Necessary and Sufficient Conditions for Robust Stability of a Class of Nonlinear Systems[1]

Jorge M. Gonçalves, Munther A. Dahleh
Department of EECS
MIT, Cambridge, MA 02139
jmg@mit.edu, dahleh@lids.mit.edu

Abstract

Input-output stability results for feedback systems are developed. Robust Stability conditions are presented for nonlinear systems with nonlinear uncertainty defined by some function (with argument equal to the norm of the input) that bounds its output norm. A *sufficient small gain theorem* for a class of these systems is presented. Then it is also shown that, for the vector spaces $(\ell_\infty, \|\cdot\|_\infty)$ and $(\ell_2, \|\cdot\|_2)$, the sufficient conditions are also *necessary* with some additional assumptions on the systems. These results capture the conservatism of the *small gain theorem* as it is applied to systems that do not need to have linear gain.

1 Introduction

This paper considers the development of necessary and sufficient conditions for the robust stabilization of certain classes of nonlinear plants. The problem of robust stabilization may be stated as follows. Given a nominal plant model and a family of possible true plants, under what condition does some compensator which stabilizes the nominal plant also stabilize every plant in the given family?

The idea that a loop of less than unity gain ensures stability of a feedback loop has been appreciated since the early days of classical control. In mathematical terms, it is related to well-known ideas on invertibility of nonlinear operators of the form $I + G_1 G_2$ where I is the identity and G_1, G_2 are nonlinear operators on Banach spaces.

The usual form of the small gain theorem assumes gain properties of the form

$$\|(Mu)_T\| \leq \gamma \|u_T\| \tag{1}$$

for the operator M where u denotes the input signal, $\gamma = \sup_{x \neq 0} \frac{\|Mx\|}{\|x\|}$, and u_T denotes the truncation of the signal u at time T. With this structure, it is shown that, if M is linear, for $p = 2, \infty$ if $\|\Delta\|_{\ell_{p-ind}} < 1$ then the feedback system of Δ and M achieves robust stability if and only if $\|M\|_{\ell_{p-ind}} \leq 1$. For details, see [1]. For a nonlinear M, the result is shown only for $p = 2$ (see [6] or [7]).

In [5] a new notion of stability for nonlinear systems is introduced. There, a generalization of (1) is given to allow more general bounding functions of the form

$$\|(Mu)_T\| \leq f(\|u_T\|)$$

where $f(\cdot)$ is a monotone function. Systems satisfying the last inequality are called monotone stable.

While conditions for sufficiency were shown with this new notion of nonlinear gain (also used by others like [8] or [9]) no results on the necessity of such conditions are known. Such results are useful to understand the degree of conservatism that the small gain theorem has. Necessity conditions for linear gain exists (see [1] or [3]). There also exists necessity conditions for nonlinear systems that have their output norm bounded by a linear function of the input norm (see [6] or [7]).

In this paper, an extension of the small gain theorem presented in [5] will be given. Using this theorem we will present sufficient conditions on M in order to guarantee robust stability. For a certain class of perturbations, these conditions on M will be simplified. Then, for the vector space $(\ell_\infty, \|\cdot\|_\infty)$ we will give conditions on M so that the sufficient conditions are also necessary with either NLTV or NLTI perturbations. For the vector field $(\ell_2, \|\cdot\|_2)$ we will also give conditions on M so that the sufficient conditions are necessary with non causal perturbations. The construction of a causal perturbation is still under investigation.

2 Preliminaries

We start by defining some standard concepts. The set of nonnegative integers is denoted by Z_+. The extended space of sequences in $\Re^n$ is denoted by ℓ_{pe} for every $1 \leq p \leq \infty$ or just by ℓ when it is obvious or when it just does not matter what $p-norm$ is being used. The restriction of f to the interval $[a, b]$ is denoted by $f|_{[a,b]}$. For every $f = \{f(0), f(1), \cdots\} \in \ell$ define $\|f\|_{p|_{[a,b]}}$ as $\|f\|_{p|_{[a,b]}} = \left(\sum_{n=a}^{b} |f(n)|^p\right)^{1/p}$.

The set of all $f \in \ell$ with $f \notin \ell_p$ is denoted by $\ell \setminus \ell_p$. Given $f \in \ell$ define $f^x = (f(0)^x, f(1)^x, f(2)^x, \ldots)$ and the support of $f \in \ell$ by $supp(f) = \{n : f(n) \neq 0\}$.

For $k \in Z_+$, S_k denotes the kth-shift (time-delay) operator on ℓ, and P_k the kth-truncation operator on ℓ. Let $H : \ell \to \ell$ be an operator. Then, H is called causal if $P_k H f = P_k H P_k f$, $\forall_{k \in Z_+}$, strictly causal if $P_k H f = P_k H P_{k-1} f$, $\forall_{k \in Z_+}$, and time invariant if $H S_1 = S_1 H$.

Let X_e and Y_e be two signal spaces. Then an operator $G : X_e \to Y_e$ provides an input-output system representation.

The following definition provides a concept of input output stability.

---

[1] Research supported in part by the Portuguese "Junta Nacional de Investigação Científica e Tecnológica" under the program "PRAXIX XXI", by the NSF under grant ECS-9157306, by the AFOSR under grant AFOSR F49620-95-0219, and by the Draper Laboratory under grant DL-H-441684

Definition 2.1 *The system G is monotone stable if there exists a monotone increasing homeomorphism $f:\Re_+\to\Re_+$ and a constant $\beta\in\Re_+$ such that*

$$\|(Gu)_T\| \le f(\|u_T\|)+\beta \tag{2}$$

for all $u\in X_e$ and $T\ge 0$.

Definition 2.2 *A nonlinear operator G is said to have finite memory if there exists an increasing integer function $FM(\cdot;G):\ Z_+\to Z_+$ with $FM(t;G)\ge t$ such that*

$$(I-P_{FM(t;G)})Gf=(I-P_{FM(t;G)})G(I-P_t)f$$

for all $f\in\ell_p$ and $t\in Z_+$. The function $FM(\cdot;G)$ is called the finite memory function associated with G.

The proof of the following proposition is done in [6] and therefore it is omitted here.

Proposition 2.1 *Let G, a nonlinear operator, have finite-memory with associated finite-memory function $FM(\cdot;G)$. Then for $f_1\in\ell_2$ with supp$(f_1)\subset[0,n]$ and $f_2\in\ell_2$ with supp$(f_2)\subset[FM(n;G)+1,\infty]$ we have $G(f_1+f_2)=Gf_1+Gf_2$*

In the following definition, assume that G is some nonlinear operator.

Definition 2.3 *Let $\eta_G(s):\ \Re_+\to\Re_+$ be a non-decreasing function such that*

$$\sup_{f\neq 0}\frac{\|G(f)\|}{\eta_G(\|f\|)}=1$$

If G is finite memory, then there always exist an f that achieves the norm. Therefore, the last definition can be understood as follows: there exists an $f\in\ell$ such that $\|G(f)\|=\eta_G(\|f\|)$; for all other $u\in\ell$, $\|G(u)\|\le\eta_G(\|u\|)$.

Consider the system in figure 1.

-bb-error = =

Figure 1: Closed loop system

Let $\mathbf{\Delta}$ denote the class of allowable perturbations. We now define the subset of $\mathbf{\Delta}$ containing elements with $\eta_\Delta(s)<ks^x$, for some given $x,k>0$ (if $k=0$ then it is obvious that the system in figure 1 is stable if and only if M is stable).

Definition 2.4 *Let $C_{\mathbf{\Delta},p,x}$ be a subset of $\mathbf{\Delta}$ defined, for some given $k>0$, as*

$$C_{\mathbf{\Delta},p,x}=\{\Delta\in\mathbf{\Delta}:\ \eta_\Delta(s)<ks^x\}$$

This is the same to say that, for every $u\in\ell$ and $T\in Z_+$, $\|\Delta(u_T)\|_p\le\eta_\Delta(\|u_T\|_p)$ where $\eta_\Delta(s)<ks^x$. This means that $\|\Delta(u_T)\|_p<k\|u_T\|_p^x$ and therefore, according to definition 2.1, all $\Delta\in C_{\mathbf{\Delta},p,x}$ are monotone stable (note that ks^x is a monotonic increasing homeomorphism for every $x,k>0$).

For perturbations $\Delta\in C_{\mathbf{\Delta},p,x}$ (for some p and x), the problem will be to find necessary and sufficient conditions on M to guarantee robust stability.

3 Sufficiency of the Small-Gain Theorem

In this section we will present a sufficient condition to achieve robust stability when the disturbance belongs to $C_{\mathbf{\Delta},p,x}$. First, in section 3.1, we will give some concepts that will be used in section 3.2 to present the general small-gain theorem. This theorem is an extension to the one presented in [5]. Finally, in section 3.3, using this theorem, a sufficient condition is given on some system M, perturbed by $\Delta\in C_{\mathbf{\Delta},p,x}$, that guarantees the robust stability of the feedback system in figure 1.

3.1 Preliminaries

Consider the system of figure 2.

-bb-error = =

Figure 2: Closed loop system

Assumption 3.1 *Let V_{1e} and V_{2e} be two signal spaces. The operators $G_1:V_{1e}\to V_{2e}$ and $G_2:V_{2e}\to V_{1e}$ are such that for all input signals $r_1\in V_{1e}$ and $r_2\in V_{2e}$ there exist unique signals $u_1,y_2\in V_{1e}$ and $u_2,y_1\in V_{2e}$.*

Definition 3.1 *Define the following function classes $M=\{f:\Re_+\to\Re_+|\ f$ is a m. i. h. of $\Re_+$ onto $\Re_+\}$, $N=\{f\in M|\ \exists_g\in M$ s.t. $f(x)\le x-g(x)\}$, and $N_y=\{f\in M|\ \exists_g\in M$ s.t. $f(x)\le x-g(x)\ \forall x\ge y\}$ where $y\ge 0$ and m. i. h. stands for monotonic increasing homeomorphism.*

So, $N\subset N_y$. Define also $M0=M\cup\{O_F\}$ and $N0_y=N_y\cup\{O_F\}$ where O_F denotes the zero function $f\equiv 0$.

Proposition 3.1 *$f\in N_y$ if and only if $\exists_g\in M$ such that $(i+g)\circ f(x)\le x$ for all $x\ge y$.*

Definition 3.2 *The feedback system in figure 2 under assumption 3.1 is called monotone stable if there exist functions $f_1,f_2:\Re_+\times\Re_+\to\Re_+$ and constants $\beta_1,\beta_2\in\Re$ such that*

$$\begin{aligned}\|y_{1T}\| &\le f_1(\|r_{1T}\|,\|r_{2T}\|)+\beta_1\\ \|y_{2T}\| &\le f_2(\|r_{1T}\|,\|r_{2T}\|)+\beta_2\end{aligned}$$

$\forall_{T\ge 0}$, $\forall_{r_1}\in\ell_{p_1e}$, $\forall_{r_2}\in\ell_{p_2e}$, and $f_1(0,\cdot)$, $f_2(0,\cdot)$, $f_1(\cdot,f_2(\cdot,0))\in M0$ with $f_1(0,0)=f_2(0,0)=0$.

3.2 General case

Let each system be monotone stable with gain functions g_1 and g_2 as in definition 2.1. This means that

$$\|y_{1T}\| \le g_1(\|u_{1T}\|)+\beta_{G_1} \tag{3}$$

$$\|y_{2T}\| \le g_2(\|u_{2T}\|)+\beta_{G_2} \tag{4}$$

The proof of the following theorem is similar to the one in [5] and therefore it will be omitted here.

Theorem 3.1 *Consider the system in figure 2. Suppose G_1 and G_2 are stable with gain functions g_1 and g_2 as in (3,4) . Suppose that assumption 3.1 holds. The feedback system is monotone stable if there exist $g \in M$ and $s^* \geq 0$ such that*

$$g_2 \circ (i + g) \circ g_1 \in N_{s^*} \tag{5}$$

Comment: In the proof of the last theorem (not done here), $s^* \neq 0$ implies that $\beta_1 \neq 0$ and $\beta_2 \neq 0$ (in definition 3.2) although both systems G_1 and G_2 have zero bias terms. The reason for is that the β_i are used to accommodate the lack of information on the closed loop system for $s \leq s^*$ (in equation (5)). Since (5) only gives us information for $s \geq s^*$, we need $\beta_1 \neq 0$ and $\beta_2 \neq 0$ in order to bound $\|y_{1T}\|$ and $\|y_{2T}\|$. Note that if $s^* = 0$ we have exactly the theorem presented in [5].

Corollary 3.1 *Consider the system in figure 2. Suppose G_1 and G_2 are stable with gain functions g_1 and g_2 as in (3,4). Suppose that assumption 3.1 holds. The feedback system is monotone stable if there exist $\rho_1, \rho_2 \in M$ and $s^* \geq 0$ such that*

$$(i + \rho_1) \circ g_2 \circ (i + \rho_2) \circ g_1 \leq s \quad \text{for all } s \geq s^* \tag{6}$$

3.3 Particular case

Consider again the system in figure 1. Assume that $\Delta \in C_{\mathbf{\Delta},p,x}$ and that M is monotone stable with gain function $m(s)$. For simplicity, let $\delta(s) = \eta_\Delta(s)$.

We will prove that it is sufficient to have $m(s) \leq (\frac{s}{k})^{\frac{1}{x}}$ (and therefore $(\delta \circ m)(s) < s$) for all $s \geq s^*$ for some $s^* \geq 0$ in order to have robust stability. This means that we do not need to find monotonically increasing functions ρ_1 and ρ_2 satisfying $(i + \rho_1) \circ \delta \circ (i + \rho_2) \circ m(s) \leq s$ in order to have closed loop stability.

Theorem 3.2 *The system in figure 1 achieves robust stability for all $\Delta \in C_{\mathbf{\Delta},p,x}$ if there exists $s^* \geq 0$ such that $m(s) \leq (\frac{s}{k})^{\frac{1}{x}}$ for all $s \geq s^*$.*

Proof: Assume there exists an $s^* \geq 0$ such that $m(s) \leq \sqrt[x]{s/k}$ for all $s \geq s^*$. Also, because $\Delta \in C_{\mathbf{\Delta},p,x}$, we have that $\delta(s) < ks^x$. Then, for a given $\Delta \in C_{\mathbf{\Delta},p,x}$ we can always find $k_1 > 0$ with $k_1 < k$ such that $\delta(s) \leq k_1 s^x$, for all $s \geq 0$. This means that $\|\Delta(u_T)\|_p \leq k_1 \|u_T\|_p^x$ and therefore Δ is monotone stable with gain function $f(s) = k_1 s^x$.

So, if we find $\rho_1, \rho_2 \in M$ and $\bar{s} \geq 0$ such that, $\forall_{s \geq \bar{s}}$,

$$(i + \rho_1) \circ m \circ (i + \rho_2) \circ f(s) \leq s \tag{7}$$

we meet all the conditions of corollary 3.1 and therefore we prove stability.

Let $\rho_1(s) = \beta s$ and $\rho_2(s) = \epsilon s$ (with $\beta > 0$ and $\epsilon > 0$). Then

$$(i+\rho_1) \circ m \circ (i+\rho_2) \circ f(s) = m(k_1 s^x (1+\epsilon))(1+\beta) \tag{8}$$

Let $\bar{s} = \left(\frac{s^*}{k_1(1+\epsilon)}\right)^{\frac{1}{x}}$. Note that when $s^* = 0$, we have $\bar{s} = 0$. Then, for any $s \geq \bar{s}$, $m(k_1 s^x(1+\epsilon)) \leq \sqrt[x]{\frac{1}{k}}\sqrt[x]{k_1 s^x (1+\epsilon)}$. Therefore, from (8)

$$(i + \rho_1) \circ m \circ (i + \rho_2) \circ f(s) \leq s \sqrt[x]{\frac{k_1}{k}(1+\epsilon)}(1+\beta)$$

and if we let $\epsilon = \frac{1 - \sqrt{\frac{k_1}{k}}}{2\sqrt{\frac{k_1}{k}}} > 0$ and $\beta = \frac{1 - \sqrt[2x]{\frac{k_1}{k}}}{2\,\sqrt[2x]{\frac{k_1}{k}}} > 0$ we actually satisfy (7) for all $s \geq \bar{s}$ which implies (from corollary 3.1) that the feedback system is stable. ■

4 ℓ_∞ Stability Robustness Necessary Conditions

Consider the system in figure 1. In [1, 2, 4] necessary conditions for stability robustness were presented for the case when M is linear time invariant. We will now extend those conditions to certain classes of nonlinear M. First, we will consider the case where the perturbation is NLTV. Then, we will prove that the necessity conditions still holds if the perturbation is NLTI.

To prove necessity, we add the following assumption on M.

Assumption 4.1 *Assume that the bound η_M defined in definition 2.3 satisfies, for all $s \geq 0$, $\eta_M(s) = \sup_{\|f\|_\infty = s} \|M(f)\|_\infty$.*

4.1 ℓ_∞ stability robustness with NLTV perturbations

Assume that $C_{\mathbf{\Delta}_{TV},\infty,x}$ represents the set of all causal NLTV perturbations according to definition 2.4. Also, M is stable, causal, and NLTV.

Theorem 4.1 *Under assumption 4.1, the system in figure 1 achieves robust stability for all $\Delta \in C_{\mathbf{\Delta}_{TV},\infty,x}$ if and only if there exists an $s^* \geq 0$ such that $\eta_M(s) \leq (\frac{s}{k})^{\frac{1}{x}}$ for all $s \geq s^*$.*

Proof: We first prove sufficiency. Assume that there exists an $s^* \geq 0$ such that $\eta_M(s) \leq (\frac{s}{k})^{\frac{1}{x}}$ for all $s \geq s^*$. This is the same to say that for any $u \in \ell$ with $\|u_T\|_\infty \geq s^*$, $\|M(u_T)\|_\infty \leq (\frac{\|u_T\|_\infty}{k})^{\frac{1}{x}}$ which means that M is monotone stable with gain function $m(s) = (\frac{s}{k})^{\frac{1}{x}}$ for every $s \geq s^*$. We can now use theorem 3.2 and conclude that the closed loop system is stable.

We now prove necessity. To simplify the proof, consider M and Δ SISO and let $m(s) = \eta_M(s)$ and $\delta(s) = \eta_\Delta(s)$.

The approach we use is to show that we can construct a destabilizing perturbation $\Delta \in C_{\mathbf{\Delta}_{TV},\infty,x}$ whenever the conditions of the theorem are not satisfied. So, assume that $\forall_{s^* \geq 0}$, $\exists_{s \geq s^*}$: $m(s) > (\frac{s}{k})^{\frac{1}{x}}$.

As in [1, 2, 4], the proof is divided in two parts: construction of an unbounded signal and construction of a destabilizing perturbation using that signal.

Construction of the unbounded signals

For simplicity assume that M has *finite-memory.* This means that there exists an increasing integer function $FM(\cdot;M)$ as in definition 2.2.

-bb-error = =

Figure 3: Construction of ξ

Assume that $N_0 = 0$ and $s_0 = 0$. The construction of ξ proceeds as follows (see figure 3). For all $n = 1, 2, 3, \cdots$, let $N_n = FM(N_{n-1}; M)$ and $s_n^* = k\left((\frac{s_{n-1}}{k})^{\frac{1}{x}} + 1\right)^x$. Then $\exists_{s_n \geq s_n^*}: \ m(s_n) > (\frac{s_n}{k})^{\frac{1}{x}}$. Choose $|\xi(t)| \leq s_n$ for $t = N_{n-1}, ..., N_n - 1$ with $\|P_{N_n-1}\xi\|_\infty = s_n$ such that m(s) is achieved. Then $\|P_{N_n-1}z\|_\infty > (\frac{s_n}{k})^{\frac{1}{x}}$ and $\|P_{N_n-1}y\|_\infty > (\frac{s_n}{k})^{\frac{1}{x}} + 1$. Therefore $\|P_{N_n-1}y\|_\infty > \left(\frac{1}{k}\|P_{N_n-1}\xi\|_\infty\right)^{\frac{1}{x}} + 1$.

Note that, in this case, $\|P_{N_n-1}\xi\|_\infty = s_n \geq s_n^* \geq k\left((\frac{s_{n-2}}{k})^{\frac{1}{x}} + 2\right)^x \geq \cdots \geq kn^x$ and $\|P_{N_n-1}y\|_\infty > n+1$. This means that both $\|\xi(t)\| \to \infty$ and $\|y(t)\| \to \infty$ as $n \to \infty$ (or as $t \to \infty$).

Because of the way $\xi(t)$ was constructed we have

$$\|P_t y\|_\infty > \left(\frac{1}{k}\|P_t\xi\|_\infty\right)^{\frac{1}{x}} \qquad (9)$$

for all t.

Construction of the destabilizing perturbation

We have $\xi = \{\xi(i)\}_{i=0}^\infty \in \ell$ and $y = \{y(i)\}_{i=0}^\infty \in \ell$ such that (9) is satisfied. Note that the inequality in (9) is equivalent to $\|P_t\xi\|_\infty < k\|P_t y\|_\infty^x$.

Constructing the destabilizing perturbation the same way as in [1, 2, 4] we have that Δ is trivial if $y = 0$: just pick Δ itself to be zero. So, assume that $y \neq 0$. Constructing $(y(i_1), y(i_2), ...)$ as in [1, 2, 4] we can now construct our Δ.

So, Δ is constructed by having (see figure 4) $\xi = \Delta(y) = \bar{\Delta}y^x$. This can be seen as a serie of two systems. The first raises every element of $y(t)$ to the power of x (and it is therefore nonlinear) while the second ($\bar{\Delta}$) is just an LTV system.

-bb-error = =

Figure 4: Structure of Δ

$\bar{\Delta}$ is a matrix constructed as in [1, 2, 4].

Each row of $\bar{\Delta}$ has at most one nonzero element which has absolute value less than k. This means that $\|\bar{\Delta}\|_{\ell_{\infty-ind}} < k$.

Now, let's see if Δ belongs to the set $C_{\mathbf{\Delta}_{TV},\infty,x}$. For every $t \geq 0$ we have $\|P_t\xi\|_\infty = \|\bar{\Delta}P_t y^x\|_\infty \leq \|\bar{\Delta}\|_{\ell_{\infty-ind}}\|P_t y^x\|_\infty < k\|P_t y^x\|_\infty = k\|P_t y\|_\infty^x$ or just that $\|P_t\xi\|_\infty < k\|P_t y\|_\infty^x$ which means that $\delta(s) \leq \|\bar{\Delta}\|_{\ell_{\infty-ind}} s^x < ks^x$. So, $\Delta \in C_{\mathbf{\Delta}_{TV},\infty,x}$. Moreover, Δ is causal and NLTV.

So, we found a bounded input that produces an unbounded output. This means that in definition 3.2 there is no monotonic increasing homeomorphism f_1 such that $\|y_{1T}\| \leq f_1(\|r_{1T}\|, \|r_{2T}\|) + \beta_1$ because there exists a bounded r_1 (with $r_2 = 0$) that produces an unbounded y_1. Therefore, we conclude that the closed loop system is unstable. ∎

Remark 4.1 *For $x = 1$ the above theorem provides a necessity proof for $\ell_\infty -$ stability of finite memory systems that satisfies*

$$\eta_M(s) = \sup_{\|f\|=s} \|M(f)\| = s$$

Moreover, the destabilizing perturbation can be LTV.

Remark 4.2 *Assumption 4.1 which states that the supremum is achieved for each s can be relaxed. Instead, let s_k be a sequence with the properties*

1. *$|s_k - s_{k-1}| \leq L$ for some $L > 0$ and*
2. *$\lim_{k\to\infty} s_k = \infty$*

The above proof can be modified with this new assumption:

$$\eta_M(s_k) = \sup_{\|f\|=s_k} \|M(f)\|$$

The proof will be omitted.

4.2 ℓ_∞ stability robustness with NLTI perturbations

Assume here that $C_{\mathbf{\Delta}_{TI},\infty,x}$ represents the set of all NLTI perturbations according to definition 2.4. The proof of the following theorem is similar to the one done in [1].

Theorem 4.2 *Under assumption 4.1, the system in figure 1 achieves robust stability for all $\Delta \in C_{\mathbf{\Delta}_{TI},\infty,x}$ if and only if there exists an $s^* \geq 0$ such that $\eta_M(s) \leq (\frac{s}{k})^{\frac{1}{x}}$ for all $s \geq s^*$.*

Proof: The proof of this theorem follows exactly as the proof of theorem 4.1 except for the construction of the destabilizing perturbation. Given the signals y and ξ, we show that a nonlinear time invariant perturbation can be constructed to destabilize the closed-loop system.

Let the signals y and ξ be given as before. Then Δ must be such that

$$\delta(s) \leq \|\bar{\Delta}\|_{\ell_{\infty-ind}} s^x < ks^x \qquad (10)$$

and $\xi = \Delta(y)$. We just need to redefine $\bar{\Delta}$. So, let $\bar{\Delta}$ be defined as follows

$$(\bar{\Delta}f)(t) = \begin{cases} k\xi(t-j), & \text{if } \exists_{j\in Z_+}: \ P_t f = P_t S_j \bar{y}, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that the new Δ is a nonlinear, time invariant, and causal system. It satisfies (10) (because $\|\bar{\Delta}\|_{\ell_{\infty-ind}} < k$) which means that $\Delta \in C_{\mathbf{\Delta}_{TI},\infty,x}$ and maps y to ξ. ∎

5 ℓ_2 Stability Robustness Necessary Conditions

Once again, we will extend the conditions for stability robustness presented in [1] to certain classes of nonlinear M.

To prove necessity, we add the following assumption on M, similar to the one in the ℓ_∞ case.

Assumption 5.1 *Assume that the bound η_M defined in definition 2.3 satisfies, for all $s \geq 0$, $\eta_M(s) = \sup_{\|f\|_2 = s} \|M(f)\|_2$.*

5.1 ℓ_2 stability robustness with non-causal perturbations

The following theorem gives a necessary and sufficient condition on the system M in figure 1 in order to guarantee that the closed loop system is stable. Here, M is assumed to be some NLTV system with its output norm bounded (to an input u) by $\eta_M(\|u\|_2)$ according to definition 2.3 and assumption 5.1.

Assume here that $C_{\mathbf{\Delta}_{NC},2,x}$ represents the set of all non causal perturbations according to definition 2.4.

Theorem 5.1 *Let $x \leq 1$. Under assumption 5.1, the system in figure 1 achieves robust stability for all $\Delta \in C_{\mathbf{\Delta}_{NC},2,x}$ if and only if there exists an $s^* \geq 0$ such that $\eta_M(s) \leq (\frac{s}{k})^{\frac{1}{x}}$ for all $s \geq s^*$.*

Proof: The proof of sufficiency follows the same way as in the proof of sufficiency of theorem 4.1.

The method of proof for necessity will again be similar to the one in [1] or in [6, 7]. Once again, for simplicity, let $m(s) = \eta_M(s)$ and $\delta(s) = \eta_\Delta(s)$. We will show that one can construct a destabilizing perturbation $\Delta \in C_{\mathbf{\Delta}_{NC},2,x}$ whenever the conditions of the theorem are not satisfied. So, assume that $\forall_{s^* \geq 0}, \exists_{s \geq s^*}: m(s) > (\frac{s}{k})^{\frac{1}{x}}$.

A particular signal $\xi \in \ell \setminus \ell_2$ is constructed for which there is an admissible Δ such that one has $(I - \Delta M)\xi \in \ell_2$. The lack of invertibility of $(I - \Delta M)$ then follows immediately.

This will be done in two steps. The first step is to construct that signal ξ. The next step is to use this signal to construct a destabilizing perturbation.

Construction of the unbounded signals

Assume that M is finite-memory. The construction of ξ proceeds as follows (assume $t_0 = 0$ and $s_0 = 1$). For any $n = 1, 2, 3, \cdots$, let $\alpha_n > 1$ and $s_n^* = \alpha_n s_{n-1}$. Then, for any $\alpha_n > 1$, $\exists_{s_n \geq s_n^*}: m(s_n) > (\frac{s_n}{k})^{\frac{1}{x}}$. Let $a_n = \frac{s_n}{s_n^*} \geq 1$ and $c_n = a_n \alpha_n > 1$. Then, $s_n = a_n s_n^* = a_n \alpha_n s_{n-1} = c_n s_{n-1} = c_n c_{n-1} s_{n-2} = c_n c_{n-1} \cdots c_2 s_1$. For simplicity, let $\|Mf_i\|_2 = \|Mf_i\|_{2|_{[0,N_i]}}$ and x_i represent some signals in ℓ_2 of appropriate length for $i = 1, 2, \cdots$. Choose $f_n \in \ell_2$, with $\|f_n\|_2 = s_n$, and an integer $N_n > 0$ such that $supp(f_n) = [0, N_n]$ and $m(s)$ is achieved. This means that

$$\|Mf_n\|_2 > \left(\frac{s_n}{k}\right)^{\frac{1}{x}} = \left(\frac{c_n c_{n-1} \cdots c_2 s_1}{k}\right)^{\frac{1}{x}}$$

Let $t_n = FM(N_n; M) + t_{n-1} + 2$. Define also $P_{t_n - 1}\xi = (f_1, 0, \cdots, f_n, 0)$. From proposition 2.1, this means that

$$P_{t_n-1} y = (Mf_1, x_1, \cdots, Mf_n, x_n) \tag{11}$$

Therefore, we have $\|P_{t_n-1}\xi\|_2$ given by

$$\|P_{t_n-1}\xi\|_2 = s_1 \sqrt{1 + \cdots + (c_n c_{n-1} \cdots c_2)^2}$$

and $\|P_{t_n-1} y\|_2$ given by

$$\begin{aligned} \|P_{t_n-1} y\|_2 &> \left(\frac{s_1}{k}\right)^{\frac{1}{x}} \sqrt{1 + \cdots + (c_n c_{n-1} \cdots c_2)^{2/x}} \\ &= \left(\frac{1}{k}\|P_{t_n-1}\xi\|_2\right)^{\frac{1}{x}} (1 - \epsilon_n) \end{aligned}$$

where

$$0 < \epsilon_n = 1 - \sqrt{\frac{1 + \cdots + (c_n c_{n-1} \cdots c_2)^{2/x}}{(1 + \cdots + (c_n c_{n-1} \cdots c_2)^2)^{1/x}}} < 1$$

It is easy to see that when $\alpha_n \to \infty$, $c_n \to \infty$, and therefore $\epsilon_n \to 0$.

Also, $s_1 > 1$ and for all $i = 1, 2, \cdots, n$, $c_i > 1$. This means that

$$\begin{aligned} \|P_{t_n-1}\xi\|_2 &= s_1 \sqrt{1 + \cdots + (c_n c_{n-1} \cdots c_2)^2} \\ &= \sqrt{n} \end{aligned}$$

Therefore, when $n \to \infty$, $\|P_{t_n-1}\xi\|_2 \to \infty$ and therefore ξ is unbounded.

Construction of the destabilizing perturbation

Given the signals y and ξ, we show that a nonlinear, non causal perturbation can be constructed to destabilize the closed-loop system.

Let the signals y and ξ be given as before. Δ must be constructed such that $\delta(s) < ks^x$ and $\xi = \Delta(y)$. Consider the perturbation defined as follows

$$(\Delta f)(k) = \begin{cases} 0, & \text{if } k < t_1, \\ \xi(k - j), & \text{if } \exists_{j \in Z_+}: P_k f = P_k S_j y, \\ 0, & \text{otherwise.} \end{cases}$$

It can be verified that Δ is a nonlinear and non causal perturbation. We notice that the maximum amplification occurs when the input signal of Δ is ξ. We also know that

$$\|P_{t_n-1}\xi\|_2 < \left(\frac{1}{1 - \epsilon_n}\right)^x k \|P_{t_n-1} y\|_2^x$$

and if take the infimum on the right side of the last inequality over ϵ_n we actually get

$$\|P_{t_n-1}\xi\|_2 < k\|P_{t_n-1} y\|_2^x$$

which means that $\delta(s) < ks^x$ and therefore $\Delta \in C_{\mathbf{\Delta}_{NC},2,x}$ and maps y to ξ.

So, Δ is constructed in such a way to have $\Delta(y) = \xi - (f_1, 0, 0, 0, \cdots) = (0, 0, f_2, 0, f_3, 0, f_4, \cdots)$.

Now, we just need to show that this is indeed a destabilizing perturbation. If we let ξ be the input to $(I - \Delta M)$ then we have

$$\begin{aligned} (I - \Delta M)(\xi) &= \xi - \Delta(M\xi) \\ &= (f_1, 0, \cdots, 0, f_n - f_n, 0, \cdots) \\ &= (f_1, 0, 0, 0, 0, \cdots) \in \ell_2 \end{aligned}$$

This implies that the system in figure 1 is not ℓ_2-*stable* because it maps a signal in ℓ_2 to a signal in $\ell \setminus \ell_2$. Therefore, as in the case of the ℓ_∞ proof, we conclude that the system is not monotone stable. This completes the proof. ■

Comment: One of the assumptions in the last theorem is $x \leq 1$. The reason that the theorem does not follow for $x > 1$ is because we assumed in the proof that M is finite memory. In fact, if $x > 1$ then M can not be finite memory. It has to be infinite memory.

Comment: In the ℓ_2 case, the construction of a causal perturbation Δ instead of a non causal one, like in the ℓ_∞ case where the conditions for stability hold for both NLTV and NLTI causal perturbations, is under investigation.

6 Concluding Remarks and Future Work

This paper presented a generalization of the small gain theorem for feedback systems. Sufficient conditions on a system M perturbed by a family of disturbances Δ were presented. Then, it was shown that those conditions are also necessary in the vector spaces $(\ell_\infty, \|\cdot\|_\infty)$ and $(\ell_2, \|\cdot\|_2)$ under appropriate assumptions on the system M. For the vector space $(\ell_\infty, \|\cdot\|_\infty)$, those conditions are necessary with either NLTV or NLTI perturbations and for the vector field $(\ell_2, \|\cdot\|_2)$, those conditions are necessary with non causal perturbations.

The sufficient conditions are a generalization of the small gain theorem presented in [5]. With this new theorem, the computation work involved in finding the function η_M of some system M is simpler since the conditions of the theorem only require that $\eta_M(s) \leq (\frac{s}{k})^{\frac{1}{x}}$ for big values of s. This way, we only need to find η_M (or a suitable upper bound of η_M) for big values of the norm of the input signal of M such that $\eta_M(s) \leq (\frac{s}{k})^{\frac{1}{x}}$. This is very useful since in most cases it very hard to find $\eta_M(s)$ for all $s \geq 0$.

In [10] results for the case where the nonlinearities are sector bounded by linear functions are given. As future work, this results can be extended to the case where the nonlinearities are sector bounded by a certain class of monotonic increasing functions. Also, as future work, in the ℓ_2 case, a causal perturbation Δ should be constructed instead of non causal one like in the ℓ_∞ case where the conditions for stability hold for both NLTV and NLTI causal perturbations.

References

[1] Munther A. Dahleh and Ignacio J. Diaz-Bobillo. *Control of Uncertain Systems: A Linear Programming Approach.* Prentice Hall, N.J., 1995.

[2] Munther A. Dahleh and M. H. Khammash. Controller design for plants with structured uncertainty. *Automatica*, 29(1), January 1993.

[3] M. Khammash and Munther A. Dahleh. Time-varying control and the robust performance of systems with structured norm-bounded perturbations. *Automatica*, 28, July 1992.

[4] M. H. Khammash and J. B. Pearson. Performance robustness of discrete-time systems with structured uncertainty. *IEEE Transactions on Automatic Control*, 36, 1991.

[5] Iven M. Y. Mareels and David J. Hill. Monotone stability of nonlinear feedback systems. *Journal of Mathematical Systems, Estimation, and Control*, 2(3):275–291, 1992.

[6] Jeff S. Shamma. The necessity of the small-gain theorem for time-varying and nonlinear systems. *IEEE Transactions on Automatic Control*, 36(10):1138–1147, October 1991.

[7] Jeff S. Shamma and Rongze Zhao. Fading-memory feedback systems and robust stability. *Automatica*, 29(1):191–200, 1993.

[8] Eduardo D. Sontag, Héctor J. Sussman, and Yudi Yang. A general result on the stabilization of linear systems using bounded controls. *Submitted to IEEE Transactions on Automatic Control*, 1994.

[9] Andrew R. Teel. A nonlinear small gain theorem for the analysis of control systems with saturation. *Submitted to IEEE Transactions on Automatic Control*, June 1994.

[10] M. Vidyasagar and C. Desoer. *Feedback Systems: Input-Output Properties.* Academic Press, N.Y., 1975.