Notes on the Computation of the Incremental Gain of Linear Systems with Saturation

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Abstract

Conditions for the non-existence of the incremental ℓ_p gain of single input discrete time linear systems with saturation is presented, as well as algebraic tests for bounding the induced ℓ_2 and $\mathcal{L}_2$ gains. These algebraic tests are found to be the same as those which would arise if one were to conservatively use a quadratic dissipation function to bound the induced 2-norm.
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1 Introduction

In this paper, the incremental gain properties of a class of single input linear systems with saturation are examined. The main justification of the study of the incremental gain is that it allows some stronger conclusions than can be drawn from the use of the induced norm in the analysis of nonlinear systems; see [3],[6] for examples of such analysis.

Unfortunately, it appears that having a finite incremental gain is too strong a condition for analysis, a fact which has been noted in the recent paper by Chitour, Liu and Sontag ([2]). That paper examines not only finite incremental gain but also more general continuity and differentiability properties of a specific class of systems (the plant must have a neutrally stable dynamics matrix) in continuous time. This present work amplifies those conclusions, as it is shown that the incremental gain does not exist for open loop unstable discrete time plants (even over some bounded signal spaces for which the induced norm exists), and an algebraic technique for bounding the ℓ_{2+} and $\mathcal{L}_{2+}$ incremental gain for some subset of the open loop stable plants is given. The inequalities which result are the same as those which would result from attempting to force a global quadratic solution to the test for the induced norm given in [4], which indicates that such algebraic tests are telling us about incremental gain magnitudes. The class of plants for which the results are applicable has been generalised at the cost of restricting our attention to the standard saturation function (as opposed to the generic class of nonlinearities studied in [2]).

2 Preliminaries

Definition 2.1: The closed loop plant operator $\Upsilon\colon w \mapsto z$ is generated by the state space equations:

$$\begin{aligned} x[k+1] &= Ax[k] + B\sigma_0(w[k] - Cx[k]), && (1)\\ z[k] &= Cx[k], && (2)\\ x[0] &= 0. && (3) \end{aligned}$$

Additionally, it is assumed that: $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$, and the linear system $[A, B, C]$ is minimal.

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The function σ_0 appearing in (1) is the standard saturation function, defined:

$$\sigma_0(x) = \begin{cases} -1 & x \leq -1, \\ x & -1 < x < 1, \\ 1 & x \geq 1. \end{cases}$$

The space norm used throughout is the Euclidean norm, denoted $\|x\|$. The signal spaces used are all ℓ_{p+} spaces (where $\|y\|_p^p = \sum_{i=0}^{\infty} \|y[i]\|^p$ for $p < \infty$, and $\|y\|_\infty = \sup_{i \in \mathbb{Z}_+}\{\|y[i]\|\}$), and the bounded subsets $\mathcal{W}_{p,\infty}(N) := \ell_{p+} \bigcap \{y: \|y\|_\infty \leq n_\infty\}$. Note that the dimension of these signal spaces is suppressed. The induced norm over ℓ_{p+} (resp. $\mathcal{W}_{p,\infty}(N)$) is defined:

$$\|\Upsilon\|_{i,p} = \sup_{w \in \ell_{p+}(\text{resp.} \mathcal{W}_{p,\infty}(N)), w \neq 0} \frac{\|\Upsilon w\|_p}{\|w\|_p}.$$

The incremental gain over ℓ_{p+} (resp. $\mathcal{W}_{p,\infty}(n_\infty)$) is defined:

$$\|\Upsilon\|_{\Delta,p} = \sup_{w,\tilde{w} \in \ell_{p+}(\text{resp. } \mathcal{W}_{p,\infty}(N)), w \neq \tilde{w}} \frac{\|\Upsilon w - \Upsilon \tilde{w}\|_p}{\|w - \tilde{w}\|_p}.$$

A development of the properties of such systems can be found in [5]. It is important to note that for any fixed system, there exists some $N^* > 0$ such that if $N < N^*$, the induced norm over $\mathcal{W}_{p,\infty}(N)$ is finite, a consequence of the discrete time counterpart of the results in [4]. Additionally, we have fixed the system to have unity negative feedback so as to avoid any technical complications with observability properties.

3 New Results

3.1 Incremental Bound Computation

We will first present a useful result which can be used to bound the incremental gain, using linear matrix inequalities. Although linear matrix inequalities have been used to bound the induced norm of such systems, the author is unaware of any references which in fact note that the quadratic potential functions derived are better suited for the incremental gain computation. And as is more fully explored in [5], the incremental gain computation via quadratic dissipation functions does not appear to be conservative, whereas they will give rise to conservative analysis for the induced norm. These results have first appeared in [5].

Definition 3.1: The matrix P is the solution to the *discrete incremental gain algebraic inequalities* for a gain γ, if:

1. P is such that $P = P^T$, and $P > 0$,
2. P is a solution to the matrix inequalities:

$$A'PA - P + C'C \leq 0, \tag{4}$$

$$\begin{bmatrix} A - BC & B \\ C & 0 \end{bmatrix}^* \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A - BC & B \\ C & 0 \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & \gamma^2 I \end{bmatrix} < 0. \tag{5}$$

These algebraic equations are soluble using standard linear matrix inequality (LMI) packages; see [1] for a study of LMI theory. The equation (5) is equivalent to the algebraic Riccati equation by the use of the Schur complement formula.

Proposition 3.2: If we define the $V(x) = x'Px$, then the following hold:

1. If P is a solution to (5), then

$$V((A - BC)x + B\omega) - V(x) - \gamma^2\omega^2 + x'C'Cx \leq \\ - \left\| (\gamma^2 I - B'PB)^{\frac{1}{2}} (\omega - (\gamma^2 I - B'PB)^{-1}(B'P(A - BC)x) \right\|^2.$$

2. If P is a solution to (4), then

$$V(Ax) - V(x) + x'C'Cx \leq 0.$$

Proof: (1.) follows from the completion of squares for discrete time, (2.) is a standard consequence of Lyapunov theory. □

Theorem 3.3: For a system of the form given by (1) - (3), if a solution to the discrete incremental gain algebraic inequalities exists for a gain γ, then $\|\mathbf{\Upsilon}\|_{\Delta,2} \leq \gamma$.
Proof: Let $\eta = x - \tilde{x}$. It is sufficient to prove that if $V(\xi) = \xi' P \xi$, that we have:

$$s[k] = V(\eta[k+1]) - V(\eta[k]) - \gamma^2 \|(w - \tilde{w})[k]\|^2 + \|C\eta[k]\|^2 \leq 0, \tag{6}$$

for all (finite) k, as (6) implies $\gamma \|w - \tilde{w}\|_2 > \|C\eta\|_2$, by standard dissipation arguments.

By using the fact that we can write $\sigma_0(a) - \sigma_0(b) = \lambda(a - b)$, with $0 \leq \lambda \leq 1$, we arrive at:

$$\begin{aligned} \eta[k+1] &= A\eta[k] + B\left[\sigma_0(w[k] - Cx[k]) - \sigma_0(\tilde{w}[k] - C\tilde{x}[k])\right] \\ &= \lambda\left[(A - BC)\eta[k] + B(w - \tilde{w})[k]\right] + (1 - \lambda)A\eta[k] \\ &=: \lambda\zeta_1 + (1 - \lambda)\zeta_2. \end{aligned}$$

Define $\kappa = \eta'[k]P\eta[k] + \gamma^2 \|(w - \tilde{w})[k]\|^2 - \|C\eta[k]\|^2$. By the application of Proposition 3.2, a consequence of inequality (5) is that $\zeta_1' P \zeta_1 \leq \kappa$. Similarly, by applying the inequality (4), we arrive at:

$$\zeta_2' P \zeta_2 \leq \eta[k]P\eta[k] - \|C\eta[k]\|^2 \leq \kappa.$$

By realising that the set $\Omega = \{\xi : \xi' P \xi \leq \kappa\}$ is convex, $\zeta_1, \zeta_2 \in \Omega \implies \eta[k+1] \in \Omega$. Therefore, $\eta'[k+1]P\eta[k+1] \leq \kappa$, implying that $s[k] \leq 0$, verifying the claim. □

It is fairly straightforward to extend this result to continuous time; this is done in [5]. The result will be presented here without proof, although it closely follows the discrete time case. The system is of the same form as the system defined in equations (1) - (3), although we of course replace $x[k+1]$ with $\dot{x}(k)$ in equation (1).

Definition 3.4: The matrix P solves the *continuous time incremental gain inequalities* for a gain γ, if:

1. P is such that $P = P'$, $P > 0$, and
2. P is a solution of the matrix inequalities:

$$A'P + PA + C'C < 0,$$

$$(A - BC)'P + P(A - BC) + \frac{1}{\gamma^2} PBB'P + C'C < 0.$$

Theorem 3.5: For a continuous time system of the given form, if there exists a matrix P which is a solution to the continuous time incremental gain inequalities for a gain γ, then

$$\|\mathbf{\Upsilon} w - \mathbf{\Upsilon}\tilde{w}\|_2 < \gamma \|w - \tilde{w}\|_2,$$

for all $w, \tilde{w} \in \mathcal{L}_{2+}$.

Remark 3.6: As $\|\mathbf{\Upsilon}\|_{i,2} \leq \|\mathbf{\Upsilon}\|_{\Delta,2}$, we have just found an algebraic test for bounding above the induced 2-norm, which may be useful in comparison with the technique given in [4]. In fact, if one were to attempt to globally fit a quadratic candididate dissipation function to the tests given in [4], it would necessarily also solve the incremental gain inequalities given in this paper. The inter-relationships between the incremental gain and induced norm are more fully explored in [5].

It is conjectured that the previously stated results are not conservative when it comes to the bounding of the incremental gain, as it is believed that the existence of the solution to the inequalities is a necessary and sufficient condition for γ to be an upper bound to the incremental gain. This will be formalised now.

Conjecture 3.7: For a system of the form given by (1) - (3), a solution to the incremental gain inequalities exists for a gain γ if and only if $\|\mathbf{\Upsilon}\|_{\Delta,2} \leq \gamma$.

Although the full proof of the reverse implication has not been completed, non-existence results have been derived which are consistent with it. These will be given in the next section.

3.2 Non-Existence of Incremental Gain

Although we do not have a full characterisation of the cases where the incremental gain does not exist, an extremely important case has been covered. It is obvious that no solution to the incremental gain inequalities will exist if the plant is either open or closed loop unstable, and it is now shown that "destabilising" signals can be constructed in this case. Of importance is the fact that the signals constructed have small infinity norm.

Theorem 3.8: Assume that $w \in \mathcal{W}_{p,\infty}(n_\infty)$, with $n_\infty > 1+\epsilon$ $(0 < \epsilon < 1)$. Then for the system described by (1) - (3), $\|\boldsymbol{\Upsilon}\|_{\Delta,p}$ does not exist over $\mathcal{W}_{p,\infty}(n_\infty)$ if:

1. $p < \infty$, and A or $A-BC$ has any eigenvalues of modulus greater than or equal to 1;
2. or if $p = \infty$ and A or $A-BC$ has any eigenvalues of modulus strictly greater than 1.

Proof: It is trivial that if the induced norm does not exist, the incremental gain is also unbounded. This will occur if the closed loop dynamics matrix $A-BC$ is unstable, or if A is unstable and n_∞ is chosen large enough (for details, see [5]). We now examine the case where $A-BC$ is stable, and the induced norm over $\mathcal{W}_{p,\infty}(n_\infty)$ is finite.

We can now make the following observations, which are standard consequences of linear system properties.

Proposition 3.9: Since $[A, B]$ is controllable, for all $y_0 \in \mathbb{R}^n$, there exists a $u \in \ell_{p+}$ such that, $\sum_{i=0}^{n-1}(A-BC)^{(n-1)-i}Bu[i] = y_0$, with $u[k] = 0$ for all $k \geq n$.

Proposition 3.10: If $p < \infty$, and A is such that it has any eigenvalues of modulus greater than or equal one, the observability of $[A, C]$ implies that there exists some $y_0 \in \mathbb{R}^n$ such that the sequence $(S_{M,p})^p := \sum_{i=0}^{M} \left\|CA^i y_0\right\|^p$, is unbounded. If $p = \infty$, and A has any eigenvectors with modulus strictly greater than 1, there exists some $y_0 \in \mathbb{R}^n$ such that the sequence $\{S_{M,\infty}\}$ is unbounded, with $S_{M,\infty}$ defined:

$$S_{M,\infty} := \max_{i \in 0,1,\ldots,m} \left\{\left\|CA^i y_0\right\|\right\}.$$

To prove the theorem, we assume the converse; i.e. that there exists some γ such that $\|\boldsymbol{\Upsilon}w - \boldsymbol{\Upsilon}\tilde{w}\|_p \leq \gamma\|w-\tilde{w}\|_p$, for all $w, \tilde{w} \in \mathcal{W}_{p,\infty}(n_\infty)$. Fix some y_0 as in Proposition 3.10, and u as the appropriate input as in Proposition 3.9. We then see that there exists some $M < \infty$ such that $S_{M,p} > \gamma^p \|u\|_p^p$, (for $p < \infty$), or $S_{M,\infty} > \gamma\|u\|_\infty$. Then choose a constant $\alpha > 0$ such that

$$\alpha^{-1} > \max\left\{\max_{i=0,\ldots,n-1}\left\{\left\|u[i] + \sum_{j=1}^{i} C(A-BC)^{j-i}u[j-i]\right\|\right\}, \frac{2}{\epsilon}\max_{i \in 0,\ldots,M-1}\left\{\left\|CA^i y_0\right\|\right\}\right\}.$$

Now define the signal w, which is causally dependent upon the induced state x:

$$w[k] = \begin{cases} 0 & 0 \leq k \leq n-1, \\ (1+\epsilon)\mathrm{sign}(-Cx[k]) + Cx[k]) & n \leq k \leq n+M-1, \text{and } \|Cx[k]\| \leq 1+\epsilon, \\ 0 & n \leq k \leq n+M-1, \text{and } \|Cx[k]\| > 1+\epsilon, \\ 0 & n+M \leq k, \end{cases}$$

having defined $\mathrm{sign}(0) = 1$.

Then, $\tilde{w}$ is defined as:

$$\tilde{w}[k] = \begin{cases} \alpha u[k] & 0 \leq k \leq n-1, \\ w[k] & k \geq n. \end{cases}$$

We can readily verify that $w, \tilde{w}$ lie in $\mathcal{W}_{p,\infty}(n_\infty)$.

We first note that by their construction, $\|w - \tilde{w}\|_p = \alpha\|u\|_p$. Define $\eta = \tilde{x} - x$. It can be verified by induction that:

$$\eta[k] = \alpha A^{k-n} y_0,$$

for k in the interval $n, \ldots, n+M$. The proof for this is omitted, but it is a result of the fact that w, $\tilde{w}$, were chosen so as to force $|w[k] - Cx[k]| \geq 1 + \epsilon$, and $|w[k] - \tilde{w}[k] - C\eta[k]| < \frac{\epsilon}{2}$ over this interval. This forces the difference system to act as if it is open loop.

If the truncation operator at time T is denoted $\mathbf{\Pi}_T$, we then have that $\|\mathbf{\Pi}_{n+M}(\mathbf{I} - \mathbf{\Pi}_{n-1})C\eta\|_p = S_{M,p}$. We then have by the defintion of $S_{M,p}$,

$$\frac{\|\mathbf{\Upsilon}\tilde{w} - \mathbf{\Upsilon}w\|_p}{\|\tilde{w} - w\|} \geq \frac{\alpha S_{M,p}}{\alpha \|u\|_p} > \gamma.$$

This contradiction proves Theorem 3.8.□

Remark 3.11: A similar construction is made in [2, Example 5] for a first order (continuous time) system. The construction in [2] uses signals which will have unbounded infinity norm as $\gamma \to \infty$, however.

We will now state, without proof, a generalisation of Theorem 3.8 which gives a further test for non-existence of the incremental gain. No *a priori* bound of the test signal sizes is given, however.

Definition 3.12: A binary sequence is a sequence whose elements each lie in the set $\{0, 1\}$.

Theorem 3.13: Fix a system $\mathbf{\Upsilon}$ of the form given by (1) - (3), $\|\mathbf{\Upsilon}\|_{\Delta,p}$ does not exist over ℓ_{p+} if there exists a binary sequence of length l, denoted $\{\alpha_1, \ldots, \alpha_l\}$, such that

$$P_\beta^l = \prod_{j=1}^{l} (\alpha_{l-j+1}A + (1 - \alpha_{l-j+1})(A - BC)),$$

has an eigenvalue of modulus greater than or equal one ($p < \infty$) or strictly greater than 1 ($p = \infty$).
Proof: In [5]. □

4 Concluding Remarks

The incremental gain properties of systems with saturation nonlinearities have been examined herein. Although it is possible to characterise the incremental gain via the use of linear matrix inequalities, it has been also verified that open loop unstable plants cannot be stabilised in an incremental gain sense (for this class of systems, at any rate). This lack of stabilisability is not conservative, as the destabilising signals have quite small size, and so the procedure of bounding input sets will not be effective.

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