Model Validation for Nonlinear and Time-Varying Systems: Improved Bounds using the $\mathcal{S}$-Procedure

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***Abstract*— This paper considers an approach to linear time-varying and nonlinear model validation put forward by R. S. Smith et al at the 39th IEEE Conference on Decision and Control in which the Yakubovich $\mathcal{S}$-procedure is applied to problems with linear time-varying perturbations. A claimed necessary and sufficient condition for invalidation is shown to be sufficient only, a tighter sufficient condition is put forward, and a similar condition for non-causal perturbations is proposed. This is illustrated by means of a deterministic numerical example.**

I. Introduction

This paper considers the invalidation of models of dynamic systems in the context of robust control. The application of the $\mathcal{S}$-procedure, a mathematical technique presented in [Yak77], is explored. This work in this paper was first presented in [Aug04] and builds heavily on that presented in [S$^+$00].

Model invalidation is the process of determining whether a given model is inconsistent with observed data. The process (or, linguistically, its opposite) is often colloquially called 'model validation'.[1] Strictly speaking, a model can never be proved valid: the model may fit all data observed to date, but it is always possibile that the future will present data which it cannot account for. The aim of invalidation experiments are twofold: firstly, they give information about a model's envelope of 'usefulness' and, secondly, if a model is not invalidated, confidence in that model is increased.

The framework used in this paper is illustrated in Figure 1. This represents an uncertain system using an upper linear fractional transformation (LFT). The linear transfer functions P_{ij} are known, as are the measured input u and output y. The exogenous noise input w and the possibly-nonlinear uncertainty dynamics Δ are unknown. The internal signal s is generally unknown, except in the rare case when both $P_{11} = P_{12} = 0$.[2] In non-trivial cases, t is always unknown, since it depends on Δ.

$$\begin{pmatrix} s \\ y \end{pmatrix} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \end{bmatrix} \begin{pmatrix} t \\ w \\ u \end{pmatrix} \tag{1}$$

At this point, we may define our a model (in)validation optimization problem: given the upper LFT parameterization of Figure 1, where P_{ij} are known, known input-output data sequences (u, y), and a scalar $\gamma_\Delta > 0$, what is the smallest value of scalar $\gamma_w \geq 0$ such that there exist a perturbation-noise pair (Δ, w) simultaneously satisfying

$$||\Delta||_\infty \leq \gamma_\Delta$$
$$||w||_2 \leq \gamma_w$$

and the input-output relation (1)?

A. The $\mathcal{S}$-procedure

The $\mathcal{S}$-procedure is stated in the following form in [S$^+$00]:

Proposition 1 ($\mathcal{S}$-Procedure) *Let $Q_0(\chi), Q_1(\chi), \ldots Q_n(\chi)$ be quadratic matrix functions of vector $\chi \in \mathbb{R}^\ell$. If there exist scalars $\tau_i \geq 0$, $i = 1, \ldots, n$, such that*

$$Q_0(\chi) - \sum_{i=1}^{n} \tau_i Q_i(\chi) \geq 0, \textit{ for all } \chi, \tag{2}$$

then, for all χ such that $Q_i(\chi) \geq 0$, $i = 1, \ldots, n$;

$$Q_0(\chi) \geq 0. \tag{3}$$

If $n = 1$ then this condition is necessary and sufficient.

The sufficiency of Proposition 1 is easy to see: if all $Q_i(\chi) \geq 0$, $i = 1, \ldots, n$ then (3) *must* hold true. The necessity when $n = 1$ is described in [Boy03] as 'not easy to prove' and the interested reader may wish to follow this up in the original sources [Yak77].[3]

The $\mathcal{S}$-procedure was applied to model validation in [S$^+$00] and forms the basis for many of the propositions in this paper. For a more detailed discussion, see [S$^+$00] or [Aug04, Ch. 6].

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[1]The terminology used in this work is consistent with much published material, though some works, e.g. [Ver00], prefer to use the term 'falsification' in place of 'invalidation'.

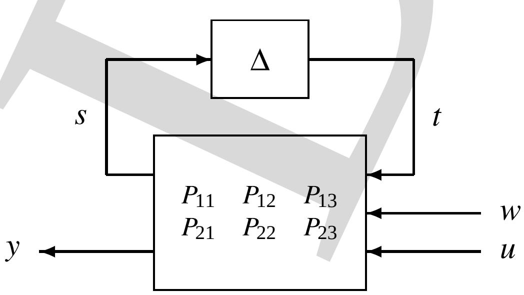

Fig. 1. A generic LFT model structure

[2]Note that this special case is of little interest, since it may be solved exactly for both LTI and LTV uncertainties using convex optimization techniques presented in [Dav96].

[3]When $n = 1$ the $\mathcal{S}$-procedure is 'lossless'.

II. Setting up the problem

Let Π_ℓ denote the ℓ-step truncation operator, applicable both to discrete-time dynamical system operators and to discrete-time sequences. Let $\Pi_k \mathcal{S}_+^m$ represent the first k elements of an infinite sequence of values in R^m (i.e. each element is an $m \times 1$ column vector).

The published literature contains a sufficient-and-necessary condition for the existence of a $\mathcal{H}_\infty$-norm bounded linear time variant (LTV) (or, equally, nonlinear) interpolant for two sequences.

Proposition 2 *[P[+]92], [P[+]94] Given sequences $u \in \Pi_l \mathcal{S}_+^m$ and $y \in \Pi_l \mathcal{S}_+^n$, and a scalar $\gamma_\Delta > 0$, there exists a stable, causal, time-varying operator Δ satisfying*

$$\|\Delta\|_\infty \leq \gamma_\Delta$$
$$\Pi_l \Delta \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{l-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{l-1} \end{bmatrix}$$

if and only if

$$\|\Pi_k y\|_2 \leq \gamma_\Delta \|\Pi_k u\|_2 \tag{4}$$

for all $k = \{1, 2, \ldots, l\}$.

There are similar necessary and sufficient conditions for the existence of a *noncausal* interpolant:

Proposition 3 *[Aug04, Appdx. C Thm. C.1] Given $u \in \Pi_\ell \mathcal{S}_+^N$, $y \in \Pi_\ell \mathcal{S}^n +$, and $\gamma \geq 0$ there exists a an infinite matrix $\Delta_{U\infty}$ with $\|\Delta_{U\infty}\| \leq \gamma$ such that*

$$\operatorname{vec} y = (\Pi_k \Delta_{U\infty}) \operatorname{vec} u$$

if and only if

$$\|y\|_2 \leq \gamma \|u\|_2$$

A. Parameterizing all interpolant signals

Following [S[+]00] we note that the nominal noise-free model is $y_{\text{nom}} = P_{23} u$, and that y_{nom} is unlikely to match y in practice, and any discrepancy must thus be accounted for by

$$y - P_{23} u = \begin{bmatrix} P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} t \\ w \end{bmatrix} \tag{5}$$

All (t, w) satisfying (5) may be parameterized

$$\begin{aligned} \begin{bmatrix} t \\ w \end{bmatrix} &= \begin{bmatrix} t_0 \\ w_0 \end{bmatrix} + R\zeta \\ &= x_0 + R\zeta \\ &= x(\zeta) \end{aligned}$$

where (t_0, w_0) is a particular solution of (5) and $R\zeta$ spans the input null space of $[P_{21}\ P_{22}]$. (With finite data sets and the corresponding lower-block Toeplitz matrices, these are easily calculated using standard linear algebra techniques.)

B. Parameterizing the noise constraint

The usual bound on the exogenous noise norm $\|w\|_2 \leq \gamma_w$ is parameterized by writing w as follows:[4]

$$\begin{aligned} w &= \begin{bmatrix} 0_{nv} & I_{nw} \end{bmatrix} x(\zeta) \\ &= \begin{bmatrix} 0_{nv} & I_{nw} \end{bmatrix} (x_0 + R\zeta) \end{aligned}$$

All noise sequences violating the bound will satisfy

$$x(\zeta)^* \begin{bmatrix} 0_{nv} & 0 \\ 0 & I_{nv} \end{bmatrix} x(\zeta) - \gamma_w^2 > 0$$

which can be re-written as

$$F_0(\gamma_w, \zeta) = \zeta^* A_0 \zeta + 2 b_0^* \zeta + c_0(\gamma_w) > 0$$

where,

$$A_0 = R^* \begin{bmatrix} 0_{nv} & 0 \\ 0 & I_{nw} \end{bmatrix} R$$
$$b_0 = R^* \begin{bmatrix} 0_{nv} & 0 \\ 0 & I_{nw} \end{bmatrix} x_0$$

and

$$c_0(\gamma_w) = x_0^* \begin{bmatrix} 0_{nv} & 0 \\ 0 & I_{nw} \end{bmatrix} x_0 - \gamma_w^2$$

C. Parameterizing the Δ constraint

Defining N_N such that

$$t = [I_{nv}, 0_{nw}] x = N_N x$$

gives expressions for the input and output of the perturbation block:

$$\begin{aligned} t &= N_N (x_0 + R\zeta) \\ s &= [P_{11}\ P_{12}](x_0 + R\zeta) + P_{13} u \end{aligned}$$

When dealing with LTV perturbations, the following projections are also helpful:

$$\Pi_k N_N : x \to \Pi_k t$$
$$\Pi_k M_N : s \to \Pi_k s$$

Recall that for there to exist an LTV system Δ satisfying $||\Delta||_\infty \leq \gamma_\Delta$ and $t = \Delta s$, it is necessary and sufficient that $\|\Pi_k t\|_2^2 \leq \gamma_\Delta^2 \|\Pi_k s\|_2^2$ for all $k = 1, 2, \ldots, N$. The energy bound at any given truncation k—referred to in [S[+]00] as the 'LTV extension condition'—can be expressed as a quadratic inequality:

$$F_k(\gamma_\Delta, \zeta) = \zeta^* A_k(\gamma_\Delta) \zeta + 2 b_k^*(\gamma_\Delta) \zeta + c_k(\gamma_\Delta) \geq 0 \tag{6}$$

[4]The subscript notation used here matches that of [S[+]00], which wisely chose ease of reading over strict notational accuracy. For '*nw*' read 'the number of data points in this truncation multiplied by the dimension of *w*' and so on.

where

$$A_k(\gamma_\Delta) = \gamma_\Delta^2 R^* \begin{bmatrix} P_{11}^* \\ P_{12}^* \end{bmatrix} (\Pi_k M_N)^* \Pi_k M_N [P_{11}\ P_{12}] R$$
$$- R^* (\Pi_k N_N)^* (\Pi_k N_N) R$$
$$b_k(\gamma_\Delta) = \gamma_\Delta^2 R^* \begin{bmatrix} P_{11}^* \\ P_{12}^* \end{bmatrix} (\Pi_k M_N)^* (\Pi_k M_N) P_{13} u$$
$$- R^* (\Pi_k N_N)^* \Pi_k N_N x_0$$
$$+ \gamma_\Delta^2 R^* \begin{bmatrix} P_{11}^* \\ P_{12}^* \end{bmatrix} (\Pi_k M_N)^* (\Pi_k M_N) [P_{11}\ P_{12}] x_0$$
$$c_k(\gamma_\Delta) = \gamma_\Delta^2 u^* P_{13}^* (\Pi_k M_N)^* \Pi_k M_N P_{13} u$$
$$- x_0^* (\Pi_k N_N)^* (\Pi_k N_N) x_0$$
$$+ \gamma_\Delta^2 x_0^* \begin{bmatrix} P_{11}^* \\ P_{12}^* \end{bmatrix} (\Pi_k M_N)^* (\Pi_k M_N) [P_{11}\ P_{12}] x_0$$
$$+ 2\gamma_\Delta^2 x_0 \begin{bmatrix} P_{11}^* \\ P_{12}^* \end{bmatrix} (\Pi_k M_N)^* \Pi_k M_N P_{13} u \tag{7}$$

III. Validation with noncausal Δ-blocks

One of the most significant constraints usually present in our mathematical models is causality: a causal system maps inputs in the past to outputs in the future.

It this section, we break from this slightly and consider a mathematically simpler problem: we will allow our perturbation Δ to have a possibly non-causal structure. Essentially, we are looking for an linear operator that satisfies $t = \Delta s$ and $||\Delta||_\infty \leq \gamma_\Delta$, but that is not constrained to be lower block-triangular.

Proposition 4 (noncausal, necessary and sufficient) *[Aug04, Ch. 6 Thm. 6.1] Given $\gamma_\Delta, \gamma_w > 0$, the LFT perturbation model satisfying $\|\Delta\|_\infty \leq \gamma_\Delta$ and $\|w\|_2 \leq \gamma_w$, is invalidated by the measured data ($y \in \mathcal{S}_N^p, u \in \mathcal{S}_N^q$) if and only if for all ζ satisfying $F_N(\gamma_\Delta, \zeta) \geq 0$;*

$$F_0(\gamma_w, \zeta) \geq 0$$

Corollary 5 *[Aug04, Ch. 6 Cor. 6.2] Given γ_Δ, $\gamma_w > 0$, the LFT perturbation model satisfying $\|\Delta\|_\infty \leq \gamma_\Delta$ and $\|w\|_2 \leq \gamma_w$, is invalidated by the measured data ($y \in \mathcal{S}_N^p, u \in \mathcal{S}_N^q$) if and only there exists $\tau \geq 0$ such that*

$$\begin{bmatrix} A_0 - \tau A_N(\gamma_\Delta) & b_0 - \tau b_N(\gamma_\Delta) \\ b_0^* - \tau b_N^*(\gamma_\Delta) & c_0(\gamma_w) - \tau c_N(\gamma_\Delta) \end{bmatrix} \geq 0$$

IV. Validation with LTV Δ-blocks

We now consider the more general case in which Δ is norm-bounded as before, but constrained to be causal. There are two lower bounds here.

A. Sufficient condition after [S+00]

The following condition is novel compared to [S+00] in that it does not claim necessity.

Proposition 6 (LTV, first sufficient condition) *[Aug04, Ch. 6 Thm. 6.3] Given $\gamma_\Delta, \gamma_w > 0$ the LFT perturbation model satisfying $\|\Delta\|_\infty \leq \gamma_\Delta$ and $\|w\|_2 \leq \gamma_w$ is invalidated by the measured data ($\hat{y} \in \mathcal{S}_L^p, \hat{u} \in \mathcal{S}_L^q$) if for any $N \in \{1, 2, \ldots, L\}$, the analogous non-causal model of Proposition 4 is invalidated by $y = \Pi_N \hat{y}$ and $u = \Pi_N \hat{u}$.*

The proposition is not *necessary* because it *might* be possible to find N (t, w) pairs, each of which satisfies the LTV extension condition at a particular N as well as the exogenous noise constraint, but this does not show that there is a *single* (t, w) pair simultaneously satisfying the LTV extension condition for *all* N.

Each separate test is a convex LMI; it is possible to implement them all simultaneously in an optimization problem, giving a lower bound on the smallest γ_w consistent with the model.

Proposition 7 *[Aug04, Ch. 6 Rem. 6.4] Given $\gamma_\Delta, \gamma_w > 0$, the conditions of Proposition 6 are failed if and only if for each $N \in \{1, 2, \ldots, L\}$ there exist separate (w_N, s_N, z_N) which do not simultaneously satisfy $\|\Pi_N w_N\|_2 \leq \gamma_w$ and $\|\Pi_N t_N\| \leq \gamma_\Delta \|\Pi_N s_N\|$.*

B. A second sufficient condition

A logical alternative to Proposition 6 is to apply the LTV extension constraint at all truncations simultaneously. This also results in a sufficient condition:

Proposition 8 (LTV, second sufficient condition) *[Aug04, Ch. 6 Thm. 6.5] Given $\gamma_\Delta, \gamma_w > 0$, the LFT perturbation model with Δ satisfying $\|\Delta\|_\infty \leq \gamma_\Delta$ and $\|w\|_2 \leq \gamma_w$ is invalid with respect to the data (y, u) if there exist $\tau_\ell \geq 0$, $\ell = 1, \ldots, N$ such that for all ζ*

$$F_0(\gamma_w, \zeta) - \sum_{\ell=1}^{N} \tau_\ell F_\ell(\gamma_\Delta, \zeta) \geq 0 \tag{8}$$

The condition is not necessary because the $\mathcal{S}$-procedure is not lossless in this case.

Corollary 9 *[Aug04, Ch. 6 Cor. 6.6] Given $\gamma_\Delta, \gamma_w > 0$, the LFT perturbation model satisfying $\|\Delta\|_\infty \leq \gamma_\Delta$ and $\|w\|_2 \leq \gamma_w$ is invalid with respect to the data (y, u) if there exist $\tau_\ell \geq 0$, $\ell = 1, \ldots, N$ such that for all ζ*

$$Q := \begin{bmatrix} A_0 - \sum_{\ell=1}^N \tau_\ell A_\ell(\gamma_\Delta) & b_0 - \sum_{\ell=1}^N \tau_\ell b_\ell(\gamma_\Delta) \\ b_0^* - \sum_{\ell=1}^N \tau_\ell b_\ell^*(\gamma_\Delta) & c_0(\gamma_w) - \sum_{\ell=1}^N \tau_\ell c_\ell(\gamma_\Delta) \end{bmatrix} \geq 0 \tag{9}$$

Note that the Corollary 9 is an LMI in γ_w and $\tau_1, \tau_2, \ldots, \tau_\ell$, again lending itself to ready computation of a lower bound on the smallest admissible γ_w.

V. Disagreement with [S+00]

The following claim is made in Section 4.4 of [S+00]. (It has been simplified a little here as we are considering but one perturbation block.)

> Proposition 10 *Given $\gamma_\Delta = \gamma_w > 0$, the LFT perturbation model satisfying $\|\Delta\|_\infty \leq \gamma_\Delta$ and $\|w\|_2 \leq \gamma_w$ is invalidated by the measured datum ($\hat{y} \in \mathcal{S}_L^p, \hat{u} \in \mathcal{S}_L^q$) (of length L) if and only if there exists a truncation $N \in \{1, 2, \ldots, L\}$ with corresponding*

truncated datum $(y,u) := (\Pi_N \hat{y}, \Pi_N \hat{u})$ *such that for all* ζ *satisfying* $F_N(\gamma_\Delta, \zeta) \geq 0$,

$$F_0(\gamma_w, \zeta) \geq 0$$

There is an obvious problem here: this is exactly the same as the lower bound of Proposition 6. The authors of [S+00] have neglected the fact that the LTV extension condition is applied to *separate* sequences at each truncation.

Sufficiency is easy enough to show. If the exogenous noise constraint and the LTV extension condition for *any* truncation cannot be simultaneously satisfied, the model is clearly invalid:

$$\begin{aligned}
&\exists N : \forall \zeta \left((F_0(\gamma_w, \zeta) \geq 0) \vee (F_N(\gamma_\Delta, \zeta) < 0) \right) \\
&\quad \rightarrow \exists N : \forall \Delta \forall w \big((||\Pi_N w||_2 > \gamma_w) \vee \\
&\qquad\qquad (||\Pi_N t||_2 > \gamma_\Delta ||\Pi_N s||_2) \big) \\
&\quad\quad \rightarrow \forall \Delta \forall w \big((||w||_2 > \gamma_w) \vee \\
&\qquad\qquad \big(\exists N : (||\Pi_N t||_2 > \gamma_\Delta ||\Pi_N s||_2) \big)
\end{aligned}$$

But the necessity? Consider the negation of the above:

$$\begin{aligned}
&\forall N \, \exists \zeta : \left((F_0(\gamma_w, \zeta) < 0) \wedge (F_N(\gamma_\Delta, \zeta) \geq 0 \right) \\
&\quad \rightarrow \forall N \, \exists \Delta \exists w : \big((||\Pi_k w||_2 \leq \gamma_w) \wedge \\
&\qquad\qquad (||\Pi_N t||_2 \leq \gamma_\Delta ||\Pi_N s||_2) \big)
\end{aligned}$$

which is not the same as

$$\exists \Delta \exists w : \left((||w||_2 \leq \gamma_w) \wedge \forall N \left(||\Pi_N t||_2 \leq \gamma_\Delta ||\Pi_N s||_2 \right) \right)$$

So, if for each truncation, we can find ζ satisfying the noise constraint and the perturbation block constraint, we can find separate (Δ, w) pairs for each truncation satisfying the exogenous noise constraint and the N-th truncation of the perturbation block constraint. However, this does *not* imply that there is a single (Δ, w) pair simultaneously satisfying the exogenous noise constraint and the noise constraint at *all* truncations of the perturbation block constraint.

The non-implication in the last line was ignored in Proposition 10, hence the claim of necessity is false.

A numerical counter-example is given in Section VII.

VI. Relating the bounds

Given our LFT model structure and $\gamma_\Delta \geq 0$,

- Let $\gamma_w^{\mathrm{NC}}(u,y)$ be the largest value of γ_w for which the data (u,y) invalidates the model structure w.r.t. non-causal perturbations using Proposition 4.
- Let $\gamma_w^{\mathrm{LTV,lb1}}(u,y)$ be the largest value of γ_w for which the data (u,y) invalidates the model structure w.r.t. LTV perturbations using Proposition 6.
- Let $\gamma_w^{\mathrm{LTV,lb2}}(u,y)$ be the largest value of γ_w for which the data (u,y) invalidates the model structure w.r.t. LTV perturbations using Proposition 8

Proposition 11 *[Aug04, Ch. 6 Thm. 6.8] Given* $u \in \mathcal{S}_L^q$, $y \in \mathcal{S}_L^p$,

$$\gamma_w^{\mathrm{NC}}(u,y) \leq \gamma_w^{\mathrm{LTV,lb1}}(u,y) \leq \gamma_w^{\mathrm{LTV,lb2}}(u,y)$$

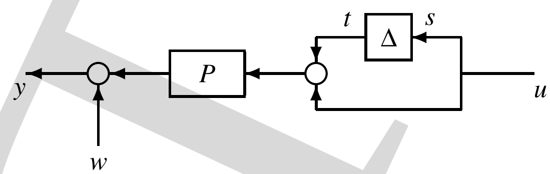

Fig. 2. Block Diagram for Numerical Example

VII. Numerical example

To illustrate the preceding sections, consider the system shown in Figure 2, where

$$P(z) = \frac{0.008264z^2 + 0.01653z + 0.008264}{z^2 - 1.636z + 0.6694}$$

Signals (y,u) are known, w is an unknown exogenous noise signal, Δ represents unknown dynamics. The output of Δ, t is unknown; the input $s = u$.

For the purposes of this example, u_k is defined as

$$u_k := \cos\left(2kT + \frac{1}{2}k^2T^2\right)$$

where $T = 0.2$, for $k = 0, 1, \ldots, 9$ and y_k was chosen to be the response to u_k of

$$P_{\mathrm{true}}(z) := \frac{0.008678z^2 + 0.01736z + 0.008678}{z^2 - 1.636z + 0.6694}$$

A. Exact LTV Invalidation Conditions

Since the input to the Δ-block is known, the LTV model non-invalidation problem is convex and readily solved.

Proposition 12 *[Aug04, Ch. 6 Rem. 6.9] Given a* $p \times q$ *linear time-invariant system* P, $\gamma_\Delta > 0$, $u \in \mathcal{S}_\ell^q$, $y \in \mathcal{S}_\ell^p$, *the smallest value of* $\gamma_w := ||w||_2$ *consistent with the model of Figure 2 with LTV uncertainty satisfying* $||\Delta||_\infty \leq \gamma_\Delta$ *is given by*

$$\gamma_w^{\mathrm{LTV}} := \min_{t \in \theta} ||y - P(u+t)||_2$$

where $\theta := \{\tau : ||\Pi_j \tau||_2 \leq \gamma_\Delta ||\Pi_j u||_2 \forall j = 1, \ldots, k\}$. *This is easily calculated by minimizing* γ_w *subject to the LMI constraints*

$$\begin{bmatrix} \gamma_\Delta^2 ||\Pi_j u||_2 & (\operatorname{vec} \Pi_j t)^* \\ \operatorname{vec} \Pi_j t & I \end{bmatrix} \geq 0$$

for all $j \in \{1, \ldots, k\}$ *and*

$$\begin{bmatrix} \gamma_w^2 & (\operatorname{vec} w)^* \\ \operatorname{vec} w & I \end{bmatrix} \geq 0$$

where $\operatorname{vec} w = \operatorname{vec} y - T_P \operatorname{vec} u - T_P \operatorname{vec} t$, *and* t *and* γ_w *are the decision variables.*

We can employ a similar technique to calculate the $\gamma_w^{\mathrm{LTV,lb1}}$ of [S+00] without using the $\mathcal{S}$-procedure.

Proposition 13 *[Aug04, Ch. 6 Rem. 6.10] Given a* $p \times q$ *linear time-invariant system* P, $\gamma > 0$, $u \in \mathcal{S}_\ell^q$, $y \in \mathcal{S}_\ell^p$ *the*

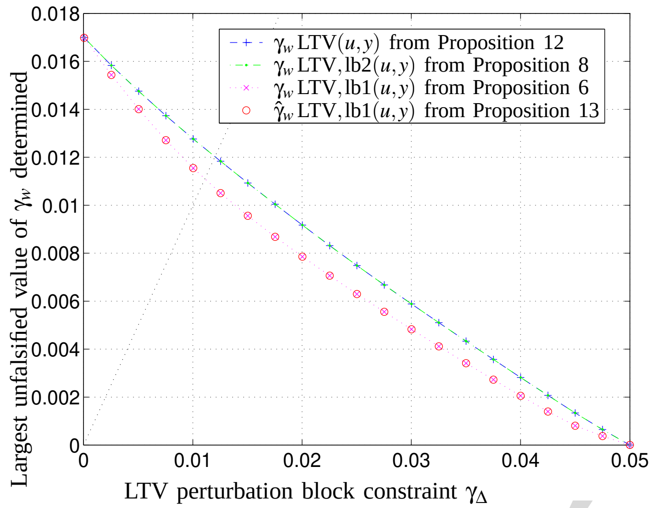

Fig. 3. Numerical example. $\gamma_w^{\text{LTV,lb1}}(u,y) = \hat{\gamma}_w^{\text{LTV,lb1}}(u,y)$ as expected, since both are intended to be the same quantity. In this case the lower bound $\gamma_w^{\text{LTV,lb2}}(u,y)$ is equal to the exact value $\gamma_w^{\text{LTV}}(u,y)$. See the text of Section VII for a more detailed discussion.

smallest value of $\gamma_w := ||w||_2$ consistent with Proposition 6 and Proposition 10 is given by

$$\hat{\gamma}_w^{\text{LTV,lb1}} := \min_{(t_1,\dots,t_k)\in\Theta} \left(\max_{j=1,\dots,k} ||\Pi_j(y - P(u+t_j))||_2 \right)$$

where $\hat{\Theta} := \{(\tau_1,\dots,\tau_k) : ||\Pi_j \tau_j||_2 \leq \gamma_\Delta ||\Pi_j u||_2 \forall j = 1,\dots,k\}$*. Again, this is easily calculated through an LMI minimization.*

B. Results

Four quantities were calculated for a range of γ_Δ: $\gamma_w^{\text{LTV}}(u,y)$, $\gamma_w^{\text{LTV,lb2}}(u,y)$, $\gamma_w^{\text{LTV,lb1}}(u,y)$ and $\hat{\gamma}_w^{\text{LTV,lb1}}(u,y)$. The results are shown in Figure 3. It can be seen that:

- $\gamma_w^{\text{LTV,lb1}}(u,y)$ and $\hat{\gamma}_w^{\text{LTV,lb1}}(u,y)$ are coincident, and smaller than $\gamma_w^{\text{LTV,lb2}}(u,y)$ and $\hat{\gamma}_w^{\text{LTV}}(u,y)$. Both the these facts are as expected: the value obtained using Proposition 10 from [S+00] is equal to our first lower bound, and our improved bound is greater than these. The latter point confirms our claim that Proposition 10 is false.
- $\gamma_w^{\text{LTV,lb2}}(u,y)$ is significantly closer to the 'true' value γ_w^{LTV}. In this case, the two actually coincide. Though we always expect $\gamma_w^{\text{LTV,lb2}}(u,y)$ to exceed $\gamma_w^{\text{LTV,lb1}}(u,y)$, it is unlikely that $\gamma_w^{\text{LTV,lb2}}(u,y)$ and γ_w^{LTV} will always coincide. It is possible that the coincidence in this example is a result of the convexity of the problem.

Most problems will not have convex solutions, but we now have a method of applying the $\mathcal{S}$-procedure to find a lower bound on the level of LTV uncertainty present. This nicely complements other available methods, which use quadratic approximations to produce upper bounds.

VIII. Conclusions

An alternative approach to LFT invalidation problems using the $\mathcal{S}$-procedure was considered. A problem was identified with a claimed necessary and sufficient condition for LTV invalidation from [S+00], which we show to be sufficient but not necessary. The techniques were adapted to produce necessary and sufficient conditions for invalidation with a noncausal perturbation structure and a tighter sufficient condition for LTV invalidation. This was illustrated using a numerical example, also serving as a counter-example to the claim of [S+00]. (In [Aug04, Chap. 6–7] the techniques are adapted for application to a system with an output offset and potentially non-zero initial state and applied to a challenging application in flight control.)

The following conclusions were drawn:

- The claimed necessary and sufficient condition of LTV invalidation proposed in [S+00] is only sufficient. A tighter sufficient condition has been found. This is useful, since we now have upper and lower bounds for LTV (in)validation problems. In a convex numerical example, the new lower bound of the smallest noise ℓ_2 norm consistent with any given perturbation size was seen to be very close to the true value from convex optimization. Model (in)validation problems are not generally convex, though.
- A necessary and sufficient condition for invalidation using noncausal perturbations has been derived. This is potentially useful in assessing the effects of causality on a given problem. (It also provides a lower bound for the LTV and LTI cases.) This is very easy to compute, requiring only a simple LMI optimization with few decision variables, but—when considered as a lower bound for the LTV case—it is less tight than the bounds described above.

IX. Acknowledgements

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