Validation of uncertain models of systems initially in motion: a closed-loop perspective

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Abstract

Previous work, such as that of Davis [2], Chen [1] and Steele and Vinnicombe [6] has resulted in sufficient conditions for the validation of models of systems initially at rest in a closed loop context using time-domain interpolation techniques. This paper extends the existing methods to deal with systems initially in motion, allowing the interpolation methods to find an initial state by 'pre-padding' the validation data with a synthetic pre-record data sequence.

1 Notation

Let $\Gamma_g x_0$ denote the response of a system G to an initial condition x_0, and for convenience define $V_s^j := \mathrm{vec}(\Pi_j s)$ (where Π_j is the j-step truncation operator). $G * u$ denotes convolution. T_x denotes the lower block Toeplitz operator applied to x.

Given a finite dimensional linear time-invariant nominal model P and a ν-gap radius β, denote the set of all similarly-dimensioned linear time-invariant systems P_i such that $\delta_\nu(P, P_i) \leq \beta$, $\mathcal{B}_\nu^{\mathrm{LTI}}(P, \beta)$.

All other notation is standard.

2 Introduction

Given experimental data and a mathematical model of the system from which it came, it is desirable to know whether the two are consistent. If they are, confidence in the model is increased. If they are not, something is wrong and a better model is needed.

In the context of closed-loop systems, a given controller C achieving a robust stability margin $b_{P,C}$ with a nominal plant P is guaranteed to achieve

$$b_{\hat{P},C} \geq b_{P,C} - \beta$$

for all $\hat{P} \in \{P_1 : \delta_\nu(P, P_1) \leq \beta\}$, where δ_ν denotes the ν-gap [7]. The model validation decision problem (MVDP) for systems initially in motion may be stated as follows:

Definition 1 (MVDP for systems initially in motion) Given a model set $\mathcal{P}^{p\times q}$, input-output data $u \in \mathcal{S}_k^q$, $y \in \mathcal{S}_k^p$, and input-output noise constraints $\mathbf{W}_u \times \mathbf{W}_y$, do there exist a system $\hat{P} \in \mathcal{P}^{p\times q}$, noise sequences $w_u, w_y \in \mathbf{W}_u \times \mathbf{W}_y$ and an initial state x_0 such that

$$y + w_y = \hat{P} * (u + w_u) + \Gamma_{\hat{P}} x_0$$

In the context of closed-loop control $\mathcal{P}^{p\times q}$ may be either $\mathcal{B}_\nu^{\mathrm{LTI}}(P, \beta)$ or $\mathcal{B}_\nu^{\mathrm{LTV}}(P, \beta)$ for some nominal system $P \in \mathcal{P}^{p\times q}$ and some $\beta \in (0, b_{\mathrm{opt}}(P))$.

2.1 Previous Work

Closed-loop model validation was considered by Davis [2], who derived sufficient and necessary conditions for validation in the gap and ν-gap metrics. These dealt with linear uncertainty, both time-invariant and time varying, using tangential Caratheodory-Fejer interpolation techniques and their LTV equivalents [4, 5]. This resulted in constraints of the form

$$\begin{bmatrix} P^*P + P^*X + X^*P + X^*X & Q^* + Y^* \\ Q + Y & \gamma I \end{bmatrix} \geq 0 \quad (1)$$

where P and Q are related to the measurements, X represents the noise at the input to the uncertainty, Y represents the noise at the output of the uncertainty and $\gamma > 0 \in \mathbb{R}$. He noted that this was non-convex in X

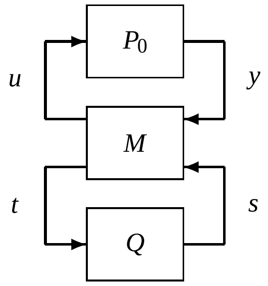

Figure 1: Parameterization of all controllers achieving $b_{P,C} > \beta$

and Y, and simplified it by setting $X = 0$. The resulting constraint

$$\begin{bmatrix} P^*P & Q^*+Y^* \\ Q+Y & \gamma I \end{bmatrix} \geq 0$$

is a linear matrix inequality in Y (and γ) and is thus convex. The condition is an approximation, and neither sufficient nor necessary for validation. Davis's work is complemented by that of Chen [1], who showed that non-convexity occured whenever a noise signal was present at the input to an uncertainty.

In more recent work, Steele and Vinnicombe [6] noted that the non-convex sufficient and necessary conditions would be convex were it not for the quadratic term in the (1,1) element of 1. Since noise signals are likely to be small, a convex constraint of the form

$$\begin{bmatrix} P^*P + P^*X + X^*P & Q^*+Y^* \\ Q+Y & \gamma I \end{bmatrix} \geq 0$$

would be a good approximation. More importantly, the Schur complement shows that it is a sufficient condition for validation: if a solution to the problem can be found, then the model is valid. (Failure to find a solution does not guarantee an invalid model, but a guarantee one way is better than no guarantee at all!) This was applied to the LTI and LTV ν-gap metric problems. This work resulted in the four propositions in Section 2.2.

2.2 Validation in the ν-gap

The set of real-rational controllers achieving $b_{P,C} > \beta$ for some nominal plant P may be parameterized in terms of a 'central controller' M [3]. Any controller

$$C_i = \mathcal{F}_\ell(M, Q)$$

where $Q \in \mathcal{RH}_\infty$ with $\|Q\|_\infty < 1$ will achieve

$$b_{P,C_i} > \beta$$

This parameterization is illustrated in Figure 1. Note that

$$\begin{bmatrix} u \\ t \end{bmatrix} = M * \begin{bmatrix} y \\ s \end{bmatrix}$$

The existence of the 'chain-scattering form' $\mathrm{ch}\,(M)$, giving

$$\begin{bmatrix} u \\ y \end{bmatrix} = \mathrm{ch}\,(M) * \begin{bmatrix} s \\ t \end{bmatrix}$$

and its inverse $(\mathrm{ch}\,(M))^{-1}$ have been shown [3].

It has been demonstrated [2] that

$$\mathcal{B}_\nu(P,\beta) = \left\{ P : \mathcal{F}(M^{-1}, \Delta), \|\Delta\|_\infty \leq 1 \right\}$$

This allows the application of time domain interpolation techniques.

Definition 2 (Standard Validation Data) Standard validation data consists of the following:

- a $p \times q$ nominal model P and a ν-gap radius β;
- the corresponding central controller M, as described in the first paragraph of Section 2.2;
- input-output data sequences $u \in \mathcal{S}_k^q$ and $y \in \mathcal{S}_k^p$;
- sequences $s \in \mathcal{S}_k^q$ and $t \in \mathcal{S}_k^p$ calculated using

$$\begin{bmatrix} s \\ t \end{bmatrix} = \mathrm{ch}\,(M)^{-1} * \begin{bmatrix} u \\ y \end{bmatrix}$$

- linear input-output noise constraints $\mathbf{W}_u$ and $\mathbf{W}_y$.

Proposition 1 (LTI Uncertainty [6]) *Given standard validation data (Definition 2) there exists a system $\hat{P} \in \mathcal{B}_\nu^{\mathrm{LTI}}(P,\beta)$ and noise sequences w_u and w_y such that $y + w_y = \hat{P} * (u + w_u)$ if and only if there exist sequences s and t such that*

$$\begin{bmatrix} T_s^*T_s + T_s^*T_{w_s} + T_{w_s}^*T_s + T_{w_s}^*T_{w_s} & T_t^* + T_{w_t}^* \\ T_t + T_{w_t} & I \end{bmatrix} \geq 0$$

and

$$\mathrm{ch}\,(M) * \begin{bmatrix} w_s \\ w_t \end{bmatrix} \in \mathbf{W}_u \times \mathbf{W}_y$$

Proposition 2 (LTV Uncertainty [6]) *Given standard validation data (Definition 2) there exists a system* $\hat{P} \in \mathcal{B}_\nu^{\text{LTV}}(P,\beta)$ *and noise sequences* w_u *and* w_y *such that* $y + w_y = \hat{P} * (u + w_u)$ *if and only if there exist sequences* s *and* t *such that*

$$\begin{bmatrix} \left(V_s^j + V_{w_s}^j\right)^* \left(V_s^j + V_{w_s}^j\right) & {V_t^j}^* + {V_{w_t}^j}^* \\ V_t^j + V_{w_t}^j & I \end{bmatrix} \geq 0$$

for all $j \in \{1, 2, \ldots, k\}$ *and*

$$\text{ch}(M) * \begin{bmatrix} w_s \\ w_t \end{bmatrix} \in \mathbf{W}_u \times \mathbf{W}_y$$

Approximating these conditions by elimination of the quadratic terms dependent on w_s yields sufficent conditions for validation:

Proposition 3 (LTI Uncertainty [6], Sufficient) *Given standard validation dat (Definition 2) there exists a system* $\hat{P} \in \mathcal{B}_\nu^{\text{LTI}}(P,\beta)$ *and noise sequences* w_u *and* w_y *such that* $y + w_y = \hat{P} * (u + w_u)$ *if there exist sequences* s *and* t *such that*

$$\begin{bmatrix} T_s^* T_s + T_s^* T_{w_s} + T_{w_s}^* T_s & T_t^* + T_{w_t}^* \\ T_t + T_{w_t} & I \end{bmatrix} \geq 0$$

and

$$\text{ch}(M) * \begin{bmatrix} w_s \\ w_t \end{bmatrix} \in \mathbf{W}_u \times \mathbf{W}_y$$

Proposition 4 (LTV Uncertainty [6], Sufficient) *Given standard validation data (Definition 2) there exists a system* $\hat{P} \in \mathcal{B}_\nu^{\text{LTV}}(P,\beta)$ *and noise sequences* w_u *and* w_y *such that* $y + w_y = \hat{P} * (u + w_u)$ *if there exist sequences* s *and* t *such that*

$$\begin{bmatrix} {V_s^j}^* V_s^j + {V_s^j}^* V_{w_s}^j + {V_{w_s}^j}^* V_s^j & {V_t^j}^* + {V_{w_t}^j}^* \\ V_t^j + V_{w_t}^j & I \end{bmatrix} \geq 0$$

for all $j \in \{1, 2, \ldots, k\}$ *and*

$$\text{ch}(M) * \begin{bmatrix} w_s \\ w_t \end{bmatrix} \in \mathbf{W}_u \times \mathbf{W}_y$$

3 Zero-padding

The most obvious approach is to try zero-padding the data and finding 'noise' sequences $u_{\text{pre}} \in \mathcal{S}_\ell^q$ and $y_{\text{pre}} \in \mathcal{S}_\ell^p$ such that the sequences $\hat{y}$ and $\hat{u}$, where

$$\hat{y}(j) = \begin{cases} 0, & 0 \leq j < \ell \\ y(j-\ell), & \ell \leq j < k \end{cases}$$

$$\hat{u}(j) = \begin{cases} 0, & 0 \leq j < \ell \\ u(j-\ell), & \ell \leq j < k \end{cases}$$

satisfy the appropriate sufficient condition for validation with noise sequences

$$w_{\hat{y}}(j) = \begin{cases} y_{\text{pre}}(j), & 0 \leq j < \ell \\ w_y(j-\ell), & \ell \leq j < k \end{cases}$$

$$w_{\hat{u}}(j) = \begin{cases} u_{\text{pre}}(j), & 0 \leq j < \ell \\ w_y(j-\ell), & \ell \leq j < k \end{cases}$$

with $w_u, w_y \in \mathbf{W}_u \times \mathbf{W}_y$. If a system is reachable, any state can be 'built-up' from an input sequence no greater in length than it's MacMillan degree, and thus this approach is equivalent to determining the middle state.

This approach is sensible, but unfortunately it does not work: a consequence of the approximations used in the sufficient conditions is that when u_{pre} and y_{pre} are decision variables, they will have no effect on the problem. They are in both cases subject to heavy constraints, and in the LTI case are forced to be zero.

Definition 3 Given $\hat{y}$ and $\hat{u}$ as above, define

$$\begin{bmatrix} \hat{s} \\ \hat{t} \end{bmatrix} := [\text{ch}(M)]^{-1} * \begin{bmatrix} \hat{u} \\ \hat{y} \end{bmatrix}$$

Theorem 5 (LTI Equivalence Theorem) *Given sequences* $s \in \mathcal{S}_k^q$, $t \in \mathcal{S}_k^p$ *with* $t_0 \neq 0$, *define* $\sigma \in \mathcal{S}_\ell^q$ *such that* $\sigma_i = 0$ *and* $\tau \in \mathcal{S}_\ell^p$ *such that* $\tau_i = 0$, *with* $\hat{s}$ *and* $\hat{t}$ *from Definition 3. Then there exist sequences* $w_\sigma \in \mathcal{S}_\ell^q$, $w_\tau \in \mathcal{S}_\ell^p$ *and* $w_s, w_t \in \mathbf{W}_s \times \mathbf{W}_t$ *such that*

$$\begin{bmatrix} T_{\hat{s}}^* T_{\hat{s}} + T_{w_{\hat{s}}}^* T_{\hat{s}} + T_{\hat{s}}^* T_{w_{\hat{s}}} & T_{\hat{t}}^* + T_{w_{\hat{t}}}^* \\ T_{\hat{t}} + T_{w_{\hat{t}}} & I \end{bmatrix} \geq 0 \quad (2)$$

where $w_{\hat{s}} = \{w_\sigma, w_s\}$ *and* $w_{\hat{t}} = \{w_\tau, w_t\}$, *if and only if there exist sequences* $w_s, w_t \in \mathbf{W}_s \times \mathbf{W}_t$ *such that*

$$\begin{bmatrix} T_s^* T_s + T_{w_s}^* T_s + T_s^* T_{w_s} & T_t^* + T_{w_t}^* \\ T_t + T_{w_t} & I \end{bmatrix} \geq 0 \quad (3)$$

and the only w_σ *and* w_τ *satisfying (2) are* $w_\sigma = \{0, 0, \ldots, 0\}$ *and and* $w_\tau = \{0, 0, \ldots, 0\}$.

Outline of proof. By partitioning matrices appropriately, it can be shown that Proposition 3 must hold true for the pre-record part of the sequence to hold true for the whole, and that this implies that the part of the w_t term in pre-record time must be zero. From this it can be deduced that Proposition 3 can only be true for the whole if the corresponding part of w_s is also zero. Proving the Theorem the other way is straightforward. □

Theorem 6 (LTV Equivalence Theorem) *Given sequences* $s \in \mathcal{S}_k^q$, $t \in \mathcal{S}_k^p$ *with* $t_0 \neq 0$, *define* $\sigma \in \mathcal{S}_\ell^q$

such that $\sigma_i = 0$ and $\tau \in \mathcal{S}_\ell^p$ such that $\tau_i = 0$, with $\hat{s}$ and $\hat{t}$ from Definition 3. Then there exist sequences $w_\sigma \in \mathcal{S}_\ell^q$, $w_\tau \in \mathcal{S}_\ell^p$ and $w_s, w_t \in \mathbf{W}_s \times \mathbf{W}_t$ such that

$$\begin{bmatrix} {V_{\hat{s}}^j}^* V_{\hat{s}}^j + {V_{\hat{s}}^j}^* V_{w_{\hat{s}}}^j + {V_{w_{\hat{s}}}^j}^* V_{\hat{s}}^j & {V_{\hat{t}}^j}^* + {V_{w_{\hat{t}}}^j}^* \\ V_{\hat{t}}^j + V_{w_{\hat{t}}}^j & I \end{bmatrix} \geq 0 \tag{4}$$

for all $j \in \{1, 2, \dots, \ell + k\}$ where $w_{\hat{s}} = \{w_\sigma, w_s\}$ and $w_{\hat{t}} = \{w_\tau, w_t\}$, if and only if there exist sequences $w_s, w_t \in \mathbf{W}_s \times \mathbf{W}_t$ such that

$$\begin{bmatrix} {V_s^j}^* V_s^j + {V_s^j}^* V_{w_s}^j + {V_{w_s}^j}^* V_s^j & {V_t^j}^* + {V_{w_t}^j}^* \\ V_t^j + V_{w_t}^j & I \end{bmatrix} \geq 0 \tag{5}$$

for all $j \in \{1, 2, \dots, k\}$, and the only w_τ satisfying (4) is the zero sequence.

Outline of proof. The proof follows near-identical lines to that of Theorem 6. However, the part of w_s in pre-record time drops out of the constraint completely instead of being forced to zero. □

4 An Alternative Approach

Another way to address this problem is to find a synthetic data sequence to use as a starting point. An initial estimate of a pre-record input sequence can be found by assuming a perfect model and finding the pre-record input consistent with the smallest input-output noise level. Definitions 4 and 5 and Theorem 7 present this formally.

Note that if a system is controllable, any state can be reached using an input sequence no greater in length than its MacMillan degree.

Definition 4 Given data sequences $u \in \mathcal{S}_k^q$, $y \in \mathcal{S}_k^p$, a positive integer ℓ and a perfect $p \times q$ model P, let T_P be the lower block Toeplitz matrix representing the first $\ell + k$ elements of the impulse response of P. Partition T_P as follows

$$T_P = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix}$$

where $T_{11} \in \mathbb{R}^{p\ell \times q\ell}$, $T_{22} \in \mathbb{R}^{pk \times qk}$ and $T_{21} \in \mathbb{R}^{pk \times q\ell}$. Define the set of *admissible sequences* $\mathbf{W}_\gamma$ as follows:

$$\mathbf{W}_\gamma = \left\{ \begin{matrix} w_u, w_y, u_{\text{pre}} : w_u \in \mathcal{S}_k^q, w_y \in \mathcal{S}_k^p, u_{\text{pre}} \in \mathcal{S}_\ell^q, \\ \text{vec}\,(y + w_y) = \begin{bmatrix} T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} \text{vec}\, u_{\text{pre}} \\ \text{vec}\,(u + w_u) \end{bmatrix} \end{matrix} \right\}$$

$\mathbf{W}_\gamma$ represents the set of input-output sequences consistent with u, y and P.

Definition 5 Given T_{21} and T_{22} from Definition 4, define Ψ as follows:

$$\Psi \in \mathbb{R}^{k(p+q) \times (\ell+k)q} := \begin{bmatrix} 0 & I \\ T_{21} & T_{22} \end{bmatrix}$$

Denote its pseudo-inverse $(\Psi^*\Psi)^{-1}\Psi^* = \Psi^\dagger \in \mathbb{R}^{(\ell+k)q \times k(p+q)}$ and partition it as follows

$$\Psi^\dagger = \begin{bmatrix} \Psi_{11}^\dagger & \Psi_{12}^\dagger \\ \Psi_{21}^\dagger & \Psi_{22}^\dagger \end{bmatrix}$$

where $\Psi_{11}^\dagger \in \mathbb{R}^{\ell q \times kq}$, $\Psi_{12}^\dagger \in \mathbb{R}^{\ell q \times kp}$, $\Psi_{21}^\dagger \in \mathbb{R}^{kq \times kq}$ and $\Psi_{22}^\dagger \in \mathbb{R}^{kq \times kp}$.

Theorem 7 *Given data sequences $u \in \mathcal{S}_k^q$, $y \in \mathcal{S}_k^p$, a $p \times q$ model P and a positive integer ℓ, let $w_u \in \mathcal{S}_k^q$ and $w_y \in \mathcal{S}_k^p$ be noise sequences, and let $u_{\text{pre}} \in \mathcal{S}_\ell^q$ be a pre-record input sequence. Define*

$$\gamma(w_u, w_y) = \left\| \begin{bmatrix} \text{vec}\, w_u \\ \text{vec}\, w_y \end{bmatrix} \right\|_2$$

Then the nominal optimal noise and pre-record input sequences

$$(w_u^\circ, w_y^\circ, u_{\text{pre}}^\circ) := \underset{w_u, w_y, u_{\text{pre}} \in \mathbf{W}_\gamma(P,u,y,\ell)}{\arg\min} \gamma(w_u, w_y)$$

(with $\mathbf{W}_\gamma(P, u, y, \ell)$ from Definition 4) are given by

$$\begin{bmatrix} \text{vec}\, u_{\text{pre}}^\circ \\ \text{vec}\, w_u^\circ \\ \text{vec}\, w_y^\circ \end{bmatrix} = \begin{bmatrix} \Psi_{11}^\dagger & \Psi_{12}^\dagger \\ \Psi_{21}^\dagger - I & \Psi_{21}^\dagger \\ T_{21}\Psi_{11}^\dagger + T_{22}\Psi_{21}^\dagger & T_{21}\Psi_{12}^\dagger + T_{22}\Psi_{22}^\dagger - I \end{bmatrix} \begin{bmatrix} \text{vec}\, u \\ \text{vec}\, y \end{bmatrix}$$

where T_{21} and T_{22} are as defined in Definition 4 and Ψ_{ij} are defined in Definition 5.

Outline of proof. The problem may be formulated as linear least-squares problem. The results follow from standard linear algebra techniques. □

Remark 1 The value of y_{pre} corresponding to u_{pre}° in Theorem 7 is given by $\text{vec}\, y_{\text{pre}}^\circ = T_{11} \text{vec}\, u_{\text{pre}}^\circ$. The corresponding initial state is easily found. ♡

This leads to a re-linearized problem:

Definition 6 (Total input-output sequences) Given measured data sequences $u \in \mathcal{S}_k^q, y \in \mathcal{S}_k^p$, pre-record sequences $u_{\text{pre}} \in \mathcal{S}_\ell^q, y_{\text{pre}} \in \mathcal{S}_\ell^p$ and noise sequences

$w_u \in \mathcal{S}_k^q$, $w_y \in \mathcal{S}_k^p$, define also the *total input-output sequences* $u_{\text{total}} \in \mathcal{S}_{\ell+k}^q$ and $y_{\text{total}} \in \mathcal{S}_{\ell+k}^p$ as follows:

$$u_{\text{total}}(j) = \begin{cases} u_{\text{pre}}(j), & 0 \le j < \ell \\ (u+w_u)(j-\ell), & \ell \le j < \ell + k \end{cases}$$

$$y_{\text{total}}(j) = \begin{cases} y_{\text{pre}}(j), & 0 \le j < \ell \\ (y+w_y)(j-\ell), & \ell \le j < \ell + k \end{cases}$$

Note that $u_{\text{total}} = \breve{u} + \breve{w}_u$ and $y_{\text{total}} = \breve{y} + \breve{w}_y$.

Definition 7 (Other extended sequences) Given

- measured data sequences $u \in \mathcal{S}_k^q, y \in \mathcal{S}_k^p$;
- nominal pre-record sequences $u_{\text{pre}}^\circ \in \mathcal{S}_\ell^q$, $y_{\text{pre}}^\circ \in \mathcal{S}_\ell^p$ and nominal noise sequences $w_u^\circ \in \mathcal{S}_k^q$, $w_y^\circ \in \mathcal{S}_k^p$,
- pre-record sequences $u_{\text{pre}} \in \mathcal{S}_\ell^q, y_{\text{pre}} \in \mathcal{S}_\ell^p$ and noise sequences $w_u \in \mathcal{S}_k^q$, $w_y \in \mathcal{S}_k^p$,

let

$$\begin{aligned} u_{\text{pre}}^\delta &:= u_{\text{pre}} - u_{\text{pre}}^\circ \\ y_{\text{pre}}^\delta &:= y_{\text{pre}} - y_{\text{pre}}^\circ \\ w_u^\delta &:= w_u - w_u^\circ \\ w_y^\delta &:= w_y - w_y^\circ \end{aligned}$$

Define the *extended nominal data sequences* $\breve{u} \in \mathcal{S}_{\ell+k}^q$ and $\breve{y} \in \mathcal{S}_{\ell+k}^p$ as follows:

$$\breve{u}(j) = \begin{cases} u_{\text{pre}}^\circ(j), & 0 \le j < \ell \\ u(j-\ell) + w_u^\circ(j-\ell), & \ell \le j < \ell + k \end{cases}$$

$$\breve{y}(j) = \begin{cases} y_{\text{pre}}^\circ(j), & 0 \le j < \ell \\ y(j-\ell) + w_y^\circ(j-\ell), & \ell \le j < \ell + k \end{cases}$$

Define also the *extended perturbation sequences* $\breve{w}_u \in \mathcal{S}_{\ell+k}^q$ and $\breve{w}_y \in \mathcal{S}_{\ell+k}^p$ as follows:

$$\breve{w}_u(j) = \begin{cases} u_{\text{pre}}^\delta(j), & 0 \le j < \ell \\ w_u^\delta(j-\ell), & \ell \le j < \ell + k \end{cases}$$

$$\breve{w}_y(j) = \begin{cases} y_{\text{pre}}^\delta(j), & 0 \le j < \ell \\ w_y^\delta(j-\ell), & \ell \le j < \ell + k \end{cases}$$

Define the *nominal chain-scattered signals* $\breve{s} \in \mathcal{S}_{\ell+k}^q$ and $\breve{t} \in \mathcal{S}_{\ell+k}^p$ as follows:

$$\begin{bmatrix} \breve{s} \\ \breve{t} \end{bmatrix} := [\text{ch}(M)]^{-1} * \begin{bmatrix} \breve{u} \\ \breve{y} \end{bmatrix}$$

Remark 2 Comparing Definitions 6 and 7, note that $u_{\text{total}} = \breve{u} + \breve{w}_u$ and $y_{\text{total}} = \breve{y} + \breve{w}_y$. ♡

Definition 8 Given noise constraints $\mathbf{W}_u$, $\mathbf{W}_y$, let $\mathbf{W}_z = \mathbf{W}_u \times \mathbf{W}_y$ and nominal input-output noise sequences $w_u^\circ \in \mathcal{S}_{\ell+k}^q$ and $w_y^\circ \in \mathcal{S}_{\ell+k}^p$, define $\mathbf{W}_{st}$ as follows:

$$\breve{\mathbf{W}}_{st} := \left\{ w_s, w_t : \begin{array}{l} w_s = \{w_{s_1}, w_{s_2}\}, \\ w_t = \{w_{t_1}, w_{t_2}\}, \\ w_{s_1} \in \mathcal{S}_\ell^q, w_{s_2} \in \mathcal{S}_k^q, \\ w_{t_1} \in \mathcal{S}_\ell^q, w_{t_2} \in \mathcal{S}_k^q, \\ \begin{bmatrix} w_u^\circ \\ w_y^\circ \end{bmatrix} + \text{ch}(M) * \begin{bmatrix} w_{s_2} \\ w_{t_2} \end{bmatrix} \in \mathbf{W}_z \end{array} \right\}$$

$\breve{\mathbf{W}}_{st}$ is the set of sequences whose final sections corrrespond to sequences in $\mathbf{W}_u \times \mathbf{W}_y$, taking account the linearization of the problem about w_u° and w_y°.

Theorem 8 (LTI uncertainty, suff. and nec.) *Given standard validation data (Definition 2) and a positive integer ℓ, let $u_{\text{pre}}^\circ \in \mathcal{S}_\ell^q$, $y_{\text{pre}}^\circ \in \mathcal{S}_\ell^p$ be nominal pre-record sequences and $w_u^\circ \in \mathcal{S}_k^q$, $w_y^\circ \in \mathcal{S}_k^p$ be nominal noise sequences calculated using Theorem 7 and Remark 1.*

Also, let $u_{\text{pre}} \in \mathcal{S}_\ell^q$, $y_{\text{pre}} \in \mathcal{S}_\ell^p$ be pre-record sequences and $w_u, w_y \in \mathbf{W}_u \times \mathbf{W}_y$ be noise sequences, all to be determined.

Then there exist a system $\hat{P} \in \mathcal{B}_\nu^{\text{LTI}}(P,\beta)$, and sequences u_{pre}, y_{pre}, w_u, w_y such that

$$y_{\text{total}} = \hat{P} * u_{\text{total}}$$

(where y_{total} and u_{total} are as defined in Definition 6) if and only if there exist sequences $\breve{w}_s$ and $\breve{w}_t$ such that

$$\begin{bmatrix} T_{\breve{s}}^* T_{\breve{s}} + T_{\breve{s}}^* T_{\breve{w}_s} + T_{\breve{w}_s}^* T_{\breve{s}} + T_{\breve{w}_s}^* T_{\breve{w}_s} & T_{\breve{t}}^* + T_{\breve{w}_t}^* \\ T_{\breve{t}} + T_{\breve{w}_t} & I \end{bmatrix} \ge 0$$

where $\breve{s} \in \mathcal{S}_{\ell+k}^q$ and $\breve{t} \in \mathcal{S}_{\ell+k}^p$ are the nominal chain scattered signals defined in Definition 7 and $(w_s, w_t) \in \breve{\mathbf{W}}_{st}\left(\mathbf{W}_u, \mathbf{W}_y, w_u^\circ, w_y^\circ\right)$ (Definition 8).

Remark 3 This is equivalent to the existence of an initial state x_0 reachable in ℓ steps such that $y + w_y = \hat{P} * (u + w_u) + \hat{\Gamma}_{\hat{P}} x_0$ ♡

Proof of Theorem 8. Follows from the definitions and Proposition 1. □

Theorem 9 (LTV uncertainty, suff. and nec.) *Given standard validation data (Definition 2) and a positive integer ℓ, let $u_{\text{pre}}^\circ \in \mathcal{S}_\ell^q$, $y_{\text{pre}}^\circ \in \mathcal{S}_\ell^p$ be nominal pre-record sequences and $w_u^\circ \in \mathcal{S}_k^q$, $w_y^\circ \in \mathcal{S}_k^p$ be nominal noise sequences calculated using Theorem 7 and Remark 1.*

Also, let $u_{\text{pre}} \in \mathcal{S}_\ell^q$, $y_{\text{pre}} \in \mathcal{S}_\ell^p$ be pre-record sequences and $w_u, w_y \in \mathbf{W}_u \times \mathbf{W}_y$ be noise sequences, all to be determined.

Then there exist a system $\hat{P} \in \mathcal{B}_\nu^{\text{LTV}}(P, \beta)$, and sequences u_{pre}, y_{pre}, w_u, w_y such that

$$y_{\text{total}} = \hat{P} * u_{\text{total}}$$

(where y_{total} and u_{total} are as defined in Definition 6) if and only if there exist sequences $\breve{w}_s$ and $\breve{w}_t$ such that

$$\begin{bmatrix} \left(V_{\breve{s}}^j + V_{\breve{w}_s}^j\right)^* \left(V_{\breve{s}}^j + V_{\breve{w}_s}^j\right) & V_{\breve{t}}^{j\,*} + V_{\breve{w}_t}^{j\,*} \\ V_{\breve{t}}^j + V_{\breve{w}_t}^j & I \end{bmatrix} \geq 0$$

for all $j \in \{1, 2, \ldots, \ell + k\}$, where $\breve{s} \in \mathcal{S}_{\ell+k}^q$ and $\breve{t} \in \mathcal{S}_{\ell+k}^p$ are the nominal chain scattered signals defined in Definition 7 and $(w_s, w_t) \in \breve{\mathbf{W}}_{st}\left(\mathbf{W}_u, \mathbf{W}_y, w_u^\circ, w_y^\circ\right)$ (Definition 8).

Remark 4 This is equivalent to the existence of an initial state x_0 reachable in ℓ steps consistent with the determined noise sequences and the measured signals. ♡

Proof of Theorem 9. Follows from the definitions and Proposition 2. □

Theorem 10 (LTI uncertainty, sufficient, convex) *Given standard validation data (Definition 2) and a positive integer ℓ, let $u_{\text{pre}}^\circ \in \mathcal{S}_\ell^q$, $y_{\text{pre}}^\circ \in \mathcal{S}_\ell^p$ be nominal pre-record sequences and $w_u^\circ \in \mathcal{S}_k^q$, $w_y^\circ \in \mathcal{S}_k^p$ be nominal noise sequences calculated using Theorem 7 and Remark 1.*

Also, let $u_{\text{pre}} \in \mathcal{S}_\ell^q$, $y_{\text{pre}} \in \mathcal{S}_\ell^p$ be pre-record sequences and $w_u, w_y \in \mathbf{W}_u \times \mathbf{W}_y$ be noise sequences, all to be determined.

Then there exist a system $\hat{P} \in \mathcal{B}_\nu^{\text{LTI}}(P, \beta)$, and sequences u_{pre}, y_{pre}, w_u, w_y such that

$$y_{\text{total}} = \hat{P} * u_{\text{total}}$$

(where y_{total} and u_{total} are as defined in Definition 6) if there exist sequences $\breve{w}_s$ and $\breve{w}_t$ such that

$$\begin{bmatrix} T_{\breve{s}}^* T_{\breve{s}} + T_{\breve{s}}^* T_{\breve{w}_s} + T_{\breve{w}_s}^* T_{\breve{s}} & T_{\breve{t}}^* + T_{\breve{w}_t}^* \\ T_{\breve{t}} + T_{\breve{w}_t} & I \end{bmatrix} \geq 0$$

where $\breve{s} \in \mathcal{S}_{\ell+k}^q$ and $\breve{t} \in \mathcal{S}_{\ell+k}^p$ are the nominal chain scattered signals defined in Definition 7 and $(w_s, w_t) \in \breve{\mathbf{W}}_{st}\left(\mathbf{W}_u, \mathbf{W}_y, w_u^\circ, w_y^\circ\right)$ (Definition 8).

Remark 5 This is effectively a sufficient condition for the existence of an initial state x_0 reachable in ℓ steps consistent with the determined noise sequences and the measured signals. ♡

Proof of Theorem 10. Follows from the definitions and Proposition 3, or alternatively from directly from Theorem 8, removing the quadratic term in w_s and using the Schur complement to demonstrate sufficiency. □

Theorem 11 (LTV uncertainty, sufficient, convex) *Given standard validation data (Definition 2) and a positive integer ℓ, let $u_{\text{pre}}^\circ \in \mathcal{S}_\ell^q$, $y_{\text{pre}}^\circ \in \mathcal{S}_\ell^p$ be nominal pre-record sequences and $w_u^\circ \in \mathcal{S}_k^q$, $w_y^\circ \in \mathcal{S}_k^p$ be nominal noise sequences calculated using Theorem 7 and Remark 1.*

Also, let $u_{\text{pre}} \in \mathcal{S}_\ell^q$, $y_{\text{pre}} \in \mathcal{S}_\ell^p$ be pre-record sequences and $w_u, w_y \in \mathbf{W}_u \times \mathbf{W}_y$ be noise sequences, all to be determined.

Then there exist a system $\hat{P} \in \mathcal{B}_\nu^{\text{LTV}}(P, \beta)$, and sequences u_{pre}, y_{pre}, w_u, w_y such that

$$y_{\text{total}} = \hat{P} * u_{\text{total}}$$

(where y_{total} and u_{total} are as defined in Definition 6) if there exist sequences $\breve{w}_s$ and $\breve{w}_t$ such that

$$\begin{bmatrix} V_{\breve{s}}^{j\,*} V_{\breve{s}}^j + V_{\breve{s}}^{j\,*} V_{\breve{w}_s}^j + V_{\breve{w}_s}^{j\,*} V_{\breve{s}}^j & V_{\breve{t}}^{j\,*} + V_{\breve{w}_t}^{j\,*} \\ V_{\breve{t}}^j + V_{\breve{w}_t}^j & I \end{bmatrix} \geq 0$$

for all $j \in \{1, 2, \ldots, \ell + k\}$, where $\breve{s} \in \mathcal{S}_{\ell+k}^q$ and $\breve{t} \in \mathcal{S}_{\ell+k}^p$ are the nominal chain scattered signals defined in Definition 7 and $(w_s, w_t) \in \breve{\mathbf{W}}_{st}\left(\mathbf{W}_u, \mathbf{W}_y, w_u^\circ, w_y^\circ\right)$ (Definition 8).

Remark 6 This is effectively a sufficient condition for the existence of an initial state x_0 reachable in ℓ steps consistent with the determined noise sequences and the measured signals. ♡

Proof of Theorem 11. Follows from the definitions and Proposition 4, or alternatively from directly from Theorem 9, removing the quadratic term in w_s and using the Schur complement to demonstrate sufficiency. □

5 Numerical Example

Synthetic data was obtained by finding first 94 elements the response of the discrete time system

$$P_{\text{true}}(z) = \frac{0.02247z + 0.02093}{z^2 - 1.764z + 0.8075}$$

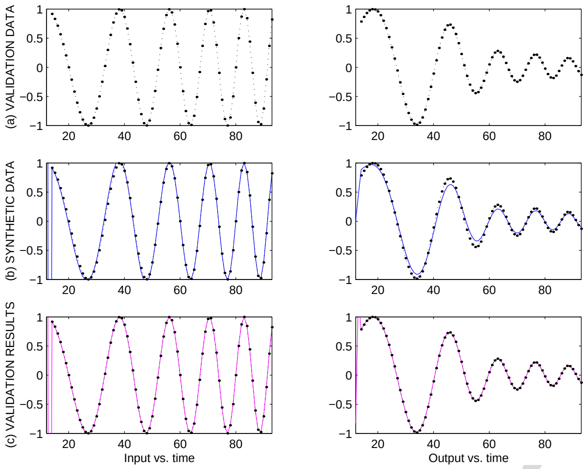

Figure 2: Numerical example. The top plots (a) show the initial data sequences. The middle plots (b) show the data including nominal pre-record and noise sequences. The bottom plots (c) show the final sequences from validation, which coincide exactly with the initial data sequences.

to the chirp signal $u_{\text{true}}(k) = \sin\frac{\pi}{30}\left(k + 0.025k^2\right)$. The first 14 samples of the data record were discarded, giving the validation data (u, y) shown in Figure 2(a).

Assuming a nominal model

$$P = \frac{0.01867z + 0.01746}{z^2 - 1.783z + 0.8187}$$

and taking the pre-input length as 2, nominal pre-input and noise sequences were found using Theorem 7 (Figure 2(b)). Theorem 10 was used to find the pre-input and noise sequences giving the smallest value of $\gamma = \left\|\begin{bmatrix} w_u \\ w_y \end{bmatrix}\right\|_2$ consistent with a system $\hat{P} \in \mathcal{B}_\nu^{\text{LTI}}(P, 0.12)$. The minimum was found to be zero, corresponding to zero noise sequences (Figure 2(c)).

Since there was no noise on the original data, and $\delta(P_{\text{true}}, P) = 0.12$, these results are as expected.

6 Conclusions

The work presented in this paper has extended existing model validation techniques to allow for non-zero initial states. Both linear time-invariant and linear time-varying cases have been considered.

At present further work to model data offsets and trends and synthesise suitable interpolant plants is being undertaken. Methods of approaching the a local minimum of the full non-convex validation conditions by successively re-linearizing about an initial solution are being developed. Finally, applications of these techniques to experimental flight test data is also being performed.

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