Model predictive control of uncertain constrained linear systems; LMI-based methods*

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Abstract

The time-varying state-feedback control of a constrained linear state-space system is addressed via Linear Matrix Inequality (LMI) based optimization methods. The constraints are specified as ellipsoidal or hyperplane constraints on the inputs and states, and the approach presented allows these to be specified without any conservativeness. The control action is specified in terms of both feedback and feedforward components. Uncertainty in the system is modeled by perturbations in a linear fractional transform (LFT) representation, and also by bounded disturbances. As a single-step design procedure the approach gives a time-varying controller capable of steering the state to a specified reference while satisfying the constraints. The method can be applied in a model predictive strategy to allow for higher performance as the state, and control input, move away from the constraint boundaries.

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1 Introduction

The control of many systems is dominated by constraints and Model Predictive Control (MPC) is a popular approach for the design of controllers respecting the constraints. MPC was initially developed in the process control industries where the control problems typically have slow dynamics giving ample time for the intersample solution of complex optimization problems. As computational speeds increase MPC is finding application in domains typically requiring higher bandwidth control, but also dominated by constraints. Aerospace and automotive applications fall within this class of problems. MPC has been studied in academic and industrial contexts for quite some time; see for example the survey papers of Rawlings [1], Mayne *et al.* [2], Chen and Allgöwer [3], Morari and Lee [4], and the detailed book of Maciejowski [5].

The closed-loop stability of MPC control systems was initially addressed by Rawlings [6] and since that time most approaches use one of several strategies to guarantee the closed-loop stability. These approaches can be loosely categorized as: using an infinite prediction horizon; including terminal cost functions or terminal constraints; and augmenting of the system with a stabilizing feedback controller. The approaches taken here can be viewed as specifying a terminal constraint set, together with a stabilizing controller over that set.

The consideration of uncertain systems is more recent. Early work, based on FIR models, appears in [7, 8, 9]. Robust linear control is a well developed field but does not directly address constrained systems. The model class typically used in robust control theory—model sets generated from a linear model and a bounded set of perturbations—has been considered in an MPC context in the work of Kothare *et al.* [10]. The work in [10] uses a conservative bounding approach for guaranteeing constraint satisfaction. The work presented in this work instead limits the class of stabilizing controllers but, it turn, gives exact constraint satisfaction. This is done by calculating the constrained control of a prespecified ellipsoidal region. Aspects of our approach are similar to that of Kouvaritakis *et al.* [11], which uses feedforward control and finds an invariant ellipse bounding the state. The feedforward component of the control is manipulated to ensure constraint satisfaction. Our work instead uses a combination of feedback and feedforward control, designed simultaneously, to ensure constraint satisfaction and closed-loop stability.

Our approach uses quadratic functionals to specify regions of state-space over which the single-step controller must operate. This is a more general specification that that used in [10, 11], and means that the regions need not be centered upon the origin. This allows us to take advantage of asymmetric constraint regions.

One potential disadvantage of our approach should be noted. The optimization results in a series of ellipses, each bounding the state at the future time-steps, and converging to an ellipse at the reference state. This is not optimal as it does not account for the effect of feedback in manipulating the size and shape of the ellipse. However, in an MPC framework, subsequent optimization recalculations do take advantage of the shrinking of an invariant ellipse under feedback.

A preliminary version of some of the work described here was presented in [12].

1.1 Notation

We will use a discrete-time formulation and the time index is denoted by a subscript k. Scalar or matrix variables that are time dependent will also be subscripted with k. The optimization results are presented as LMI constraints of the form $S = S^T \leq 0$. Because the LMI is always symmetric the lower diagonal elements will usually not be explicitly indicated. For example,

$$S = \begin{bmatrix} S_{11} & S_{12} \\ \bullet & S_{22} \end{bmatrix},$$

where $\bullet$ is taken to represent $S_{21} = S_{12}^T$.

2 Problem description

The objective is to control the state of a linear system from an initial state, x_0, to a desired reference, which for notational simplicity we take to be the origin. A discrete-time framework is used and we consider that the state is measured at each time-step.

Uncertainty in the model is incorporated via a linear fractional perturbation structure, described by the equations,

$$\begin{aligned} x_{k+1} &= A\,x_k + B\,u_k + B_d\,d_k + B_p\,p_k, && (1) \\ q_k &= C_q\,x_k + D_{qu}\,u_k + D_{qd}\,d_k, && (2) \\ p_k &= (\Delta q)_k. && (3) \end{aligned}$$

The operator, Δ, is block diagonal,

$$\Delta = \begin{bmatrix} \Delta_1 & & 0 \\ & \ddots & \\ 0 & & \Delta_m \end{bmatrix},$$

and is assumed to be norm bounded by one. This bound is without loss of generality as any scalings can be included in C_q and B_p. One interpretation is that at each time-step, k, the perturbation blocks, Δ_l can be viewed as an unknown, time-varying, matrix with $\bar{\sigma}(\Delta_l) \leq 1$. Denote the projection onto the components associated with Δ_l by Π_l. The norm bound on each Δ_l implies that,

$$(\Pi_l p_k)^T (\Pi_l p_k) \leq (\Pi_l q_k)^T (\Pi_l q_k), \quad \text{for all } l = 1, \ldots, m, \quad \text{and for all } k. \tag{4}$$

The above will be used to create equivalent matrix conditions to specify the set of perturbations.

This formulation can also be viewed as replacing the state-space matrices, (A,B), by $(A, B) \in (\mathcal{A}, \mathcal{B})$, where

$$(\mathcal{A}, \mathcal{B}) = \{ (A + B_p \Delta C_q, B + B_p \Delta D_{qu}) \mid \bar{\sigma}(\Delta_l) \leq 1 \}. \tag{5}$$

This perturbation framework for modeling uncertainty is widely used in robust control theory. Refer to Doyle and Packard [13, 14], or the MATLAB μ-Tools manual [15] for more details. Linear matrix inequality methods using this framework are discussed in detail in Boyd *et al.* [16], and also considered by Kothare *et al.* [10] in their work of robust LMI techniques for MPC. We will largely follow the notation in [10] for ease of comparison.

The state equation, (1), also contains a disturbance input, d_k, which is modeled as coming from a bounded set, $d_k \in \mathcal{D}$. To begin we will consider this to be specified by an l_2 norm bound on d_k at each time, k.

$$\mathcal{D} = \left\{ d_k \;\middle|\; d_k^T d_k \leq 1 \right\}. \tag{6}$$

Again the unity bound is without loss of generality as scalings can be included in B_d. It is also possible to include more general ellipsoidal or hyperplane bounds on d_k.

Our approach is based on maintaining the state within a series of invariant ellipses. We define an ellipse, with size and shape defined by $P > 0$, and with center z_k,

$$\mathcal{P}_k := \left\{ x \;\middle|\; (x - z_k)^T P^{-1} (x - z_k) \right\}. \tag{7}$$

It is assumed that the initial state is within a known ellipse, $x_0 \in \mathcal{P}_0$.

The robust MPC approach will be presented in three parts. We begin, in Section 3, by considering only the nominal case: $\Delta = 0$ and $d_k = 0$. We first determine a series of feedback and feedforward controls that will take an initial state, x_0, lying within a prespecified ellipse, and move the state to the origin.[1] This consists of two parts; an N step procedure to move the prespecified ellipse to the origin, and a terminal controller to give asymptotic convergence. The feedforward component of this work is similar to that of Löfberg [17], which relies on MPC recalculation to provide feedback.

If at a subsequent time-step, there is an opportunity to recalculate the future controls, this can be used to implement a model predictive approach. Details are given in Section 4, and applying an MPC method allows the control to become more aggressive as the state approaches the origin. Section 5 extends the methods to handle the complete robust control model given in Equations (1) to (3). Including a feedback component in the future controls will allow us to handle the effects of perturbations and future disturbances.

3 Nominal ellipsoidal control

In the nominal case the dynamics are given by,

$$x_{k+1} = A\,x_k \; + \; B\,u_k. \tag{8}$$

Given a measured initial state, x_0, we choose an ellipse, $\mathcal{P}_0$, defined by,

$$\mathcal{P}_0 := \left\{ \, x \;\middle|\; (x - z_0)^T P^{-1} (x - z_0) \leq 1, \; P = P^T > 0 \, \right\},$$

such that $x_0 \in \mathcal{P}_0$. The ellipse is centered at z_0, and may be chosen in a manner that reflects our uncertainty in the measurement of x_0. Alternatively, we may partition the state-space into a number of ellipses in advance of attempting the control design. In either case we consider the matrix P to be fixed. Assume also that $\mathcal{P}_0$ does not violate any of the state constraints to be specified later.

We consider this problem in the context of a standard quadratic control cost. Given $Q = Q^T > 0$ and $R = R^T > 0$, the control cost is defined as,

$$V(x,u) \; := \; \sum_{k=0}^{\infty} x_k^T Q x_k + u_k^T R u_k. \tag{9}$$

The control problem is made more interesting by the inclusion of input and state constraints. Consider M_u hyperplane constraints on the input of the form,

$$g_j^T u_k \leq h_j, \quad j = 1, \ldots, M_u, \text{ and } k = 0, 1, \ldots \tag{10}$$

[1] All of the results presented here are easily extended to any reference state, x_{ref}, for which there exists an equilibrium input, v_{ref}, satisfying, $x_{\mathrm{ref}} = Ax_{\mathrm{ref}} + Bv_{\mathrm{ref}}$.

In an analogous manner, we have M_x constraints on the state,

$$r_i^T x_k \leq s_i, \quad i = 1, \ldots, M_x, \text{ and } k = 0, 1, \ldots \tag{11}$$

Output constraints can be transformed into state constraints by post-multiplying by an output "C" matrix.

The formal specification of the nominal problem is as follows.

Problem 1 (Nominal design problem) *Consider a system described by (8). Given an initial ellipse, $\mathcal{P}_0$, and an initial state, $x_0 \in \mathcal{P}_0$, find a sequence of controls, u_k, $k = 0, 1, \ldots$, such that for all $x_0 \in \mathcal{P}_0$, the resulting state and input trajectories satisfy the following:*

a) $\lim_{k \longrightarrow \infty} x_k = 0$*;*

b) x_k *satisfies the state constraints (11), for all* $k > 0$*;*

c) u_k *satisfies the input constraints (10); for all* $k \geq 0$*.*

and minimizes the cost, $V(x, u)$*.*

A combination of N feedback and feedforward controls will be used to move the center of the initial ellipse from z_0 to the origin. This is then followed by a transition controller for one time-step and then a stabilizing terminal controller to give convergence to the origin with bounded cost. The transitional controller is used to tighten the bound of the performance cost. It can be viewed as optional and in the MPC context detailed in Section 4, it may provide little additional benefit.

The control signal is specified in the form,

$$u_k = \begin{cases} K_k \left(x_k - z_k \right) + v_k, & k = 0, \ldots, N-1 \\ K_N \, x_k & k = N \\ K_\infty \, x_k & k > N. \end{cases}$$

If the feedforward components, v_k, are chosen to satisfy,

$$z_{k+1} = A \, z_k + B \, v_k,$$

with $z_N = 0$, this has the effect of generating a series of ellipses,

$$\mathcal{P}_k := \left\{ x \mid (x - z_k)^T P^{-1} (x - z_k) \leq 1 \right\},$$

with the Nth ellipse,

$$\mathcal{P}_N := \left\{ x \mid x^T P^{-1} x \leq 1 \right\},$$

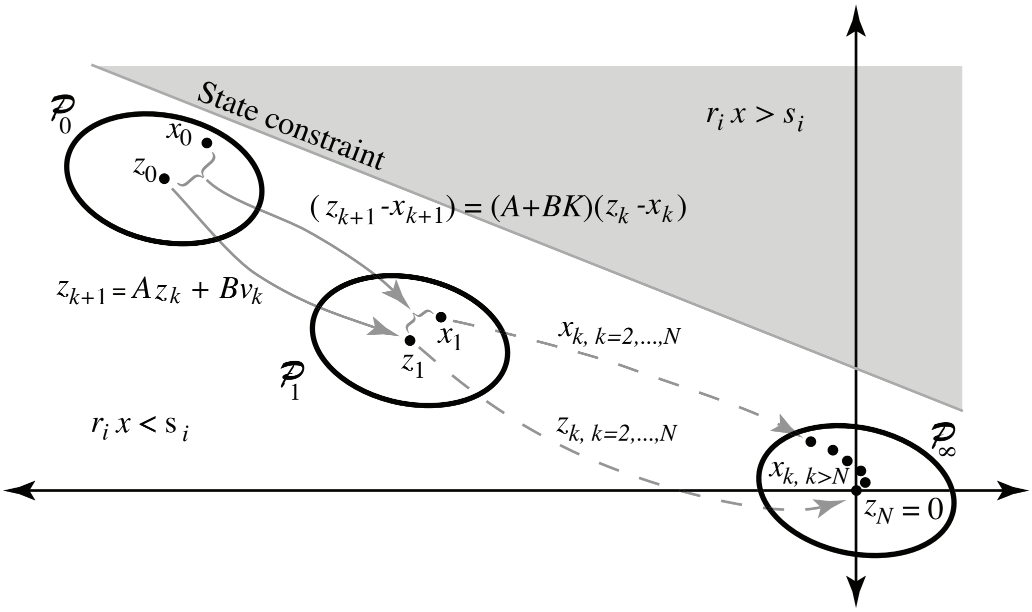

Figure 1: Nominal ellipsoidal control. The feedforward component moves the ellipse centers, z_k, to the origin. The feedback component maintains the state within the corresponding ellipse and gives convergence to the origin in the final ellipse.

centered at the origin. It is a simple matter to show that this then gives,

$$x_{k+1} - z_{k+1} \; = \; (A + BK_k)(x_k - z_k). \tag{12}$$

The purpose of the feedback term, K_k, is to maintain the next state, x_{k+1} within the shifted ellipse, $\mathcal{P}_{k+1}$. For $k \geq N$, the ellipse is centered at the origin and the terminal controller gives the required convergence. This control strategy is illustrated in Figure 1.

In this description we give a time-varying control gain, K_k. This is not essential and the calculations are significantly simplified by simply choosing a fixed value of K. In an MPC context this may not be conservative as K is recalculated at each time-step in any case.

We leave open the problem of choosing a value for N. In the unconstrained case, observability implies that the choice of N equal to the state dimension is sufficient. In the presence of constraints the required value for N is problem dependent.

In the following the design formulation is presented via three theorems. Theorem 2 gives sufficient LMI conditions on the design variables to move the state from ellipse to ellipse with bounded cost and constraint satisfaction. It applies for time-steps $k = 0, \dots, N-1$. The control design for $k = N$ is addressed in Theorem 10, and Theorem 11 gives the analogous result for the terminal control design ($k > N$).

Theorem 2 (Nominal ellipse-to-ellipse) *Given a nominal system described by (8) and an ellipse size and shape specified by $P = P^T > 0$. If there exist auxiliary variables, $0 \leq \xi_k \leq 1$, $\beta_k \geq 0$, $\eta_{k,j} \geq 0$, $j = 1, \ldots, M_u$, $\zeta_{k,i} \geq 0$, $i = 1, \ldots, M_x$, and design variables, K_k, v_k, z_{k+1}, and γ_k such that the linear constraint,*

a)

$$z_{k+1} \;=\; A\, z_k \;+\; B\, v_k; \tag{13}$$

and the $2 + M_u + M_x$ linear matrix inequality constraints,

b)

$$S_k := \begin{bmatrix} -\xi_k P^{-1} & (A + BK_k)^T \\ \bullet & -P \end{bmatrix} \leq 0; \tag{14}$$

c)

$$T_k := \begin{bmatrix} -\beta_k P^{-1} & 0 & (A + BK_k)^T & K_k^T \\ \bullet & -\gamma_k + \beta_k & z_{k+1}^T & v_k^T \\ \bullet & \bullet & -Q^{-1} & 0 \\ \bullet & \bullet & \bullet & -R^{-1} \end{bmatrix} \leq 0; \tag{15}$$

d)

$$U_{k,j} := \begin{bmatrix} -\eta_{k,j} P^{-1} & g_j^T K_k \\ \bullet & \eta_{k,j} + 2 g_j^T v_k - 2 h_j \end{bmatrix} \leq 0; \quad j = 1, \ldots, M_u; \tag{16}$$

e)

$$X_{k,i} := \begin{bmatrix} -\zeta_{k,i} P^{-1} & (A + BK_k)^T r_i \\ \bullet & \zeta_{k,i} + 2 r_i^T z_{k+1} - 2 s_i \end{bmatrix} \leq 0; \quad i = 1, \ldots, M_x; \tag{17}$$

are satisfied, then given

$$x_k \in \mathcal{P}_k \;=\; \left\{\, x \;\middle|\; (x - z_k)^T P^{-1} (x - z_k) \leq 1 \,\right\},$$

the control, $u_k = K_k(x_k - z_k) + v_k$, maps x_k to x_{k+1} such that K_k, u_k and x_{k+1} satisfy the following conditions:

f) $A + BK_k$ *is Hurwitz;*

g) $x_{k+1} \in \mathcal{P}_{k+1} \;=\; \left\{ x \;\middle|\; (x - z_{k+1})^T P^{-1}(x - z_{k+1}) \le 1 \right\}$*;*

h) $x_{k+1}^T Q x_{k+1} + u_k^T R u_k \le \gamma_k$.

i) u_k *satisfies the* M_u *input constraints, (10); and*

j) x_{k+1} *satisfies the* M_x *state constraints, (11);*

The proof of this theorem will be given as a series of lemmas, and these lemmas in turn rely on two well known results—the S-procedure and Schur complements—which we present here for completeness. These are behind many LMI results and much more detail can be found in Boyd *et al.* [16].

Lemma 3 (S-procedure) *Given* $m + 1$ *quadratic functions of a variable* $x \in \mathcal{R}^n$*,*

$$F_i(x) = x^T A_i x + 2b_i^T x + c_i, \qquad i = 0, \ldots, m,$$

where $A_i = A_i^T$*. If there exists,* $\tau_i \ge 0$*, such that,*

$$\text{for all } x, \quad F_0(x) - \sum_{i=1}^{m} \tau_i F_i(x) \le 0,$$

then $F_0(x) \le 0$ *for all* x *such that* $F_i(x) \le 0$*,* $i = 1, \ldots, m$*.*

If $m = 1$ *and there exists an* $\hat{x}$ *such that* $F_1(\hat{x}) < 0$*, then this condition is necessary and sufficient.*

The Schur complement formula below is stated for non-strict inequalities. A proof of this generalization is given by Boyd *et al.* [16, p. 28].

Lemma 4 (Schur complement) *Given* $Q = Q^T$ *and* $R = R^T$*, the condition,*

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \le 0,$$

is equivalent to

$$R \le 0, \quad Q - SR^{\dagger}S^T \le 0, \quad S(I - RR^{\dagger}) = 0,$$

where $R^{\dagger}$ *denotes the Moore-Penrose inverse of* R*.*

The following lemma is useful in forming LMIs from quadratic form constraints. A proof of a similar lemma can be found in [18].

Lemma 5 *Given a quadratic functional defined by $F(x) = x^T X x + 2y^T x + z$. The quadratic constraint $F(x) \leq 0$ is satisfied for all x if and only if the matrix $\begin{bmatrix} X & y \\ y^T & z \end{bmatrix}$ is negative semidefinite.*

We now proceed with the lemmas addressing the LMI conditions of the theorem.

Lemma 6 (Nominal stability) *If there exists $0 \leq \xi_k \leq 1$ such that $S_k \leq 0$ (Equation 14) then $A+BK_k$ is Hurwitz, and $x_{k+1} \in \mathcal{P}_{k+1}$, where z_{k+1} is given by (13).*

Proof of Lemma 6: Applying the Schur complement (Lemma 4) to the condition (14) shows (14) to be equivalent to,

$$-\xi_k P^{-1} + (A + BK_k)^T P^{-1}(A + BK_k) \leq 0,$$

As $0 < \xi_k \leq 1$ and $P > 0$, this implies that,

$$(A + BK_k)^T P^{-1}(A + BK_k) - P^{-1} < 0,$$

The matrix $P^{-1} > 0$ can therefore be seen as a solution to the discrete Lyapunov equation proving the stability of $A + BK_k$.

The condition that $x_k \in \mathcal{P}_k$ is equivalent to the quadratic functional condition,

$$(x_k - z_k)^T P^{-1}(x_k - z_k) - 1 \leq 0.$$

Under the action of the control, $x_{k+1} - z_{k+1} = (A + BK_k)(x_k - z_k)$, and so $x_{k+1} \in \mathcal{P}_{k+1}$ is equivalent to,

$$(x_k - z_k)^T (A + BK_k)^T P^{-1}(A + BK_k)(x_k - z_k) - 1 \leq 0.$$

By the S-procedure, the requirement that $x_k \in \mathcal{P}_k$ implies that $x_{k+1} \in \mathcal{P}_{k+1}$ is equivalent to the existence of $\xi_k \geq 0$ such that,

$$(x_{k+1} - z_{k+1})^T P^{-1}(x_{k+1} - z_{k+1}) - 1 - \xi_k \left((x_k - z_k)^T P^{-1}(x_k - z_k) - 1\right) \leq 0.$$

Expressing this as a quadratic functional, and applying Lemma 5 gives, as an equivalent LMI condition,

$$\begin{bmatrix} (A + BK_k)^T P^{-1}(A + BK_k) - \xi_k P^{-1} & 0 \\ 0 & \xi_k - 1 \end{bmatrix} \leq 0, \quad \text{and } \xi_k > 0.$$

Note that this condition can be decoupled to give, $0 \le \xi_k \le 1$, and,

$$(A + BK_k)^T P^{-1}(A + BK_k) - \xi_k P^{-1} \le 0.$$

Application of the Schur complement shows that this is equivalent to $S_k \le 0$. ∎

Lemma 7 (Nominal ellipse-to-ellipse cost) *For all $x_k \in \mathcal{P}_k$, the control, $u_k = K_k (x_k - z_k) + v_k$, with v_k satisfying (13), gives x_{k+1} such that,*

$$x_{k+1}^T Q x_{k+1} + u_k^T R u_k \le \gamma_k, \tag{18}$$

if and only if $T_k \le 0$ (Equation 15).

Proof of Lemma 7: As $u_k = K_k(x_k - z_k) + v_k$ and $x_{k+1} = (A + BK_k)(x_k - z_k) + z_{k+1}$, the condition expressed in (18) is equivalent the quadratic functional constraint,

$$\begin{aligned} &((A + BK_k)(x_k - z_k) + z_{k+1})^T Q ((A + BK_k)(x_k - z_k) + z_{k+1}) \\ &\qquad + (K_k(x_k - z_k) + v_k)^T R (K_k(x_k - z_k) + v_k) - \gamma_k \le 0. \end{aligned} \tag{19}$$

Requiring (19) to hold for all $x_k \in \mathcal{P}_k$, is equivalent—via the S-procedure—to the existence of $\beta_k \ge 0$, such that,

$$\begin{aligned} &((A + BK_k)(x_k - z_k) + z_{k+1})^T Q ((A + BK_k)(x_k - z_k) + z_{k+1}) \\ &+ (K_k(x_k - z_k) + v_k)^T R (K_k(x_k - z_k) + v_k) - \gamma_k \\ &\qquad - (x_k - z_k)\beta_k P^{-1}(x_k - z_k) + \beta_k \le 0, \end{aligned} \tag{20}$$

for all x_k. This can be expressed quadratic functional constraint in $(x_k - z_k)$,

$$\begin{aligned} &F(x_k - z_k) \ := \\ &(x_k - z_k)^T \left((A + BK_k)^T Q(A + BK_k) + K_k^T R K_k - \beta_k P^{-1}\right) \\ &+ 2 \left(z_{k+1}^T Q + v_k^T R K_k\right) (x_k - z_k) \\ &\qquad + z_{k+1}^T Q z_{k+1} + v_k^T R v_k - \gamma_k + \beta_k \le 0. \end{aligned} \tag{21}$$

Requiring $F(x_k - z_k) \le 0$ for all x_k is equivalent to requiring $F(x) \le 0$ for all x. Using Lemma 5 to formulate an LMI and applying the Schur complement twice shows that this is equivalent to $T_k \le 0$. ∎

Lemma 8 (Nominal state constraint) *For all $x_k \in \mathcal{P}_k$, the control $u_k = K_k(x_k - z_k) + v_k$ generates x_{k+1} satisfying the state constraint,*

$$r_i^T x_{k+1} \leq s_i, \tag{22}$$

if and only if there exists $\zeta_{k,i} \geq 0$ such that $X_{k,i} \leq 0$ (Equation 15).

Proof of Lemma 8: The hyperplane constraint, (22) can be expressed in terms of x_k as,

$$2r_i^T(A + BK_k)(x_k - z_k) + 2r_i^T z_{k+1} - 2s_i \leq 0.$$

Using the S-procedure, this is satisfied for all $x_k \in \mathcal{P}_k$ if and only if there exists $\zeta_{k,i} \geq 0$ such that,

$$\begin{aligned} 2r_i^T(A + BK_k)(x_k - z_k) + 2r_i^T z_{k+1} - 2s_i \\ - \zeta_{k,i}\left((x_k - z_k)^T P^{-1}(x_k - z_k) - 1\right) \leq 0. \end{aligned}$$

This is a quadratic functional constraint of the form $F(x_k - z_k) \leq 0$, and is satisfied for all x_k if and only if $F(x) \leq 0$ is satisfied for all x. By Lemma 5 this is then equivalent to $X_{k,i} \leq 0$. ■

Lemma 9 (Input constraint) *For all $x_k \in \mathcal{P}_k$, the control $u_k = K_k(x_k - z_k) + v_k$ satisfies the input constraint,*

$$g_j^T u_k \leq h_j, \tag{23}$$

if and only if there exists $\eta_{k,j} \geq 0$ such that $U_{k,j} \leq 0$ (Equation 16).

Proof of Lemma 9: Substituting $u_k = K_k(x_k - z_k) + v_k$ into (23) and following the line of argument given in the proof of Lemma 8 gives the desired result. ■

Proof of Theorem 2: This is simply a matter of identifying the appropriate lemmas for each part. Lemma 6 shows that the constraints *a)* and *b)* imply *f)* and *g)*. Lemma 7 shows that the constraints *a)* and *c)* imply *h)*. Lemma 9 shows that constraint *d)*, when applied to each of the M_u input constraints, implies *i)*. Lemma 8 shows that the constraint, *e)*, when applied to each of the M_x state constraints, implies *j)*. ■

Analogous results, for both the input and state constraints, hold if the hyperplane constraints, (22) and (23), are replaced by ellipsoidal constraints. The corresponding LMI condition is easily derived using the arguments presented in Lemma 8. It is interesting to note that Lemmas 8 and 9 are both necessary and sufficient. Moreover, for multiple constraints, multiple LMI conditions are applied and all constraints are satisfied if and only if all LMIs are satisfied. This exact constraint specification differs significantly from most other LMI-based approaches in model predictive control.

Theorem 2 gives a means of calculating a controller, K_k, and a feedforward input, v_k that will take all x_k in the ellipse $\mathcal{P}_k$ and move it to the ellipse $\mathcal{P}_{k+1}$, centered at z_{k+1}. The input signal and all $x_{k+1} \in \mathcal{P}_{k+1}$ satisfy hyperplane or ellipsoidal bounds. We will apply this approach N times to achieve an ellipse centered at the origin (i.e. $z_N = 0$). At $k = N$, $x_k \in \mathcal{P}_N$ and a transition controller, bounding only the input cost is applied for a single time-step. The details of this are specified in Theorem 10 below. The motivation for this step is to give a closer bound on the quadratic cost function (Equation 9).

Theorem 10 (Nominal transitional ellipse) *Given a nominal system described by (8) and an ellipse size and shape specified by $P = P^T > 0$. Assume also that the quadratic input cost weight satisfies $R \geq 0$. If there exists auxiliary variables, $\zeta_{N,j}$, $j = 1, \ldots, M_u$, $\beta_N \geq 0$, $\xi_N \geq 0$, and design variables, K_N and $\gamma_N > 0$ such that:*

a)

$$S_N := \begin{bmatrix} -\xi_N P^{-1} & (A + BK_N)^T \\ \bullet & -P \end{bmatrix} \leq 0; \tag{24}$$

b)

$$T_N := \begin{bmatrix} -\beta_N P & K_N^T \\ \bullet & -R^{-1} \end{bmatrix} \leq 0; \tag{25}$$

c)

$$\beta_N \leq \gamma_N; \tag{26}$$

d)

$$U_{N,j} := \begin{bmatrix} -\zeta_{N,j} P^{-1} & g_j^T K_N \\ \bullet & \zeta_{N,j} - 2h_j \end{bmatrix} \leq 0, \quad j = 1, \ldots, M_u, \tag{27}$$

then for all $x_k \in \mathcal{P}_N$ the control, $u_N = K_N\, x_N$, gives

e) $A + BK_N$ is Hurwitz and $x_{N+1} \in \mathcal{P}_N$;

f) $u_N^T R u_N \le \gamma_N$*;*

g) u_N*, satisfies the* M_u *input constraints, (10).*

Proof of Theorem 10: If $S_N \le 0$ (Equation 24) Lemma 6 implies that $A + BK_N$ is Hurwitz, and, as $z_N = 0$, $x_{N+1} \in \mathcal{P}_N$, implying part e). For all $x_N \in \mathcal{P}_N$, the cost bound $u_N^T R u_N \le \gamma_N$ is equivalent—via the S-procedure—to the condition that,

$$x_N^T K_N^T R K_N x_N - \gamma_N - x_N^T \beta_N P x_N + \beta_N \le 0,$$

for all x_N. Application of the Schur complement procedure shows the equivalence between conditions b) and c) and the bound in part f). Lemma 9, with $z_N = 0$ and $v_N = 0$, shows that part d) is equivalent to part g). ■

For $k \ge N$, the ellipse bounding the state is centered at the origin. For $k > N$, a terminal controller is applied to give asymptotic convergence to the origin with bounded quadratic cost. Theorem 11 below gives the conditions required for the calculation of the terminal controller.

Theorem 11 (Nominal terminal ellipse) *Given a nominal system described by (8) and an ellipse size and shape specified by* $P = P^T > 0$*. Assume also that the quadratic cost weightings satisfy* $Q > 0$ *and* $R \ge 0$*. If there exists auxiliary variables,* $\eta_{\infty,j}$*,* $j = 1, \ldots, M_u$*, and design variables,* K_∞ *and* $\gamma_\infty > 0$ *such that:*

a)

$$T_\infty := \begin{bmatrix} -P^{-1} & (A + BK_\infty)^T & I & K_\infty^T \\ \bullet & -P^{-1} & 0 & 0 \\ \bullet & \bullet & -\gamma_\infty Q^{-1} & 0 \\ \bullet & \bullet & \bullet & -\gamma_\infty R^{-1} \end{bmatrix} \le 0; \qquad (28)$$

b)

$$U_{\infty,j} := \begin{bmatrix} -\eta_{\infty,j} P^{-1} & g_j^T K_\infty \\ \bullet & \eta_{\infty,j} - 2h_j \end{bmatrix} \le 0, \quad j = 1, \ldots, M_u, \qquad (29)$$

then for all $x_k \in \mathcal{P}_\infty$ *the control,* $u_k = K_\infty x_k$*, gives state and input trajectories satisfying the following conditions.*

c) $A + BK_\infty$ *is Hurwitz and* $x_k \in \mathcal{P}_\infty$*, for all k;*

d) $\lim_{k \longrightarrow \infty} x_k = 0$*;*

e) $\sum_{l=k}^{\infty} x_l^T Q x_l + u_l^T R u_l \ \leq \ \gamma_\infty$*;*

f) u_k*, satisfies the* M_u *input constraints, (10) for all* k*.*

Proof of Theorem 11: Applying the Schur complement lemma twice shows that $T_\infty \leq 0$ is equivalent to,

$$(A + BK_\infty)^T P^{-1}(A + BK_\infty) - P^{-1} \leq \frac{-1}{\gamma_\infty}\left(Q + K_\infty^T R K_\infty\right). \qquad (30)$$

As $\gamma_\infty > 0$, $Q > 0$, and $R \geq 0$,

$$(A + BK_\infty)^T P^{-1}(A + BK_\infty) - P^{-1} < 0. \qquad (31)$$

The matrix $P^{-1} > 0$ can be interpreted as the solution to a discrete Lyapunov equation proving that $A + BK_\infty$ is Hurwitz. It also follows that the ellipse $x_k^T P^{-1} x_k \leq 1$ is invariant under $x_{k+1} = (A + BK_\infty)x_k$ proving part c). The strict inequality in (31) shows that d) also holds.

To show part e), define a positive definite function, $V(x) := x^T P^{-1} x$. Now,

$$V(x_{l+1}) - V(x_l) = x_l^T\left((A + BK_\infty)^T P^{-1}(A + BK_\infty) - P^{-1}\right)x_l,$$

which implies, by (30),

$$V(x_{l+1}) - V(x_l) \leq \frac{-1}{\gamma_\infty} x_l^T \left(Q + K_\infty^T R K_\infty\right) x_l.$$

Sum both sides of this equation from $l = k$ to $l \longrightarrow \infty$ to get,

$$\lim_{l \longrightarrow \infty} V(x_l) - V(k) \leq \frac{-1}{\gamma_\infty} \sum_{l=k}^{\infty} x_l^T \left(Q + K_\infty^T R K_\infty\right) x_l.$$

By part d) above, and the positive definiteness of the function $V(x)$,

$$\lim_{l \longrightarrow \infty} V(x_l) = 0,$$

giving,

$$\frac{1}{\gamma_\infty} \sum_{l=k}^{\infty} x_l^T \left(Q + K_\infty^T R K_\infty\right) x_l \leq V(x_k).$$

As, for all $x_k \in \mathcal{P}_\infty$, $V(x_k) \leq 1$, this gives, part e) as required. Applying Lemma 9 to c), with $\eta_{\infty,j}$ replacing $\eta_{k,j}$, K_∞ replacing K_k, and $v_k = 0$, implies that g) is satisfied. ∎

The above results naturally suggest an optimization-based algorithm for solving Problem 1.

Problem 12 *[Nominal Ellipse Optimization] Solve the following LMI optimization problem for the design variables: K_k, v_k,*

$$\gamma^* = x_0^T Q x_0 + \min_{\gamma_k, \gamma_\infty} \left(\sum_{k=0}^{N} \gamma_k + \gamma_\infty \right),$$

subject to the LMI constraints:

$$S_k \leq 0, \quad k = 0, \ldots, N-1, \qquad \textit{(Eqn. 14)}$$

$$T_k \leq 0 \quad k = 0, \ldots, N-1, \qquad \textit{(Eqn. 15)}$$

$$U_{k,j} \leq 0, \quad k = 0, \ldots, N-1, \quad j = 1, \ldots, M_u, \qquad \textit{(Eqn. 16)}$$

$$X_{k,i} \leq 0, \quad k = 0, \ldots, N-1, \quad i = 1, \ldots, M_x, \qquad \textit{(Eqn. 17)}$$

$$S_N \leq 0, \qquad \textit{(Eqn. 24)}$$

$$T_N \leq 0 \qquad \textit{(Eqn. 25)}$$

$$\beta_N \leq \gamma_N \qquad \textit{(Eqn. 26)}$$

$$U_{N,j} \leq 0, \quad j = 1, \ldots, M_u, \qquad \textit{(Eqn. 27)}$$

$$T_\infty \leq 0, \qquad \textit{(Eqn. 28)}$$

$$U_{\infty,j} \leq 0, \quad j = 1, \ldots, M_u, \qquad \textit{(Eqn. 29)}$$

and the linear constraint:

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} z_0 + \begin{bmatrix} B & & & 0 \\ AB & B & & \\ \vdots & & \ddots & \\ A^{N-1}B & & \cdots & B \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}, \qquad (32)$$

with $z_N = 0$.

The control strategy involves using feedforward control to shift the initial ellipse, $\mathcal{P}_0$, to the origin. To make this a well posed problem we assume that the terminal shifted ellipse, $\mathcal{P}_\infty$, does not violate the state constraints. Stated formally,

Assumption A1 For all $x \in \mathcal{P}_\infty$, x satisfies the state constraints, $r_i^T x \leq s_i$, $i = 1, \dots, M_x$.

This assumption may be relaxed in an MPC context (see Section 4). It is practically useful, although not strictly necessary, to also assume that the initial ellipse, $\mathcal{P}_0$, also satisfies the state constraints.

Theorem 13 *Under assumption A1, if, for a given N, there exists a solution to Problem 12, then, for all $x_0 \in \mathcal{P}_0$, the control,*

$$u_k = \begin{cases} K_k(x_k - z_k) + v_k, & k = 0, \dots, N-1 \\ K_N\, x_k & k = N \\ K_\infty\, x_k & k > N \end{cases}$$

solves Problem 1 with cost $V(x,u) \leq \gamma^*$.

Proof of Theorem 13: A solution to Problem 12 gives design variables satisfying the conditions of Theorems 2, 10 and 11. The linear constraint in (32) ensures that v_k is chosen such that the after the control actions, $k = 0$ to $k = N-1$, the ellipse center, z_N, is at the origin. As each of the feedback controllers, K_k, K_N, and K_∞ is Hurwitz, $\lim x_{k \longrightarrow \infty} = 0$ giving condition a). Input state constraint satisfaction (condition c) in Problem 1) is given by input constraint satisfaction in each of Theorems 2, 10 and 11. State constraint satisfaction (condition d) in Problem 1) is ensured for $k = 0, \dots, N-1$, by state constraint satisfaction in Theorem 2. For $k \geq N$, state constraint satisfaction is given by Assumption A1.

To show the cost bound express γ^* as,

$$\gamma^* = x_0^T Q x_0 + \sum_{k=0}^{N-1} \left(x_{k+1}^T Q x_{k+1} + u_k^T R u_k \right) \\ + u_N^T R u_N + \sum_{k=N+1}^{\infty} \left(x_k^T Q x_k + u_k^T R u_k \right). \quad (33)$$

By Theorem 2, the first summation is bounded by $\sum_{k=0}^{N-1} \gamma_k$. Theorem 10 gives the bound on the u_N term as γ_N, and Theorem 11 bounds the remaining infinite summation by γ_∞. ■

The algorithm presented in Problem 12 gives a time-varying feedback gain. It is a simple matter to pose this problem for a fixed feedback gain, K, and doing so reduces the number of LMI constraints from $3+2N+(N+2)M_u+NM_x$ to $3+N+(N+1)M_u+NM_x$ (i.e. $N+M_u$ fewer constraints). Note also that imposing time-varying input or state constraints can be done with no additional computational cost. The additional degrees of freedom in the time-varying design problem may make it feasible in cases where the time-invariant one is not. An alternative method of calculating time-varying controllers is discussed an the end of the following section.

4 Model predictive control

The approach described in Section 3 has a potentially significant conservative aspect: the computation assumes that the ellipse, specified by $P > 0$, is of the same size and shape at each future time instance. This assumption is required to give a convex optimization problem. In reality, the feedback gain, K_k, will shrink and reshape the ellipse. In an MPC context, the optimization problem is resolved at each time-step and the knowledge that x_{k+1} is in a smaller ellipse may be exploited.

Two further aspects require description in an MPC context: the size and shape of the new ellipse, and a guarantee that feasibility of the optimization will be maintained from time-step to time-step. To adequately describe these aspects a "conditional" notation is introduced. The ellipse,

$$\mathcal{P}_{k|n} := \left\{ x_k \;\middle|\; (x_k - z_{k|n})^T P_{k|n}^{-1} (x_k - z_{k|n}) \le 1 \right\},$$

is calculated at time n, and contains the state at time k, where $k \ge n$. A similar notation applies to the control input;

$$u_{k|n} = K_{k|n}\left(x_k - z_{k|n}\right) + v_{k|n}, \tag{34}$$

is the actuation calculated at time-step n, to be applied at time-step k.

The MPC approach is described in terms of a time index n as follows:

1. Initialize $n = 0$, and choose $P_{0|0}$ such that $x_0 \in \mathcal{P}_{0|0}$ and $\mathcal{P}_{N|0}$ satisfies the state constraints.

2. Solve Problem 12 to calculate,

$$u_{k|n} = \begin{cases} K_{k|n}(x_k - z_{k|n}) + v_{k|n}, & k = n, \ldots, n+N-1 \\ K_{N|n}\, x_k & k = n+N \\ K_{\infty|n}\, x_k & k > n+N \end{cases}$$

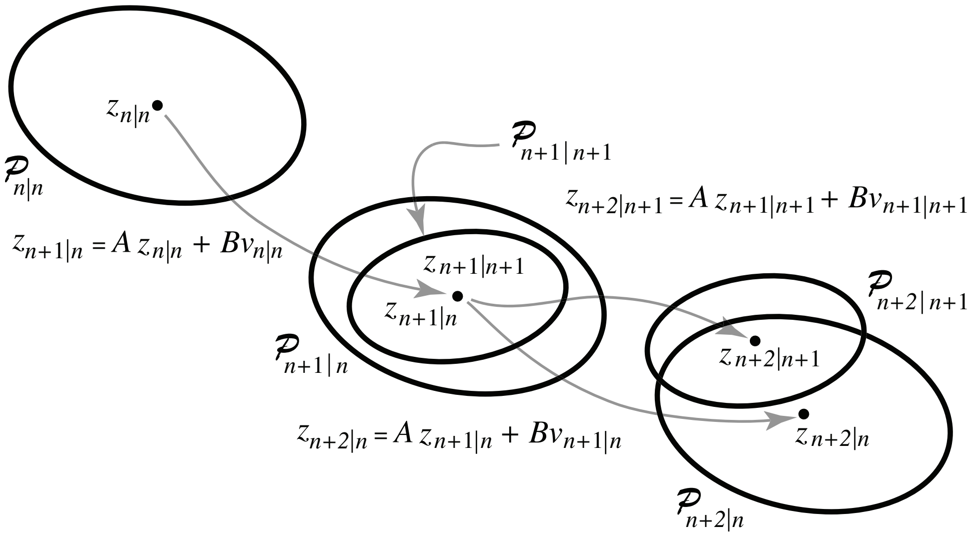

Figure 2: The evolution of bounding ellipses in the MPC approach. At each time-step a recalculation may shrink the size of the current and future bounding ellipses.

3. Apply the actuation, $u_{n|n}$, to the plant.

4. $n = n + 1$:

5. Given $\mathcal{P}_{n-1|n-1}$ from the previous time-step, calculate a new ellipse $\mathcal{P}_{n|n}$ containing x_n (see Theorem 14 below).

6. Go to step 2.

Step 5. in the above exploits the knowledge of the previous feedback step to calculate a smaller ellipse containing the new state. This concept is illustrated in Figure 2. The method of calculation is given in Theorem 14 below.

Theorem 14 *If* $(A + BK_{n|n})^{-1}$ *exists then for all* $x_n \in \mathcal{P}_{n|n}$*, the control in (34) gives* $x_{n+1} \in \mathcal{P}_{n+1|n+1}$ *where,*

$$P_{n+1|n+1} = (A + BK_{n|n})^T P_{n|n} (A + B_{n|n}). \tag{35}$$

Proof of Theorem 14: This follows immediately from (12). ■

Note that the recalculation changes $P_{n+1|n+1}$; it does not change the ellipse center and so $z_{n+1|n+1} = z_{n+1|n}$. However, in general, $z_{k|k} \neq z_{k|n}$ for $k > n+1$ as future ellipse centers are recalculated in Step 2 of the algorithm above.

The invertibility assumption in Theorem 14 is required to maintain the assumption that all ellipses are described by a matrix $P > 0$. If this is not satisfied then the next ellipse is a thin set in the state-space. This may happen if the control is "deadbeat" in one or more state directions. To avoid numerical problems in such cases one can expand the collapsed directions slightly by instead using,

$$P_{n+1|n+1} = (A + BK_{n|n})^T P_{n|n}(A + B_{n|n}) + \epsilon U_\perp U_\perp^T,$$

where $U_\perp$ is a basis for the null space of $A + BK_{n|n}$.

An important aspect of the application of optimization based methods is the feasibility of subsequent optimization steps. Regions of the state-space can be tested for feasibility off-line, but some assurance that a problem that is feasible at the initial time remains feasible is required. Theorem 15 below addresses this point.

Theorem 15 *If at time, $n = 0$, the ellipsoidal control design optimization (Problem 12) is feasible, then it is feasible for all $n > 0$.*

Proof of Theorem 15: This follows simply from the fact that $\mathcal{P}_{n+1|n+1} \subset \mathcal{P}_{n|n}$, the solution at time-step n satisfies all of the constraints of the design at time-step $n + 1$. ∎

Note that the MPC concept described above may be applied off-line in an iterative fashion to optimize the calculation of a time-varying control. Doing this takes advantage of the knowledge the the feedback reduces the size of the subsequent ellipses and may lead to a lower quadratic cost.

5 Robust ellipsoidal control

In the robust case we must contend with the fact that both the perturbations, Δ, and the disturbances, d_k, influence the transition to the next state, x_{k+1}. The formal specification of the robust problem is as follows.

Problem 16 (Robust design problem) *Consider the system defined by Equations (1), (2) and 3, with perturbation constraints, (4), and a bounded disturbance, $d_k \in \mathcal{D}$ (6). Given an initial ellipse, $\mathcal{P}_0$, and an initial state, $x_0 \in \mathcal{P}_0$, find a sequence of controls, u_k, $k = 0, 1, \ldots,$ such that for all $x_0 \in \mathcal{P}_0$, the resulting state and input trajectories satisfy the following:*

a) $x_k \in \mathcal{P}_\infty$ for all $k \geq N$.

b) x_k satisfies the state constraints (11), for all $k > 0$;

c) u_k satisfies the input constraints (10); for all $k \geq 0$.

Furthermore, if $d_k = 0$,

d) $\lim_{k \longrightarrow \infty} x_k = 0$,

and the cost function, $V(x, u)$, is minimized.

The problem is again broken up into three parts: an N step implementation keeping $x_k \in \mathcal{P}_k$ and moving the ellipses to the origin at step $k = N$; a transitional step bounding the actuation cost at $k = N$; and a terminal controller maintaining $x_k \in \mathcal{P}_k$ for all $k > N$ with bounded cost. To do this, we replace each of the LMI constraints in Problem 12 with an LMI constraint guaranteeing the same objective in the presence of perturbations, Δ, and disturbances, d_k.

Because of the effects of perturbations and the disturbances, it is not necessarily true that the feedforward component has the effect,

$$z_{k+1} \;=\; A\, z_k \;+\; B\, v_k. \tag{36}$$

However, we can still use the nominal case in (36) as a definition of the next ellipse center, z_{k+1}. The effects of the perturbations and disturbances will then be reflected in the feedback controller K_k required to maintain the invariant ellipsoid.

The following lemma is a direct replacement for Lemma 6.

Lemma 17 (Robust stability) *Given, a perturbed state-space system described by (1), (2), (3), the block diagonal perturbation constraints (4), and a bounded disturbance, $d_k \in \mathcal{D}$ (Equation 6). If there exists, $0 \leq \xi_k \leq 1$, $\nu_k \geq 0$, and $\Lambda_k = \operatorname{diag}(\lambda_{1,k} I, \cdots, \lambda_{m,k} I) > 0$, such that,*

$$S_k^R := \begin{bmatrix} -\xi_k P^{-1} & 0 & 0 & 0 & (A + BK_k)^T & (C_q + D_{qu}K_k)^T \\ \bullet & -\nu_k I & 0 & 0 & B_d^T & D_{qd}^T \\ \bullet & \bullet & -\Lambda_k^{-1} & 0 & \Lambda_k^{-1} B_p^T & 0 \\ \bullet & \bullet & \bullet & \xi_k + \nu_k - 1 & 0 & (C_q z_k + D_{qu} v_k)^T \\ \bullet & \bullet & \bullet & \bullet & -P & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\Lambda_k^{-1} \end{bmatrix} \leq 0, \tag{37}$$

then $A + BK_k$ is Hurwitz, and the control,

$$u_k = K_k(x_k - z_k) + v_k, \tag{38}$$

results in $x_{k+1} \in \mathcal{P}_{k+1}$, where z_{k+1} is defined by (36).

The notation S_k^R is used to distinguish the robust LMI condition from the equivalent nominal LMI condition, S_k.

Proof of Lemma 17: Pre- and post-multiplying S_k^R by, $\begin{bmatrix} I & 0 & 0 & 0 & I & 0 \end{bmatrix}$ and its transpose implies that,

$$\begin{bmatrix} -\xi_k P^{-1} & (A+BK_k)^T \\ (A+BK) & -P \end{bmatrix} \leq 0.$$

A Schur complement operation shows that this is equivalent to,

$$(A+BK_k)^T P^{-1}(A+BK_k) - \xi_k P^{-1} \leq 0,$$

and as $0 \leq \xi_k \leq 1$, P^{-1} is the solution to a Lyapunov equation proving that $A+BK_k$ is Hurwitz.

Note that under the operation of the control in (38),

$$x_{k+1} - z_{k+1} = (A+BK_k)(x_k - z_k) + B_d d_k + B_p p_k.$$

the requirement that $x_{k+1} \in \mathcal{P}_{k+1}$ is therefore equivalent to the quadratic functional,

$$F_0 =$$

$$\begin{bmatrix} (x_k - z_k) \\ d_k \\ p_k \end{bmatrix}^T \begin{bmatrix} (A+BK_k)^T \\ B_d^T \\ B_p^T \end{bmatrix} P^{-1} \begin{bmatrix} (A+BK_k) & B_d & B_p \end{bmatrix} \begin{bmatrix} (x_k - z_k) \\ d_k \\ p_k \end{bmatrix} - 1 \leq 0.$$

Each of the conditions implied by $x_k \in \mathcal{P}_k$, $d_k \in \mathcal{D}$, and the perturbation constraints can also be expressed as quadratic functionals of the vector variable, $\begin{bmatrix} (x_k - z_k)^T & d_k^T & p_k^T \end{bmatrix}^T$. The requirement that $x_k \in \mathcal{P}_k$ is equivalent to,

$$F_1 = \begin{bmatrix} (x_k - z_k) \\ d_k \\ p_k \end{bmatrix}^T \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} P^{-1} \begin{bmatrix} I & 0 & 0 \end{bmatrix} \begin{bmatrix} (x_k - z_k) \\ d_k \\ p_k \end{bmatrix} - 1 \leq 0.$$

The requirement that $d_k \in \mathcal{D}$ is equivalent to,

$$F_2 = \begin{bmatrix} (x_k - z_k) \\ d_k \\ p_k \end{bmatrix}^T \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} I \begin{bmatrix} 0 & I & 0 \end{bmatrix} \begin{bmatrix} (x_k - z_k) \\ d_k \\ p_k \end{bmatrix} - 1 \leq 0.$$

Under the action of the control in (38),

$$q_k = (C_q + D_{qu}K_k)(x_k - z_k) + D_{qd}d_k + C_q z_k + D_{qu}v_k.$$

Each of the m perturbation constraints,

$$(\Pi_l p_k)^T (\Pi_l p_k) \leq (\Pi_l q_k)^T (\Pi_l q_k), \quad \text{for all } l = 1, \ldots, m,$$

is therefore equivalent to,

$$\begin{aligned}
F_{3l} = \begin{bmatrix} (x_k - z_k) \\ d_k \\ p_k \end{bmatrix}^T & \left(\begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \Pi_l^T \Pi_l \begin{bmatrix} 0 & 0 & I \end{bmatrix} \right. \\
& \left. - \begin{bmatrix} (C_q + D_{qu} K_k)^T \\ D_{qd}^T \\ 0 \end{bmatrix} \Pi_l^T \Pi_l \begin{bmatrix} (C_q + D_{qu} K_k) & D_{qd} & 0 \end{bmatrix} \right) \begin{bmatrix} (x_k - z_k) \\ d_k \\ p_k \end{bmatrix} \\
& - 2(C_q z_k + D_{qu} v_k)^T \Pi_l^T \Pi_l \begin{bmatrix} (C_q + D_{qu} K_k) & D_{qd} & 0 \end{bmatrix} \begin{bmatrix} (x_k - z_k) \\ d_k \\ p_k \end{bmatrix} \\
& \qquad - (C_q z_k + D_{qu} v_k)^T \Pi_l^T \Pi_l (C_q z_k + D_{qu} v_k) \leq 0.
\end{aligned}$$

Now, via the S-procedure, the requirements of the lemma are met if there exists, $\xi_k \geq 0$, $\nu_k \geq 0$ and $\lambda_{l,k} > 0$ for $l = 1, \ldots, m$, such that,

$$F_0 - \xi_k F_1 - \nu_k F_2 - \sum_{l=1}^{m} \lambda_{l,k} F_{3l} \leq 0. \tag{39}$$

To simplify the last summation, define,

$$\Lambda_k := \sum_{l=1}^{m} \lambda_{l,k} \Pi_l^T \Pi_l = \operatorname{diag}(\lambda_{1,k} I, \cdots, \lambda_{m,k} I).$$

Note that we impose the slightly stricter requirement that $\lambda_{l,k} > 0$ in order to be able to express the final LMI in terms of Λ_k^{-1}. The quadratic functional, (39) is a function of $\begin{bmatrix} (x_k - z_k)^T & d_k^T & p_k^T \end{bmatrix}^T$ which is required to hold for all x_k, d_k and p_k. This is equivalent to requiring it to hold as a function of an arbitrary argument, and by Lemma 5, we can express this as a negative semidefinite matrix constraint. Applying the Schur complement operation twice gives the matrix constraint,

$$\begin{bmatrix}
-\xi_k P^{-1} & 0 & 0 & 0 & (A + BK_k)^T & (C_q + D_{qu} K_k)^T \\
\bullet & -\nu_k I & 0 & 0 & B_d^T & D_{qd}^T \\
\bullet & \bullet & -\Lambda_k & 0 & B_p^T & 0 \\
\bullet & \bullet & \bullet & \xi_k + \nu_k - 1 & 0 & (C_q z_k + D_{qu} v_k)^T \\
\bullet & \bullet & \bullet & \bullet & -P & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & -\Lambda_k^{-1}
\end{bmatrix} \leq 0,$$

Pre- and post-multiplying the above by, $\mathrm{diag}(I, I, \Lambda_k^{-1}, I, I, I)$ shows it to be equivalent to $S_k^R \leq 0$. ■

Unlike the nominal ellipse-to-ellipse constraints, S_k, the robust constraints, S_k^R each depend on the ellipse centers, z_k, and feedforward controls, v_k. This is due to the fact that the perturbations in the system dynamics are reflected in the mapping of the ellipse centers under the feedforward control. One consequence of this is that there is no reduction in complexity in posing a time-invariant control design problem over a time-varying control design problem.

The following lemma is a robust replacement for Lemma 7 and bounds the cost in moving from ellipse $x_k \in \mathcal{P}_k$ to ellipse $x_{k+1} \in \mathcal{P}_{k+1}$.

Lemma 18 (Robust ellipse-to-ellipse cost) *Given, a perturbed state-space system described by (1), (2), (3), the block diagonal perturbation constraints (4), and a bounded disturbance, $d_k \in \mathcal{D}$ (Equation 6). If there exists, $0 \leq \beta_k \leq 1$, $\alpha_k \geq 0$, and $\Psi_k = \mathrm{diag}(\psi_{1,k}I, \cdots, \psi_{m,k}I) > 0$, such that,*

$$T_k^R := \begin{bmatrix} -\beta_k P & 0 & 0 & 0 & (A+BK_k)^T & K_k^T & (C_q + D_{qu}K_k)^T \\ \bullet & -\alpha_k I & 0 & 0 & B_d^T & 0 & D_{qd}^T \\ \bullet & \bullet & -\Psi_k^{-1} & 0 & \Psi_k^{-1}B_p^T & 0 & 0 \\ \bullet & \bullet & \bullet & \beta_k + \alpha_k - \gamma_k & z_{k+1}^T & v_k^T & (D_{qu}v_k + C_q z_k)^T \\ \bullet & \bullet & \bullet & \bullet & -Q^{-1} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -R^{-1} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\Psi_k^{-1} \end{bmatrix} \leq 0, \quad (40)$$

then for all $x_k \in \mathcal{P}_k$, the control, $u_k = K_k(x_k - z_k) + v_k$, gives x_{k+1} such that,

$$x_{k+1}^T Q x_{k+1} + u_k^T R u_k \leq \gamma_k, \quad (41)$$

Proof of Lemma 18: The cost bound in (41) is equivalent to the quadratic functional,

$$F_0 :=$$
$$\begin{bmatrix}(x_k - z_k)\\ d_k \\ p_k\end{bmatrix}^T \left(\begin{bmatrix}(A+BK_k)^T \\ B_d^T \\ B_p^T\end{bmatrix} Q \begin{bmatrix}(A+BK_k) & B_d & B_p\end{bmatrix} + \begin{bmatrix}K_k^T \\ 0 \\ 0\end{bmatrix} R \begin{bmatrix}K_k & 0 & 0\end{bmatrix} \right) \begin{bmatrix}(x_k - z_k)\\ d_k \\ p_k\end{bmatrix}$$
$$+ 2\left(z_{k+1}^T Q \begin{bmatrix}(A+BK_k) & B_d & B_p\end{bmatrix} + v_k^T R \begin{bmatrix}K_k & 0 & 0\end{bmatrix}\right) \begin{bmatrix}(x_k - z_k)\\ d_k \\ p_k\end{bmatrix}$$
$$+ z_{k+1}^T Q z_{k+1} + v_k^T R v_k^T - \gamma_k \leq 0.$$

The remainder of the proof now follows the line of argument given in the proof of Lemma 17 by defining the auxiliary variables and applying the S-procedure to give,

$$F_0 - \beta_k F_1 - \alpha_k F_2 - \sum_{l=1}^{m} \psi_{l,k} F_{3l} \leq 0. \qquad \blacksquare$$

The input constraints, (10), are imposed on the current control and therefore specified as a function of $x_k \in \mathcal{P}_k$. A robust version of the input constraint LMI, $U_{k,j}$ (see Equation 16), is not required; the input constraint requirement given in Lemma 9 is also relevant to the robust case. However, the state constraints are imposed on x_{k+1} and therefore require the robust version of Lemma 8 given below.

Lemma 19 (Robust state constraint) *If there exists $\zeta_{k,i} \geq 0$, $\rho_{k,i} \geq 0$ and $\Theta_{k,i} = \mathrm{diag}(\theta_{1,k,i} I, \cdots, \theta_{m,k,i} I) > 0$, such that,*

$$X_{k,i}^R :=$$
$$\begin{bmatrix} -\zeta_{k,i} P^{-1} & 0 & 0 & (A+BK_k)^T r_i & (C_q + D_{qu} K_k)^T \\ \bullet & -\rho_{k,i} I & 0 & B_d^T r_i & D_{qd}^T \\ \bullet & \bullet & -\Theta_{k,i}^{-1} & \Theta_{k,i}^{-1} B_p^T r_i & 0 \\ \bullet & \bullet & \bullet & \zeta_{k,i} + \rho_{k,i} + 2 r_i^T z_{k+1} - 2 s_i & (C_q + D_{qu} v_k)^T \\ \bullet & \bullet & \bullet & \bullet & -\Theta_{k,i}^{-1} \end{bmatrix} \leq 0, \quad (42)$$

for all $i = 1, \ldots, M_x$*, then, for all* $x_k \in \mathcal{P}_k$*, the control,* $u_k = K_k\,(x_k - z_k) + v_k$*, generates* x_{k+1} *satisfying the state constraint,*

$$r_i^T x_{k+1} \leq s_i, \quad \textit{for all} \quad i = 1, \ldots, M_x. \tag{43}$$

Proof of Lemma 19: Each state constraint in (43) is equivalent to the linear functional,

$$F_0 \;=\; 2r_i^T \begin{bmatrix}(A + BK_k) & B_d & B_p\end{bmatrix} \begin{bmatrix} x_k - z_k \\ d_k \\ p_k \end{bmatrix} + 2r_i^T z_{k+1} - 2s_i \leq 0.$$

The argument again proceeds as in Lemma 17 by defining the auxiliary variables and applying the S-procedure to give,

$$F_0 - \zeta_{k,i} F_1 - \rho_{k,i} F_2 - \sum_{l=1}^{m} \theta_{l,k,i} F_{3l} \leq 0. \qquad \blacksquare$$

The transitional case ($k = N$) is again simplified by the conditions, $z_N = 0$ and $v_N = 0$. The following replaces Theorem 10 in the robust case. It is presented without proof as it follows from straightforward simplification of Lemma 17. Because the transitional cost does not depend on x_{k+1} there is no need to consider perturbations or disturbances in developing the LMIs bounding the cost.

Theorem 20 (Robust transitional ellipse) *Given, a perturbed state-space system described by (1), (2), (3), the block diagonal perturbation constraints (4), and a bounded disturbance,* $d_k \in \mathcal{D}$ *(Equation 6), and an ellipse size and shape specified by* $P = P^T > 0$*. Assume also that the quadratic input cost weight satisfies* $R \geq 0$*.*

If there exists auxiliary variables, $\xi_N \geq 0$*,* $\nu_N \geq 0$*,* $\Lambda_N = \mathrm{diag}(\lambda_{1,N} I, \cdots, \lambda_{m,N} I) > 0$*,* $\beta_N \geq 0$*,* $\zeta_{N,j} \geq 0$*,* $j = 1, \ldots, M_u$*, and design variables,* K_N *and* $\gamma_N > 0$ *such that:*

a)

$$S_N^R :=$$

$$\begin{bmatrix} -\xi_N P^{-1} & 0 & 0 & (A + BK_N)^T & (C_q + D_{qu} K_N)^T \\ \bullet & -\nu_N I & 0 & B_d^T & D_{qd}^T \\ \bullet & \bullet & -\Lambda_N^{-1} & \Lambda_N^{-1} B_p^T & 0 \\ \bullet & \bullet & \bullet & -P & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Lambda_N^{-1} \end{bmatrix} \leq 0; \tag{44}$$

b)

$$\xi_N + \nu_N \leq 1; \tag{45}$$

c)

$$T_N := \begin{bmatrix} -\beta_N P & K_N^T \\ \bullet & -R^{-1} \end{bmatrix} \leq 0; \tag{46}$$

d)

$$\beta_N \leq \gamma_N; \tag{47}$$

e)

$$U_{N,j} := \begin{bmatrix} -\zeta_{N,j} P^{-1} & g_j^T K_N \\ \bullet & \zeta_{N,j} - 2h_j \end{bmatrix} \leq 0, \quad j = 1, \ldots, M_u, \tag{48}$$

then for all $x_k \in \mathcal{P}_N$ *the control,* $u_N = K_N\, x_N$*, gives*

e) $A + BK_N$ *is Hurwitz and* $x_{N+1} \in \mathcal{P}_N$*;*

f) $u_N^T R u_N \leq \gamma_N$*;*

g) u_N *satisfies the* M_u *input constraints, (10).*

We now consider the robust terminal control case. The ellipse centers and feedforward control inputs are both zero for $k \geq N$. We will again use assumption A1 to ensure that all $x \in \mathcal{P}_k$, $k > N$ satisfy the state constraints.

The presence of a disturbance input, d_k, complicates the choice of performance function. As $d_k \neq 0$, it is no longer true that $\lim_{k \longrightarrow \infty} x_k = 0$. Furthermore any quadratic cost function, such as $V(x, u)$ in Equation 9, will no longer be monotonically decreasing. The previous robust stability condition—that for all $x_k \in \mathcal{P}_\infty$, and all $d_k \in \mathcal{D}$, and all perturbations, $\bar{\sigma}\,(\Delta_l) \leq 1$, the next state satisfies $x_{k+1} \in \mathcal{P}_\infty$—can be enforced and does give a degree of performance.

In the following we present an LMI which can be used to minimize the cost function, $V(x, u)$ in (9), in the case where $d_k = 0$ for $k > N$, but the system is still subject to dynamic perturbations, Δ. Note that if this particular LMI is not employed then it would suffice to use $K_\infty = K_N$ to satisfy the robust stability and input constraints.

Theorem 21 (Robust terminal ellipse) *Given, a perturbed state-space system described by (1), (2), (3), the block diagonal perturbation constraints (4), and a bounded disturbance, $d_k \in \mathcal{D}$ (Equation 6), and an ellipse size and shape specified by $P = P^T > 0$. Assume also that the quadratic cost weightings satisfy $Q > 0$ and $R \geq 0$.*

If there exists auxiliary variables, $\xi_\infty \geq 0$, $\nu_\infty \geq 0$, $\Lambda_\infty = \mathrm{diag}(\lambda_{1,\infty} I, \cdots, \lambda_{m,\infty} I) > 0$, $\Psi_\infty = \mathrm{diag}(\psi_{1,\infty} I, \cdots, \psi_{m,\infty} I) > 0$, $\eta_{\infty,j} \geq 0$, $j = 1, \ldots, M_u$, and design variables, K_∞ and $\gamma_\infty > 0$ such that:

a)

$$S^R_\infty := \begin{bmatrix} -\xi_\infty P^{-1} & 0 & 0 & (A+BK_\infty)^T & (C_q + D_{qu}K_\infty)^T \\ \bullet & -\nu_\infty I & 0 & B_d^T & D_{qd}^T \\ \bullet & \bullet & -\Lambda_\infty^{-1} & \Lambda_\infty^{-1} B_p^T & 0 \\ \bullet & \bullet & \bullet & -P & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Lambda_\infty^{-1} \end{bmatrix} \leq 0; \quad (49)$$

b)

$$\xi_\infty + \nu_\infty - 1 \leq 0; \quad (50)$$

c)

$$T^R_\infty := \begin{bmatrix} -P^{-1} & 0 & (A+BK_\infty)^T & (C_q + D_{qu}K_\infty)^T & I & K_\infty^T \\ \bullet & -\Psi_\infty^{-1} & \Psi_\infty^{-1} B_p^T & 0 & 0 & 0 \\ \bullet & \bullet & -P & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\Psi_\infty^{-1} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\gamma_\infty Q^{-1} & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\gamma_\infty R^{-1} \end{bmatrix} \leq 0; \quad (51)$$

d)

$$U_{\infty,j} := \begin{bmatrix} -\eta_{\infty,j} P^{-1} & g_j^T K_\infty \\ \bullet & \eta_{\infty,j} - 2h_j \end{bmatrix} \leq 0, \quad j = 1, \ldots, M_u, \quad (52)$$

then for all $x_k \in \mathcal{P}_\infty$ the control, $u_k = K_\infty x_k$, gives state and input trajectories satisfying the following conditions.

e) $A + BK_\infty$ *is Hurwitz and* $x_k \in \mathcal{P}_\infty$, *for all* k;

f) u_k, *satisfies the* M_u *input constraints, (10) for all* k.

Furthermore, if $d_k = 0$ *for* $k > N$, *then*

g) $\lim_{k \longrightarrow \infty} x_k = 0$;

h) $\sum_{l=k}^{\infty} x_l^T Q x_l + u_l^T R u_l \ \leq \ \gamma_\infty$;

Proof of Theorem 21: The robust stability LMI, S_∞^R, (condition *a)*) is the same as that in the $k = N$ case and as before implies condition *e)*. The input constraint LMI, $U_{\infty,j}$, (condition *d)*) is the same as the prior time-steps and Lemma 9 shows that condition *d)* implies condition *f)*.

Now consider the $d_k = 0$ case. The Hurwitz property given by S_∞^R implies that x_k decays to zero (condition *g)*). It remains to show that T_∞^R (condition *c)*) gives the required cost bound. We again define a positive definite function, $V(x) := x^T P^{-1} x$ and use an argument similar to that in the proof of Theorem 11. To do this we need to develop the LMI constraint that implies,

$$V(x_{k+1}) - V(x_k) + x_k^T \frac{1}{\gamma_\infty} \left(Q + K_\infty^T R K_\infty \right) x_k \leq 0. \tag{53}$$

In the $d_k = 0$ case we have, in the closed-loop,

$$x_{k+1} = (A + BK_\infty) x_k + B_p p_k,$$

which allows us to express (53) as the quadratic functional,

$$\begin{aligned} F_0 &= \\ &\begin{bmatrix} x_k \\ p_k \end{bmatrix}^T \left(\begin{bmatrix} (A + BK_\infty) \\ B_p^T \end{bmatrix} P^{-1} \begin{bmatrix} (A + BK_\infty) & B_p^T \end{bmatrix} - \begin{bmatrix} P^{-1} & 0 \\ 0 & 0 \end{bmatrix} \right. \\ &\qquad\qquad \left. + \begin{bmatrix} \gamma_\infty^{-1}(Q + K_\infty^T R K_\infty) & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x_k \\ p_k \end{bmatrix}. \end{aligned} \tag{54}$$

Under the assumption $d_k = 0$, each of the m perturbation constraints,

$$\left(\Pi_l p_k\right)^T \left(\Pi_l p_k\right) \leq \left(\Pi_l q_k\right)^T \left(\Pi_l q_k\right), \quad \text{for all } l = 1, \ldots, m,$$

is equivalent to,

$$
\begin{aligned}
F_{3l} = \begin{bmatrix} x_k \\ p_k \end{bmatrix}^T \Bigg(\begin{bmatrix} 0 \\ I \end{bmatrix} \Pi_l^T \Pi_l \begin{bmatrix} 0 & I \end{bmatrix} \\
- \begin{bmatrix} (C_q + D_{qu} K_\infty)^T \\ 0 \end{bmatrix} \Pi_l^T \Pi_l \begin{bmatrix} (C_q + D_{qu} K_\infty) & 0 \end{bmatrix} \Bigg) \begin{bmatrix} x_k \\ p_k \end{bmatrix}.
\end{aligned}
$$

Applying the S-procedure gives the following sufficient condition for (54) for all perturbations. The condition is,

$$
F_0 - \sum_{l=1}^{m} \psi_{l,\infty} F_{3l} \leq 0, \tag{55}
$$

and a series of Schur complement operations shows that this is equivalent to T_∞^R. From this point the argument in bounding the cost is equivalent to that given in Theorem 11. Therefore condition *c)* implies condition *h)*. ∎

We can now put each of these ellipse optimization steps together to get a solution to the robust problem.

Problem 22 *[Robust Ellipse Optimization] Solve the following LMI optimization problem for the design variables: K_k, v_k,*

$$
\gamma^* = x_0^T Q x_0 \; + \min_{\gamma_k, \gamma_\infty} \left(\sum_{k=0}^{N} \gamma_k \; + \; \gamma_\infty \right),
$$

subject to the LMI constraints:

$$
\begin{array}{llll}
S_k^R \leq 0, & k = 0, \dots, N-1, & & \textit{(Eqn. 37)} \\
T_k^R \leq 0 & k = 0, \dots, N-1, & & \textit{(Eqn. 40)} \\
U_{k,j} \leq 0, & k = 0, \dots, N-1, & j = 1, \dots, M_u, & \textit{(Eqn. 16)} \\
X_{k,i}^R \leq 0, & k = 0, \dots, N-1, & i = 1, \dots, M_x, & \textit{(Eqn. 42)} \\
S_N^R \leq 0, & & & \textit{(Eqn. 44)} \\
\xi_N + \nu_N \leq 1 & & & \textit{(Eqn. 45)} \\
T_N \leq 0 & & & \textit{(Eqn. 46)} \\
\beta_N \leq \gamma_N & & & \textit{(Eqn. 47)} \\
U_{N,j} \leq 0, & j = 1, \dots, M_u, & & \textit{(Eqn. 48)} \\
S_\infty^R \leq 0, & & & \textit{(Eqn. 49)} \\
\xi_\infty + \nu_\infty \leq 1, & & & \textit{(Eqn. 50)} \\
T_\infty^R \leq 0, & & & \textit{(Eqn. 51)} \\
U_{\infty,j} \leq 0, & j = 1, \dots, M_u, & & \textit{(Eqn. 52)}
\end{array}
$$

and the linear constraint:

$$
\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} z_0 + \begin{bmatrix} B & & & 0 \\ AB & B & & \\ \vdots & & \ddots & \\ A^{N-1}B & & \cdots & B \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix},
$$

with $z_N = 0$.

6 Computational issues

The formulation of this problem in terms of LMI constraints shows that the resulting optimization is convex. General purpose LMI solvers can be computationally demanding, and application of this approach to MPC control will likely require the development of specialized code. The potential for efficient code can be seen by noting that most of the LMIs presented differ in only several entries enabling efficient low rank gradient updating.

The initial ellipse defined by P_k is not a variable in the LMI optimization, which raises the question of how it should be chosen. One approach is to design a state feedback controller for the unconstrained problem. This feedback controller specifies an invariant ellipse which can then be scaled so that the resulting feedback gains, and the states contained within the ellipse, satisfy the input and state constraints. We should also note that in the presence of the disturbances, $d_k \in \mathcal{D}$, there is a minimum size to $\mathcal{P}_k$. This can be expressed by the requirement that there exists $0 \leq \alpha \leq 1$, such that,

$$\begin{bmatrix} -\alpha I & B_d^T \\ B_d & -P \end{bmatrix} \leq 0. \tag{56}$$

If (56) is not satisfied then there exists a disturbance, $d_k \in \mathcal{D}$ that would move the state from the origin—where the control would be $u_k = 0$—to outside of $\mathcal{P}_k$ in a single step.

There are two features of this approach which are attractive. The first is that the solution of the LMI problem generates a series of local controllers, K_k, and a sequence of feedforward inputs, $v(k)$, that is a feasible solution for every subsequent problem. This solution can be used as an initialization for the optimization at subsequent time-steps. It can also be used as a contingency solution if a subsequent optimization does not converge in sufficient time.

The second important feature arises from the convexity of the problem. At each subsequent time-step, the objective of the optimization need only be to improve the performance of the design by recalculating K_k and $v(k)$. It is not necessary to calculate the optimal K_k and $v(k)$ in order to derive benefit from the MPC approach. This means that the early termination of an optimization method will yield some performance improvement in the control design problem.

7 Conclusions

The solution of the constrained control problem presented here is base upon two underlying ideas. The first is that once an invariant ellipse has been specified, all of the desired control properties (ellipsoidal invariance, stability, constraint satisfaction, and performance cost) can be expressed linearly with respect to the feedback and feedforward control gains. This immediately leads to convex optimization problems for the calculation of the control action. This also means the input and state constraints are applied directly to the design variables, K_k and v_k, allowing multiple constraints without introducing conservativeness into the design.

The second key point is that the feedback component of the design allows us to provide rigorous robustness results with respect to both exogenous disturbances and dynamic uncertainty. Many MPC approaches calculate a feedforward control and use the MPC recalculation as the feedback step. This makes it harder to give specific robust performance guarantees. Employing both feedback and feedforward enables us to guarantee ellipsoidal invariance and constraint satisfaction in the presence of uncertainty.

At each time step there is no requirement that same plant state-space matrices (A, B) be used in the calculation. This allows us to extend the approach to a variety of nonlinear constrained control problems by considering different linearizations in different regions of the state-space. The optimization problem is now complicated by the fact that the appropriate linearization will depend on z_k. The inclusions of perturbations may alleviate this problem somewhat as long as we can "cover" the nonlinear linear behaviors of the system by the perturbed set of linear behaviors. The extensions to nonlinear and time-varying systems are an area of future work.

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References

[1] J. B. Rawlings, "Tutorial overview of model predictive control," *IEEE Control Sys. Magazine*, pp. 38–52, June 2000.

[2] D. Q. Mayne, J. B. R. anc C. V. Rao, and P. O. M. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, pp. 789–814, 2000.

[3] H. Chen and F. Allgöwer, "Nonlinear model predictive control schemes with guaranteed stability," in *NATO AM on Nonlinear Model Based*

Process Control (R. Berber and C. Kravaris, eds.), pp. 465–494, Norwell, MA: Kluwer, 1998.

[4] M. Morari and J. H. Lee, "Model predictive control: past, present and future," in *Proc. Joint 6th Int. Symp. Process Systems Engineering (PSE '97) and 30th European Symp. Compuer Aided Process Systems Engineering (ESCAPE 7)*, 1997.

[5] J. Maciejowski, *Predictive Control with Constraints.* England: Prentice-Hall, 2002.

[6] J. B. Rawlings and K. R. Muske, "The stability of constrained receding horizon control," *IEEE Trans. Auto. Control*, vol. 38, no. 10, pp. 1512–1516, 1993.

[7] H. Genceli and M. Nikolaou, "Robust stability analysis of constrained l_1-norm model predictive control," *AIChE J.*, vol. 39, pp. 1954–1965, 1993.

[8] E. Polak and T. H. Yang, "Moving horizon control of linear systems with input saturation and plant uncertainty—1: robustness," *Int. J. of Control*, vol. 53, pp. 613–638, 1993.

[9] Z. Q. Zheng and M. Morari, "Robust stability of constrained model predictive control," in *Proc. Amer. Control Conf.*, pp. 379–383, 1993.

[10] M. Kothare, V. Balakrishnan, and M. Morari, "Robust constrained model predictive control using linear matrix inequalities," *Automatica*, vol. 32, no. 10, pp. 1361–1379, 1996.

[11] B. Kouvaritakis, J. A. Rossiter, and J. Schuurmans, "Efficient robust predictive control," *IEEE Trans. Auto. Control*, vol. 45, no. 8, pp. 1545–1549, 2000.

[12] R. Smith, "Robust model predictive control of constrained linear systems," in *Proc. Amer. Control Conf.*, vol. 1, pp. 245–250, June 2004.

[13] J. Doyle and A. K. Packard, "Uncertain multivariable systems from a state space perspective," in *Proc. Amer. Control Conf.*, vol. 3, pp. 2147–2152, 1987.

[14] A. K. Packard and J. C. Doyle, "The complex structured singular value," *Automatica*, vol. 29, no. 1, pp. 71–109, 1993.

[15] G. Balas, J. Doyle, K. Glover, A. Packard, and R. Smith, *μ-Analysis and Synthesis Toolbox (μ-Tools).* The MathWorks, Inc., Natick, MA, 1993.

[16] S. P. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory.* Philadelphia: SIAM, 1994.

[17] J. Löfberg, "A convex relaxation of a minimax MPC controller," Tech. Rep. LiTH-ISY-R-2338, Linköpings universitet, February 2001. available via `http://www.control.isy.liu.se`.

[18] G. Dullerud and R. Smith, "A nonlinear functional approach to LFT model validation," *Syst. and Control Letters*, vol. 47, no. 1, pp. 1–11, 2002.