ij}(k)^2
\end{aligned}$$

Also, from Appendix B, the following expression for $\gamma_{ijij}(k,\nu,\nu+k)$ is derived.

$$\begin{aligned}\gamma_{ijij}(k,\nu,\nu+k) &= \mathbb{E}y_i(0)y_j(k)y_i(\nu)y_j(\nu+k)\\ &= \gamma_{ij}(k)^2+\gamma_{ii}(\nu)\gamma_{jj}(\nu)+\gamma_{ij}(\nu+k)\gamma_{ji}(\nu-k)\\ &\quad+\sum_{a=-\infty}^{\infty}\sum_{q,r,s,t=1}^{p}A_{iq}(a)A_{jr}(a+k)A_{is}(a+\nu)A_{jt}(a+\nu+k)\\ &\quad\times\begin{pmatrix}\gamma^{(e)}_{qrst}(0,0,0)-\gamma^{(e)}_{qr}(0)\gamma^{(e)}_{st}(0)\\ -\gamma^{(e)}_{qs}(0)\gamma^{(e)}_{rt}(0)-\gamma^{(e)}_{rs}(0)\gamma^{(e)}_{rs}(0)\end{pmatrix}\end{aligned}$$

Hence, the following bounds the original expression:

$$\begin{aligned}&\frac{d+1}{N(d)\epsilon^2}\sum_{k=0}^{d}\sum_{\nu=-(N(d)-1)}^{(N(d)-1)}\sum_{i,j=1}^{p}\frac{N(d)-|\nu|}{N(d)}\Bigg\{\gamma_{ii}(\nu)\gamma_{jj}(\nu)+\gamma_{ij}(\nu+k)\gamma_{ji}(\nu-k)\\ &+\sum_{q,r,s,t=1}^{p}\sum_{a=-\infty}^{\infty}A_{iq}(a)A_{jr}(a+k)A_{is}(a+\nu)A_{jt}(a+\nu+k)\begin{pmatrix}\gamma^{(e)}_{qrst}(0,0,0)-\gamma^{(e)}_{qr}(0)\gamma^{(e)}_{st}(0)\\ -\gamma^{(e)}_{qs}(0)\gamma^{(e)}_{rt}(0)-\gamma^{(e)}_{rs}(0)\gamma^{(e)}_{rs}(0)\end{pmatrix}\Bigg\}\end{aligned}$$

It is possible to show that the individual parts of the above expression are each bounded.

$$\begin{aligned}\left|\sum_{\nu=-(N(d)-1)}^{N(d)-1}\sum_{i,j=1}^{p}\gamma_{ii}(\nu)\gamma_{jj}(\nu)\right| &= \sum_{\nu=-(N(d)-1)}^{N(d)-1}\left(\sum_{i=1}^{p}\gamma_{ii}(\nu)\right)^2\\ &\quad\text{from Lemma 6.28}\\ &\leqslant \sum_{\nu=-(N(d)-1)}^{N(d)-1}p\sum_{i=1}^{p}\gamma_{ii}(\nu)^2\\ &\leqslant \sum_{\nu=-(N(d)-1)}^{N(d)-1}p\sum_{i,j=1}^{p}\gamma_{ij}(\nu)^2\\ &= \sum_{\nu=-(N(d)-1)}^{N(d)-1}p\,\|\Gamma(\nu)\|_2^2\\ &\quad\text{adding some non-negative terms gives}\\ &\leqslant p\sum_{\nu=-\infty}^{\infty}\|\Gamma(\nu)\|_2^2\\ &= p\left\|\Phi(e^{i\theta})\right\|_2^2\\ &\leqslant p^2\left\|\Phi(e^{i\theta})\right\|_\infty^2.\end{aligned}$$

Also,

$$\begin{aligned}\left|\sum_{\nu=-(N(d)-1)}^{N(d)-1}\sum_{i,j=1}^{p}\gamma_{ij}(\nu+k)\gamma_{ji}(\nu-k)\right| &= \sum_{\nu=-(N(d)-1)}^{N(d)-1}\left|\sum_{i,j=1}^{p}\gamma_{ij}(\nu+k)\gamma_{ji}(\nu-k)\right|\\ &\leqslant \sum_{\nu=-(N(d)-1)}^{N(d)-1}\left(\sum_{i,j=1}^{p}\gamma_{ij}(\nu+k)^2\right)^{\frac{1}{2}}\left(\sum_{i,j=1}^{p}\gamma_{ji}(\nu-k)^2\right)^{\frac{1}{2}}\\ &= \sum_{\nu=-(N(d)-1)}^{N(d)-1}\|\Gamma(\nu+k)\|_2\,\|\Gamma(\nu-k)\|_2\end{aligned}$$

$$
\begin{aligned}
&\leqslant \left(\sum_{\nu=-(N(d)-1)}^{N(d)-1} \|\Gamma(\nu+k)\|_2^2\right)^{\frac{1}{2}} \left(\sum_{\nu=-(N(d)-1)}^{N(d)-1} \|\Gamma(\nu-k)\|_2^2\right)^{\frac{1}{2}} \\
&\leqslant \sum_{\nu=-\infty}^{\infty} \|\Gamma(\nu-k)\|_2^2 \\
&= \left\|\Phi(e^{i\theta})\right\|_2^2 \\
&\leqslant p\left\|\Phi(e^{i\theta})\right\|_\infty^2 .
\end{aligned}
$$

From standing assumptions it is clear that

$$
\left|\gamma^{(e)}_{qrst}(0,0,0) - \gamma^{(e)}_{qr}(0)\gamma^{(e)}_{st}(0) - \gamma^{(e)}_{qs}(0)\gamma^{(e)}_{rt}(0) - \gamma^{(e)}_{rs}(0)\gamma^{(e)}_{rs}(0)\right| = K < \infty
$$

for some constant $K > 0$. Hence the only term remaining to be dealt with is

$$
\begin{aligned}
&\left|\sum_{k=0}^{d}\sum_{\nu=-(N(d)-1)}^{N(d)-1}\sum_{i,j=1}^{p}\sum_{a=-\infty}^{\infty}\sum_{q,r,s,t=1}^{p} A_{iq}(a)A_{jr}(a+k)A_{is}(a+\nu)A_{jt}(a+\nu+k)\right| \\
&= \left|\sum_{k=0}^{d}\sum_{\nu=-(N(d)-1)}^{N(d)-1}\sum_{i,j=1}^{p}\sum_{a=-\infty}^{\infty}\left(\sum_{q=1}^{p}A_{iq}(a)\right)\left(\sum_{r=1}^{p}A_{jr}(a+k)\right)\left(\sum_{s=1}^{p}A_{is}(a+\nu)\right)\left(\sum_{t=1}^{p}A_{jt}(a+\nu+k)\right)\right| \\
&\leqslant \sum_{k=0}^{d}\sum_{i,j=1}^{p}\sum_{a=-\infty}^{\infty}\left|\sum_{q=1}^{p}A_{iq}(a)\right|\left|\sum_{r=1}^{p}A_{jr}(a+k)\right|\left(\sum_{\nu=-(N(d)-1)}^{N(d)-1}\left|\sum_{s=1}^{p}A_{is}(a+\nu)\right|\left|\sum_{t=1}^{p}A_{jt}(a+\nu+k)\right|\right)
\end{aligned}
$$

and since the last terms are all positive, then

$$
\begin{aligned}
&\leqslant \sum_{k=0}^{d}\sum_{i,j=1}^{p}\left(\sum_{a=-\infty}^{\infty}\left|\sum_{q=1}^{p}A_{iq}(a)\right|\left|\sum_{r=1}^{p}A_{jr}(a+k)\right|\right)\left(\sum_{\nu=-\infty}^{\infty}\left|\sum_{q=1}^{p}A_{iq}(\nu)\right|\left|\sum_{r=1}^{p}A_{jr}(\nu+k)\right|\right) \\
&= \sum_{k=0}^{d}\sum_{i,j=1}^{p}\left(\sum_{a=-\infty}^{\infty}\left|\sum_{q=1}^{p}A_{iq}(a)\right|\left|\sum_{r=1}^{p}A_{jr}(a+k)\right|\right)^2 \\
&\leqslant \sum_{k=0}^{d}\sum_{i,j=1}^{p}\left(\sum_{a=-\infty}^{\infty}\sum_{q=1}^{p}|A_{iq}(a)|\sum_{r=1}^{p}|A_{jr}(a+k)|\right)^2 \\
&\leqslant \sum_{k=0}^{d}\sum_{i,j=1}^{p}\left(\sum_{a=-\infty}^{\infty}\sum_{r,s=1}^{p}|A_{rs}(a)|\sum_{r,s=1}^{p}|A_{rs}(a+k)|\right)^2
\end{aligned}
$$

hence from Lemma 6.29 with $a_i = \|A(i)\|_1$

$$
\leqslant \sum_{k=0}^{d}\sum_{i,j=1}^{p}\left(\sum_{a=-\infty}^{\infty}\left(\sum_{r,s=1}^{p}|A_{rs}(a)|\right)^2\right)^2
$$

then using the norm inequality of Lemma 6.30

$$
\begin{aligned}
&\leqslant \sum_{k=0}^{d}\sum_{i,j=1}^{p}\left(\sum_{a=-\infty}^{\infty}p^2\sum_{r,s=1}^{p}|A_{rs}(a)|^2\right)^2 \\
&= p^6\sum_{k=0}^{d}\left(\sum_{a=-\infty}^{\infty}\|A(a)\|_2^2\right)^2
\end{aligned}
$$

applying Lemma 6.31 to the norms, and noting the independence of the bracketed term from k,

$$
\leqslant p^6(d+1)\left(\lambda_{\min}^{-1}(\Gamma_e(0))\sum_{a=-\infty}^{\infty}\left\|A(a)\Gamma_e(0)^{\frac{1}{2}}\right\|_2^2\right)^2
$$

and hence from Lemma B.1 it is clear that

$$= \quad p^6(d+1)\lambda_{\min}^{-2}(\Gamma_e(0))\mathrm{tr}\ (\Gamma_y(0))^2$$

Since $(y(t))_{t\in\mathbb{Z}}$ is full rank, then $\lambda_{\min}(\Gamma_e(0)) > 0$ as $\Gamma_e(0)$ is Hermitian. Also, from the assumptions made about the process

$$\Gamma_y(0) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(e^{i\theta})\, d\theta > \alpha I_p$$

and hence $\mathrm{tr}\ (\Gamma_y(0)) > p\alpha > 0$.

Hence

$$\begin{aligned}
&\mathbb{P}\left(d^\kappa \sum_{k=0}^{d} \left\|\Gamma(k) - \hat{\Gamma}^{(N)}(k)\right\|_\sigma \geqslant \epsilon\right) \\
\leqslant\ & \frac{d^{2\kappa}(d+1)}{N(d)\epsilon^2}\left\{Kp^6(d+1)\lambda_{\min}^{-2}(\Gamma_e(0))\mathrm{tr}\ (\Gamma_y(0))^2 + (d+1)p(p+1)\left\|\Phi(e^{i\theta})\right\|^2\right\} \\
=\ & \frac{d^{2\kappa}(d+1)^2}{N(d)}\epsilon^{-2}\left\{Kp^6\lambda_{\min}^{-2}(\Gamma_e(0))\mathrm{tr}\ (\Gamma_y(0))^2 + p(p+1)\left\|\Phi(e^{i\theta})\right\|^2\right\}
\end{aligned}$$

Let the coefficient of $(d+1)^2/N(d)$ be called $M(\epsilon)$, which has been shown to be positive.

This implies

$$\begin{aligned}
\sum_{d=1}^{\infty}\mathbb{P}\left(d^\kappa \sum_{k=0}^{d} \left\|\Gamma(k) - \hat{\Gamma}^{(N)}(k)\right\|_\sigma \geqslant \epsilon\right) &\leqslant \sum_{d=1}^{\infty}\frac{d^{2\kappa}(d+1)^2}{N(d)}M(\epsilon) \\
&< \infty
\end{aligned}$$

under the assumptions of the Theorem. Hence from the Borel-Cantelli Theorem only finitely many of the above described events occur (with probability 1), and hence for each ϵ there exists a $d(\epsilon)$ (and hence $N(d(\epsilon))$) such that

$$D^\kappa \sum_{k=0}^{D} \left\|\Gamma(k) - \hat{\Gamma}^{(N(D))}(k)\right\|_\sigma < \epsilon,\ \forall D > d(\epsilon) \text{ w.p. } 1\ [\mathbb{P}].$$

Hence the first part of the result is established.

To obtain the second part note that given $K > 0$ then

$$\begin{aligned}
\mathbb{P}\left(\sum_{k=0}^{d} k^2 \left\|\Gamma(k) - \hat{\Gamma}^{(N)}(k)\right\|_2^2 > K^2\right) &\leqslant \frac{1}{K^2}\sum_{k=0}^{d} k^2\mathbb{E}\left\|\Gamma(k) - \hat{\Gamma}^{(N)}(k)\right\|_2^2 \\
&\leqslant \frac{d^2}{K^2}\sum_{k=0}^{d}\mathbb{E}\left\|\Gamma(k) - \hat{\Gamma}^{(N)}(k)\right\|_2^2.
\end{aligned}$$

Comparing this to (6.7) in the previous analysis, then

$$\mathbb{P}\left(\sum_{k=0}^{d} k^2 \left\|\Gamma(k) - \hat{\Gamma}^{(N)}(k)\right\|_2^2 > K^2\right) \leqslant \frac{d^2(d+1)}{N(d)}M(K).$$

Hence, following the above reasoning, such events occur only finitely often if the above is a summable sequence in d, as required. ∎

Corollary 6.32.1 (Convergence of Spectral-Density Estimate) *Under the assumptions F.2, B.2 and E.1 with $\kappa \geqslant 0$ then*

$$\lim_{d\to\infty}\left\|d^\kappa\left(\Phi(e^{i\theta}) - \hat{\Phi}_d(e^{i\theta})\right)\right\|_\infty = 0 \quad \text{w.p. } 1\ [\mathbb{P}].$$

Proof :

$$\begin{aligned} d^{\kappa}\left\|\Phi(e^{i\theta})-\hat{\Phi}_d(e^{i\theta})\right\|_{\infty} &\leqslant 2\left\|d^{\kappa}\sum_{k=d+1}^{\infty}\Gamma(k)e^{ik\theta}\right\|_{\infty}+\left\|d^{\kappa}\sum_{k=-d}^{d}\left(\Gamma(k)-\hat{\Gamma}^{(N(d))}(k)\right)e^{ik\theta}\right\|_{\infty} \\ &\leqslant 2\sum_{k=d+1}^{\infty}d^{\kappa}\left\|\Gamma(k)\right\|_{\sigma}+2\sum_{k=0}^{d}d^{\kappa}\left\|\Gamma(k)-\hat{\Gamma}^{(N(d))}(k)\right\|_{\sigma} \end{aligned}$$

By assumption B.2 the first of these terms converges to zero with d increasing. By Lemma 6.32 the second term converges to zero. ■

Corollary 6.32.2 (Convergence of Hankel Operator Estimate) *Under the assumptions F.2, B.2 and E.1*

$$\lim_{d\to\infty} d^{\kappa}\left\|R\mathbf{H}_{\Phi}-R\mathbf{H}_{\hat{\Phi}_d}\right\|_{op}=0 \quad \text{w.p. } 1\ [\mathbb{P}].$$

Proof :

$$\begin{aligned} d^{\kappa}\left\|R\mathbf{H}_{\Phi}-R\mathbf{H}_{\hat{\Phi}_d}\right\|_{op} &\leqslant d^{\kappa}\left\|R\right\|_{op}\left\|\mathbf{H}_{\Phi}-\mathbf{H}_{\hat{\Phi}_d}\right\|_{op} \\ &= d^{\kappa}\left\|\mathbf{H}_{\Phi-\hat{\Phi}_d}\right\|_{op} \\ &\leqslant \left\|d^{\kappa}\left(\Phi(e^{i\theta})-\hat{\Phi}_d(e^{i\theta})\right)\right\|_{\infty} \end{aligned}$$

which tends to zero as $d\to\infty$ with probability 1, by Corollary 6.32.1. ■

The following Lemmas will be used to establish that the approximation to the spectral-density is eventually itself a spectral-density.

Lemma 6.33 (Singular Values of Operators) *If A and B are bounded operators on a Hilbert space H then*

$$\sigma_{m+n-1}(A+B)\leqslant\sigma_m(A)+\sigma_n(B).$$

Proof : See Corollary 1.5 Partington [103]. ■

Corollary 6.33.1 *If $F,\hat{F}:\mathbb{C}\to\mathbb{C}^{p\times p}$ then*

$$\sigma_{\min}\left(\hat{F}(\theta)\right)\geqslant\sigma_{\min}\left(F(\theta)\right)-\sigma_{\max}\left(F(\theta)-\hat{F}(\theta)\right)$$

■

Lemma 6.34 (Weyl) *Let $A,B\in\mathbb{C}^{n\times n}$ be Hermitian, then for $k=1,\dots,n$*

$$\lambda_k(A)+\lambda_{\min}(B)\leqslant\lambda_k(A+B)\leqslant\lambda_k(A)+\lambda_{\max}(B).$$

Proof : See 4.3.1 p181 Horn & Johnson [58]. ■

Proposition 6.35 (Convergence of Toeplitz Operators) *Under assumptions F.2, B.2 and E.1 ($\kappa=0$) there exists $D\in\mathbb{N}$ such that for all $d\geqslant D$, $\mathbf{T}_{\hat{\Phi}_d}$ is boundedly invertible. Also*

$$\lim_{\substack{d\geqslant D\\ d\to\infty}}\left\|\mathbf{T}_{\Phi}^{-1}-\mathbf{T}_{\hat{\Phi}_d}^{-1}\right\|_{op}=0 \quad \text{w.p. } 1\ [\mathbb{P}].$$

Proof : Given $0 < \epsilon \leqslant \frac{\alpha}{2}$, from the convergence of Corollary 6.32.1 there exists $D \in \mathbb{N}$ such that for all $d \geqslant D$

$$\left\| \Phi(e^{i\theta}) - \hat{\Phi}_d(e^{i\theta}) \right\|_\infty < \epsilon.$$

As $\Phi(e^{i\theta})$ is positive-definite and $\hat{\Phi}_d(e^{i\theta})$ is, by construction, Hermitian then applying Lemma 6.34 to $A = \hat{\Phi}_d(e^{i\theta}) - \Phi(e^{i\theta})$ and $B = \Phi(e^{i\theta})$ with $k = p$ gives

$$\begin{aligned}
\lambda_{\min}\left(\hat{\Phi}_d(e^{i\theta})\right) &\geqslant \lambda_{\min}\left(\Phi(e^{i\theta})\right) + \lambda_{\min}\left(\hat{\Phi}_d(e^{i\theta}) - \Phi(e^{i\theta})\right) \\
&\geqslant \alpha - \epsilon \\
&\geqslant \alpha/2 > 0
\end{aligned}$$

since $\lambda_k\left(\Phi(e^{i\theta})\right) = \sigma_k\left(\Phi(e^{i\theta})\right) \geqslant \alpha$ by assumption, and it is known (1.5.1 p.144 Marcus & Minc [84]) that

$$\sigma_{\min}(A) \leqslant |\lambda_k(A)| \leqslant \sigma_{\max}(A), \ \forall k = 1, \ldots, n.$$

Thus $\hat{\Phi}_d(e^{i\theta})$ is positive-definite and hence invertibility of $\mathbf{T}_{\hat{\Phi}_d}$ is given by Lemmas 2.33 and 2.45.

Finally,

$$\begin{aligned}
\left\| \mathbf{T}_\Phi^{-1} - \mathbf{T}_{\hat{\Phi}_d}^{-1} \right\|_{op} &= \left\| \mathbf{T}_{\Phi^{-1}} - \mathbf{T}_{\hat{\Phi}_d^{-1}} \right\|_{op} \quad \text{from Lemma 2.45} \\
&= \left\| \mathbf{T}_{\left(\Phi^{-1} - \hat{\Phi}_d^{-1}\right)} \right\|_{op} \\
&= \left\| \Phi(e^{i\theta})^{-1} - \hat{\Phi}_d(e^{i\theta})^{-1} \right\|_\infty \quad \text{by properties of Toeplitz operators} \\
&\leqslant \left\| \Phi(e^{i\theta})^{-1} \right\|_\infty \left\| \Phi(e^{i\theta}) - \hat{\Phi}_d(e^{i\theta}) \right\|_\infty \left\| \hat{\Phi}_d(e^{i\theta})^{-1} \right\|_\infty \\
&\leqslant \alpha^{-1}\epsilon \left[\inf_{\theta \in [0,2\pi)} \sigma_{\min}\left(\hat{\Phi}_d(e^{i\theta})\right) \right]^{-1} \quad \text{since } \left\| A^{-1} \right\|_{op} = \sigma_{\min}^{-1}(A) \\
\text{which from Corollary 6.33.1} &\leqslant \alpha^{-1}\epsilon \left[\inf_{\theta \in [0,2\pi)} \left\{ \sigma_{\min}\left(\Phi(e^{i\theta})\right) - \sigma_{\max}\left(\Phi(e^{i\theta}) - \hat{\Phi}_d(e^{i\theta})\right) \right\} \right]^{-1} \\
&\leqslant \alpha^{-1}\epsilon \left[\inf_{\theta \in [0,2\pi)} \sigma_{\min}\left(\Phi(e^{i\theta})\right) - \sup_{\theta \in [0,2\pi)} \sigma_{\max}\left(\Phi(e^{i\theta}) - \hat{\Phi}_d(e^{i\theta})\right) \right]^{-1} \\
&\leqslant \alpha^{-1}\epsilon(\alpha - \epsilon)^{-1} \\
&\leqslant \frac{2\epsilon}{\alpha^2}
\end{aligned}$$

Hence the Toeplitz operator inverses converge. ∎

The following Proposition refers to the spectral factors of Φ_d defined in §6.2.2.

Proposition 6.36 (Convergence of Spectral Factors) *Under assumptions F.2, B.2, B.4, E.1, and E.2 where $\kappa = 0$ there exists $D \in \mathbb{N}$ such that*

$$\lim_{\substack{d \geqslant D \\ d \to \infty}} \left\| W - \hat{W}_d \right\|_\infty = 0, \ \lim_{\substack{d \geqslant D \\ d \to \infty}} \left\| V - \hat{V}_d \right\|_\infty = 0$$

and hence that

$$\lim_{\substack{d \geqslant D \\ d \to \infty}} \left\| \mathbf{T}_W - \mathbf{T}_{\hat{W}_d} \right\|_{op} = 0, \ \lim_{\substack{d \geqslant D \\ d \to \infty}} \left\| \mathbf{T}_V - \mathbf{T}_{\hat{V}_d} \right\|_{op} = 0$$

all with probability 1 $[\mathbb{P}]$.

Under assumptions F.2, B.2, B.4, E.1 where $\kappa = 2l$, $l \geqslant 1$ there exists $D \in \mathbb{N}$ such that

$$\lim_{\substack{d \geqslant D \\ d \to \infty}} d^l \left\| W - \hat{W}_d \right\|_\infty = 0, \quad \lim_{\substack{d \geqslant D \\ d \to \infty}} d^l \left\| V - \hat{V}_d \right\|_\infty = 0$$

and hence that

$$\lim_{\substack{d \geqslant D \\ d \to \infty}} d^l \left\| \mathbf{T}_W - \mathbf{T}_{\hat{W}_d} \right\|_{op} = 0, \quad \lim_{\substack{d \geqslant D \\ d \to \infty}} d^l \left\| \mathbf{T}_V - \mathbf{T}_{\hat{V}_d} \right\|_{op} = 0$$

all with probability 1 $[\mathbb{P}]$.

Proof : Under both sets of assumptions, from Proposition 6.35, there exists $D_1 \in \mathbb{N}$ such that $\hat{\Phi}_d(e^{i\theta}) > 0 \ \forall d \geqslant D_1$. Also, from Corollary 6.32.1, for some $D_2 \in \mathbb{N}$

$$\left\| \Phi - \hat{\Phi}_d \right\|_\infty \leqslant \frac{\alpha}{\alpha + \beta} \tag{6.8}$$

for all $d \geqslant D_2$. Letting $D = \max(D_1, D_2)$, note that Lemma 2.33 implies that all of the objects under consideration are well-defined.

Under the condition that (6.8) holds then Theorem 2 Anderson [4] and p79 Rozanov [112] show that

$$d^l \left\| V - \hat{V}_d \right\|_\infty \leqslant \bar{K}_d \left\| d^{2l}(\Phi - \hat{\Phi}_d) \right\|_\infty^{\frac{1}{2}}$$

where $\bar{K}_d$ depends upon only α, β and the $\mathcal{L}^2_{p \times p}$-norm of the derivatives of Φ and $\hat{\Phi}_d$. Hence, from Corollary 6.32.1, to establish

$$\left\| d^l(V - \hat{V}_d) \right\|_\infty \to 0 \text{ as } d \to \infty$$

it suffices to show that $\sup \left\{ \bar{K}_d \,|\, d > D \right\} < \infty$. From the analysis given in [4] it is seen that it is sufficient to show that the derivatives of Φ and $\hat{\Phi}_d$ are bounded in $\mathcal{L}^2_{p \times p}$ for all $d \geqslant D$. Hence,

$$\begin{aligned}
\left\| \frac{d\Phi(e^{i\theta})}{d\theta} \right\|_2^2 &= \left\| \frac{d}{d\theta} \sum_{k=-\infty}^{\infty} \Gamma(k) e^{ik\theta} \right\|_2^2 \\
&= \left\| i \sum_{k=-\infty}^{\infty} k\Gamma(k) e^{ik\theta} \right\|_2^2 \\
&= \operatorname{tr} \sum_{k=-\infty}^{\infty} k^2 \Gamma(k)\Gamma(k)^* < \infty
\end{aligned}$$

by assumption B.4 on Φ.

$$\begin{aligned}
\left\| \frac{d\hat{\Phi}_d(e^{i\theta})}{d\theta} \right\|_2 &= \left\| \frac{d}{d\theta} \left(\Phi(e^{i\theta}) - \hat{\Phi}_d(e^{i\theta}) \right) - \frac{d\Phi(e^{i\theta})}{d\theta} \right\|_2 \\
&\leqslant \left\| \frac{d}{d\theta} \left(\Phi(e^{i\theta}) - \hat{\Phi}_d(e^{i\theta}) \right) \right\|_2 + \left\| \frac{d\Phi(e^{i\theta})}{d\theta} \right\|_2 \\
&= \left(2 \sum_{k=d+1}^{\infty} \| k\Gamma(k) \|_2^2 + \sum_{k=-d}^{d} k^2 \left\| \Gamma(k) - \hat{\Gamma}^{(N)}(k) \right\|_2^2 \right)^{\frac{1}{2}} + \left\| \frac{d\Phi(e^{i\theta})}{d\theta} \right\|_2 \\
&\leqslant \sqrt{2} \left(\sum_{k=0}^{d} k^2 \left\| \Gamma(k) - \hat{\Gamma}^{(N)}(k) \right\|_2^2 \right)^{\frac{1}{2}} + 3 \left\| \frac{d\Phi(e^{i\theta})}{d\theta} \right\|_2
\end{aligned}$$

If $\kappa = 0$ then assumption E.2 implies the boundedness of the first term (from Proposition 6.32).

If $\kappa = 2l > 0$ assumption E.1 implies $\left\| d^{2l}(\Phi - \hat{\Phi}_d) \right\|_\infty \to 0$ and implies assumption E.2, hence bounding the first term.

In both cases assumption B.4 bounds the second term.

The result for W and $\hat{W}_d$ is obtained by consideration of Φ^T (c.f. Lemma 5.3).

Finally, the operator norm convergence result follows from the familiar equality between operator norms and the $\mathcal{L}^\infty_{p\times p}$-norms of the symbols. ■

Proposition 6.37 (Convergence of Toeplitz Operators) *Under either F.3 or F.4 there exists $D \in \mathbb{N}$ such that*

$$\lim_{\substack{d \geqslant D \\ d\to\infty}} \left\| d^l(\mathbf{T}_V^{-1} - \mathbf{T}_{\hat{V}_d}^{-1}) \right\|_{op} = 0 \quad \text{w.p. } 1\ [\mathbb{P}].$$

Proof : A similar argument to that in Proposition 6.35 is used. Given $\frac{\sqrt{\alpha}}{2} > \epsilon > 0$, take $D \in \mathbb{N}$ such that $\left\| d^l(V - \hat{V}_d) \right\|_\infty < \epsilon$ for all $d > D$ (possible by previous theorem).

Firstly, the following shows that the operators mentioned are invertible. From Corollary 6.33.1

$$\begin{aligned}
\inf_{\theta\in[0,2\pi)} \sigma_{\min}\left(\hat{V}_d(e^{i\theta})\right) &\geqslant \inf_{\theta\in[0,2\pi)} \left\{\sigma_{\min}\left(V(e^{i\theta})\right) - \sigma_{\max}\left(V(e^{i\theta}) - \hat{V}_d(e^{i\theta})\right)\right\} \\
&\geqslant \inf_{\theta\in[0,2\pi)} \sigma_{\min}\left(V(e^{i\theta})\right) - \sup_{\theta\in[0,2\pi)} \sigma_{\max}\left(V(e^{i\theta}) - \hat{V}_d(e^{i\theta})\right) \\
&\geqslant \sqrt{\alpha} - \frac{\sqrt{\alpha}}{2} \quad (d \geqslant 1) \\
&= \frac{\sqrt{\alpha}}{2} > 0
\end{aligned}$$

Since $\det\left(\hat{V}_d(z)\right) \neq 0$ in $\bar{\mathbb{D}}$ and is continuous, then it is bounded away from zero. Hence the inverse is bounded on $\bar{\mathbb{D}}$, and is analytic in $\mathbb{D}$ since $\hat{V}_d(z)$ is analytic in $\mathbb{D}$. Hence the Toeplitz operator with this symbol is boundedly invertible from Lemma 2.45 using the simple factorisation $\hat{V}_d = \hat{V}_d I_p$.

$$\begin{aligned}
d^l \left\| \mathbf{T}_V^{-1} - \mathbf{T}_{\hat{V}_d}^{-1} \right\|_{op} &= d^l \left\| \mathbf{T}_{V^{-1}} - \mathbf{T}_{\hat{V}_d^{-1}} \right\|_{op} \\
&= d^l \left\| \mathbf{T}_{\left(V^{-1} - \hat{V}_d^{-1}\right)} \right\|_{op} \\
&= d^l \left\| V^{-1} - \hat{V}_d^{-1} \right\|_\infty \\
&\leqslant \left\| V^{-1} \right\|_\infty \left\| d^l(V - \hat{V}_d) \right\|_\infty \left\| \hat{V}_d^{-1} \right\|_\infty \\
&\leqslant \frac{1}{\sqrt{\alpha}} \epsilon \left\| \hat{V}_d^{-1} \right\|_\infty \\
&= \frac{1}{\sqrt{\alpha}} \epsilon \left[\inf_{\theta\in[0,2\pi)} \sigma_{\min}\left(\hat{V}_d(e^{i\theta})\right) \right]^{-1}
\end{aligned}$$

Hence, by the above inequalities

$$d^l \left\| \mathbf{T}_V^{-1} - \mathbf{T}_{\hat{V}_d}^{-1} \right\|_{op} \leqslant \frac{2\epsilon}{\alpha}$$

And so probability 1 convergence of $\hat{V}_d$ implies probability 1 convergence of the Toeplitz operators. ■

Note 6.38 *This result rather misrepresents the results in Anderson [4] which imply the convergence of the inverses of the spectral factors first, then the spectral factors themselves.*

■

Theorem 6.39 (Convergence of Canonical Correlation Operator Estimates) *Under assumptions F.3 or F.4 there exists $D \in \mathbb{N}$ such that*

$$\lim_{\substack{d \geqslant D \\ d \to \infty}} \left\| d^l (R\mathbf{H}_{V^{-1}W^*} - \mathbf{T}_{\bar{\hat{V}}_d}^{-1} R\mathbf{H}_{\hat{\Phi}_d} \mathbf{T}_{\hat{W}_d}^{-1}) \right\|_{op} = 0 \ \text{ w.p. } 1 \ [\mathbb{P}].$$

Proof : Given $0 < \epsilon \leqslant \min(\frac{\alpha}{2}, \frac{\sqrt{\alpha}}{2})$, then from Corollary 6.32.1 and Proposition 6.36 there exists $D \in \mathbb{N}$ such that for all $d \geqslant D$

$$\max\left\{ d^l \left\| \Phi(e^{i\theta}) - \hat{\Phi}_d(e^{i\theta}) \right\|_\infty, d^l \left\| V - \hat{V}_d \right\|_\infty, d^l \left\| W - \hat{W}_d \right\|_\infty \right\} < \epsilon.$$

This guarantees that all the quantities mentioned below exist.

Then,

$$\begin{aligned}
\left\| d^l (R\mathbf{H}_{V^{-1}W^*} - \mathbf{T}_{\bar{\hat{V}}_d}^{-1} R\mathbf{H}_{\hat{\Phi}_d} \mathbf{T}_{\hat{W}_d}^{-1}) \right\|_{op} &= \left\| d^l (\mathbf{T}_{\bar{V}}^{-1} R\mathbf{H}_\Phi \mathbf{T}_W^{-1} - \mathbf{T}_{\bar{\hat{V}}_d}^{-1} R\mathbf{H}_{\hat{\Phi}_d} \mathbf{T}_{\hat{W}_d}^{-1}) \right\|_{op} \quad \text{from Lemma 2.46} \\
&= \left\| d^l \left(\left(\mathbf{T}_{\bar{V}}^{-1} - \mathbf{T}_{\bar{\hat{V}}_d}^{-1} \right) R\mathbf{H}_\Phi \mathbf{T}_W^{-1} + \mathbf{T}_{\bar{\hat{V}}_d}^{-1} \left(R\mathbf{H}_\Phi - R\mathbf{H}_{\hat{\Phi}_d} \right) \mathbf{T}_W^{-1} + \mathbf{T}_{\bar{\hat{V}}_d}^{-1} R\mathbf{H}_{\hat{\Phi}_d} \left(\mathbf{T}_W^{-1} - \mathbf{T}_{\hat{W}_d}^{-1} \right) \right) \right\|_{op} \\
&\leqslant \left\| d^l (\mathbf{T}_{\bar{V}}^{-1} - \mathbf{T}_{\bar{\hat{V}}_d}^{-1}) \right\|_{op} \left\| \mathbf{H}_\Phi \right\|_{op} \left\| \mathbf{T}_W^{-1} \right\|_{op} + \left\| \mathbf{T}_{\bar{\hat{V}}_d}^{-1} \right\|_{op} \left\| d^l (\mathbf{H}_\Phi - \mathbf{H}_{\hat{\Phi}_d}) \right\|_{op} \left\| \mathbf{T}_W^{-1} \right\|_{op} \\
&\quad + \left\| \mathbf{T}_{\bar{\hat{V}}_d}^{-1} \right\|_{op} \left\| \mathbf{H}_{\hat{\Phi}_d} \right\|_{op} \left\| d^l (\mathbf{T}_W^{-1} - \mathbf{T}_{\hat{W}_d}^{-1}) \right\|_{op} \qquad (6.9)
\end{aligned}$$

To bound the norm of the Hankel operators note that the Hankel norm is bounded above by the $\mathcal{L}^\infty$-norm. A bound for the difference of the inverses of the Toeplitz operators was derived in the proof of Proposition 6.37. Other bounds are obtained from the boundedness of $\Phi(e^{i\theta})$, and noting that

$$\begin{aligned}
\left\| \mathbf{T}_{\hat{V}_d}^{-1} \right\|_{op} &\leqslant \left\| \mathbf{T}_V^{-1} - \mathbf{T}_{\hat{V}_d}^{-1} \right\|_{op} + \left\| \mathbf{T}_V^{-1} \right\|_{op} \\
&\leqslant \frac{2\epsilon}{\alpha} + \left\| V^{-1} \right\|_\infty \\
&\leqslant \frac{2\epsilon}{\alpha} + \frac{1}{\sqrt{\alpha}}
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\| d^l (R\mathbf{H}_{V^{-1}W^*} - \mathbf{T}_{\bar{\hat{V}}_d}^{-1} R\mathbf{H}_{\hat{\Phi}_d} \mathbf{T}_{\hat{W}_d}^{-1}) \right\|_{op} &\leqslant \frac{2\epsilon}{\alpha}.\beta.\frac{1}{\sqrt{\alpha}} + \left(\frac{2\epsilon}{\alpha} + \frac{1}{\sqrt{\alpha}} \right).\epsilon.\frac{1}{\sqrt{\alpha}} + \left(\frac{2\epsilon}{\alpha} + \frac{1}{\sqrt{\alpha}} \right).(\beta + \epsilon).\frac{2\epsilon}{\alpha} \\
&= \frac{\epsilon}{\alpha\sqrt{\alpha}} \left(2\beta + 2\epsilon + \sqrt{\alpha} + \frac{2}{\sqrt{\alpha}} (2\epsilon + \sqrt{\alpha})(\beta + \epsilon) \right) \\
&\leqslant \epsilon \frac{10\beta + 2\sqrt{\alpha}}{\alpha\sqrt{\alpha}}
\end{aligned}$$

Hence the operators converge in the norm. ■

Note that throughout this section the Hankel operators are being considered as vector Hankel operators, not matrix Hankel operators. Note 2.40 shows that these two interpretations possess the same spectrum and set of eigenvectors save for multiplicity. Hence the approach of §6.2.2 gives valid estimates for the spectral points of the Matrix Hankel operator with the same symbol. The only difference is the multiplicity of the eigenvalues.

Theorem 6.40 (Consistency of σ_i and s_i Estimates) *Let $(y(t))_{t\in\mathbb{Z}}$ be a stochastic process satisfying conditions F.3 with spectral-density $\Phi(e^{i\theta})$ and normalised right spectral factor*

$$V(z) = C\left(z^{-1}I - A_v\right)^{-1} K + V$$

where the equivalent continuous-time transfer function

$$V_c(s) = \tilde{C}\left(sI - \tilde{A}_v\right)^{-1} \tilde{K} + \tilde{V}$$

is an element of $\mathrm{M}_{\mathrm{n}}^{\mathrm{p}}[\omega]$ and $\left(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V}\right)$ is an element of the minimum-phase canonical form of Corollary 4.62.1 given by the parameters

$$\left\{s_i, \sigma_i, \check{B}_i, \check{D} \,|\, i = 1, \ldots, n\right\}.$$

If $\mathcal{C}_d$ and

$$\left\{\hat{s}_i^{(d)}, \hat{\sigma}_i^{(d)}, \hat{B}_i^{(d)} \,|\, i = 1, \ldots, n,\ d \in \mathbb{N}\right\}$$

are constructed by the algorithm described in §6.2.2 then $\hat{s}_i^{(d)}$ and $\hat{\sigma}_i^{(d)}$ are strongly consistent estimators of s_i and σ_i, $i = 1, \ldots, n$.

Proof : From Lemma 2.13 it is clear that the operator $R\mathbf{H}_{V^{-1}W^*}$ is compact. The results of §6.1 and §6.3 show that

$$\|R\mathbf{H}_{V^{-1}W^*} - \mathcal{C}_d\|_{op} \to 0 \text{ as } k \to \infty$$

(see Theorem 6.39).

Order the eigenvalues with respect to decreasing modulus

$$\lambda(R\mathbf{H}_{V^{-1}W^*}) = \{\lambda_i \,|\, i = 1, \ldots, n\} \cup \{0\}.$$

Then let

$$0 < \epsilon < \frac{1}{3}\min\left(\inf\{|\lambda_i - \lambda_j| \,|\, i, j = 1, \ldots, n,\ i \neq j\}, \{|\lambda_n|\}\right)$$

and

$$\Delta_i^{(\epsilon)} := \{|z - \lambda_i| < \epsilon\}.$$

From Theorem 6.18 1), the spectra of $R\mathbf{H}_{V^{-1}W^*}$ and $\mathcal{C}_d$ converge in the Hausdorff metric. Hence, $\exists N \in \mathbb{N}$ such that

$$\lambda(\mathcal{C}_d) \subseteq \bigcup_{i=1}^{n} \Delta_i^{(\epsilon)} \cup \{|z| < \epsilon\},\ \forall d > N$$

since

$$\sup_{f\in\lambda(\mathcal{C}_d)} \inf_{g\in\lambda(R\mathbf{H}_{V^{-1}W^*})} |f - g| < \epsilon$$

If furthermore $N > N_2$ as given in Theorem 6.18 2) then the multiplicity of such eigenvalues must be one.

Hence

$$\lambda_i(\mathcal{C}_d) \to \lambda_i(R\mathbf{H}_{V^{-1}W^*}),\ i = 1, \ldots, n \text{ as } d \to \infty$$

without ambiguity in the ordering. Note, that from Newburgh's result Theorem 6.4 and the probability 1 norm-convergence of $\mathcal{C}_d$, all of the above eigenvalues converge w.p. 1 [$\mathbb{P}$].

Also, again from the relationship between P and Σ (see Theorem 4.51)

$$\hat{\sigma}_i^{(d)} = \frac{|\lambda_i(\mathcal{C}_d)|}{\sqrt{1 - \lambda_i(\mathcal{C}_d)^2}}$$

and so if

$$\epsilon < 1 - \max\left\{|\lambda_i(R\mathbf{H}_{V^{-1}W^*})| \mid i = 1, \ldots, n\right\}$$

the continuity of the above function gives $\hat{\sigma}_i^{(d)}$ consistent for σ_i.

Finally, since

$$\epsilon < |\lambda_n|$$

then

$$\text{signum}\,(\lambda_i(\mathcal{C}_d)) = \text{signum}\,(\lambda_i(R\mathbf{H}_{V^{-1}W^*}))$$

and hence $\hat{s}_i^{(d)}$ is a consistent estimator of s_i. ■

It is also of interest to consider the behaviour of the remainder of the spectrum of $\mathcal{C}_d$ as $d \to \infty$ as the previous result only deals with the first n eigenvalues of $\mathcal{C}_d$. Of course, this operator will in general have very large rank. From the construction in the proof, all of the remaining spectrum must lie in the disc

$$\{|z| < \epsilon\}.$$

Note 6.41 *The consistency of the estimates of σ_i and s_i, $i = 1, \ldots, n$ does not strictly depend on knowledge of n. If the "true model" has greater order then the above estimates are still consistent although they ignore further useful information contained in the spectrum. If the order of the "true model" is $n_2 < n$ then the last $n - n_2$ eigenvalues of the approximants will converge to 0.*

■

It remains to show the method for estimating the $\breve{B}_i$ parameters of the realisation of the right spectral factor of the spectral-density. The method hinges on the observation of the realisation of the Schmidt vectors of $\mathbf{H}_{V^{-1}W^*}$. Under the same assumptions as Theorem 6.40 it is clear from Theorems 5.19 and 5.20 that the normalised eigenvectors of $R\mathbf{H}_{V^{-1}W^*}$ are

$$u_i(z) = \sqrt{\frac{2}{\sigma_i}} s_i \breve{K}^T \left((1-z)I_n - (1+z)\breve{A}_v^T\right)^{-1} e_i, \; i = 1, \ldots, n. \tag{6.10}$$

Hence, if

$$\breve{K} = \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix}$$

then

$$u_i(-1) = \frac{1}{\sqrt{2\sigma_i}} s_i \begin{bmatrix} b_{i1} \\ \vdots \\ b_{ip} \end{bmatrix}. \tag{6.11}$$

σ_i and s_i have consistent estimators described in the previous Theorem. Hence this quantity should provide a means of estimating the remaining parameters, as demonstrated in the next result.

Theorem 6.42 (Consistency of the B-estimates) *Under the assumptions of Theorem 6.40 and further assuming E.1 with $\kappa \geqslant 1$ then*

$$\hat{B}_i^{(d)} = \sqrt{2\hat{\sigma}_i{}^{(d)}}\,\hat{s}_i^{(d)}\hat{u}_i^{(d)}(-1)$$

where $\hat{u}_i^{(d)}$ is defined by (6.6), is a strongly consistent estimator of $\breve{B}_i$.

Proof : From Theorems 6.39 (with $l = 0$) and 6.20 it is clear that

$$\lim_{d\to\infty} \left\| \hat{u}_i^{(d)}(z) - u_i(z) \right\|_2 = 0.$$

Also, since $\hat{u}_i^{(d)}$ is an eigenvector of $R\mathbf{H}_{\hat{V}_d^{-1}\hat{W}_d^*}$ then Lemma 2.14 shows that $\hat{u}_i^{(d)}$ is a polynomial of degree at most d. Thus $\hat{u}_i^{(d)} - u_i(z)$ is the difference of a polynomial and a continuous rational function and so is continuously differentiable. Furthermore, it is the case (e.g. following the proof of Proposition 6.36) that

$$\left\| \frac{d\hat{u}_i^{(d)}(e^{i\theta})}{d\theta} \right\|_2 \leqslant d \left\| \hat{u}_i^{(d)}(e^{i\theta}) \right\|_2 .$$

Applying Theorem 6.19 with $l \geqslant 1$ (see assumption E.1) implies that

$$\lim_{d\to\infty} \left\| \frac{d\hat{u}_i^{(d)}}{d\theta}(e^{i\theta}) - \frac{du_i}{d\theta}(e^{i\theta}) \right\|_2 = 0$$

and hence Lemma 6.21 demonstrates that

$$\hat{u}_i^{(d)}(-1) \to u_i(-1) \text{ as } d \to \infty \text{ w.p. 1 } [\mathbb{P}].$$

The function

$$\begin{aligned} f : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{p\times 1} &\to \mathbb{R}^{p\times 1} \\ f : (a, b, c) &\mapsto \sqrt{2a}bc \end{aligned}$$

is continuous. Also

$$f\left(\sigma_i, s_i, u_i(-1)\right) = \begin{bmatrix} b_{i1} \\ \vdots \\ b_{ip} \end{bmatrix}$$

and hence by Lemma 6.23

$$f\left(\hat{\sigma}_i^{(d)}, \hat{s}_i^{(d)}, \hat{u}_i^{(d)}(-1)\right) = \hat{B}_i^{(d)} \to \begin{bmatrix} b_{i1} \\ \vdots \\ b_{ip} \end{bmatrix} \text{ w.p. 1 } [\mathbb{P}] \text{ as } d \to \infty.$$

(Note that the independence of the arguments is not assumed in Lemma 6.23.) ■

6.3 Finite Section Approximations: Stage 2

At steps 3) and 4) of §6.2.2 it was assumed that it is possible to use operators acting upon $\mathcal{H}^2_{p\times p}$ directly. In this section a more realistic alternative is suggested: that only the restriction to a finite-dimensional subspace may be directly manipulated. Consideration of the algorithm suggested above suggests that the natural finite-dimensional subspaces would be spaces of polynomials of bounded degree.

6.3.1 Details of Stage 2

The following steps are taken to estimate the appropriate parameters:

1) Follow steps 1) and 2) from §6.2.2.

2) From the coefficients of $\hat{\Phi}_d(e^{i\theta})$, $\hat{V}_d(e^{i\theta})$ and $\hat{W}_d(e^{i\theta})$ the following matrices are formed:

$$X'_d = \begin{bmatrix} \hat{\Gamma}^{(N)}(1) & \hat{\Gamma}^{(N)}(2) & \hat{\Gamma}^{(N)}(3) & \hat{\Gamma}^{(N)}(4) & \cdots \\ \hat{\Gamma}^{(N)}(2) & \hat{\Gamma}^{(N)}(3) & \hat{\Gamma}^{(N)}(4) & \ddots & \ddots \\ \hat{\Gamma}^{(N)}(3) & \hat{\Gamma}^{(N)}(4) & \ddots & \hat{\Gamma}^{(N)}(d) & \ddots \\ \hat{\Gamma}^{(N)}(4) & \ddots & \hat{\Gamma}^{(N)}(d) & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \in \mathbb{C}^{p(d+1)\times p(d+1)}$$

$$Y'_d = \begin{bmatrix} \hat{A}_{d,0} & \hat{A}_{d,1} & \hat{A}_{d,2} & \hat{A}_{d,3} & \cdots \\ 0 & \hat{A}_{d,0} & \hat{A}_{d,1} & \hat{A}_{d,2} & \ddots \\ 0 & 0 & \hat{A}_{d,0} & \hat{A}_{d,1} & \ddots \\ 0 & 0 & 0 & \hat{A}_{d,0} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \end{bmatrix} \in \mathbb{C}^{p(d+1)\times p(d+1)}$$

$$Z'_d = \begin{bmatrix} \hat{B}_{d,0} & 0 & 0 & 0 & \cdots \\ \hat{B}_{d,1} & \hat{B}_{d,0} & 0 & 0 & \ddots \\ \hat{B}_{d,2} & \hat{B}_{d,1} & \hat{B}_{d,0} & 0 & \ddots \\ \hat{B}_{d,3} & \hat{B}_{d,2} & \hat{B}_{d,1} & \hat{B}_{d,0} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \end{bmatrix} \in \mathbb{C}^{p(d+1)\times p(d+1)}$$

3) The approximate correlation operator

$$\mathcal{C}'_d = {Y_d}'^{-1} X'_d {Z_d}'^{-1}$$

is formed. The first n eigenvalues, ranked by magnitude, are

$$\left\{\lambda_i^{(d)'} \mid i = 1, \ldots, n\right\} \subseteq \lambda\left(\mathcal{C}'_d\right)$$

and the normalised eigenvectors are $\left\{u_i^{(d)'} \mid i = 1, \ldots, n\right\}$ where

$$\begin{aligned} u_i^{(d)'} &= \begin{bmatrix} u_{i,0}^{(d)'} \\ \vdots \\ u_{i,d}^{(d)'} \end{bmatrix} \in \mathbb{R}^{(p\times(d+1))\times 1} \\ \mathcal{C}'_d u_i^{(d)'} &= \lambda_i^{(d)'} u_i^{(d)'}. \end{aligned}$$

4) The estimates

$$\begin{aligned} \hat{s}_i^{(d)'} &= \mathrm{signum}(\lambda_i^{(d)'}) \\ \hat{\sigma}_i^{(d)'} &= \frac{|\lambda_i^{(d)'}|}{\sqrt{1 - \lambda_i^{(d)'2}}} \end{aligned}$$

$i = 1, \ldots n$ and the $\hat{B}_i^{(d)}$ parameters are calculated from

$$\hat{B}_i^{(d)'} = \sqrt{2\hat{\sigma}_i^{(d)'}}\,\hat{s}_i^{(d)'} \left(\sum_{j=0}^{d} (-1)^j\, u_{i,j}^{(d)'}\right)$$

6.3.2 Analysis of Stage 2

It is now shown that using X'_d, Y'_d and Z'_d (finite truncations of the operators used in §6.2.2) produces the same results as using the full-blown operators as described in the previous section. In fact the finite-dimensional approximation is exact. Such methods are very commonly used, and there are many deep results known regarding their use. An excellent summary of the state of the art of finite section approximations is to be found in Böttcher & Silbermann [13].

Let the space of $p \times p$-matrix polynomials of degree k be denoted

$$\mathcal{P}^k_{p\times p} := \left\{ f : \mathbb{C} \to \mathbb{C}^{p\times p} \mid f(z) = \sum_{i=0}^{k} A_i z^i,\ A_i \in \mathbb{R}^{p\times p} \right\} \tag{6.12}$$

and the orthogonal projection from $\mathcal{H}^2_{p\times p}(\mathbb{D})$ onto $\mathcal{P}^k_{p\times p}$ will be denoted $\mathbf{P}_k$. Also, let the operator $A\,|_{H_1}$ be the restriction of A to some subspace H_1 of the domain of A.

Thus define finite-rank approximations to the following operators:

$$\begin{aligned} T_{d,m} &:= \mathbf{P}_m \mathbf{T}^{-1}_{\bar{\tilde{V}}_d} \mathbf{P}_m \\ \tilde{T}_{d,m} &:= \mathbf{P}_m \mathbf{T}^{-1}_{\hat{W}_d} \mathbf{P}_m \\ H_{d,m} &:= \mathbf{P}_m R \mathbf{H}_{\hat{\Phi}_d} \mathbf{P}_m \end{aligned}$$

Theorem 6.43 (Equivalence of Finite Dimensional Approximation) *Under assumptions F.3*

$$\mathbf{T}^{-1}_{\bar{\tilde{V}}_d} R \mathbf{H}_{\hat{\Phi}_d} \mathbf{T}^{-1}_{\hat{W}_d} = T_{d,m} H_{d,m} \tilde{T}_{d,m}$$

if $m \geqslant d$.

Proof : Consider the following decomposition of the difference of these operators

$$\begin{aligned} &\mathbf{T}^{-1}_{\bar{\tilde{V}}_d} R \mathbf{H}_{\hat{\Phi}_d} \mathbf{T}^{-1}_{\hat{W}_d} - T_{d,m} H_{d,m} \tilde{T}_{d,m} = \\ &\quad \left(\mathbf{T}^{-1}_{\bar{\tilde{V}}_d} - T_{d,m} \right) R \mathbf{H}_{\hat{\Phi}_d} \mathbf{T}^{-1}_{\hat{W}_d} + T_{d,m} \left(R \mathbf{H}_{\hat{\Phi}_d} - H_{d,m} \right) \mathbf{T}^{-1}_{\hat{W}_d} + T_{d,m} H_{d,m} \left(\mathbf{T}^{-1}_{\hat{W}_d} - \hat{T}_{d,m} \right) \end{aligned}$$

Bearing in mind the form of these operators it is easily seen that, given $m \geqslant d$,

$$\operatorname{im}(R\mathbf{H}_{\hat{\Phi}_d}) \leqslant \mathcal{P}^d_{p\times p} \leqslant \mathcal{P}^m_{p\times p} \leqslant \ker(\mathbf{T}_{\tilde{V}^{-1}_{d,k}} - \mathbf{P}_m \mathbf{T}_{\tilde{V}^{-1}_{d,k}} \mathbf{P}_m).$$

Also note that $R\mathbf{H}_{\hat{\Phi}_d} = \mathbf{P}_m R \mathbf{H}_{\hat{\Phi}_d} \mathbf{P}_m$ if $m \geqslant d$, and hence the second term is null.

Finally,

$$\operatorname{im}\left(\mathbf{T}_{\hat{W}^{-1}_d} - \mathbf{P}_m \mathbf{T}_{\hat{W}^{-1}_d} \mathbf{P}_m \right) \leqslant \left(\mathcal{P}^m_{p\times p} \right)^{\perp} = \ker \mathbf{P}_m \leqslant \ker H_{d,m},$$

and hence the third term is also zero. ■

Lemma 6.44 (Block Inversion Formula) *Let H_1 and H_2 be vector spaces (possibly infinite-dimensional) and $X \in \operatorname{End}(H_1 \oplus H_2)$ with block representation*

$$X \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} Ah_1 + Bh_2 \\ Dh_2 \end{bmatrix}$$

where $A \in \operatorname{End}(H_1)$, $B : H_2 \to H_1$, $D \in \operatorname{End}(H_2)$.

If X is invertible then it has inverse

$$X^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{bmatrix}$$

■

Corollary 6.44.1 (Inverse of Truncated Toeplitz Operator) *If V is a unit in $\mathcal{H}^{\infty}_{p\times p}$ then*

$$\mathbf{P}_k\mathbf{T}_V^{-1}\mathbf{P}_k = \left(\mathbf{P}_k\mathbf{T}_V\mathbf{P}_k\mid_k\right)^{-1} \oplus 0_{\left(\mathcal{P}^m_{p\times p}\right)^\perp}$$

Proof : Consider $H_1 = \mathcal{P}^m_{p\times p}$ and $H_2 = \left(\mathcal{P}^m_{p\times p}\right)^\perp$ and $X = \mathbf{T}_V$ then

$$X = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

where

$$\begin{aligned} A &= \mathbf{P}_k\mathbf{T}_V\mid_k \\ &= \mathbf{P}_k\mathbf{T}_V\mathbf{P}_k\mid_k \\ B &= \mathbf{P}_k\mathbf{T}_V\mid_{(\mathcal{P}^k_{p\times p})^\perp} \\ C &= \mathbf{P}_{(\mathcal{P}^k_{p\times p})^\perp}\mathbf{T}_V\mid_{(\mathcal{P}^k_{p\times p})^\perp} \end{aligned}$$

hence from Lemma 6.44

$$\mathbf{P}_kX^{-1}\mathbf{P}_k = \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

■

Remark 6.45 (Matrix Inversion) *The significance of this result is that the operators $T_{d,m}$ and $\tilde{T}_{d,m}$ may be calculated by inversion of a $p(d+1)\times p(d+1)$-matrix, as opposed to calculating the inverse of a matrix polynomial. An algorithm to perform the latter is presented in Kucera [77] Chapter 7 §14.*

■

The following finite operator is now seen to be the approximant that is of interest

$$\mathcal{C}'_d \;:=\; \left(\mathbf{P}_d\mathbf{T}_{\bar{\hat{V}}_d}\mathbf{P}_d\mid_d\right)^{-1}\left(\mathbf{P}_dR\mathbf{H}_{\hat{\Phi}_d}\mathbf{P}_d\mid_d\right)\left(\mathbf{P}_d\mathbf{T}_{\hat{W}_d}\mathbf{P}_d\mid_d\right)^{-1}.$$

This has the finite matrix representation (increasing in dimension as $d\to\infty$) given in §6.3.1 and may be calculated using only linear algebraic operations.

Theorem 6.46 (Consistency of Finite Section Scheme) *Under the conditions described in Theorem 6.42 the estimates $\hat{s}_i^{(d)'}$, $\hat{\sigma}_i^{(d)'}$ and $\hat{B}_i^{(d)'}$, $i = 1,\ldots,n$ are consistent for s_i, σ_i and $\check{B}_i$ respectively as $d\to\infty$.*

Proof : From Theorem 6.43 it is clear that the operator

$$\begin{aligned} \mathbf{T}^{-1}_{\bar{\hat{V}}_d}R\mathbf{H}_{\hat{\Phi}_d}\mathbf{T}^{-1}_{\hat{W}_d} &= T_{d,m}H_{d,m}\tilde{T}_{d,m} \\ &= \left(\left(\mathbf{P}_k\mathbf{T}_{\bar{\hat{V}}_d}\mathbf{P}_k\mid_k\right)^{-1}\oplus 0_X\right)\left(\left(\mathbf{P}_kR\mathbf{H}_{\hat{\Phi}_d}\mathbf{P}_k\mid_k\right)\oplus 0_X\right)\left(\left(\mathbf{P}_k\mathbf{T}_{\hat{W}_d}\mathbf{P}_k\mid_k\right)^{-1}\oplus 0_X\right) \\ &\qquad \text{from Corollary 6.44.1 where } X = \left(\mathcal{P}^m_{p\times p}\right)^\perp \\ &= \mathcal{C}'_d\oplus 0_{\left(\mathcal{P}^m_{p\times p}\right)^\perp}. \end{aligned}$$

Hence, the spectral properties of $\mathcal{C}_d$ and $\mathcal{C}'_d$ are identical.

Finally, note that

$$u_i^{(d)}(z) = \sum_{j=0}^{d} u_{i,j}^{(d)'}z^j \text{ (sic)}$$

showing that $\hat{B}_i^{(d)'} = \hat{B}_i^{(d)}$. ■

6.4 Algorithm Implementation: Stage 3

The remaining unrealisable objects manipulated in the previous section are the spectral factors $\hat{V}_d$ and $\hat{W}_d$ which are assumed to be obtained directly from $\hat{\Phi}_d$. Unfortunately this is not possible for any but the simplest of spectral densities.

Since no simple computation provides the polynomial spectral factors required, it is necessary to resort to an iterative scheme of some sort to provide a sequence of approximate spectral factors

$$\tilde{V}_{d,k} \to \hat{V}_d \text{ and } \tilde{W}_{d,k} \to \hat{W}_d \text{ as } k \to \infty$$

where $\tilde{V}_{d,k}$ and $\tilde{W}_{d,k}$ are polynomials of degree at most d.

A variety of algorithms for such calculations exist, ranging from Whittle [124] and Youla & Kazanjian [134] to Johnson [68]. The most relevant of these use only the Fourier coefficients of $\hat{\Phi}_d$ and may be categorized as of Bauer type e.g. see [134] and the references therein, or of Newton-Raphson type e.g. see Wilson [129] for the univariate case and Wilson [130] for the MIMO case. Wilson's results will be discussed in §6.4.2. Johnson [68] gives a very interesting set of algorithms with varying rates of convergence. This final paper also gives an overview of the available methods.

6.4.1 Details of Final Algorithm

The following steps are taken to estimate the appropriate parameters:

1) Follow steps 1) and 2) from §6.2.2. Note that $\hat{V}_d$ and $\hat{W}_d$ are not directly available.

2) Approximate spectral factors are constructed via an appropriate algorithm e.g. that described below in §6.4.2. Such an iterative procedure produces estimates

$$\begin{aligned}\tilde{V}_{d,k}(z) &= \sum_{l=0}^{d} \tilde{A}_{d,k,l} z^l \\ \tilde{W}_{d,k}(z) &= \sum_{l=0}^{d} \tilde{B}_{d,k,l} z^l\end{aligned}$$

such that

$$\tilde{V}_{d,k} \to \hat{V}_d \text{ and } \tilde{W}_{d,k} \to \hat{W}_d \text{ as } k \to \infty \text{ in } \mathcal{H}^\infty.$$

3) From the coefficients of $\hat{\Phi}_d(e^{i\theta})$, $\tilde{V}_{d,k}$ and $\tilde{W}_{d,k}$ the following matrices are formed:

$$\tilde{X}_{d,k} = \begin{bmatrix} \hat{\Gamma}^{(N)}(1) & \hat{\Gamma}^{(N)}(2) & \hat{\Gamma}^{(N)}(3) & \hat{\Gamma}^{(N)}(4) & \cdots \\ \hat{\Gamma}^{(N)}(2) & \hat{\Gamma}^{(N)}(3) & \hat{\Gamma}^{(N)}(4) & \ddots & \ddots \\ \hat{\Gamma}^{(N)}(3) & \hat{\Gamma}^{(N)}(4) & \ddots & \hat{\Gamma}^{(N)}(d) & \ddots \\ \hat{\Gamma}^{(N)}(4) & \ddots & \hat{\Gamma}^{(N)}(d) & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \in \mathbb{C}^{p(d+1)\times p(d+1)}$$

$$\tilde{Y}_{d,k} = \begin{bmatrix} \tilde{A}_{d,k,0} & \tilde{A}_{d,k,1} & \tilde{A}_{d,k,2} & \tilde{A}_{d,k,3} & \cdots \\ 0 & \tilde{A}_{d,k,0} & \tilde{A}_{d,k,1} & \tilde{A}_{d,k,2} & \ddots \\ 0 & 0 & \tilde{A}_{d,k,0} & \tilde{A}_{d,k,1} & \ddots \\ 0 & 0 & 0 & \tilde{A}_{d,k,0} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \end{bmatrix} \in \mathbb{C}^{p(d+1)\times p(d+1)}$$

$$\tilde{Z}_{d,k} = \begin{bmatrix} \tilde{B}_{d,k,0} & 0 & 0 & 0 & \cdots \\ \tilde{B}_{d,k,1} & \tilde{B}_{d,k,0} & 0 & 0 & \ddots \\ \tilde{B}_{d,k,2} & \tilde{B}_{d,k,1} & \tilde{B}_{d,k,0} & 0 & \ddots \\ \tilde{B}_{d,k,3} & \tilde{B}_{d,k,2} & \tilde{B}_{d,k,1} & \tilde{B}_{d,k,0} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \end{bmatrix} \in \mathbb{C}^{p(d+1)\times p(d+1)}$$

4) The approximate correlation operator

$$\tilde{\mathcal{C}}_{d,k} = \tilde{Y}_{d,k}^{-1} \tilde{X}_{d,k} \tilde{Z}_{d,k}^{-1}$$

is formed. The first n eigenvalues, ranked by magnitude, are

$$\left\{ \tilde{\lambda}_i^{(d,k)} \,|\, i = 1, \ldots, n \right\} \subseteq \lambda\left(\tilde{\mathcal{C}}_{d,k} \right)$$

and the normalised eigenvectors are $\left\{ \tilde{u}_i^{(d,k)} \,|\, i = 1, \ldots, n \right\}$ where

$$\begin{aligned} \tilde{u}_i^{(d,k)} &= \begin{bmatrix} \tilde{u}_{i,0}^{(d,k)} \\ \vdots \\ \tilde{u}_{i,d}^{(d,k)} \end{bmatrix} \in \mathbb{R}^{(p\times(d+1))\times 1} \\ \tilde{\mathcal{C}}_{d,k} \tilde{u}_i^{(d,k)} &= \tilde{\lambda}_i^{(d,k)} \tilde{u}_i^{(d,k)}. \end{aligned}$$

5) The estimates

$$\begin{aligned} \tilde{s}_i^{(d,k)} &= \mathrm{signum}(\tilde{\lambda}_i^{(d,k)}) \\ \tilde{\sigma}_i^{(d,k)} &= \frac{|\tilde{\lambda}_i^{(d,k)}|}{\sqrt{1 - \tilde{\lambda}_i^{(d,k)2}}} \end{aligned}$$

$i = 1, \ldots n$ and the $\tilde{B}_i^{(d,k)}$ parameters are calculated from

$$\tilde{B}_i^{(d,k)} = \sqrt{2\tilde{\sigma}_i^{(d,k)}}\,\tilde{s}_i^{(d,k)} \left(\sum_{j=0}^{d} (-1)^j \, \tilde{u}_{i,j}^{(d,k)} \right)$$

6.4.2 Wilson's Algorithm

The algorithm described by Wilson in [129, 130, 131] has been one of the most thoroughly investigated spectral factorisation algorithms e.g. see [65, 120, 78] and [68]. Its error analysis appears more complete than other methods and hence provides a starting point to examine the difficulties involved in utilising an approximate spectral factorisation scheme. Only the univariate scheme will be described, since all of the relevant problems already occur for this case.

The algorithm is iterative, at each step providing estimates of the $(d+1)$ coefficients of the polynomial spectral factors described in (6.4a) and (6.4b) of the spectral-density given in (6.2).

The approximate spectral factors of $\hat{\Phi}_d$ to be formed below are of the form

$$\tilde{V}_{d,k}(z) = \sum_{l=0}^{d} \tilde{A}_{d,k,l} z^l$$

and hence at each step (k increasing) an approximate spectral-density may be formed

$$\tilde{\Phi}_{d,k}(e^{i\theta}) := \sum_{l=-d}^{d} \tilde{\Gamma}_{d,k}(l) e^{il\theta} := \tilde{V}_{d,k}(e^{i\theta}) \tilde{V}_{d,k}^T(e^{-i\theta}).$$

The iteration proceeds from an initial estimation of the parameters of the simple form

$$\tilde{P}_{d,0} := \begin{bmatrix} \tilde{A}_{d,0,0} \\ \vdots \\ \tilde{A}_{d,0,d} \end{bmatrix} := \begin{bmatrix} \sqrt{\hat{\Phi}_d(1)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Given the kth estimate of the parameters $\tilde{P}_{d,k}$, the next estimate is formed from the equation

$$\tilde{P}_{d,k+1} := T_d\left(\tilde{P}_{d,k}\right)^{-1} \left(\hat{\Gamma}_d + \tilde{\Gamma}_{d,k}\right) \tag{6.13}$$

where

$$T_d\left(\begin{bmatrix} X_0 \\ \vdots \\ X_d \end{bmatrix}\right) := \begin{bmatrix} X_0 & X_1 & X_2 & \cdots & X_d \\ X_1 & X_2 & \cdots & X_d & 0 \\ X_2 & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ X_d & 0 & \cdots & \cdots & 0 \end{bmatrix} + \begin{bmatrix} X_0 & X_1 & X_2 & \cdots & X_d \\ 0 & X_0 & X_1 & \cdots & \vdots \\ 0 & 0 & X_0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & X_1 \\ 0 & 0 & 0 & \cdots & X_0 \end{bmatrix} \tag{6.14}$$

and

$$\hat{\Gamma}_d := \begin{bmatrix} \hat{\Gamma}_d^{(N)}(0) \\ \vdots \\ \hat{\Gamma}_d^{(N)}(d) \end{bmatrix}, \tilde{\Gamma}_{d,k} := \begin{bmatrix} \tilde{\Gamma}_{d,k}(0) \\ \vdots \\ \tilde{\Gamma}_{d,k}(d) \end{bmatrix} \in \mathbb{R}^{d+1\times d+1}.$$

By an ingenious argument Wilson [129, 131] has shown that this algorithm is convergent for any polynomial spectral-density $\hat{\Phi}_d$. However, despite comments suggesting otherwise (e.g. p231 [131]) only local properties of the convergence have been published (as would be expected from such an algorithm).

The error analysis proceeds by considering

$$E_{d,k+1} := \begin{bmatrix} e_{d,k,0} \\ \vdots \\ e_{d,k,d} \end{bmatrix} := \tilde{P}_{d,k} - \hat{P}_d := \tilde{P}_{d,k} - \begin{bmatrix} \hat{A}_{d,0} \\ \vdots \\ \hat{A}_{d,d} \end{bmatrix}$$

to show that

$$E_{d,k+1} = T_d\left(\tilde{P}_{d,k}\right)^{-1} F_{d,k} \tag{6.15}$$

where

$$F_{d,k} = \begin{bmatrix} f_{d,k,0} \\ \vdots \\ f_{d,k,d} \end{bmatrix}, \; f_{d,k,l} := \sum_{j=0}^{d-l} e_{d,k,j} e_{d,k,j+l}.$$

From (6.15) Wilson has shown that

$$\|E_{d,k+1}\|_\infty \leqslant (d+1) \frac{\left\|T_d\left(\hat{P}_d\right)^{-1}\right\|_{op,\infty}}{1 - 2(d+1)\left\|T_d\left(\hat{P}_d\right)^{-1}\right\|_{op,\infty} \|E_{d,k}\|_\infty} \|E_{d,k}\|_\infty^2 \tag{6.16}$$

from which may be concluded the local convergence of the algorithm, at a quadratic rate.

This characteristic of the error bounds is common to most error analyses and implies that it is unreasonable to hope for a globally convergent scheme for spectral factorization with a guaranteed rate. In the following section it will be shown that only local results can be obtained by the method employed in this Chapter.

Wilson's analysis is deficient for present purposes in several respects, although perfectly adequate for the purposes pursued in his work. The first of these deficiencies is that the neighbourhood for which quadratic convergence occurs contracts to a point as $d \to \infty$. For this reason a rederivation of the above error bound (6.16) will be undertaken in the next subsection.

Further problems are the assumption that $T_d\left(\hat{P}_d\right)$ is invertible and the fact that the $\|.\|_\infty$-norm of the Fourier coefficients of the spectral factor does not provide very strong convergence of the spectral factors as $d \to \infty$.

6.4.3 Analysis of Final Algorithm

To demonstrate the effects of incorporating approximation of the spectral factors into the algorithm proposed in the previous sections consider the approximate canonical correlation Hankel operator that will be utilised when $\tilde{V}_{d,k}$ and $\tilde{W}_{d,k}$ are calculated, namely

$$\tilde{\mathcal{C}}_{d,k} := \left(\mathbf{P}_d \mathbf{T}_{\bar{\tilde{V}}_{d,k}} \mathbf{P}_d \mid_d\right)^{-1} \left(\mathbf{P}_d \mathbf{H}_{\hat{\Phi}_d} \mathbf{P}_d \mid_d\right) \left(\mathbf{P}_d \mathbf{T}_{\tilde{W}_{d,k}} \mathbf{P}_d \mid_d\right)^{-1}. \tag{6.17}$$

The asymptotic properties of the approximation scheme now depend on the limit of $\tilde{\mathcal{C}}_{d,k}$ as both $d \to \infty$ and $k \to \infty$.

It is necessary to consider in what sense this limit might exist, and what it might mean. It is clear that

$$\lim_{d\to\infty}\left(\lim_{k\to\infty} \tilde{\mathcal{C}}_{d,k}\right) = \lim_{d\to\infty} \mathcal{C}'_d = \mathcal{C} \tag{6.18}$$

if $\tilde{V}_{d,k} \to \hat{V}_d$ in $\mathcal{H}^\infty$. However, such a result is of no practical consequence since it ignores the practicality that

$$\lim_{k\to\infty} \tilde{V}_{d,k} = \hat{V}_d$$

is not available for any $d > 0$.

It is also clear that for current purposes

$$\lim_{d\to\infty} \tilde{V}_{d,k}$$

does not exist, as no assumption has been made regarding the ability to implement algorithms involving the calculation of spectral factors of spectral densities with infinitely many non-zero Fourier coefficients. Hence

$$\lim_{k\to\infty}\left(\lim_{d\to\infty}\tilde{V}_{d,k}\right)$$

is not of interest either.

A much stronger form of limit is the double limit

$$\lim_{\substack{d\to\infty\\k\to\infty}}\tilde{\mathcal{C}}_{d,k}$$

which, if it exists, is equal to $\mathcal{C}$, since the iterated limit (6.18) exists. This subsection discusses conditions for its existence. The approach taken below seeks to circumvent some of the shortcomings of Wilson's method. Furthermore, the results below suggest a possible method of establishing local results.

It is not the case that the B-parameters may be estimated from the above results. This may be shown by considering the following limit, which will be assumed to exist:

$$\begin{aligned}&\lim_{\substack{d\to\infty\\k\to\infty}}\left\|d^l\left(\mathcal{C}-\tilde{\mathcal{C}}_{d,k}\right)\right\|=0\\ \Leftrightarrow\quad &\forall\epsilon>0\ \exists N\in\mathbb{N}\text{ such that }\forall d,k>N\ \left\|d^l\left(\mathcal{C}-\tilde{\mathcal{C}}_{d,k}\right)\right\|<\epsilon.\end{aligned}\tag{6.19}$$

Such an expression cannot hold $\forall d>N$ unless either $l=0$ or the bracketed term is identically zero $\forall d>N'\geqslant N$. This shows that the results involving the estimation of the B-parameters (Theorems 6.42 and 6.46) can no longer hold. §6.5 describes one possible way round this difficulty.

It is also clear from this analysis that it is necessary to be able to guarantee a particular accuracy of the estimate of $\hat{V}_d$ i.e. that

$$\left\|\hat{V}_d-\tilde{V}_{d,k}\right\|_\infty\leqslant f(\epsilon)\ \forall k\geqslant K\ \forall d\in\mathbb{N}.$$

This condition is a very strong constraint on the error analysis of an algorithm. One simple way to demonstrate this result would be to bound the initial error of the algorithm and then to show that the error decreases at some particular rate independent of d. The following error analysis unsuccessfully attempts to find a non-shrinkng region of quadratic convergence.

Many other limits could be explored e.g. $d=k\to\infty$, but as will be shown in the next section, all suffer from similar problems encountered in examination of the double limit.

The basis for the following analysis is a generalisation of the result (6.16) which will lead to a region of quadratic convergence which is independent of the order of the spectral factors, that is d. In the following results the notation established in the previous section will be used.

Lemma 6.47 (Generalisation of Wilson's Bound [129])

$$\|E_{d,k+1}\|_p\leqslant\frac{\left\|T_d\left(\hat{P}_d\right)^{-1}\right\|_{op,p}}{1-\left\|T_d\left(\hat{P}_d\right)^{-1}\right\|_{op,p}\left\|T_d\left(\hat{P}_d\right)-T_d\left(\tilde{P}_{d,k}\right)\right\|_{op,p}}\|F_{d,k}\|_p\tag{6.20}$$

for any vector norm $\|.\|_p$ *and induced operator norm* $\|.\|_{op.p}$, $p\in[1,\infty]$.

Proof : From (6.15)

$$\begin{aligned}E_{d,k+1}&=T_d\left(\tilde{P}_{d,k}\right)^{-1}F_{d,k}\\&=\left\{I_{d+1}+T_d\left(\hat{P}_d\right)^{-1}\left(T_d\left(\tilde{P}_{d,k}\right)-T_d\left(\hat{P}_d\right)\right)\right\}^{-1}T_d\left(\hat{P}_d\right)^{-1}F_{d,k}\\ \Rightarrow T_d\left(\hat{P}_d\right)^{-1}F_{d,k}&=\left\{I_{d+1}+T_d\left(\hat{P}_d\right)^{-1}\left(T_d\left(\tilde{P}_{d,k}\right)-T_d\left(\hat{P}_d\right)\right)\right\}E_{d,k+1}\end{aligned}$$

Hence, since

$$1 - \|A\|_{op,p}\,\|B\|_{op,p} \leqslant \|I + AB\|_{op,p} \leqslant 1 + \|A\|_{op,p}\,\|B\|_{op,p}$$

the result follows. ■

Lemma 6.48

$$\|F_{d,k}\|_1 \leqslant \|E_{d,k}\|_1^2$$

Proof :

$$\begin{aligned}
\|F_{d,k}\|_1 &= \sum_{l=0}^{d} |f_{d,k,l}| \\
&= \sum_{l=0}^{d} \left| \sum_{j=0}^{d-l} e_{d,k,j} e_{d,k,j+l} \right| \\
&\leqslant \sum_{l=0}^{d} \left| \sum_{m=0}^{d} e_{d,k,l} e_{d,k,m} \right| \\
&\leqslant \sum_{l=0}^{d} \sum_{m=0}^{d} |e_{d,k,l}|\,|e_{d,k,m}| \\
&= \left(\sum_{l=0}^{d} |e_{d,k,l}| \right)^2 \\
&= \|E_{d,k}\|_1^2
\end{aligned}$$

■

Lemma 6.49 (1-Norm Version of Wilson's Inequality)

$$\|E_{d,k+1}\|_1 \leqslant \frac{\left\| T_d\left(\hat{P}_d\right)^{-1} \right\|_{op,1}}{1 - 2\left\| T_d\left(\hat{P}_d\right)^{-1} \right\|_{op,1} \|E_{d,k}\|_1} \|E_{d,k}\|_1^2 \tag{6.21}$$

Proof : This follows from the two previous results and the observation that the operator norm induced by the l^1 norm is the maximum column sum of a matrix, and therefore

$$\begin{aligned}
\left\| T_d\left(\hat{P}_d\right) - T_d\left(\tilde{P}_{d,k}\right) \right\|_{op,1} &= \|T_d\left(E_{d,k}\right)\|_{op,1} \\
&\leqslant 2\,\|E_{d,k}\|_1 \ \text{(see (6.14))}
\end{aligned}$$

■

The following observation allows the further investigation of Wilson's algorithm, and the solution to some of the problems mentioned above.

Lemma 6.50

$$T_d\left(\hat{P}_d\right) = \mathbf{P}_{d+1}\left(R\mathbf{H}_{z^{-1}\hat{V}_d^* \hat{V}_d^{-1}} + I \right) \mathbf{T}_{\hat{V}_d}\,|_{d+1} \tag{6.22}$$

Proof : Consideration of the definition (6.14) shows that

$$\begin{aligned} T_d\left(\hat{P}_d\right) &= \mathbf{P}_{d+1}\left(R\mathbf{H}_{z^{-1}\hat{V}_d^*} + \mathbf{T}_{\hat{V}_d}\right)\mathbf{P}_{d+1}\mid_{d+1} \\ &= \mathbf{P}_{d+1}\left(R\mathbf{H}_{z^{-1}\hat{V}_d^*}\mathbf{T}_{\hat{V}_d}^{-1} + I\right)\mathbf{T}_{\hat{V}_d}\mathbf{P}_{d+1}\mid_{d+1} \\ &= \mathbf{P}_{d+1}\left(R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}} + I\right)\mathbf{T}_{\hat{V}_d}\mathbf{P}_{d+1}\mid_{d+1} \end{aligned}$$

■

Thus the matrix of interest in the inequality (6.21) may be related to a Hankel operator similar to that previously studied. As will be seen below the z^{-1} component does makes a substantial difference to the behaviour of the operator.

Consider the space $\mathcal{H}^2$ decomposed as $\mathcal{P}_{1\times1}^{d+1} \oplus \left(\mathcal{P}_{1\times1}^{d+1}\right)^{\perp}$. Then

$$\begin{aligned} &\left(R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}} + I\right)\mathbf{T}_{\hat{V}_d} \\ &= \begin{bmatrix} \mathbf{P}_{d+1}\left(R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}} + I_{d+1}\right)\mid_{d+1} & 0 \\ 0 & I_{\left(\mathcal{P}_{1\times1}^{d+1}\right)^{\perp}} \end{bmatrix} \begin{bmatrix} \mathbf{P}_{d+1}\mathbf{T}_{\hat{V}_d}\mid_{d+1} & 0 \\ \mathbf{P}_{\left(\mathcal{P}_{1\times1}^{d+1}\right)^{\perp}}\mathbf{T}_{\hat{V}_d}\mid_{d+1} & \mathbf{P}_{\left(\mathcal{P}_{1\times1}^{d+1}\right)^{\perp}}\mathbf{T}_{\hat{V}_d}\mid_{\left(\mathcal{P}_{1\times1}^{d+1}\right)^{\perp}} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{P}_{d+1}\left(R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}} + I_{d+1}\right)\mathbf{T}_{\hat{V}_d}\mid_{d+1} & 0 \\ \mathbf{P}_{\left(\mathcal{P}_{1\times1}^{d+1}\right)^{\perp}}\mathbf{T}_{\hat{V}_d}\mid_{d+1} & \mathbf{P}_{\left(\mathcal{P}_{1\times1}^{d+1}\right)^{\perp}}\mathbf{T}_{\hat{V}_d}\mid_{\left(\mathcal{P}_{1\times1}^{d+1}\right)^{\perp}} \end{bmatrix} \end{aligned} \tag{6.23}$$

This result provides a condition for the invertibility of the T_d matrices appearing in the Wilson algorithm.

Lemma 6.51 (Invertibility of T_d) *$T_d\left(\hat{P}_d\right)$ is invertible if and only if $-1 \notin \lambda\left(R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}}\right)$.*

Proof : From (6.23) it is clear that $\left(R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}} + I\right)\mathbf{T}_{\hat{V}_d}$ is invertible if and only if $\mathbf{P}_{d+1}\left(R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}} + I_{d+1}\right)\mathbf{T}_{\hat{V}_d}\mid_{d+1}$ and $\mathbf{P}_{\left(\mathcal{P}_{1\times1}^{d+1}\right)^{\perp}}\mathbf{T}_{\hat{V}_d}\mid_{\left(\mathcal{P}_{1\times1}^{d+1}\right)^{\perp}}$ are both invertible.

From the definition of $\hat{V}_d$ (assuming d sufficiently large to guarantee $\hat{\Phi}_d$ is a spectral-density) then $\mathbf{T}_{\hat{V}_d}$ is invertible, and hence the second of the above terms is invertible.

$$\begin{aligned} \mathbf{P}_{d+1}\left(R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}} + I_{d+1}\right)\mathbf{T}_{\hat{V}_d}\mid_{d+1} \text{ is invertible} &\Leftrightarrow \left(R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}} + I\right)\mathbf{T}_{\hat{V}_d} \text{ is invertible} \\ &\Leftrightarrow R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}} + I \text{ is invertible} \\ &\Leftrightarrow -1 \notin R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}}. \end{aligned}$$

This last step follows from the fact that the rank of $R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}}$ is $d+1$ and hence has only point spectrum, from Kroneckers Theorem. ■

Unfortunately, the following result shows that there is no simple way to rule out this spectral point.

Lemma 6.52 (Norm of Hankel Operator)

$$\left\|R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}}\right\|_{op} = 1.$$

Proof : $z^{-1}\hat{V}_d^*\hat{V}_d^{-1}$ is an inner function, hence $\operatorname{dist}\left(z^{-1}\hat{V}_d^*\hat{V}_d^{-1}, \mathcal{H}^\infty\right) \leqslant 1$.

$$\operatorname{dist}\left(z^{-1}\hat{V}_d^*\hat{V}_d^{-1}, \mathcal{H}^\infty\right) = \left\|R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}}\right\|_{op} \text{ from Nehari's Theorem}$$

and

$$\begin{aligned} \operatorname{dist}\left(z^{-1}\hat{V}_d^*\hat{V}_d^{-1}, \mathcal{H}^\infty\right) < 1 \quad &\Leftrightarrow \quad \mathbf{T}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}} \text{ is left invertible, see Nikol'skiĭ [94] p203} \\ &\Leftrightarrow \quad \mathbf{T}_{z^{-1}\hat{V}_d^*} \text{ is left invertible.} \end{aligned}$$

Noting that

$$\mathbf{T}_{z^{-1}\hat{V}_d^*}1 = \mathbf{P}_+ z^{-1}\hat{V}_d^* = 0$$

it is clear that $\mathbf{T}_{z^{-1}\hat{V}_d^*}$ is not left invertible. Hence the result. ■

From the assumptions on the spectral-density $\hat{\Phi}_d$ and the results of Hannan and Poskitt [47] and Green [41] it is clear that $\left\|\mathbf{H}_{\hat{V}_d^*\hat{V}_d^{-1}}\right\|_{op} < 1$. Hence, from Nikol'skiĭ [94] p203 it is clear that $R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}}$ must attain its norm. The following observation, which may be found in Jonckheere & Helton [69] §III Lemma 2, establishes this.

Lemma 6.53 (Eigenvector of $R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}}$ [69]) *$\hat{V}_d$ is an eigenvector of $R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}}$ with eigenvalue 1.*

Proof :

$$R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}}\hat{V}_d = R\mathbf{P}_- z^{-1}\hat{V}_d^* = Rz^{-1}\hat{V}_d^* = \hat{V}_d.$$

■

It may be possible to prove that 1 is the only spectral point of unit modulus from a state-space argument employing the assumptions about $\hat{V}_d$. However, for current purposes it will suffice to make the assumption C.1.

Hence, since $\hat{V}_d \to V$ in $\mathcal{H}^\infty$ then $\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}}$ approximates $\mathbf{H}_{z^{-1}V_d^*V_d^{-1}}$ in the induced operator norm on $\mathcal{L}^2$. Since this limit is finite-rank, as are all of the approximants, it is possible to bound $\left\|\left(z^{-1}\hat{V}_d^*\hat{V}_d^{-1} + I\right)^{-1}\right\|_{op,2}$ in terms of $\left(\max\left\{|\lambda_i - 1| : \lambda_i \in \lambda\left(\mathbf{H}_{z^{-1}V_d^*V_d^{-1}}\right)\right\}\right)^{-1}$. However, this is insufficient for the present context where the induced $\mathcal{L}^1$-norm is under consideration.

$$\begin{aligned} \left\|T_d\left(\hat{P}_d\right)^{-1}\right\|_{op,1} &\leqslant \left\|\left(R\mathbf{H}_{z^{-1}V_d^*V_d^{-1}} + I\right)^{-1}\mathbf{T}_{\hat{V}_d}^{-1}\right\|_{op,1} \\ &\leqslant \left\|\left(R\mathbf{H}_{z^{-1}V_d^*V_d^{-1}} + I\right)^{-1}\right\|_{op,1}\left\|\mathbf{T}_{\hat{V}_d^{-1}}\right\|_{op,1} \end{aligned}$$

$$\left\|\mathbf{T}_{\hat{V}_d^{-1}}\right\|_{op,1} \leqslant \left\|\hat{V}_d^{-1}\right\|_{A_{1\times 1}(\mathbb{T})}$$

From Wiener's Theorem e.g. p367 Rudin [113]

$$\left\|\hat{V}_d\right\|_{A_{1\times 1}(\mathbb{T})} = \sum_{k=-\infty}^{\infty}\left|\hat{A}_{d,k}\right| < \infty$$

and

$$\hat{V}_d(e^{i\theta}) \neq 0, \ \forall\theta \in [0, 2\pi)$$

then

$$\left\|\hat{V}_d^{-1}\right\|_{A_{1\times 1}(\mathbb{T})} < \infty.$$

This can also be bounded uniformly in d by assuming $V \in A_{1\times 1}(\mathbb{T})$ from Anderson's results [4] relating the norms of spectral factors to those of the spectral densities.

However, the first term is not so easily handled. The author is unable to provide any method to bound this term uniformly for all d. Hence, it is necessary to make the following conjecture:

Conjecture 1 ($\mathcal{L}^1$-Boundedness of $\left(R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}}+I\right)^{-1}$)

$$\exists K \in \mathbb{R}_+, D \in \mathbb{N} \text{ such that } \left\| \left(R\mathbf{H}_{z^{-1}V_d^*V_d^{-1}}+I\right)^{-1} \right\|_{op,1} < K,\ \forall d \geqslant D.$$

■

If this conjecture hold true then it is clear that a local result may be found showing the consistency of the approximation scheme for the s_i and σ_i parameters.

6.5 Other Considerations

In the three algorithms presented in §6.2.2, §6.3.1 and §6.4.1 the B-parameters have been estimated by approximating the Schmidt vectors of the canonical correlation Hankel operator evaluated at the point $z = -1$. The Schmidt vectors take on a very simple form at this point, but for any point on $\mathbb{T}$ the convergence of the approximate canonical correlation Hankel operator is obtained at the price of substantially greater growth rates in the amount of data used at each iteration. This increased growth rate vitiates any attempt to obtain convergence of the algorithmic implementation of the estimates of the B-parameters in §6.4.1, as was noted in §6.4.3.

If B-estimates can be obtained by approximating the Schmidt vectors inside the unit circle, then only the proof of Conjecture 1 impedes the estimation of all of the parameters of the minimum-phase canonical form developed previously. Only partial results have been established regarding the construction of such estimates. The two results below show first that there is generically an invertible relationship between the estimated values of the Schmidt vectors and the B-parameters. The second result shows an algebraic problem whose solution gives the possibility of showing the inverse relationship to be injective.

Only the SISO case will be considered for notational simplicity.

Let $V_c \overset{s}{=} \left(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V}\right)$ be the continuous-time equivalent system to $V_d \overset{s}{=} (A_v, K, C, V)$ which is a realisation of the innovations representation (3.2) of the "true model" of the underlying stochastic process. Further, let V_c be an element of the canonical form described in Theorem 4.60.

It has been shown in Theorems 5.19 and 5.20 that the normalised eigenvectors of $R\mathbf{H}_{V_d^{-1}W_d^*}$ are of the form

$$u_i(z) = \sqrt{\frac{2}{\sigma_i}} s_i \tilde{K}^T \left((1-z)I_n - (1+z)\tilde{A}_v^T\right)^{-1} e_i \tag{6.24}$$

where e_i is the ith member of the natural basis of $\mathbb{R}^n$.

Consider the mapping

$$\begin{aligned} \Xi : \mathcal{S} \times \mathbb{R}_+^n &\rightarrow \mathbb{R}^{2n} \\ \Xi : \begin{pmatrix} s_1\sigma_1 \\ \vdots \\ s_n\sigma_n \\ b_1 \\ \vdots \\ b_n \end{pmatrix} &\rightarrow \begin{pmatrix} s_1\sigma_1 \\ \vdots \\ s_n\sigma_n \\ u_1(0) \\ \vdots \\ u_n(0) \end{pmatrix} \end{aligned} \tag{6.25}$$

where

$$\mathcal{S} = \left\{ (s_1\sigma_1, \ldots, s_n\sigma_n) \mid \sigma_1 > \ldots > \sigma_n > 0,\ s_i \in \{\pm 1\},\ i = 1, \ldots, n \right\}.$$

From Theorem 4.60 it is clear that $u_i(0)$ is well defined for all $s_i \in \{\pm 1\}$, $i = 1, \ldots, n$, and $\sigma_1 > \ldots > \sigma_n > 0$. Also, it is clear that in the union of these cones the function Ξ is continuously differentiable.

Letting $\theta = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$, $\phi_1^T = (s_1\sigma_1, \ldots, s_n\sigma_n)$ and $\phi_2^T = (b_1, \ldots, b_n)$ then the matrix of partial derivatives

$$D_\theta \Xi = [y_{ij}]_{1 \leqslant i,j \leqslant 2n}$$

where

$$y_{ij} = \frac{\partial \Xi(\theta)_j}{\partial \theta_i}$$

is of the form

$$D_\theta \Xi = \begin{bmatrix} I_n & X(\phi_1, \phi_2) \\ 0 & Y(\phi_1, \phi_2) \end{bmatrix}.$$

This is invertible if and only if its determinant is non-zero, if and only if

$$\det Y(\phi_1, \phi_2) \neq 0.$$

$$Y(\phi_1, \phi_2) = \left[\frac{\partial u_j(0)}{\partial b_i}\right]_{1 \leqslant i,j \leqslant n},$$

so writing

$$U = \begin{bmatrix} u_1(0), & \cdots, & u_n(0) \end{bmatrix} = \sqrt{2}\phi_2^T \Sigma^{-\frac{1}{2}} S \left(I_n - \tilde{A}_v(\phi_1, \phi_2)\right)^{-1} \tag{6.26}$$

then

$$Y(\phi_1, \phi_2) = \left[\frac{\partial}{\partial b_1}, \cdots, \frac{\partial}{\partial b_n}\right]^T U.$$

Hence,

$$\begin{aligned}
\frac{\partial}{\partial b_i} U &= \sqrt{2} \frac{\partial}{\partial b_i} \phi_2^T S \Sigma^{-\frac{1}{2}} \left(I_n - \tilde{A}_v(\phi_1, \phi_2)\right)^{-1} \\
&= \sqrt{2} \left(\phi_2^T S \Sigma^{-\frac{1}{2}} \left\{ \frac{\partial}{\partial b_i} \left(I_n - \tilde{A}_v(\phi_1, \phi_2)\right)^{-1} \right\} + e_i^T S \Sigma^{-\frac{1}{2}} \left(I_n - \tilde{A}_v(\phi_1, \phi_2)\right)^{-1} \right) \\
&= \sqrt{2} \left(e_i^T S \Sigma^{-\frac{1}{2}} - \phi_2^T S \Sigma^{-\frac{1}{2}} \left(I_n - \tilde{A}_v(\phi_1, \phi_2)\right)^{-1} \left(\frac{\partial}{\partial b_i} \tilde{A}_v(\phi_1, \phi_2) \right) \right) \left(I_n - \tilde{A}_v(\phi_1, \phi_2)\right)^{-1}.
\end{aligned}$$

Let

$$\psi_i = \phi_2^T S \Sigma^{-\frac{1}{2}} \operatorname{adj}\left(I - \tilde{A}_v(\phi_1, \phi_2)\right) \frac{\partial}{\partial b_i} \tilde{A}_v(\phi_1, \phi_2)$$

and

$$\Psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_n \end{bmatrix}.$$

Hence,

$$Y(\phi_1, \phi_2) = \sqrt{2} S \Sigma^{-\frac{1}{2}} \left(S \Sigma^{-\frac{1}{2}} - \frac{1}{\det(I_n - \tilde{A}_v)} \Psi \right) \left(I_n - \tilde{A}_v(\phi_1, \phi_2)\right)^{-1}$$

and so

$$\det Y(\phi_1,\phi_2) \neq 0 \quad \Leftrightarrow \quad \det\left(S\Sigma^{-\frac{1}{2}} - \frac{1}{\det(I_n - \tilde{A}_v)}\Psi\right)$$
$$\Leftrightarrow \quad \left(\frac{1}{\det(I_n - \tilde{A}_v)}\right)^n \det\left(\det(I - \tilde{A}_v)S\Sigma^{-\frac{1}{2}} - \Psi\right) \neq 0.$$

Since the first term of this is non-zero and finite for all permissible values of ϕ_1 and ϕ_2 and the second is a non-zero rational-function in ϕ_1 and ϕ_2 then the numerators and denominators are zero only on an algebraic variety in $\mathbb{R}^{2n}$.

Hence it is clear that on the intersection of $\mathcal{S} \times \mathbb{R}^n_+$ and the complement of the algebraic surface discussed above i.e. for almost all ϕ_1 and ϕ_2, the Jacobian of Ξ is non-zero. Hence at every point in that set (it is open) the inverse function theorem shows the local existence of a continuously differentiable inverse function.

Hence it has been shown that:

Theorem 6.54 *The function Ξ defined in (6.25) is locally invertible almost everywhere in $\mathcal{S}\times\mathbb{R}^{2n}_+$, and the inverse is continuously differentiable.*

■

Theorem 6.54 does not provide any upper bound on the cardinality of

$$\Xi^{-1}\left((s_1\sigma_1, \ldots, s_n\sigma_n, u_1(0), \ldots, u_n(0))^T\right)$$

which may be much more than 1.

To obtain more information regarding the properties of Ξ^{-1} it is necessary to attempt to use the details of the problem more extensively.

From (6.26) with $z = 0$ it is clear that

$$\sqrt{2}\tilde{K}^T S\Sigma^{-\frac{1}{2}}\left(I - \tilde{A}_v^T\right)^{-1} \quad = \quad U$$

which, from examination of the form of $\tilde{A}_v$, assuming s_i and σ_i parameters are already estimated, and assuming estimates of $u_i(0)$ are available, then this is a set of n simultaneous quadratic equations in n unknowns.

Hence, if the set of equations is consistent (the previous result shows that they are generically) then there will exist at most 2^n solutions.

These quadratic equations can be shown to be a set of n simultaneous hyperbolic equations

$$b_i f_i(\tilde{K}) = u_i(0), \; i = 1, \ldots, n.$$

Such an approach is amenable to symbolic solution. Some preliminary investigation of this approach has failed to find any problems for which multiple real positive solutions exist for any b_i.

6.6 Conclusions

This Chapter has analysed a naive algorithm for estimating the parameters appearing in the previously derived time-series model. It has been shown that the theoretical properties of the estimation procedure are good, and various spectral estimation results have been derived in passing.

The consistency of the whole scheme is hampered by the necessity of obtaining approximate spectral factors by an iterative procedure whose properties are not sufficiently strong. Several results describing the properties of Wilson's algorithm were derived in §6.4.3.

Chapter 7

Estimation of Parameters: ARMAX systems

In the previous Chapter an algorithm for the estimation of the parameters describing a finite-dimensional time-series model was presented and analysed. The resulting algorithm was not entirely successful due to the involvement of a spectral factorisation step for which no algorithm converged sufficiently fast.

By contrast, this Chapter considers a more complicated model class i.e. ARMAX systems, but a standard estimation scheme is applied. This standard method is used as a basis for an iterative scheme which sucessfully avoids a spectral factorisation operation, and therefore consistency of the algorithm may be demonstrated.

The basic motivation for the proposed algorithm is the same as in the previous Chapter. Thus a series of operators are constructed which approximate another operator whose spectral properties are related to the parameters of the underlying model. However, instead of a spectral estimate and spectral factorisation, at each step an ARX estimation is performed. ARX estimates are described in the next section, and the important convergence results of Ljung and Wahlberg are presented. These results will provide the necessary basis for the subsequent convergence proofs.

7.1 ARX Model Estimation

7.1.1 Review

A stochastic process $(y(t))_{t\in\mathbb{Z}}$ generated from a white noise sequence $(e(t))_{t\in\mathbb{Z}}$ and input sequence $(u(t))_{t\in\mathbb{Z}}$ by a *vector difference equation* (see Chapter 1 of Hannan & Deistler [46])

$$y(t)+\sum_{k=1}^{n}A(k)y(t-k)=\sum_{l=0}^{n}B(l)u(t-l)+e(t) \tag{7.1}$$

is called an *ARX process* i.e. auto-regressive on y with exogenous inputs.

The simplicity and tractability of such models derives from the fact that they may be rewritten in the following *regression form*

$$y(t)=\theta^{n^T}\phi_t^n+e(t) \tag{7.2}$$

where

$$\begin{aligned}\theta^n &:= \begin{bmatrix} A(1) & \cdots & A(n) & B(0) & \cdots & B(n)\end{bmatrix}^T \in \mathbb{R}^{(2n+1)\times 1}\\ \phi_t^n &= \begin{bmatrix} -y(t-1) & \cdots & -y(t-n) & u(t) & \cdots & u(t-n)\end{bmatrix}^T \in \mathbb{R}^{(2n+1)\times 1}.\end{aligned}$$

The ARX estimation problem is to find the best $\hat{y}(\hat{\theta}^n,t)$ where

$$\begin{aligned}\hat{y}(\hat{\theta}^n,t) &= \hat{\theta}^{n^T}\phi_t^n\\ &= -\sum_{k=1}^{n}\hat{A}_n(k)y(t-k)+\sum_{l=0}^{n}\hat{B}_n(l)u(t-l)\end{aligned}$$

given knowledge of $(u(t))_{t=1}^N$ and $(y(t))_{t=1}^N$. A solution is said to be optimal if it minimises the error functional

$$V_N(\theta) = \frac{1}{N}\sum_{t=n+1}^{N}\|y(t)-\hat{y}(\theta,t)\|_2^2.$$

It is a simple exercise to show that the unique minimum of this criterion function occurs at

$$\hat{\theta}_N^n=\begin{bmatrix}\hat{A}_n(1) & \ldots & \hat{A}_n(n) & \hat{B}_n(0) & \ldots & \hat{B}_n(n)\end{bmatrix}^T=R^n(N)^{-1}f^n(N)\in\mathbb{R}^{(2n+1)\times 1} \tag{7.3}$$

where

$$R^n(N) := \frac{1}{N}\sum_{t=n+1}^{N} \phi_t^n \phi_t^{n^T} \tag{7.4}$$

$$f^n(N) := \frac{1}{N}\sum_{t=n+1}^{N} \phi_t^n y(t). \tag{7.5}$$

This solution is known e.g. Caines [16] Chapter 6, to be consistent for the true parameters θ as $N \to \infty$.

Also, let

$$\hat{\lambda}_N^n = V_N\left(\hat{\theta}_N^n\right) = \frac{1}{N}\sum_{t=n+1}^{N} \left\| y(t) - \hat{\theta}^{n^T}\phi_t^n \right\|_2^2. \tag{7.6}$$

For the full analysis of the multivariate case Chapter 6 of [16] gives a excellent summary of these results.

7.1.2 Working Assumptions

The estimation results sought in this Chapter will be for ARMAX systems, as described in §3.1.2. However, many of the subsidiary results hold for more general models than these. For this reason, and because it fits more naturally with the results of Ljung & Wahlberg, a more general model structure is used.

Throughout this Chapter it will be assumed that the "true model" is of the form

$$\sum_{k=0}^{\infty} \hat{A}_o(k) y(t-k) = \sum_{l=0}^{\infty} \hat{B}_o(l) u(t-l) + e(t)$$

where

$$A_o(z) = \sum_{k=0}^{\infty} \hat{A}_o(k) z^k$$

$$B_o(z) = \sum_{k=0}^{\infty} \hat{B}_o(k) z^k$$

are analytic in $\mathbb{D}$, continuous on $\mathbb{T}$, and $A_o(z)^{-1}$ is analytic in $\mathbb{D}$. Without loss of generality it may also be assumed that $A_o(0) = 1$. $e(t)$ is assumed to be white noise of variance λ_o.

This may also be written as

$$y(t) = \sum_{k=0}^{\infty} \hat{G}_o(k) u(t-k) + \sum_{l=0}^{\infty} \hat{H}_o(l) e(t-l)$$

where

$$G_o(z) = \sum_{k=0}^{\infty} \hat{G}_o(k) z^k = \frac{B_o(z)}{A_o(z)}$$

and

$$H_o(z) = \frac{1}{A_o(z)}.$$

Hence the spectral-density of the noise signal may be defined as

$$\begin{aligned}\Phi_o(e^{i\theta}) &= H_o(e^{i\theta}) H_o(e^{-i\theta}) \\ &= \frac{1}{A_o(e^{i\theta}) A_o(e^{-i\theta})}.\end{aligned}$$

In an analogous manner to the previous Chapter, a scheme for estimation based upon finite data and finite calculations is proposed. Before proceeding to describe the assumptions on the deterministic and stochastic parts of the underlying system which are necessary, some notation and definitions are introduced which allow these two classes to be treated in a similar manner.

Given sequences $(u(t))_{t\in\mathbb{Z}}$ and $(y(t))_{t\in\mathbb{Z}}$, with p and q components respectively, which may be paths of deterministic or stochastic processes, then the following may be defined

$$R_{uy}^N(\tau) := \begin{cases} \frac{1}{N}\sum_{t=1}^{N-\tau} u(t)y(t+\tau)^T, & 0 \leqslant \tau < N \\ \frac{1}{N}\sum_{t=-\tau+1}^{N} u(t)y(t+\tau)^T, & 0 \geqslant \tau > -N \\ 0 & \text{otherwise} . \end{cases} \tag{7.7}$$

$R_{uy}^N(\tau)$ is obviously closely related to the estimated covariance $\hat{\Gamma}^{(N)}(k)$ given in (6.3) although different weightings have been used in this case. The definition also applies to non-stochastic sequences.

Where it exists, the following is also defined

$$R_{uy}(\tau) := \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} u(t)y(t+\tau)^T. \tag{7.8}$$

This is a generalisation of the lagged covariance between two processes. Indeed if $u(t)y^T(t+\tau)$ is an ergodic process e.g. strictly stationary and has

$$\int_\Omega \left\| u(t)y^T(t+\tau)\right\|_1 \, d\mathbb{P}(\omega) < \infty,$$

then this is exactly the lagged covariance.

Definition 7.1 (Quasi-Stationarity [80]) *Let $(f_n)_{n=1}^\infty$ be a decreasing sequence of scalars, with $f_n \to 0$ as $n \to \infty$. The vector sequence $(u(t))_{t\in\mathbb{Z}}$ is said to be f_n-*quasi-stationary *if*

$$\sup_{|\tau|\leqslant N} \left\| R_{uu}(\tau) - R_{uu}^N(\tau)\right\|_2 \leqslant C_1 f_N$$

and

$$\frac{1}{N}\sum_{t=-N}^{t=N} \|u(t)\|_2^2 \leqslant C_2$$

for N sufficiently large, where C_1 and C_2 are positive constants.

■

The following is an obvious generalisation of the spectral-density of a stochastic process.

Definition 7.2 (Power Spectral-Density) *For an f_n-quasi-stationary sequence $(u(t))_{t\in\mathbb{Z}}$, the* power spectral-density *is defined as*

$$\Phi_{uu}(e^{i\theta}) = \sum_{t=-\infty}^{\infty} R_{uu}(t)e^{it\theta}$$

where this sum exists.

■

In the following assumptions κ will be assumed to be a non-negative integer. The assumptions which will be used in this Chapter are:

G Assumptions on the Function $d(N)$.

G.1 $d(N) \to \infty$ as $n \to \infty$.

G.2 $\dfrac{d^{2+2\kappa}(N)\log N}{N} \to 0$ as $n \to \infty$.

H Assumptions on the "True System"

H.1 $\lim_{d\to\infty} d^{\kappa} \sum_{k=d+1}^{\infty} \left(\left|\hat{A}_o(k)\right| + \left|\hat{B}_o(k)\right|\right) = 0$[1].

I Assumptions on the Innovations Sequence $(e(t))_{t\in\mathbb{Z}}$.

I.1 $\mathbb{E}e_i^{4+\alpha}(t) < \infty$, for some $\alpha > 0$.

J Assumptions on the input sequence $(u(t))_{t\in\mathbb{Z}}$.

J.1 $(u(t))_{t\in\mathbb{Z}}$ is $\left(\frac{\log N}{N}\right)^{\frac{1}{2}}$-quasi-stationary.

J.2 $|u(t)| < C,\ \forall t \in \mathbb{Z},\ C > 0$.

J.3 If $\Phi_{uu}(e^{i\theta}) = V_u(e^{i\theta})V_u^T(e^{-i\theta})$ and $V_u(e^{i\theta}) = \sum_{k=0}^{\infty} \hat{V}_u(k)e^{ik\theta}$ then $\sum_{k=0}^{\infty}\left|\hat{V}_u(k)\right| < \infty$.

K Experimental Conditions

K.1 $\gamma I_2 \geqslant \begin{bmatrix} \Phi_{uu} & \Phi_{ue} \\ \Phi_{eu} & \Phi_{ee} \end{bmatrix} \geqslant \delta I_2 > 0$.

L Summary Conditions

L.1 All of G.1, G.2, H.1, I.1, J.1, J.2, J.3 and K.1.

7.1.3 Ljung & Wahlberg's Result

Whether or not the "true model" is of the form (7.1) the parameter estimate $\hat{\theta}_N^n$ may be formed from (7.3) although this no longer has the interpretation as a linear least squares best predictor.

From $\hat{\theta}_N^{d(N)}$ the transfer functions

$$A_d(z) = \sum_{k=0}^{d} \hat{A}_d(k)z^k \tag{7.9}$$

where $A_d(0) = 1$, and

$$B_d(z) = \sum_{k=0}^{d} \hat{B}_d(k)z^k \tag{7.10}$$

may be formed. The following result, modified from Ljung and Wahlberg [80] demonstrates the behaviour of these estimates. It plays a similar role in the Chapter as does Proposition 6.32 in Chapter 6.

---

[1]See the footnote on p123

Theorem 7.3 ([80]) *Under assumptions L.1, where $\kappa \geqslant 0$ and where $A_d(z)$ and $B_d(z)$ are given by the equations (7.3), (7.9) and (7.10) then*

$$\lim_{d\to\infty} \left\| d^\kappa \begin{bmatrix} A_d(z) - A_o(z) \\ B_d(z) - B_o(z) \end{bmatrix} \right\|_\infty \to 0 \textit{ as } d \to \infty \text{ w.p. 1 } [\mathbb{P}].$$

Also, with $\hat{\lambda}_N^d$ defined by (7.6) then

$$\lim_{N\to\infty} \left| \hat{\lambda}_N^d - \lambda_o \right| = 0 \text{ w.p. 1 } [\mathbb{P}].$$

Proof : By Theorem 3.1 [80] the above expression

$$d^\kappa \left\| \begin{bmatrix} A_d(z) - A_o(z) \\ B_d(z) - B_o(z) \end{bmatrix} \right\|_\infty = d^\kappa O(m(N)) \text{ w.p. 1 } [\mathbb{P}] \tag{7.11}$$

where

$$m(N) = d(N) \left(\frac{\log N}{N} \right)^{\frac{1}{2}} [1 + n(N)] + n(N),$$

and

$$n(d) = \sum_{k=d+1}^{\infty} \left(\left| \hat{A}_o(k) \right| + \left| \hat{B}_o(k) \right| \right)$$

From the assumption H.1 $d^\kappa n(d) \to 0$ as $d \to \infty$. Hence the second part of the first term is also bounded. Finally,

$$\frac{d^{2+2\kappa}(N) \log N}{N} \to 0 \text{ as } N \to \infty$$

by assumption G.2

Also, from page 20 of [80]

$$\left| \hat{\lambda}_N^d - \lambda_o \right| = O\left((1 + d(N)) \left(\frac{\log N}{N} \right)^{\frac{1}{2}} + \left(\frac{d(N) \log N}{N} \right)^{\frac{1}{2}} (1 + n(N)) + n(N) \right).$$

The first term is bounded by the preceding comments, and the second term is bounded by $m(N)$. The last term converges to zero by assumption H.1. ■

Corollary 7.3.1 *Assuming conditions L.1 hold, and using the same notation as Theorem 7.3 then*

$$\lim_{d\to\infty} d^\kappa \left\| \Phi_o^{-1} - A_d A_d^* \right\|_\infty = 0 \text{ w.p. 1 } [\mathbb{P}].$$

■

7.2 Estimation of the Noise Model

7.2.1 Outline of Scheme

Given $(y(t))_{t=1}^N$ and $(u(t))_{t=1}^N$ and a function d on the natural numbers satisfying G.2 with $\kappa \geqslant 1$ then:

1) The matrix $R^d_{reg}(N)$ is formed from the data, by

$$R^d_{reg}(N) := \begin{cases} R^d(N), & \text{if } \sigma_{\min}(R^d(N)) < \frac{\delta}{2} \\ R^d(N) + \frac{\delta}{2} I_d, & \text{otherwise} \end{cases}$$

where δ is given in K.1.

2) The coefficients $(\hat{A}_{d(N)}(k))_{k=1}^{d(N)}$ and $(\hat{B}_{d(N)}(l))_{l=0}^{d}$ are calculated from (7.3) using $R^d_{reg}(N)$ in place of $R^d(N)$.

3) The spectral-density

$$\hat{\Psi}_d(e^{i\theta}) := \sum_{k=-d}^{d} \psi_d(k) e^{-ik\theta} \tag{7.12}$$

is formed, where

$$\psi_d(k) := \sum_{t=-\infty}^{\infty} \hat{A}_d(t) \hat{A}_d(t+k)$$

where $\hat{A}_d(k)$ assumed to be zero if $k \notin [0, d]$.

4) From the coefficients calculated above the following matrices are formed

$$X_d = \begin{bmatrix} \psi_d(1) & \psi_d(2) & \psi_d(3) & \psi_d(4) & \cdots \\ \psi_d(2) & \psi_d(3) & \psi_d(4) & \psi_d(5) & \ddots \\ \psi_d(3) & \psi_d(4) & \psi_d(5) & \psi_d(6) & \ddots \\ \psi_d(4) & \psi_d(5) & \psi_d(6) & \psi_d(7) & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \in \mathbb{C}^{(d+1)\times(d+1)}$$

$$Y_d = \begin{bmatrix} \hat{A}_d(0) & \hat{A}_d(1) & \hat{A}_d(2) & \hat{A}_d(3) & \cdots \\ 0 & \hat{A}_d(0) & \hat{A}_d(1) & \hat{A}_d(2) & \ddots \\ 0 & 0 & \hat{A}_d(0) & \hat{A}_d(1) & \ddots \\ 0 & 0 & 0 & \hat{A}_d(0) & \ddots \\ \vdots & \ddots & \ddots & \ddots & \end{bmatrix} \in \mathbb{C}^{(d+1)\times(d+1)}$$

5) The approximate correlation operator

$$\mathcal{D}_d = Y_d^{-1} X_d Y_d^{-T}$$

is formed. The first n eigenvalues, ranked by magnitude, are

$$\left\{ \lambda_i^{(d)} \mid i = 1, \ldots, n \right\} \subseteq \lambda\left(\mathcal{D}_d\right)$$

and the normalised eigenvectors are $\left\{ u_i^{(d)} \mid i = 1, \ldots, n \right\}$ where

$$\begin{aligned} u_i^{(d)} &= \begin{bmatrix} u_{i,0}^{(d)} \\ \vdots \\ u_{i,d}^{(d)} \end{bmatrix} \in \mathbb{R}^{(d+1)\times 1} \\ \mathcal{D}_d u_i^{(d)} &= \lambda_i^{(d)} u_i^{(d)}. \end{aligned}$$

6) The estimates

$$\begin{aligned} \hat{s}_i^{(d)} &= \operatorname{signum}(\lambda_i^{(d)}) \\ \hat{\sigma}_i^{(d)} &= \frac{|\lambda_i^{(d)}|}{\sqrt{1-\lambda_i^{(d)^2}}} \end{aligned}$$

$i = 1, \ldots n$. The $\hat{B}_i^{(d)}$ parameters are calculated from

$$\hat{B}_i^{(d)} = \sqrt{2\sigma_i^{(d)}} s_i^{(d)} \left(\sum_{j=0}^{d} (-1)^j \, u_{i,j}^{(d)} \right),$$

and $\hat{\lambda}_N^d$ is calculated from (7.6).

7.2.2 Consistency of Noise Model Estimates

The basic method of the algorithm is the same as in the previous Chapter in that it attempts to approximate an operator with known spectral properties. Lemma 2.47 shows that

$$\tilde{\mathbf{H}}_{H_o^{-1}H_o^*} R = \mathbf{T}_{H_o^*} \left(R\mathbf{H}_{\Phi_o^{-1}} \right) \mathbf{T}_{H_o^T}$$

which is seen to be equal to

$$R\mathbf{H}_{\left(H_o^{-1}H_o^*\right)(z^{-1})}.$$

In Proposition 5.17 it is shown that if H_o has the realisation described in §5.1.1 (this assumes that H_o is rational) then

$$\left(H_o^{-1}H_o^*\right)(z) = K_o^T(zI - A_z^T)^{-1}L_o^T + V^{-1}W^T + \hat{C}_d(z^{-1}I - \hat{A}_d)^{-1}\hat{B}_d$$

and hence the anti-stable part of $(H_o^{-1}H_o^*)(z^{-1})$ has the realisation $\hat{C}_d(zI - \hat{A}_d)^{-1}\hat{B}_d$ which is known to be Lyapunov balanced with Gramian Σ and sign symmetric with respect to S (see Corollary 5.17.1).

The construction of an algorithm to approximate $\tilde{\mathbf{H}}_{H_o^{-1}H_o^*} R$ will, from the same arguments as were applied in the last Chapter, give estimates of some of the parameters describing the noise process.

Despite the differences in the problem formulation between this Chapter and the last there remain many similarities which allow the application of the results of the last Chapter. To see this let

$$\Psi_o = \Phi_o^{-1}$$

and rewrite the operator of interest as

$$\tilde{\mathbf{H}}_{H_o^{-1}H_o^*} R = \mathbf{T}_{A_o^*}^{-1} \left(R\mathbf{H}_{\Psi_o} \right) \mathbf{T}_{A_o^T}^{-1}.$$

This establishes the relationship to the canonical correlation operator.

Theorem 7.4 *Under conditions L.1 where $\kappa \geqslant 0$ and where $\hat{\Psi}_d$ is formed from the equations (7.12) and (7.3) then $\exists D \in \mathbb{N}$ such that*

$$\begin{aligned} \lim_{\substack{d\to\infty \\ d\geqslant D}} d^\kappa \left\| \tilde{\mathbf{H}}_{H_o^{-1}H_o^*} R - \mathbf{T}_{\hat{A}_d^*}^{-1} \left(R\mathbf{H}_{\hat{\Psi}_d} \right) \mathbf{T}_{\hat{A}_d}^{-1} \right\|_{op} &= 0 \\ \lim_{\substack{d\to\infty \\ d\geqslant D}} d^\kappa \left\| \tilde{\mathbf{H}}_{H_o^{-1}H_o^*} R - \left(\mathbf{P}_d \mathbf{T}_{\hat{A}_d^*} \mathbf{P}_d \,|_d \right)^{-1} \left(\mathbf{P}_d R\mathbf{H}_{\hat{\Psi}_d} \mathbf{P}_d \,|_d \right) \left(\mathbf{P}_d \mathbf{T}_{\hat{A}_d} \mathbf{P}_d \,|_d \right)^{-1} \right\|_{op} &= 0. \end{aligned}$$

Proof : First it is necessary to show the existence of all of the operators mentioned. Note that

$$\gamma^{\frac{1}{2}} \geqslant \left|A_o(e^{i\theta})\right| \geqslant \delta^{\frac{1}{2}} \geqslant 0$$

and is analytic in $\mathbb{D}$ with no zeros in $\mathbb{D}$.

Since $\hat{A}_d \to A_o$ in $\mathcal{L}^\infty$, by Theorem 7.3 then $\exists D \in \mathbb{N}$ such that $\forall d \geqslant D$

$$\sup_{z\in\mathbb{T}} \left|A_o(z) - \hat{A}_d(z)\right| < \epsilon$$

and since $\hat{A}_d$ is analytic (in $\mathbb{C}$) $\forall d \in \mathbb{N}$ then $\hat{A}_d$ has no zeros in $\mathbb{D}$, by Rouché's Theorem. Hence $\mathbf{T}_{\hat{A}_d}$ is invertible $\forall d \geqslant D$.

The proof of the first statement follows that of Theorem 6.39, *mutatis mutandis.*

The proof of the second statement follows Theorem 6.43 and Corollary 6.44.1. ∎

The consistency of the estimators of s_i and σ_i follows by exactly the same arguments as appear in §6.3.2.

Theorem 7.5 *Given an ARMAX system of the form (3.7) where the spectral-density of the noise process $\Phi_{ee}(e^{i\theta})$ has normalised right spectral factor*

$$V_d(z) = C(z^{-1}I - A_v)^{-1}K + V$$

where the equivalent continuous-time transfer function

$$V_c(s) = \tilde{C}\left(sI - \tilde{A}_v\right)^{-1}\tilde{K} + \tilde{V}$$

is an element of $\mathrm{M}^{\mathrm{p}}_{\mathrm{n}}[\omega]$ and $\left(\tilde{A}_v, \tilde{K}, \tilde{C}, \tilde{V}\right)$ is an element of the minimum-phase canonical form of Corollary 4.62.1 given by the parameters

$$\left\{s_i, \sigma_i, \check{B}_i, \check{D} \,|\, i = 1, \ldots, n\right\}.$$

Also let $\mathcal{D}_d$ and

$$\left\{\hat{s}_i^{(d)}, \hat{\sigma}_i^{(d)}, \hat{B}_i^{(d)}, \hat{\lambda}_N^d \,|\, i = 1, \ldots, n,\ d \in \mathbb{N}\right\}$$

be constructed by the algorithm described in §7.2.1.

If conditions L.1 hold with $\kappa \geqslant 1$ then $\hat{s}_i^{(d)}$, $\hat{\sigma}_i^{(d)}$, $\hat{B}_i^{(d)}$ and $\hat{\lambda}_N^d$ are strongly consistent estimators of s_i, σ_i, $\check{B}_i$ and $\check{D}$, $i = 1, \ldots, n$.

Proof : The proof of Theorem 6.46 applies here once it is noted that the stable part of $V_o^{-1}W_o^*$ has a balanced sign symmetric realisation with Gramian P and sign matrix S. The convergence of $\hat{\lambda}_N^d$ to $\check{D}$ is shown in Theorem 7.3. ∎

7.3 Estimation of the Deterministic Model

The estimation method used for the deterministic part of the model is analogous to that for the stochastic component. The algorithm is based upon the observation that

$$G_o = A_o^{-1}B_o$$

and hence that

$$\begin{aligned}
\tilde{\mathbf{H}}_{G_o}R &= \mathbf{P}_+\mathbf{M}_{A_o^{-1}B_o}\mathbf{P}_-R \\
&= \mathbf{P}_+\mathbf{M}_{A_o^{-1}}\mathbf{P}_+\mathbf{P}_+\mathbf{M}_{B_o}\mathbf{P}_-R \\
&= \mathbf{T}_{A_o^{-1}}\tilde{\mathbf{H}}_{B_o}R \\
&= \mathbf{T}_{A_o}^{-1}R\mathbf{H}_{B_o(z^{-1})}
\end{aligned} \tag{7.13}$$

The anti-Hankel operator $\tilde{\mathbf{H}}_{G_o}$ is used in this context since $\mathbf{H}_{G_o}$ is identically zero, as G_o is a stable symbol.

7.3.1 Outline of Scheme: Deterministic Part

1) Follow the steps 1) and 2) of §7.2.1.

2) Form the matrix Y as in 4) of §7.2.1 and let

$$X_d = \begin{bmatrix} \hat{B}_d(1) & \hat{B}_d(2) & \hat{B}_d(3) & \hat{B}_d(4) & \cdots \\ \hat{B}_d(2) & \hat{B}_d(3) & \hat{B}_d(4) & \hat{B}_d(5) & \ddots \\ \hat{B}_d(3) & \hat{B}_d(4) & \hat{B}_d(5) & \hat{B}_d(6) & \ddots \\ \hat{B}_d(4) & \hat{B}_d(5) & \hat{B}_d(6) & \hat{B}_d(7) & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \in \mathbb{C}^{(d+1)\times(d+1)}.$$

3) Form the approximate Hankel operator

$$\mathcal{E}_d = Y_d^{-1} X_d.$$

The first n eigenvalues, ranked by magnitude, are

$$\left\{\mu_i^{(d)} \mid i = 1, \ldots, n\right\} \subseteq \lambda\left(\mathcal{E}_d\right)$$

and normalised eigenvectors are $\left\{v_i^{(d)} \mid i = 1, \ldots, n\right\}$ where

$$\begin{aligned} v_i^{(d)} &= \begin{bmatrix} v_{i,0}^{(d)} \\ \vdots \\ v_{i,d}^{(d)} \end{bmatrix} \in \mathbb{R}^{(d+1)\times 1} \\ \mathcal{D}_d v_i^{(d)} &= \mu_i^{(d)} v_i^{(d)}. \end{aligned}$$

4) The estimates

$$\begin{aligned} \hat{s}_i^{(d)} &= \text{signum}(\mu_i^{(d)}) \\ \hat{\sigma}_i^{(d)} &= \frac{|\mu_i^{(d)}|}{\sqrt{1 - \mu_i^{(d)2}}} \end{aligned}$$

$i = 1, \ldots n$ and the $\hat{B}_i^{(d)}$ parameters are calculated from

$$\hat{B}_i^{(d)} = \sqrt{2\sigma_i^{(d)}} s_i^{(d)} \left(\sum_{j=0}^{d} (-1)^j \, v_{i,j}^{(d)}\right)$$

7.3.2 Consistency of Deterministic Model Estimates

Theorem 7.6 *Given that assumptions L.1 hold and that $\hat{A}_d$ and $\hat{B}_d$ are given as described above then*

$$\begin{aligned} \lim_{\substack{d\to\infty \\ k\to\infty}} \left\| \tilde{\mathbf{H}}_{G_o} R - \mathbf{T}_{\hat{A}_d}^{-1}\left(R\mathbf{H}_{\hat{B}_d(z^{-1})}\right)\right\|_{op} &= 0. \\ \lim_{\substack{d\to\infty \\ d\geqslant D}} d^{\kappa} \left\| \tilde{\mathbf{H}}_{G_o} R - \left(\mathbf{P}_d \mathbf{T}_{\hat{A}_d} \mathbf{P}_d\,|_d\right)^{-1} \left(\mathbf{P}_d R \mathbf{H}_{\hat{B}_d(z^{-1})} \mathbf{P}_d\,|_d\right)\right\|_{op} &= 0. \end{aligned}$$

Proof : From (7.13) and various familiar inequalities

$$\begin{aligned}\left\|\mathbf{T}_{\hat{A}_d}^{-1}\left(R\mathbf{H}_{\hat{B}_d(z^{-1})}\right)-\tilde{\mathbf{H}}_{G_o}R\right\|_{op} &= \left\|\mathbf{T}_{\hat{A}_d}^{-1}\left(R\mathbf{H}_{\hat{B}_d(z^{-1})}\right)-\mathbf{T}_{A_o}^{-1}\left(R\mathbf{H}_{B_o(z^{-1})}\right)\right\|_{op}\\ &= \left\|\left(\mathbf{T}_{\hat{A}_d}^{-1}-\mathbf{T}_{A_o}^{-1}\right)\left(R\mathbf{H}_{\hat{B}_d(z^{-1})}\right)+\mathbf{T}_{A_o}^{-1}R\left(\mathbf{H}_{\hat{B}_d(z^{-1})}-\mathbf{H}_{B_o(z^{-1})}\right)\right\|_{op}\\ &\leqslant \left\|\mathbf{T}_{\hat{A}_d}^{-1}-\mathbf{T}_{A_o}^{-1}\right\|_{op}\left\|\hat{B}_d(z^{-1})\right\|_\infty+\left\|\mathbf{T}_{A_o}^{-1}\right\|_{op}\left\|\hat{B}_d(z^{-1})-B_o(z^{-1})\right\|_\infty .\end{aligned}$$

Also the proof of Proposition 6.37 demonstrates that

$$\lim_{\substack{d\geqslant D\\ d\to\infty}}\left\|\mathbf{T}_{\hat{A}_d}^{-1}-\mathbf{T}_{A_o}^{-1}\right\|_{op}=0.$$

The proof of the first statement is then established upon noting that $\hat{B}_d(z^{-1})$ is bounded in $\mathcal{L}^\infty$; $\left\|\mathbf{T}_{A_o}^{-1}\right\|_{op}=\|A_o^{-1}\|_\infty<\gamma^{-\frac{1}{2}}$ and

$$\lim_{\substack{d\geqslant D\\ d\to\infty}}\left\|B_o-\hat{B}_d\right\|_\infty=0$$

by Theorem 7.3.

To establish the second part consider:

$$\begin{aligned}&\mathbf{T}_{\hat{A}_d}^{-1}\left(R\mathbf{H}_{\hat{B}_d(z^{-1})}\right)-\left(\mathbf{P}_m\mathbf{T}_{\hat{A}_d}^{-1}\mathbf{P}_m\right)\left(\mathbf{P}_mR\mathbf{H}_{\hat{B}_d(z^{-1})}\mathbf{P}_m\right)\\ &\quad=\left(\mathbf{T}_{\hat{A}_d}^{-1}-\left(\mathbf{P}_m\mathbf{T}_{\hat{A}_d}^{-1}\mathbf{P}_m\right)\right)\left(R\mathbf{H}_{\hat{B}_d(z^{-1})}\right)+\left(\mathbf{P}_m\mathbf{T}_{\hat{A}_d}^{-1}\mathbf{P}_m\right).\end{aligned}$$

Since

$$\mathrm{im}R\mathbf{H}_{\hat{B}_d(z^{-1})}\leqslant\left(\mathcal{P}_{p\times p}^d\right)^\perp=\ker\left(\mathbf{T}_{\hat{A}_d}^{-1}-\mathbf{P}_m\mathbf{T}_{\hat{A}_d}^{-1}\mathbf{P}_m\right)$$

and the second term is identically zero then the two terms are equal.

This completes the proof. ∎

Finally application of Corollary 6.44.1 gives the final form of the required result.

Theorem 7.7 *Let (A,B,C) be a realisation of*

$$G_d(z)=C(z^{-1}I-A)^{-1}B$$

the deterministic component of an ARMAX system of the form (3.7) where the equivalent continuous-time transfer function

$$G_c(s)=\tilde{C}\left(sI-\tilde{A}\right)^{-1}\tilde{B}$$

is an element of $\mathrm{C}_{\mathbf{n}}^{\mathbf{m},\mathbf{m}}[sym]$ and $\left(\tilde{A},\tilde{B},\tilde{C}\right)$ is an element of the symmetric stable canonical form of Theorem 4.44 given by the parameters

$$\left\{s_i,\sigma_i,\check{B}_i\,|\,i=1,\ldots,n\right\}.$$

Also let $\mathcal{E}_d$ and

$$\left\{\hat{s}_i^{(d)},\hat{\sigma}_i^{(d)},\hat{B}_i^{(d)}\,|\,i=1,\ldots,n,\ d\in\mathbb{N}\right\}$$

be constructed by the algorithm described in §7.3.1.

If conditions L.1 hold with $\kappa\geqslant1$ then $\hat{s}_i^{(d)}$, $\hat{\sigma}_i^{(d)}$ and $\hat{B}_i^{(d)}$ are strongly consistent estimators of s_i, σ_i and $\check{B}_i$, $i=1,\ldots,n$.

∎

7.4 Conclusions

This Chapter has shown that it is possible to obtain strongly consistent estimates of the parameters of an ARMAX process from a series of ARX estimates of the process. Combining these results with Hendry's results of [57] show that this is a basis for an efficient estimation algorithm i.e. one which achieves the Cramer-Rao lower bound on the variance of the estimates.

Chapter 8

Conclusions

This thesis has provided approaches to several of the standard problems which arise in system identification. Basic to each approach are some of the properties of balanced realisations.

This Chapter describes the conclusions and contributions of this thesis, suggests areas of work to consolidate the results above, and suggests areas of further research which are natural continuations of the work.

8.1 Contributions

The main contributions of this thesis are:

- Four novel forms of balanced realisation have been introduced, namely left and right minimum-phase balanced realisations in both discrete and continuous time versions. Their existence and uniqueness properties have been demonstrated, and the interrelations between the realisations explained. It has been demonstrated that this form of realisation characterises minimum-phase systems. The properties of augmentability and reducibility have been established for these balanced realisations (Chapter 4).

- A canonical form has been derived for minimum-phase systems based on minimum-phase balancing. This canonical form has been shown to provide a parametrisation of the set of minimum-phase systems (Chapter 4).

- Other canonical forms for symmetric positive-real, symmetric stable and symmetric minimum-phase systems have been proposed. These canonical forms also provide parametrisations of the respective sets of transfer functions (Chapter 4).

- A model reduction scheme has been proposed for minimum-phase balanced models and a relative error bound has been established for this scheme (Chapter 4).

- The relationships between the balanced state-space realisations of the spectral factors, spectral summand and the phase factor corresponding to a spectral-density have been described. Moreover, the parameters of the various canonical forms have been related one to another. It has been shown that the eight transfer functions mentioned above (four continuous and four discrete equivalents) may be simultaneously balanced, each in a different manner, to give each in its canonical form. From this the realisations and parameters may be related (Chapter 5).

- Expressions for the eigenvalues and eigenvectors of the flipped canonical correlation Hankel operator $R\mathbf{H}_{V^{-1}W^{*}}$ have been obtained in terms of the parameters of the minimum-phase parametrisation of the continuous-time right spectral factor (Chapter 5).

- Proposition 6.32 provided a generalisation of a result of Caines [15] describing the rate of convergence of ensembles of estimated lagged covariances of an observed time series to their true values (Chapter 6).

- A uniformly convergent spectral estimation scheme has been established for time-series models (Chapter 6).

- An estimation scheme for the parameters describing a time-series has been proposed. It has been shown that this fails to provide a consistent scheme since it incorporates a spectral factorisation step for which no sufficiently powerful results are available (Chapter 6).

- An analysis of Wilson's algorithm for spectral factorisation obtained new results describing conditions for the steps of the algorithm to be well posed, and a conjecture under which the properties of the algorithm would provide a restricted consistency result (Chapter 6).

- A consistent estimation scheme for the parameters describing the stochastic and deterministic components of an ARMAX system has been described in Chapter 7.

Other contribution of this thesis are:

- Connections between the many representations of both stochastic and deterministic systems have been presented. Special attention has been paid to the development of the theory of canonical correlations between the past and future of a stochastic process and the relationship with state-space representations and Hankel operators (Chapter 2).

- The connections between spectral factors, spectral summands and phase factors have been explored in terms of the state-space realisations of these transfer functions. (Chapter 5).

8.2 Further Work

There are two areas where further work is necessary to consolidate that described in Chapter 6 of this thesis:

- Conjecture 1, regarding the boundedness of the operator $\left(R\mathbf{H}_{z^{-1}\hat{V}_d^*\hat{V}_d^{-1}} + I\right)^{-1}$ over $\mathcal{L}^1$, $\forall d \geqslant D \in \mathbb{N}$ remains to be established. It may be possible to exploit the nature of the objects under consideration to obtain a bound on $\left\|\hat{V}_d^*\hat{V}_d^{-1}\right\|_{A_{1\times 1}(\mathbb{T})}$.

- No algorithm has been suggested for the estimation of the D term appearing in Chapter 6. It is clear that the scalar case estimator is the standard deviation of the model error for the estimated system. An example of a similar result is found in Ljung & Wahlberg [80]. No results have been obtained for such an estimator in the context described in Chapter 6.

8.3 Further Topic of Research

The following items are a selection of the most relevant ways in which the results and investigations of this thesis may be extended:

- The algorithms described in Chapters 6 and 7 should be implemented. This would allow the testing of the small sample properties of the proposed algorithms. Also, this would allow comparison with other estimation methods, in terms of computational efficiency.

- The continuity of these parameter estimates with respect to $\mathcal{L}^\infty$-norm perturbations of the system might be established. This would provide further evidence that these are suitable parameterisations for system identification.

- Further investigation of spectral factorisation algorithms should be carried out so as to extend the range of the results of Chapter 6. A starting point for this would be the analysis which appears in Johnson [68].

Appendix A

Canonical Forms

Theorem A.1 (Canonical Form for Positive-Real Transfer Functions) $G \in \mathrm{TP}_{\mathrm{n}}^{\mathrm{m}}$ *if and only if there exists an unique realisation* (A, B, C, D) *given by the following set of parameters:*

$$\begin{array}{llllll}
p_1 > \ldots > p_k > 0 & , & \textit{where} & p_i \in (0,1) \subseteq \mathbb{R} \ , \\
n_1 \ , \ \ldots \ , \ n_k & , & " & n_i \in \mathbb{N} \ ; \ \sum_{j=1}^{k} n_j = n \ , \\
r_1 \ , \ \ldots \ , \ r_k & , & " & r_i \in \mathbb{N} \ ; \ r_j \leqslant n_j \ , \\
\check{U}_1 \ , \ \ldots \ , \ \check{U}_k & , & " & \check{U}_i \in \mathbb{R}^{r_i \times m}, \ \textit{s.t.} \ \check{U}_i \check{U}_i^T = I_{r_i} \\
\check{A}_1 \ , \ \ldots \ , \ \check{A}_i & , & " & \check{A}_i \ r_i - \textit{balanced}, \\
\check{B}_1 \ , \ \ldots \ , \ \check{B}_k & , & " & \check{B}_i \ \textit{positive-upper-triangular and} \ \mathrm{rank}(\check{B}_i) = r_i, \\
\check{D}, & & \check{D} + \check{D}^T > 0. &
\end{array}$$

The matrices (A, B, C, D) *are partitioned according to* $(n_i)_{i=1}^k$ *and the individual blocks are given by*

$$\begin{aligned}
B_i &= \begin{bmatrix} \check{B}_i \\ 0 \end{bmatrix} \\
A_{ii} &= \check{A}_i + B_i(D+D^T)^{-1}C_i - \frac{1+p_i^2}{2p_i} B_i(D+D^T)^{-1}B_i^T \\
A_{ij} &= B_i(D+D^T)^{-1}C_j - \frac{1}{p_j^2 - p_i^2}\left\{p_j(1-p_i^2)B_i(D+D^T)^{-1}B_j^T - p_i(1-p_j^2)C_i^T(D+D^T)^{-1}C_j\right\} \\
C_i^T &= \sqrt{B_i(D+D^T)^{-1}B_i^T}\, U_i(D+D^T)^{1/2} \\
D &= \check{D} \\
U_i &= \begin{bmatrix} \check{U}_i \\ 0 \end{bmatrix} \in \mathbb{R}^{m \times m}
\end{aligned}$$

where

$$\sqrt{\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} \sqrt{A} & 0 \\ 0 & 0 \end{bmatrix}.$$

Furthermore, (A, B, C, D) *is positive-real balanced with Gramian* $\Sigma = \mathrm{diag}(p_1, \ldots, p_k)$*.*

Proof : Consider the (i,i)th blocks of the equations (4.1) and (4.2):

$$\begin{aligned}
p_i\left(A_{ii} + A_{ii}^T\right) - p_i B_i(D+D^T)^{-1}C_i - p_i C_i^T(D+D^T)^{-1}B_i^T + p_i^2 C_i^T(D+D^T)^{-1}C_i & \\
+ B_i(D+D^T)^{-1}B_i^T &= 0 \quad (A.1) \\
p_i\left(A_{ii} + A_{ii}^T\right) - p_i B_i(D+D^T)^{-1}C_i - p_i C_i^T(D+D^T)^{-1}B_i^T + p_i^2 B_i(D+D^T)^{-1}B_i^T & \\
+ C_i^T(D+D^T)^{-1}C_i &= 0 \quad (A.2)
\end{aligned}$$

Addition of these two equations gives:

$$\begin{aligned}
2p_j\left(A_{ii} + A_{ii}^T\right) &= 2p_i B_i(D+D^T)^{-1}C_i + 2p_i C_i^T(D+D^T)^{-1}B_i^T \\
&\quad - (1+p_i^2)C_i^T(D+D^T)^{-1}C_i - (1+p_i^2)B_i(D+D^T)^{-1}B_i^T \\
A_{ii} &= \check{A} + B_i(D+D^T)^{-1}C_i \\
&= -\frac{1+p_i^2}{4p_i}\left\{C_i^T(D+D^T)^{-1}C_i + B_i(D+D^T)^{-1}B_i^T\right\} \qquad (A.3)
\end{aligned}$$

Subtraction of these two gives

$$\begin{aligned}(p_i^2-1)C_i^T(D+D^T)^{-1}C_i &= (p_i^2-1)B_i(D+D^T)^{-1}B_i^T \\ C_i^T\sqrt{D+D^T}^{-1} &= \sqrt{B_i(D+D^T)^{-1}B_i^T}\begin{bmatrix}\check{U}_i \\ 0\end{bmatrix}. \end{aligned} \tag{A.4}$$

where $\check{U}_i \in \mathbb{R}^{r_i \times m}$ and $\check{U}_i\check{U}_i^T = I_{r_i}$.

Substitution of (A.4) into (A.3) gives

$$A_{ii} = \check{A} + B_i(D+D^T)^{-1}C_i - \frac{1+p_i^2}{2p_i}B_i(D+D^T)^{-1}B_i^T \tag{A.5}$$

The (i,j)th blocks of the equations (4.1) and (4.2):

$$\begin{aligned}p_j\left(A_{ij}-B_i(D+D^T)^{-1}C_j\right)+p_i\left(A_{ji}^T-C_i^T(D+D^T)^{-1}B_j^T\right)+p_ip_jC_i^T(D+D^T)^{-1}C_j& \\ +B_i(D+D^T)^{-1}B_j^T &= 0 \quad (A.6) \\ p_j\left(A_{ji}^T-C_i^T(D+D^T)^{-1}B_j^T\right)+p_i\left(A_{ij}-B_i(D+D^T)^{-1}C_j\right)+p_ip_jB_i(D+D^T)^{-1}B_j^T& \\ +C_i^T(D+D^T)^{-1}C_j &= 0 \quad (A.7)\end{aligned}$$

Taking $p_j\times$(A.6)$-p_i\times$(A.7) gives

$$(p_j^2-p_i^2)A_{ij}-(p_j^2-p_i^2)B_i(D+D^T)^{-1}C_j-p_i(1-p_j^2)C_i^T(D+D^T)^{-1}C_j+p_j(1-p_i^2)B_i(D+D^T)^{-1}B_j^T=0$$

Hence,

$$A_{ij} = B_i(D+D^T)^{-1}C_j - \frac{1}{p_j^2-p_i^2}\left\{p_j(1-p_i^2)B_i(D+D^T)^{-1}B_j - p_i(1-p_j^2)C_i^T(D+D^T)^{-1}C_j\right\}$$

The properties of augmentability and reducibility are established in Ober [95]. ■

Remark A.2 *In the single-input single-output case the canonical form takes on a much simplified, and more elegant form as the following calculations show:*

$$\begin{aligned}B_i &= \begin{bmatrix}\check{b}_i \\ 0 \\ \vdots \\ 0\end{bmatrix} \\ C_i &= s_iB_i^T \\ A_{ii} &= \check{A}_i - \frac{(1-s_ip_i)^2}{4dp_i}B_iB_i^T \\ A_{ij} &= \frac{s_j(p_j^2-p_i^2)-p_j(1-p_i^2)+s_is_jp_i(1-p_j^2)}{p_j^2-p_i^2}B_i(D+D^T)^{-1}B_j \\ &= \frac{s_j(p_j^2-p_i^2)+p_ip_j(p_i-s_is_jp_j)+(s_is_jp_i-p_j)}{2d(p_j^2-p_i^2)}B_iB_j^T \\ &= \left(s_j+\frac{(p_i-s_is_jp_j)(p_ip_j+s_is_j)}{p_j^2-p_i^2}\right)\frac{B_iB_j^T}{2d} \\ &= \left(s_j-\frac{1+p_ip_js_is_j}{p_j+s_is_jp_i}\right)\frac{B_iB_j^T}{2d}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1 - s_j p_j - s_i p_i + p_i p_j s_i s_j}{p_j + s_i s_j p_i} \frac{B_i B_j^T}{2d} \\
&= -\frac{(1 - s_i p_i)(1 - s_j p_j) B_i B_j^T}{2d(p_j + s_i s_j p_i)}
\end{aligned}
$$

where $s_i \in \{\pm 1\}$ *takes the place of* U_i*,* D *is a scalar* d*.*

$\blacksquare$

Appendix B

Higher Order Moments

It will be assumed throughout this Appendix that the following hold:
A.1, A.3, A.2, D.2, and B.1.

$$y(t) = \sum_{k=0}^{\infty} A(k)e(t-k)$$

$$y_i(t) = \sum_{k=0}^{\infty}\sum_{j=1}^{p} A_{ij}(k)e_j(t-k) \tag{B.1}$$

Lemma B.1 (Covariances in terms of Wold Decomposition)

$$\gamma_{ij}^{(y)}(t) = \begin{cases} \sum_{k=0}^{\infty}\left[A(k)\Gamma_e(0)A^T(k+t)\right]_{ij} & t>0 \\ \sum_{k=0}^{\infty}\left[A(k)\Gamma_e(0)A^T(k)\right]_{ij} & t=0 \\ \sum_{k=0}^{\infty}\left[A(k-t)\Gamma_e(0)A^T(k)\right]_{ij} & t<0 \end{cases} \tag{B.2}$$

$$= \sum_{k=-\infty}^{\infty}\left[A(k)\Gamma_e(0)A^T(k+t)\right]_{ij} \tag{B.3}$$

■

Lemma B.2 (Fourth order central moments in terms of Wold Decomposition) *Using unsuperscripted symbols for the covariances of the process* $(y(t))_{t\in\mathbb{Z}}$*, it is clear that*

$$\begin{aligned}\gamma_{ijkl}(t_1,t_2,t_3) &= \mathbb{E}y_i(0)y_j(t_1)y_k(t_2)y_l(t_3) \\ &= \sum_{a=-\infty}^{\infty}\sum_{q,r,s,t=1}^{p} A_{iq}(a)A_{jr}(a+t_1)A_{ks}(a+t_2)A_{lt}(a+t_3) \\ &\quad\times\begin{pmatrix} \gamma_{qrst}^{(e)}(0,0,0)-\gamma_{qr}^{(e)}(0)\gamma_{st}^{(e)}(0) \\ -\gamma_{qs}^{(e)}(0)\gamma_{rt}^{(e)}(0)-\gamma_{rs}^{(e)}(0)\gamma_{rs}^{(e)}(0)\end{pmatrix} \\ &\quad+\gamma_{ij}(t_1)\gamma_{kl}(t_3-t_2)+\gamma_{ik}(t_2)\gamma_{jl}(t_3-t_1)+\gamma_{il}(t_3)\gamma_{jk}(t_2-t_1)\end{aligned} \tag{B.4}$$

Proof :

$$\begin{aligned}&\mathbb{E}y_i(0)y_j(t_1)y_k(t_2)y_l(t_3) = \\ &\quad\mathbb{E}\sum_{a,b,c,d=-\infty}^{\infty}\sum_{q,r,s,t=1}^{p} A_{iq}(a)e_q(-a)A_{jr}(b)e_r(t_1-b)A_{ks}(c)e_s(t_2-c)A_{lt}(d)e_t(t_3-d) \\ &= \sum_{a,b,c,d=-\infty}^{\infty}\sum_{q,r,s,t=1}^{p} A_{iq}(a)A_{jr}(b)A_{ks}(c)A_{lt}(d)\mathbb{E}e_q(-a)e_r(t_1-b)e_s(t_2-c)e_t(t_3-d)\end{aligned}$$

Only terms with innovations occurring in two pairs together or all four together in the time variable give a non-zero contribution. Hence the following four terms are obtained.

$$\begin{aligned}&= \sum_{a=-\infty}^{\infty}\sum_{q,r,s,t=1}^{p} A_{iq}(a)A_{jr}(a+t_1)A_{ks}(a+t_2)A_{lt}(a+t_3)\mathbb{E}e_q(-a)e_r(-a)e_s(-a)e_t(-a) \\ &\quad[\text{ here } -a=t_1-b=t_2-c=t_3-d] \\ &\quad+\sum_{\substack{a,c=-\infty \\ a-c\neq t_2}}^{\infty}\sum_{q,r,s,t=1}^{p} A_{iq}(a)A_{jr}(a+t_1)A_{ks}(c)A_{lt}(c+t_3-t_2)\mathbb{E}e_q(-a)e_r(-a)e_s(-c)e_t(-c)\end{aligned}$$

$$\left[\text{here}\quad \begin{array}{c} t_1 - b = -a \\ t_3 - d = t_2 - c \\ t_2 - c \neq -a \end{array}\right]$$

$$+ \sum_{\substack{a,b=-\infty \\ a-b\neq t_1}}^{\infty} \sum_{q,r,s,t=1}^{p} A_{iq}(a)A_{jr}(b)A_{ks}(a+t_2)A_{lt}(b+t_3-t_1)\mathbb{E}e_q(-a)e_r(-b)e_s(-a)e_t(-b)$$

$$\left[\text{here}\quad \begin{array}{c} t_2 - c = -a \\ t_3 - d = t_1 - b \\ t_1 - b \neq -a \end{array}\right]$$

$$+ \sum_{\substack{a,b=-\infty \\ a-b\neq t_1}}^{\infty} \sum_{q,r,s,t=1}^{p} A_{iq}(a)A_{jr}(b)A_{ks}(b+t_2-t_1)A_{lt}(a+t_3)\mathbb{E}e_q(-a)e_r(-b)e_s(-b)e_t(-a)$$

$$\left[\text{here}\quad \begin{array}{c} t_3 - d = -a \\ t_1 - b = t_2 - c \\ t_1 - b \neq -a \end{array}\right]$$

Adding and subtracting the excluded terms in the last three summands in combination with the assumption on the fourth order moments gives

$$\begin{aligned}
= &\sum_{a=-\infty}^{\infty} \sum_{q,r,s,t=1}^{p} A_{iq}(a)A_{jr}(a+t_1)A_{ks}(a+t_2)A_{lt}(a+t_3)\gamma^{(e)}_{qrst}(0,0,0) \\
&+ \sum_{a=-\infty}^{\infty} \sum_{q,r=1}^{p} A_{iq}(a)\gamma_{qr}(0)A_{jr}(a+t_1) \sum_{c=-\infty}^{\infty} \sum_{s,t=1}^{p} A_{ks}(c)\gamma_{st}(0)A_{lt}(c+t_3-t_2) \\
&- \sum_{a=-\infty}^{\infty} \sum_{q,r,s,t=1}^{p} A_{iq}(a)A_{jr}(a+t_1)A_{ks}(a+t_2)A_{lt}(a+t_3)\gamma_{qr}(0)\gamma_{st}(0) \\
&+ \sum_{a=-\infty}^{\infty} \sum_{q,s=1}^{p} A_{iq}(a)\gamma_{qs}(0)A_{ks}(a+t_2) \sum_{b=-\infty}^{\infty} \sum_{r,t=1}^{p} A_{jr}(b)\gamma_{rt}(0)A_{lt}(b+t_3-t_1) \\
&- \sum_{a=-\infty}^{\infty} \sum_{q,r,s,t=1}^{p} A_{iq}(a)A_{jr}(a+t_1)A_{ks}(a+t_2)A_{lt}(a+t_3)\gamma_{qs}(0)\gamma_{rt}(0) \\
&+ \sum_{a=-\infty}^{\infty} \sum_{q,t=1}^{p} A_{iq}(a)\gamma_{qt}(0)A_{lt}(a+t_3) \sum_{b=-\infty}^{\infty} \sum_{s,r=1}^{p} A_{jr}(b)\gamma_{rs}(0)A_{ks}(b+t_2-t_1) \\
&- \sum_{a=-\infty}^{\infty} \sum_{q,r,s,t=1}^{p} A_{iq}(a)A_{jr}(a+t_1)A_{ks}(a+t_2)A_{lt}(a+t_3)\gamma_{rs}(0)\gamma_{rs}(0) \\
= &\sum_{a=-\infty}^{\infty} \sum_{q,r,s,t=1}^{p} A_{iq}(a)A_{jr}(a+t_1)A_{ks}(a+t_2)A_{lt}(a+t_3) \left(\begin{array}{c} \gamma^{(e)}_{qrst}(0,0,0) - \gamma^{(e)}_{qr}(0)\gamma^{(e)}_{st}(0) \\ -\gamma^{(e)}_{qs}(0)\gamma^{(e)}_{rt}(0) - \gamma^{(e)}_{rs}(0)\gamma^{(e)}_{rs}(0) \end{array} \right) \\
&+ \sum_{a=-\infty}^{\infty} \left[A(a)\Gamma_e(0)A^T(a+t_1)\right]_{ij} \sum_{c=-\infty}^{\infty} \left[A(c)\Gamma_e(0)A^T(c+t_3-t_2)\right]_{kl} \\
&+ \sum_{a=-\infty}^{\infty} \left[A(a)\Gamma_e(0)A^T(a+t_2)\right]_{ik} \sum_{b=-\infty}^{\infty} \left[A(b)\Gamma_e(0)A^T(b+t_3-t_1)\right]_{jl} \\
&+ \sum_{a=-\infty}^{\infty} \left[A(a)\Gamma_e(0)A^T(a+t_3)\right]_{il} \sum_{b=-\infty}^{\infty} \left[A(b)\Gamma_e(0)A^T(b+t_2-t_1)\right]_{jk}
\end{aligned}$$

which from Lemma B.1 gives

$$= \sum_{a=-\infty}^{\infty} \sum_{q,r,s,t=1}^{p} A_{iq}(a)A_{jr}(a+t_1)A_{ks}(a+t_2)A_{lt}(a+t_3) \begin{pmatrix} \gamma^{(e)}_{qrst}(0,0,0) - \gamma^{(e)}_{qr}(0)\gamma^{(e)}_{st}(0) \\ -\gamma^{(e)}_{qs}(0)\gamma^{(e)}_{rt}(0) - \gamma^{(e)}_{rs}(0)\gamma^{(e)}_{rs}(0) \end{pmatrix}$$
$$+ \gamma_{ij}(t_1)\gamma_{kl}(t_3-t_2) + \gamma_{ik}(t_2)\gamma_{jl}(t_3-t_1) + \gamma_{il}(t_3)\gamma_{jk}(t_2-t_1)$$

■

Lemma B.3 (Alternative form for $\gamma_{ijij}(k,\nu,\nu+k)$)

$$\begin{aligned}
\gamma^{(y)}_{ijij}(k,\nu,\nu+k) &= \mathbb{E}y_i(0)y_j(k)y_i(\nu)y_j(\nu+k) \\
&= \gamma_{ij}(k)^2 + \gamma_{ii}(\nu)\gamma_{jj}(\nu) + \gamma_{ij}(\nu+k)\gamma_{ji}(\nu-k) && \text{(B.5)} \\
&\quad + \sum_{a=-\infty}^{\infty} \sum_{q,r,s,t=1}^{p} A_{iq}(a)A_{jr}(a+k)A_{is}(a+\nu)A_{jt}(a+\nu+k) && \text{(B.6)} \\
&\quad \times \begin{pmatrix} \gamma^{(e)}_{qrst}(0,0,0) - \gamma^{(e)}_{qr}(0)\gamma^{(e)}_{st}(0) \\ -\gamma^{(e)}_{qs}(0)\gamma^{(e)}_{rt}(0) - \gamma^{(e)}_{rs}(0)\gamma^{(e)}_{rs}(0) \end{pmatrix} \\
&= \gamma_{ij}(k)^2 + \gamma_{ii}(\nu)\gamma_{jj}(\nu) + \gamma_{ij}(\nu+k)\gamma_{ji}(\nu-k) \\
&\quad + \sum_{a=-\infty}^{\infty} \sum_{q,r,s,t=1}^{p} A_{iq}(a)A_{jr}(a+k)A_{is}(a+\nu)A_{jt}(a+\nu+k)\gamma_{qrst}(0,0,0) \\
&\quad - \sum_{a=-\infty}^{\infty} \left[A(a)\Gamma_e(0)A^T(a+k)\right]_{ij} \left[A(a+\nu)\Gamma_e(0)A^T(a+\nu+k)\right]_{ij} \\
&\quad - \sum_{a=-\infty}^{\infty} \left[A(a)\Gamma_e(0)A^T(a+\nu)\right]_{ii} \left[A(a+k)\Gamma_e(0)A^T(a+\nu+k)\right]_{jj} \\
&\quad - \sum_{a=-\infty}^{\infty} \left[A(a)\Gamma_e(0)A^T(a+\nu+k)\right]_{ij} \left[A(a+k)\Gamma_e(0)A^T(a+\nu)\right]_{ij} && \text{(B.7)}
\end{aligned}$$

■

Appendix C

Notation

This section contains a description, and some definitions, of most of the notation contained in the main body of the thesis.

The symbol := will be taken to mean that the expression on the right hand side is a definition of the symbols appearing on the left.

C.1 General

Let S be a subset of a metric space, A a set, a, n whole numbers, and x a real number.

Symbols	Meaning
$\bar{S}, S^{\mathbb{C}}$	The closure and the complement of S in the appropriate space.
$\mathbb{R}, \mathbb{C}, \mathbb{T}, \mathbb{D}, i\mathbb{R}$	The real line, complex plane, unit circle, open unit disc and imaginary axis.
$\mathbb{Z}, \mathbb{Z}_+, \mathbb{Z}_-$	The set of integers, positive integers and non-positive integers.
$\Re z, \Im z$	The real and imaginary parts of z.
$\mathbb{C}_\pm$	$:= \{z \in \mathbb{C} \mid \pm \Re z > 0\}$ - the open right/left half-plane.
δ_{ij}	$:= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$.
$M(s)$	$:= \frac{1-s}{1+s}$ is the "bilinear transform" (see (2.30)).
$f(x) = O_a(g(x))$	$\exists K > 0, \epsilon > 0,\ \lvert f(x)\rvert \leqslant K\lvert g(x)\rvert\ \forall x \in (a-\epsilon, a+\epsilon)$
$f(x) = o_a(g(x))$	$\lim_{x\to a} f(x)/g(x) = 0$
$f(x) \sim_a g(x)$	$\exists K, L > 0, \exists \epsilon > 0$ such that $L\lvert g(x)\rvert \leqslant \lvert f(x)\rvert \leqslant K\lvert g(x)\rvert, \forall x \in (a-\epsilon, a+\epsilon)$ (as in above two cases, omission of a indicates $a = 0$)
$f(x) \sim_\infty g(x)$	$\exists K, L, N > 0$, such that $L\lvert g(x)\rvert \leqslant \lvert f(x)\rvert \leqslant K\lvert g(x)\rvert, \forall x > N$
$\lim_{x \nearrow x_0} f(x)$	The limit of f(x) as x approaches x_0 from below.
$\lim_{x \searrow x_0} f(x)$	The limit of f(x) as x approaches x_0 from above.
$\text{signum}(x)$	$:= \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$
$a \bmod n$	$:= r + 1$ where $(a-1) = bn + r,\ b \in \mathbb{Z},\ 0 \leqslant r < n$.
$\text{card}(A)$	The cardinality of A.

C.2 Matrices

Let A be a $p \times p$ real matrix, B a $m \times n$ complex matrix, C a $p \times p$ complex matrix and P a $p \times p$ complex positive-definite matrix. Let A_i be complex $p_i \times p_i$ matrices , $i = 1, \ldots, k$.

Symbols	Meaning
$\mathbb{R}^{p\times m}, \mathbb{C}^{p\times m}$	Sets of real and complex $p \times m$ matrices.
$\mathrm{GL}_n(\mathbb{R})$	The group of $n \times n$ non-singular real matrices.
$\mathrm{O}_n(\mathbb{R})$	The group of $n \times n$ orthogonal real matrices.
$\mathrm{O}_{m\times n}(\mathbb{R})$	$:= \left\{U \in \mathbb{R}^{m\times n} \,:\, UU^T = I_m \text{ if } m \leqslant n, \text{ or } U^TU = I_n \text{ if } n \leqslant m\right\}.$
$\mathrm{P}_m(\tau_1, \tau_2)$	$:= \left\{ U \in \mathrm{O}_{(\tau_1+\tau_2)\times m}(\mathbb{R}) \;\middle|\; \begin{array}{c} U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \ U_i \in \mathrm{O}_{\tau_i\times m}(\mathbb{R}), \\ U_i \text{ positive-upper-triangular}, \ i = 1,2 \end{array} \right\}.$
$\mathrm{diag}(A_1, \ldots, A_k)$	The block diagonal square matrix of size $\sum_{i=1}^k p_i$.
$[B]_{ij}$	The (i,j)th element of the matrix B, $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$.
$\pi(A)$	The number of eigenvalues of A in $\mathbb{C}_+$, counting multiplicity.
$\nu(A)$	The number of eigenvalues of A in $\mathbb{C}_-$, counting multiplicity.
$\delta(A)$	The number of eigenvalues of A on $i\mathbb{R}$, counting multiplicity.
$\mathrm{In}(A)$	$:= (\pi(A), \delta(A), \nu(A))$ - the inertia of a matrix.
$\mathrm{sig}(C)$	$:= \pi(C) - \nu(C)$. The *signature* of C.
$\mathrm{tr}\,(C)$	$:= \sum_{i=1}^p [C]_{ii}$. The *trace* of C.
$\det(C)$	The determinant of C.
$P^{1/2} = \sqrt{P}$	The positive-definite square root of P.
$B > 0, B \geqslant 0$	B is a positive-definite (positive-semi-definite) Hermitian matrix.
$M(C)$	$:= (I - C)(I + C)^{-1}$ is the "bilinear transform" for matrices (see §2.2.3).

C.3 Stochastic Processes

The following are the symbols used to describe the probabilistic and covariance structures used in this dissertation.

Symbols	Meaning
$(\Omega, \mathcal{F}, \mathbb{P})$	The underlying probability space on which all random variables are defined (see §2.3).
$\mathbb{P}$	The probability measure on the underlying probability space (see §2.3).
w.p. 1 $[\mathbb{P}]$	The event so predicated happens for all but a measure zero subset of the probability space Ω.
$\mathbb{E}(X)$	The expectation of the random variable X (see (2.37)).
$(x\|X)$	The orthogonal projection of x onto the space X (see Definition 2.22).
$\mathrm{Var}(X)$	The variance of the random variable X.
$\Gamma_e(k)$	The kth covariance of the process $(e(t))_{t\in\mathbb{Z}}$ (see Definition 2.20).
$\hat{\Gamma}^{(N)}(k)$	The estimate of the kth covariance of the underlying process using N data points. See 6.3 for definition.
$\gamma_{ij}(r)$	$:= \mathbb{E}y_i(0)y_j(r)$ (see (2.38).
$\gamma_{ijkl}(r,s,t)$	$:= \mathbb{E}y_i(0)y_j(r)y_k(s)y_l(t)$ (see (2.38).
$\Phi(e^{i\theta})$	The spectral-density function of $(y(t))_{t\in\mathbb{Z}}$ (see Definition 2.31).
$\hat{\Phi}_d(e^{i\theta})$	The spectral-density function estimate of order d. See equation (6.2) for the definition.
V, W	The right and left spectral factors of $\Phi(e^{i\theta})$.
$\hat{V}_d(e^{i\theta}), \hat{W}_d(e^{i\theta})$	The right and left spectral factors of $\hat{\Phi}_d(e^{i\theta})$. See §5.1.2 and §6.2.2.
$\tilde{V}_{d,k}(e^{i\theta}), \tilde{W}_{d,k}(e^{i\theta})$	Approximants to $\hat{V}_d(e^{i\theta})$ and $\hat{W}_d(e^{i\theta})$. See §6.3 for definition.
$J(x,y)$	The mutual information between x and y. See Definition 2.49.

C.4 Norms, Operators and Linear Spaces

Let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ be complex vector spaces and let K and $H_0, H_1 \leqslant H_2$ be Hilbert spaces, and let $A : H_2 \to K$ be a bounded linear operator.

Symbols	Meaning
$\mathcal{U} \leqslant \mathcal{V}$	$\mathcal{U}$ is a subspace of $\mathcal{V}$.
$\mathcal{V} \oplus \mathcal{W}$	The (external) direct sum of $\mathcal{V}$ and $\mathcal{W}$.
$H_0 \bigoplus H_1$	The inner direct sum of orthogonal subspaces of H_2.
$\bigvee_{i \in I} x_i, \bigvee_H \{x_i \mid i \in I\}$	The closed linear span in H of the set $\{x_i \mid i \in I\}$.
$H_1^{\perp}$	The orthogonal complement of H_1 in H_2.
$L(\mathcal{V}, \mathcal{W})$	The set on linear maps from $\mathcal{V}$ to $\mathcal{W}$.
$\mathrm{End}(K)$	The set of bounded (in operator norm) elements in $L(K, K)$.
I_K	The identity mapping on K.
0_K	The zero mapping on K.
$A\mid_{H_1}$	The restriction of A to the space H_1 i.e. the map $H_1 \to K$ given by $A\mid_{H_1} x = Ax, \ \forall x \in H_1$.
$A\mid_m$	$:= A\mid_{\mathcal{P}^m_{p\times p}}$.
$\mathbf{P}_{H_1}$	The orthogonal projection of H_2 onto H_1. See §2.2.4.
$\mathbf{P}_+$	The Riesz projection of $\mathcal{L}^2_{p\times p}$ onto $\mathcal{H}^2_{p\times p}$.
$\mathbf{P}_-$	$:= \left(I_{\mathcal{L}^2_{p\times p}} - \mathbf{P}_+\right)$.
$\mathbf{P}_k$	$:= \mathbf{P}_{\mathcal{P}^k_{p\times p}}$.
$\mathbf{H}_\phi$	A Hankel operator with symbol ϕ. See Definition 2.10.
$\mathbf{T}_\phi$	A Toeplitz operator with symbol ϕ. See Definition 2.44.
R	The flip operator. See Definition 2.15.
$R(T, z)$	The resolvent operator function. See Definition 6.5.
$\mathbf{P}(T; \lambda)$	The spectral projection operator of T associated with eigenvalue λ. See Definition 6.7.
$\mathrm{Lap}_{p\times q}$	The Laplace transform (2.18).
$\mathrm{Hilb}_{p\times q}$	The Hilbert isomorphism (see §2.2.4).
$\mathbf{C}_{\hat{\phi}}, \phi * u$	A convolution operator (see (2.33, §2.2.1).
$S_{p\times q}$	Shift operator (see §2.2.1).
$\mathbf{Trunc}^s_{p\times q}$	A truncation operator (see §2.2.1).
$\mathbf{M}_\phi$	A multiplication operator (see §2.2.4).
$[f]_\pm$	$:= \mathbf{P}_\pm f$.

In this dissertation the following conventions are followed in the naming of norms. Let $X = [x_{ij}]^{1\leqslant i\leqslant p}_{1\leqslant j\leqslant m} \in \mathbb{C}^{p\times m}$, $x \in \mathbb{C}^{p\times 1}$ and $f : \mathbb{T} \to \mathbb{C}^{p\times p}$. Also let $(u(t))_{t\in\mathbb{Z}}$ be a sequence with $u(t) \in \mathbb{R}^{p\times q}$.

Symbols	Meaning
$\langle x, y\rangle_H$	The inner product on the space H e.g. see p11, (2.9), (2.11), (2.40), and (2.42).
$\lVert x\rVert_r$	$:= \left(\sum_{i=1}^{p} \lvert x_i^r\rvert\right)^{\frac{1}{r}},\ r\in[1,\infty)$.
$\lVert x\rVert_\infty$	$:= \max\{\lvert x_i\rvert, i=1,\dots,p\}$
$\lVert X\rVert_\sigma$	The maximum singular value norm of a complex matrix X.
$\lVert X\rVert_2$	$:= \left(\sum_{i=1}^{p}\sum_{j=1}^{m} \lvert x_{ij}\rvert^2\right)^{\frac{1}{2}}$
$\lVert X\rVert_1$	$:= \sum_{i=1}^{p}\sum_{j=1}^{m} \lvert x_{ij}\rvert$
$\lVert X\rVert_{op,1}$	$:= \max_i \sum_{j=1}^{m} \lvert x_{ij}\rvert$, the maximum row sum
$\lVert X\rVert_{op,\infty}$	$:= \max_j \sum_{i=1}^{p} \lvert x_{ij}\rvert$, the maximum column sum
$\lVert A\rVert_{op}$	The operator norm of A. This is the same as $\lVert X\rVert_{op}$, but is used to distinguish finite matrices.
$\lVert f\rVert_\infty$	$:= \text{ess sup}_{z\in\mathbb{T}} \lVert f(z)\rVert_\sigma$.
$\lVert f\rVert_2$	$:= \left(\frac{1}{2\pi}\int_{\mathbb{T}} \left\lVert f(e^{i\theta})\right\rVert_2^2 \, d\theta\right)^{\frac{1}{2}}$
$\lVert f\rVert_p$	$:= \left(\frac{1}{2\pi}\int_{\mathbb{T}} \left\lVert f(e^{i\theta})\right\rVert_p^p \, d\theta\right)^{\frac{1}{p}}$
$\left\lVert \sum_{k=-\infty}^{\infty} A_k e^{ik\theta}\right\rVert_{A_{p\times p}(\mathbb{T})}$	$:= \sum_{k=-\infty}^{\infty} \lVert A_k\rVert_1$
$\left\lVert (u(t))_{t\in\mathbb{Z}}\right\rVert_2$	See (2.2).
$\lVert f\rVert_{C^0}$	$:= \sup_{\mathbb{T}} \left\lVert f(e^{i\theta})\right\rVert_\sigma$.
$\text{dist}(f,\mathcal{H}^\infty)$	$:= \inf_{g\in\mathcal{H}^\infty} \lVert f-g\rVert_\infty$.

The following are used to describe the various modes of convergence of operators. Let T and $T_n,\ n\in\mathbb{N}$ be bounded operators.

Symbols	Meaning
$T_n \xrightarrow{s} T$	T_n are stably convergent to T. See Definition 6.6.
$T_n \xrightarrow{cc} T$	T_n exhibit collectively compact convergence to T. See definition 6.9.
$T_n \xrightarrow{ss} T$	T_n are strongly stably convergent to T. See Definition 6.11.
$T_n \xrightarrow{\sigma} T$	T_n are radially convergent to T. See Definition 6.14.
$T_n \xrightarrow{u\sigma} T$	T_n are uniformly radially convergent to T. See Definition 6.15.

The following symbols are used to describe the spectra and related notions of operators.

Symbols	Meaning
$\sigma_{\max}(A), \sigma_{\min}(A), \sigma_k(A)$	The minimum, maximum and kth singular value of the operator A. The ordering is from greatest to smallest in $\mathbb{R}$.
$\lambda_k(A)$	The kth eigenvalues of A, where if A is Hermitian, the ordering is from greatest to smallest in $\mathbb{R}$, else the ordering is lexicographic in decreasing modulus and increasing argument of the eigenvalues.
$\lambda_{\min}(H), \lambda_{\max}(H)$	The minimum and maximum eigenvalues of a Hermitian operator H, where the ordering is the usual one on the real numbers.
$\lambda_\sigma(A)$	The spectral radius of A. See Definition 6.13.
$\mathrm{tr}\,(A)$	The trace of the operator A.
$\rho(A)$	The resolvent set of A. See Definition 6.5.

The following are the symbols used to denote spaces of functions. Let $\mathcal{T}$ be a topological space.

Symbols	Meaning
$\mathcal{A}_{p\times p}(\mathbb{D})$	The disc algebra of continuous functions on $\mathbb{T}$ which are analytic in $\mathbb{D}$. The norm is the C_0 norm on $\mathbb{T}$.
$A_{p\times p}(\mathbb{T})$	The Wiener algebra of $p\times p$ continuous functions with summable fourier coefficients.
$\mathcal{P}^k_{p\times p}$	$:= \left\{ f \mid f(z) = \sum_{i=0}^k A_i z^i,\ A_i \in \mathbb{C}^{p\times p} \right\}$ The space of $p\times p$-matrix polynomials of degree not greater than k.
$C^0_{p\times p}(\mathcal{T}),\ C^0_{p\times p}$	The spaces of continuous functions from $\mathcal{T}$ and from $\mathbb{T}$ to $\mathbb{C}^{p\times p}$.
$C^r_{p\times p}(\mathcal{T}),\ C^r_{p\times p}$	The spaces of r-times continuously differentiable functions from $\mathcal{T}$ and from $\mathbb{T}$ to $\mathbb{C}^{p\times}$
$\mathcal{L}^\infty_{p\times q}$	See (2.5) and (2.14).
$\mathcal{L}^2_{p\times q}$	See (2.8).
$\mathcal{L}^2_{p\times p}(F)$	See (2.41).
$\mathcal{H}^\infty_{p\times q}$	See (2.6) and (2.15).
$\mathcal{H}^2_{p\times q}$	See (2.7) and (2.17).
$\mathcal{H}^2_{p\times p}(F)$	See p35.
$l^2_{p\times q}$	See (2.1).

The following describe operations on the above operators and Hilbert spaces. Let $u = (u(t))_{t\in\mathbb{Z}} \in l^2_{p\times q}$ and $v = (v(t))_{t\in\mathbb{Z}} \in l^2_{q\times r}$.

Symbols	Meaning
$u * v$	The convolution of u and v. See §2.1.

C.5 Transfer Functions and State-Space

In this dissertation the following conventions are followed in the naming of classes of transfer functions.

The following conventions are used throughout:

- Any class of the form $\mathrm{X}_{\mathrm{n}}^{\mathrm{p,m}}$ is a set of $p\times m$ continuous-time state-space systems with an n-dimensional state-space.

- Any class of the form $\mathrm{TX}_{\mathrm{n}}^{\mathrm{p,m}}$ is a set of continuous-time transfer functions with realisations from $\mathrm{X}_{\mathrm{n}}^{\mathrm{p,m}}$.

- Any class of the form $\mathrm{DX}_{\mathrm{n}}^{\mathrm{p,m}}$ is a set of $p \times m$ discrete-time state-space systems with an n-dimensional state-space.

- Any class of the form $\mathrm{TDX}_{\mathrm{n}}^{\mathrm{p,m}}$ is a set of continuous-time transfer functions with realisations from $\mathrm{DX}_{\mathrm{n}}^{\mathrm{p,m}}$.

- Four other classes of systems of the form $\mathrm{X}_{\mathrm{n}}^{\mathrm{p}}$ are $p \times p$ systems with n-dimensional state-space.

- $\mathrm{TX}_{\mathrm{n}}^{\mathrm{p}}[sym]$ is a set of $p \times p$ symmetric continuous-time transfer functions with an n-dimensional state-space.

Assume that G is a real, rational, matrix function.

Symbols	Meaning
$\mathcal{P} \sim \mathcal{Q}$	State-space equivalence (see §2.2.1
$S(A,B,C,D)T$	$:= (SAT, SB, CT, D)$
$(A,B,C,D)^T$	$:= (A^T, C^T, B^T, D^T)$
$(A,B,C,D)^{-1}$	$:= (A - BD^{-1}C, -BD^{-1}, D^{-1}C, D^{-1})$
$G \stackrel{s}{=} (A,B,C,D)$	$G(s) := \begin{cases} C(sI_n - A)^{-1}B + D \\ C(sI - A)^{-1}B + D \end{cases}$
$[G]_-$	$:= \mathbf{P}_- G$ - the anti-stable part of G whether discrete or continuous-time.
$G^*(s)$	$:= (G(-s))^T$
$G^*(z)$	$:= (G(z^{-1}))^T$
$\mathrm{L}_{\mathrm{n}}^{\mathrm{p,m}}$	The set of minimal realisations of $p \times m$ transfer functions with McMillan degree n. See §2.2.1.
$\mathrm{C}_{\mathrm{n}}^{\mathrm{p,m}}$	The set of stable minimal realisations of $p \times m$ transfer functions with McMillan degree n. See §2.2.1.
$\mathrm{M}_{\mathrm{n}}^{\mathrm{p}}$	The set of $p \times p$ continuous-time minimum-phase transfer function of McMillan degree n (see Definition 3.1).
$\mathcal{I}_{\mathrm{n}}^{\mathrm{p}}$	$:= \{(A,B,C,D) \in \mathrm{DM}_{\mathrm{n}}^{\mathrm{p}} \mid D > 0\}$. The class of models for time series defined in §3.1.1.
$\mathcal{P}_{\mathrm{n}}^{\mathrm{p}}$	$:= \{(A,B,C,D) \in \mathrm{DM}_{\mathrm{n}}^{\mathrm{p}} \mid D = I_p\}$. The class of predictors for models in $\mathcal{I}_{\mathrm{n}}^{\mathrm{p}}$ (see §3.1.1).
$\mathrm{I}_b^a G(s)$	The Cauchy-Index of G on the interval $s \in [a,b]$. See Definition 4.14.
$\mathrm{P}_{\mathrm{n}}^{\mathrm{p}}[\alpha], \mathrm{M}_{\mathrm{n}}^{\mathrm{p}}[\alpha]$	Spectral summands and factors corresponding to symmetric spectral densities, but with simple block structure on B-matrix. See Definition 4.57.
$\mathrm{P}_{\mathrm{n}}^{\mathrm{p}}[\omega], \mathrm{M}_{\mathrm{n}}^{\mathrm{p}}[\omega]$	Spectral summands and factors corresponding to symmetric spectral densities, but with no repeated characteristic values. See Definition 4.57.

C.6 Acronyms

Acronym	Meaning
SISO	Single-Input Single-Output
MIMO	Multiple-Input Multiple-Output
OLHP	Open Left Half-Plane
ORHP	Open Right Half-Plane
CLHP	Closed Left Half-Plane
CRHP	Closed Right Half-Plane

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