How good are deterministic models for analyzing congestion control in delayed stochastic networks?

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***Abstract*— We investigate the regime where instability in deterministic fluid flow models for congestion control analysis in data networks corresponds to a significant increase in the variance of the flow in stochastic networks. This is shown to be the case when there are large number of packets in flight with small queue thresholds. The analysis is carried out by modelling an M/M/1 queue with delayed feedback as a stochastic hybrid system and analyzing the transient probability distribution of the states with partial differential equations. We also introduce a deterministic nonlinear dynamic queue model that captures the dynamics of the stochastic feedback system. Most of the literature on congestion control analysis using deterministic models, is currently based on queueing models that are valid in one of the extreme cases of negligible queueing delays relative to propagation delays (these are modelled with static functions) or never emptying queues (modelled as integrators). The proposed model is shown to be valid both in these extreme conditions, as well as intermediate regimes of large delays, emptying queues and significant queue dynamics.**

***Index Terms*— Communication Networks, Robust Control, Complex Systems.**

I. INTRODUCTION

Internet congestion control may be approached as a specific example of a wider class of problems that try to analyze the dynamic behaviour of highly interconnected dynamical systems. In this case, dynamics are induced by the way the control law is implemented at the users or resources (primal or dual algorithms respectively) as well as by the nature of the congestion control signals being fed back within the network. People within the control community have tended to use deterministic models to mathematically analyze the network behaviour (e,g. the primal algorithm in [1], [2] with stability analysis in [3], [4] and the dual in [5], [6]). A significant work has been carried out that tries to justify the use of deterministic models in a many flows regime, such as the convergence results in [7], [8] and the models suggested in such a limiting regime in [9], [10]. Attempts have also been made to take into account stochastic variability as the TCP parameters vary. In [1], [11], Kelly considers brownian perturbation of linearized source laws and analyzes the coefficient of variation of the flows at steady state. Nevertheless, delayed analysis of the stochastic models is left as an open problem. In [12] a poisson counter is used to model the generation of congestion signals and the ensemble average of the resulting stochastic differential equation is approximated by an ODE. However, queues are assumed never to empty, as a means of justifying the use of integrator dynamics for the queue length. This is not a valid assumption in the case of large delays and capacities.

Our aim in this paper is to investigate the effect that stochastic variability in the network has in the dynamic behaviour of the sending rates (particularly at the onset of limit cycles in a deterministic model) under various regimes, including the more realistic conditions of large delays and resource capacities. A stochastic network appears to be more appropriate to consider, in the limiting case of a large scale network with many users, due to randomness in the number of users present as well as the tasks they perform. It is therefore crucial to establish whether stochastic effects dominate the dynamic behaviour predicted by the deterministic models e.g. how large are delay induced limit cycles in the sending rate relative to the stochastic variation? Within this context deterministic models need to be refined so as to capture the dynamics induced by the stochastic character of the network. We therefore propose a more sophisticated model for the queue dynamics that behaves satisfactorily in all regimes and not just the two extreme cases of negligible queueing delays, or non-emptying queues.

In the paper we analyze the simplest form of a stochastic network with deterministic feedback and delay i.e. an M/M/1 queue with fixed service rate (capacity) and delayed arrival rate that increases or decreases linearly based on whether the queue exceeds a given threshold. The key idea is that resources may be expected to be able to control overall arrival rate but not the precise timing of arrivals. Despite the simplicity of the model, it is going to reveal the general trends in the dynamic response, since a network of queues is more likely to lead to averaging out of the stochastic effects. The time varying poisson source is also not an unreasonable assumption, in spite of the well known bursty, self-similar nature of Internet traffic presented in the pioneering article [13]. Recent work in [14] indicates that multiplexing in large capacity links leads to poisson traffic. This is based on the limit

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theorem that the superposition of independent marked point processes tends to a poisson process, even though individual point processes may have long range dependence (see [15]).

The paper is structured as follows. We present first the stochastic model and how it can be approximated by different deterministic models. The stochastic model is then analyzed and simulated as a stochastic hybrid system. The simulation results are confirmed by deriving and solving numerically the corresponding Kolmogorov equations for the probability distribution of the system states. The main observation is that instability in the deterministic model results in a substantial increase in the coefficient of variation of the sending rate in the stochastic model only when there are large number of packets in flight and small queue thresholds. We then investigate the extent to which the dynamics of the stochastic model are being captured by the deterministic ones. It turns out that the more sophisticated deterministic queueing model being proposed captures the dynamics of the sending rate in all regimes, including large and small queueing delays as well as emptying queues.

II. Description of the models

A. The stochastic model

The block diagram of the stochastic model mentioned above is given below

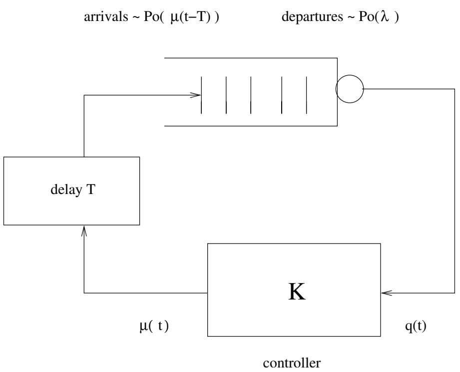

Fig. 1. Stochastic model.

$\mu(t)$ is the arrival rate and $q(t)$ the queue length. The control law K is

$$\frac{d\mu(t)}{dt} = \begin{cases} k & \text{if } q(t) < N \\ -k & \text{if } q(t) \geq N \\ 0 & \text{if } q(t) \geq N \text{ and } \mu(t) = 0 \end{cases} \tag{1}$$

B. Deterministic models

1) Static model: The deterministic fluid flow models used in the literature correspond to feedback systems where each resource communicates back to the user a price (this is taken to be a function of the flow through it) by means of the end destination node. An optimization framework can be adopted such that by appropriate dynamics at the resources or the users, the equilibrium flows reached maximize an aggregate user utility (see [1]). Within this formulation, the dynamics in the model in figure 1 are at the user, and a static price being fed back can be taken as the probability the queue length exceeds a threshold N at steady state, given arrival rate μ and departure rate λ i.e.

$$p(\mu) = \left(\frac{\mu}{\lambda}\right)^N \tag{2}$$

Hence the deterministic fluid flow model can be described by the following ode

$$\begin{aligned} \frac{d\mu(t)}{dt} &= [1 - p(\mu(t-T))]k - kp(\mu(t-T)) \\ &= k[1 - 2p(\mu(t-T))] \end{aligned} \tag{3}$$

Linearising (3) about its equilibrium arrival rate $\mu_0 = p^{-1}(0.5) = \lambda(0.5)^{1/N}$ gives the following delayed ode for the perturbations $\Delta\mu$ about μ_0

$$\begin{aligned} \frac{d\Delta\mu(t)}{dt} &= -2kp'(\mu_0)\Delta\mu(t-T) \\ &= -\frac{2kN}{\lambda}(0.5)^{\frac{N-1}{N}}\Delta\mu(t-T) \end{aligned} \tag{4}$$

We know that for the delayed differential equation of the form $\dot{u}(t) = -\alpha u(t-T)$, $u(t)$ converges to 0 as $t \to \infty$ if and only if $\alpha T < \pi/2$ (proved using Nyquist criterion). Applying this result to this case gives a critical gain k_c for stability of

$$k_c = \frac{\lambda\pi}{2NT}(0.5)^{\frac{1}{N}} = \frac{\pi\mu_0}{2NT}$$

2) Dynamic model: The static price model is based on the assumption that queueing delays are much smaller than propagation delays, so that queue dynamics are averaged out. Nevertheless at the onset of instability this might not be a valid assumption, so a more sophisticated dynamic model needs to be considered. We propose a dynamic price model that gives the probability of exceeding the threshold N at time t given that at this instance the probability the queue is of size q is $x_q(t)$ (assuming again constant arrival and departure rates μ and λ respectively). This can be described by the following state space model

$$\dot{x} = A(\mu)x \quad p(\mu) = Cx \quad \text{where}$$

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_q \\ \vdots \end{bmatrix}, \quad A = \begin{bmatrix} -\mu & \lambda & 0 & 0 & \dots \\ \mu & -\lambda-\mu & \lambda & 0 & \\ 0 & \mu & -\lambda-\mu & \ddots & \\ 0 & 0 & \ddots & \ddots & \\ \vdots & & & & \end{bmatrix},$$

$$C = [\ 0_{1\times N} \quad 1 \quad 1 \quad \dots \quad 1\ ] \tag{5}$$

Note that in practice μ will be varying so this price function will behave as a nonlinear dynamical system. As discussed in [16], the linearization of this price function reduces to the static price (2) in the case of short, low load queues and to integrator dynamics (assumed in [6]) for the case of long, high load queues.

III. Analysis of the transient response of the Stochastic Model

A. Stochastic hybrid system formulation

The stochastic system in figure 1 can be analyzed as a Piecewise Deterministic Markov Process (PDMP) or Stochastic Hybrid System (SHS), as these are introduced in [17] and defined using hybrid system notation in [18] respectively. We first give a formal PDMP definition for the system and describe an algorithm that can be used to generate its executions. We then explain why such a formulation is valid and give precise expressions for the associated parameters.

Omitting the delay, the system in figure 1 is characterized by the following:

- A countable set of discrete states $Q = \{0,1,2,\ldots\}$. These are the possible queue lengths.
- A map $X : Q \to [0,\infty)$ mapping each discrete state to a set of continuous states. These are the controller states which in this case is the arrival rate μ, so $X(q) = \mu$.
- A vector field $f : \mathbb{S} \to \mathbb{R}$, where $\mathbb{S} = \bigcup_{q\in Q}\{q\}\times X(q)$ is the hybrid state space. The flow of f is a function $\Phi : \mathbb{S}\times\mathbb{R} \to \mathbb{S}$ with $\Phi(\alpha,t) = \begin{bmatrix} \Phi_Q(\alpha,t) \\ \Phi_X(\alpha,t) \end{bmatrix}$ such that for $\alpha = (q,x)$, $\Phi(\alpha,0) = \alpha$ and for all $t \in \mathbb{R}$, $\Phi_Q(\alpha,t) = q$ and $\frac{d}{dt}\Phi_X(\alpha,t) = f(\Phi(\alpha,t))$. f describes the evolution in time of the continuous states and is given in (1) by interpreting $\Phi_X(\alpha,t) = \mu(t)$ with $\mu(0) = x$.
- A transition rate $\zeta : \mathbb{S} \to \mathbb{R}_+$. This will determine the distribution function for the time between stochastic transitions i.e.

$$F(\alpha,t) = 1 - exp\left(-\int_0^t \zeta(\Phi(\alpha,\tau))d\tau\right) \qquad (6)$$

  So ζ is defined as

$$\zeta((q,\mu)) = \begin{cases} \mu+\lambda & \text{if } q \neq 0 \\ \mu & \text{if } q = 0 \end{cases} \qquad (7)$$

- A transition probability measure $R : \mathscr{B}(\overline{\mathbb{S}})\times\mathbb{S} \to [0,1]$, where $\mathscr{B}(\overline{\mathbb{S}}) = \sigma\left(\bigcup_{q\in Q}\{q\}\times\mathscr{B}(q)\right)$, $\sigma(.)$ denotes the generated σ-algebra and $\mathscr{B}(q)$ the Borel σ-algebra of $X(q)$. It has the interpretation that $R(A,\alpha)$ gives the probability that a hybrid state $\alpha \in \mathbb{S}$ will have a transition into set A. This is defined here as

$$R((q+1,\mu),(q,\mu)) = \begin{cases} \frac{\mu}{\mu+\lambda} & \text{for } q \neq 0 \\ 1 & \text{for } q = 0 \end{cases}$$

$$R((q-1,\mu),(q,\mu)) = \frac{\lambda}{\mu+\lambda} \qquad \text{for } q \neq 0$$

  otherwise $R = 0$

- $Init : \mathscr{B}(\bar{\mathbb{S}}) \to [0,1]$ is an initial probability measure in $(\bar{\mathbb{S}},\mathscr{B}(\bar{\mathbb{S}}))$.

We now present an algorithm for generating executions of the PDMP when a delay T is introduced between the queue and the controller. So we need to define initial hybrid states within an interval [-T,0]. For convenience we take these identical to those generated at $t = 0$.

Algorithm I

1) Initialization. Set $t = 0$ and generate $\alpha_0 = (q_0,\mu_0)$ according to $Init$. Set $\alpha_t = \alpha_0$ for $t \in [-T,0]$.
2) Set $\hat{\alpha} = (q(t-T),\mu(t))$.
3) Sample Dt from a distribution with distribution function $F(\hat{\alpha},Dt)$ as defined in (6).
4) Set $\mu(t') = \Phi_X(\hat{\alpha},t'-t)$, $q(t') = q(t)$ for all $t' \in [t,t+Dt)$.
5) Set α_{t+Dt} according to $R(.,(q(t),\Phi_X(\hat{\alpha},Dt)))$.
6) Set $t = t+Dt$ and go to step 2.

The PDMP formulation is feasible due to the fact that the sum of two Poisson distributed random variables is also Poisson distributed, with parameter the sum of the parameters of the two variables. Here, we superimpose the stochastic arrivals and departures and treat them as a single point process with transition rate $\mu(t)+\lambda$. If $q = 0$ only arrivals are allowed so the transition rate is $\mu(t)$. The distribution function for the time between stochastic transitions is derived using (6). Hence sampling from this distribution can readily be done using the inverse cumulative density function method i.e. $Dt = F^{-1}(\alpha,\xi)$ where ξ is uniformly distributed in the interval $[0,1]$.

More precisely

$$\begin{aligned}\int_0^{Dt}\zeta(\Phi(\alpha_{t_1},\tau))d\tau &= \int_0^{Dt}\mu(t_1+\tau)d\tau + \lambda Dt \\ &= \int_0^{Dt}[\mu(t_1)+\tau\dot{\mu}]^+ d\tau + \lambda Dt\end{aligned}$$

Note that this function takes different forms depending on whether the term in the square brackets saturates to zero. This occurs when $\dot{\mu}$ is negative and Dt exceeds $-\mu(t_1)/\dot{\mu}$. After simplifying the algebra the sampling rule becomes the following:

Let $\xi \sim U[0,1]$ and $\xi_0 = e^{[\frac{\mu(t_1)}{\dot{\mu}}(\lambda+\frac{\mu(t_1)}{2})]}$

1) if $\dot{\mu} > 0$ or $\xi > \xi_0$
   $Dt = \frac{1}{\dot{\mu}}\left[\sqrt{(\mu(t_1)+\lambda)^2 - 2\dot{\mu}ln\xi} - (\mu(t_1)+\lambda)\right]$
2) if $\dot{\mu} < 0$ and $\xi < \xi_0$
   $Dt = \frac{1}{\lambda}\left[\mu(t_1)^2/2\dot{\mu} - ln\xi\right]$
3) if $\dot{\mu} = 0$ $Dt \sim Exp(\lambda+\mu(t_1))$

B. Analysis with PDE's

As shown in (5), the vector p of probabilities of the length q of an M/M/1 queue with constant arrival rate μ at time t

(note the change of notation from x to p), obeys an ode of the form:

$$\dot{p} = Ap$$

In the case of the feedback system in figure 1 the arrival rate evolves as a random process, since it is the output of controller K, with input the queue length q(t), which is a random process. Since $q(t)$ and $\mu(t)$ are not independent random variables, we consider their joint distribution function p, the evolution of which is described by a pde. The pde can be derived in the following way, which is a generalization of the Kolmogorov differential equations for continuous time Markov processes:

Let $p_q(\mu,t)$ the probability the queue is of length q at time t when the arrival rate is μ.Then

$$\begin{aligned} p_q(\mu(t+dt),t+dt) \;=\; & \mu(t)dt\,p_{q-1}(\mu(t),t)+ \\ & \lambda dt\,p_{q+1}(\mu(t),t)+ \\ & [1-(\mu(t)+\lambda)dt]p_q(\mu(t),t) \end{aligned} \tag{8}$$

Doing a Taylor expansion on $p_q(\mu(t+dt),t+dt)$ and letting $dt \to 0$ reduces the equation above to the following pde

$$\begin{aligned} \dot{\mu}\frac{\partial p_q(\mu(t),t)}{\partial \mu} + \frac{\partial p_q(\mu(t),t)}{\partial t} = & \mu(t)dt\,p_{q-1}(\mu(t),t)+ \\ & +\lambda p_{q+1}(\mu(t),t)+ \\ & -(\mu(t)+\lambda)p_q(\mu(t),t) \end{aligned} \tag{9}$$

Writing the system of equations in matrix form gives

$$M\frac{\partial p}{\partial \mu} + \frac{\partial p}{\partial t} = A(\mu)p \tag{10}$$

where $A(\mu)$ is as defined in (5),

$$M = diag(\dot{\mu}(q)), \qquad \frac{\partial p}{\partial \mu} = \begin{bmatrix} \frac{\partial p_1(\mu,t)}{\partial \mu} \\ \frac{\partial p_2(\mu,t)}{\partial \mu} \\ \vdots \\ \frac{\partial p_q(\mu,t)}{\partial \mu} \\ \vdots \end{bmatrix},$$

$$\frac{\partial p}{\partial t} = \begin{bmatrix} \frac{\partial p_1(\mu,t)}{\partial t} \\ \frac{\partial p_2(\mu,t)}{\partial t} \\ \vdots \\ \frac{\partial p_q(\mu,t)}{\partial t} \\ \vdots \end{bmatrix}, \quad \dot{\mu}(q) = \begin{cases} k & \text{if } q < N \\ -k & \text{if } q \geq N \\ 0 & \text{if } q \geq N \text{ and } \mu = 0 \end{cases} \tag{11}$$

Each extra state added into the controller will also evolve as a random process and therefore the dimensions of the pde will increase. A convenient way to approximate a delay T between the controller and the queue is by means of a first order Pade approximation

$$e^{-sT} \approx \frac{1-\frac{T}{2}s}{1+\frac{T}{2}s}$$

or in state space form

$$\begin{aligned} \dot{x} &= \frac{-2}{T}x + u \\ y &= \frac{4}{T}x - u \end{aligned}$$

Similarly with (8) and (9) the left hand side of (8) now becomes $p_q(\mu(t+dt),x(t+dt),t+dt)$ and its first order Taylor expansion is

$$\begin{aligned} & p_q(\mu,x,t+dt) + d\mu\frac{\partial p_q(\mu,x,t)}{\partial \mu} + dx\frac{\partial p_q(\mu,x,t)}{\partial x} + \\ & dt\frac{\partial p_q(\mu,x,t)}{\partial t} = \\ & p_q(\mu,x,t+dt) + \dot{\mu}dt\frac{\partial p_q(\mu,x,t)}{\partial \mu} + \\ & \underbrace{(\frac{-2}{T}x+\mu)}_{\dot{x}}dt\frac{\partial p_q(\mu,x,t)}{\partial x} + dt\frac{\partial p_q(\mu,x,t)}{\partial t} \end{aligned}$$

Therefore the pde now is

$$M_1\frac{\partial p}{\partial \mu} + M_2\frac{\partial p}{\partial x} + \frac{\partial p}{\partial t} = A(\mu)p$$

$$\text{where} \quad M_1 = diag(\dot{\mu}(q)), \quad M_2 = diag(\frac{-2}{T}x+\mu) \tag{12}$$

IV. Stochastic variability Vs deterministic instability

We want to investigate how the variability in μ varies with the parameters of the stochastic system which are controller gain, number of packets in flight and queue threshold. We use steady state sample statistics to investigate this behaviour, as the pde's described above require very fine discretization grids to avoid error accumulation in the numerical solver. Note that the data are presented in nondimensional form such that the capacity λ is not a relevant parameter (this is equivalent to the fact that a scaling in time will not affect the relative behaviour of the system thus making capacity redundant). The feedback gain is normalized by the critical gain k_c that causes instability in the deterministic model with static price.

From the plots in figure 2 it is apparent that the onset of limit cycles in the deterministic models is an important parameter for the stochastic network, whenever there are large number of packets in flight with small queue thresholds. In figure 2 we see this effect for 3750 packets in flight which is realistic if we consider capacities∼100Mbit/s, packet size∼kbit, and delays at least∼10msec. Large delays tend to average out stochastic effects and hence the onset of limit cycles turns out to have a more dominant effect. In addition, the variability in μ appears to increase linearly with the controller gain k well below k_c in all cases, thus justifying the use for analysis of linearized deterministic models with brownian perturbation in this regime.

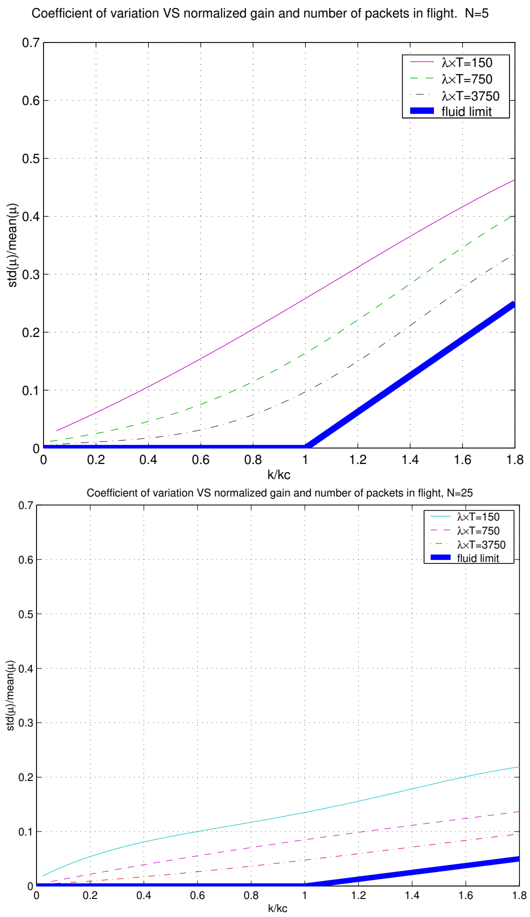

Fig. 2. Coefficient of variation of μ vs no. of packets in flight, normalized controller gain and queue threshold.

V. Comparing the Stochastic and Deterministic Models

We now examine the extent to which the dynamics of the stochastic model are being captured by the deterministic models in section II-B. Note that the ensemble average is not a valid parameter for evaluating such a performance in a jump process, due to the randomness in the times between jumps. This will tend to average out the trajectories to very small amplitudes, even though there are significant dynamics in them, because of the random phase difference between individual trajectories.

It appears from the simulations in figure 5 that the response of the deterministic model with dynamic price captures these dynamics very well, even when the queueing delay is comparable to the propagation delay (stochastic effects are more evident in this regime). As a step further, we try to find ways that the stochastic model can be approximated as a deterministic model+noise. This is illustrated in the block diagram in figure 3 which is equivalent to figure 1. We assume that at steady state, the arrival rate μ feeding the noise process is approximately constant and equal to the steady state value predicted by the deterministic $F-K$ feedback system. An explicit expression for the power

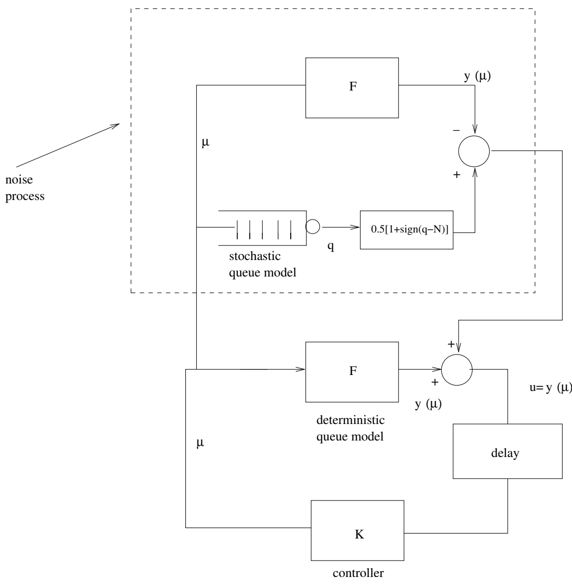

Fig. 3. The diagram shows how one can treat the stochastic effects as noise into the deterministic model. The control law K is $\frac{d\mu}{dt} = (1-u(t))k - ku(t)$ and the system F is the deterministic model. The queue has poisson arrivals at rate μ, poisson departures at rate λ and length q. The operator $sign(x)$ gives as output -1 if $x < 0$ and 1 if $x \geq 0$.

spectrum of such a noise process can easily be derived analytically (omitted from paper due to length limits) and appears to be similar to that of band limited noise.

We now compare the following cases:

1) Stochastic model (as in figure 1).
2) Linear model + noise (as in figure 3 with F being the linearization of the static price (2)).
3) Nonlinear model + noise (as in figure 3 with F being the dynamic price model (5)).

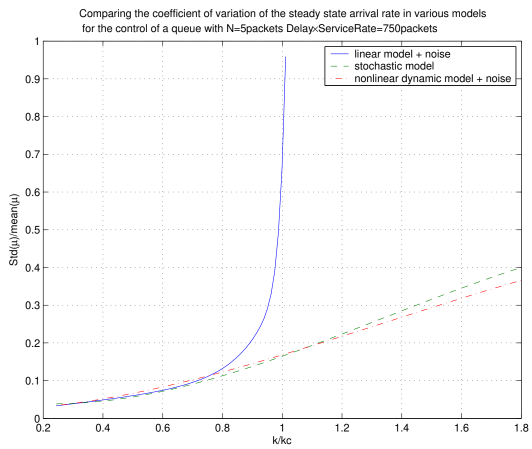

Fig. 4. Coefficient of variation of the arrival rate μ with increasing normalized controller gain k/k_c.

Figure 4 shows the sample standard deviation (from time responses) of the steady state arrival rate in the 3 cases mentioned above. We used Delay×Capacity= 750 packets, N = 5packets. We observe that the nonlinear dynamic

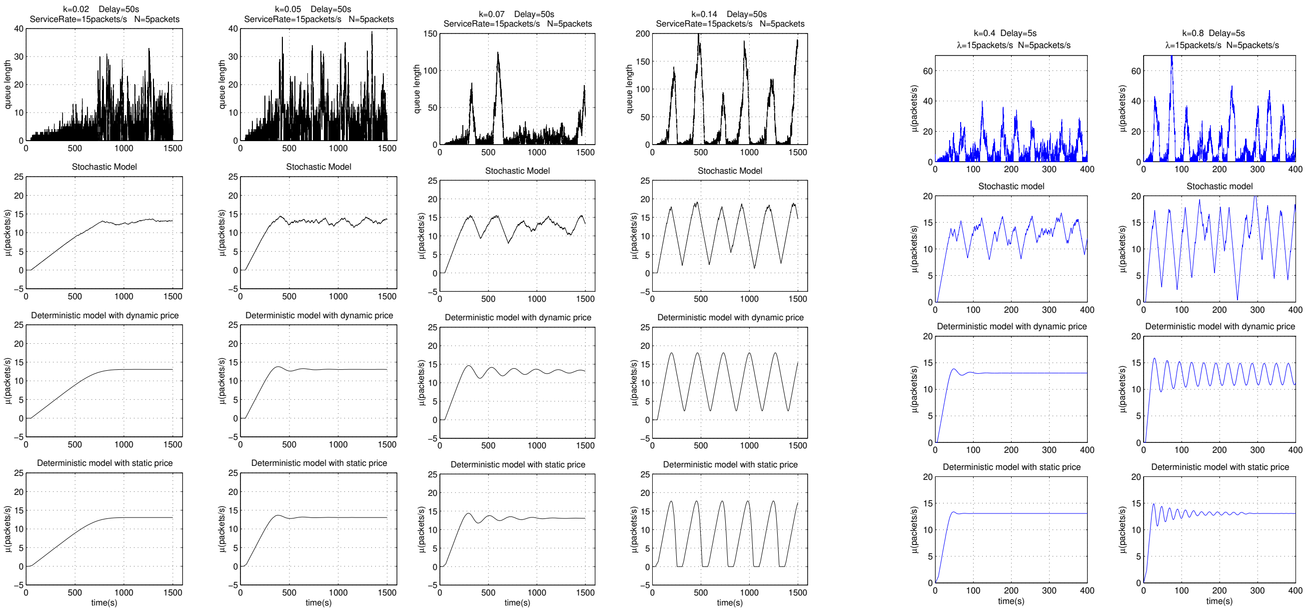

Fig. 5. Comparing the stochastic and deterministic models in stable and unstable regimes. Plots in the left correspond to a critical gain for instability $k_c = 0.08$. Plots in the right correspond to a case where the queue lag is comparable to the propagation delay with $k_c = 0.8$.

model+noise approximates first and second order statistics of the stochastic model very well. Furthermore, it is apparent in this regime, that the use of linearized models for predicting the coefficient of variation is only valid for gains well below the onset of instability.

VI. Conclusions

We have shown how feedback control of queues can be analyzed as a stochastic hybrid system and have derived pde's that describe the transient probability distribution of the system states in the presence of delays. This has been used as a basis for simulating a simplified stochastic feedback system that enables us identify the regime where instability in deterministic models (in the form of limit cycles) corresponds to a substantial increase in the variation of the flow rate in a stochastic network. Furthermore, a more sophisticated deterministic nonlinear dynamic queue model is suggested. This captures the queue dynamics not only in the extreme cases of negligible queueing delays (relative to propagation delays) or queues which never empty, but also in intermediate regimes of large delays and significant queueing dynamics.

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