A Method for Robust Receding Horizon Output Feedback Control of Constrained Systems*

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Abstract

This paper considers output feedback control of linear discrete-time systems with convex state and input constraints and subject to bounded state disturbances and output measurement errors. We show that the non-convex problem of finding a constraint admissible affine output feedback policy, to be used in conjunction with a fixed linear state observer, can be converted to an equivalent convex problem. When used in the design of a time-varying robust receding horizon control (RHC) law, we derive conditions under which the resulting closed-loop system is guaranteed to satisfy the system constraints for all time, given an initial state estimate and bound on the state estimation error. When the state estimation error bound matches the minimal robust positively invariant (mRPI) set for the system error dynamics, we show that this control law is actually time-invariant, but its calculation generally requires solution of an infinite-dimensional optimisation problem. Finally, using an invariant outer approximation to the mRPI error set, we develop a time-invariant control law that can be computed by solving a finite-dimensional, tractable optimisation problem at each time step, which guarantees that the closed-loop system satisfies the constraints for all time.

Keywords: Robustness; output feedback; constrained control; predictive control

1 Introduction

This paper considers the problem of output feedback control of linear discrete-time systems with mixed state and input constraints, subject to bounded disturbances on the states and measurements. The main aim is to provide a method for efficient calculation of feedback policies that ensure that the state and input constraints are satisfied for all time, while ensuring that the domain of attraction of the resulting closed-loop system is as large as possible.

The problem of formulating robust control policies that guarantee constraint satisfaction is a long-standing one in the control literature [9, 10, 43], and various methods have been

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devised for its solution; these include methods based on robust dynamic programming [8], set invariance [11], ℓ_1 control [15], reference governors [17] and predictive control [4, 12, 27].

A variety of techniques have been proposed for the *off-line* calculation of output feedback control laws which robustly satisfy system constraints for this type of problem. These include methods employing dynamic linear control laws [15, 40], set valued observers coupled with static nonlinear set-valued maps [38, 39], and controllers based on robust dynamic programming [1, 31]; these methods typically suffer from very high computational complexity or excessive conservativeness.

Within the predictive control literature, in which a finite-horizon optimal control problem is solved *on-line* at each time instant and applied to the plant in receding horizon fashion, techniques for guaranteeing stability and constraint satisfaction for *undisturbed* systems via state feedback are now well established; see, for example, the excellent surveys in [16, 29, 32]. More problematic has been the development of robust receding horizon control policies (RHC) for uncertain systems, where one wishes to guarantee constraint satisfaction for *all possible* realizations of the system uncertainty. It is now generally accepted that, in order to provide a reasonable domain of attraction, optimisation must be performed over a sequence of *feedback policies*, rather than over fixed *input sequences*, otherwise problems of infeasibility may quickly arise [4, 29]. Unfortunately, optimisation over *arbitrary* nonlinear feedback policies is generally computationally intractable, leading to optimisation problems whose size grows exponentially with the problem data [37, 42].

For robust predictive control using output feedback, a common ad-hoc approach is to employ an observer and substitute the resulting state estimate in place of the true system state in conjunction with a standard predictive control scheme [16, 32]. However, in order to ensure that the region of attraction is as large as possible while guaranteeing robust constraint satisfaction, an explicit model of the estimation error seems necessary, and a number of control schemes based on error set membership estimation [7, 36] have been proposed [3, 14]. When the system dynamics are linear, a common approach is to employ a combination of a fixed linear observer and associated estimation error set with a fixed stabilising linear control law, to which a sequence of input perturbations is calculated at each time instant. Variations on this theme have been proposed in [25, 35, 44], and may be considered the output feedback counterparts to the state feedback methods proposed in [2, 13, 24].

A related technique from the predictive control literature is to define a 'tube' of trajectories based on a controlled invariant set [23], within which the true state of the system is guaranteed to remain, and to treat the problem as one of steering this set to the origin, where the initial reference state (the 'centre' of the tube at the initial time) is treated as a decision variable. The invariant set from which the tube is constructed is typically determined *off-line* by defining a fixed linear feedback law (see [30] for the state feedback and [28] for the output feedback case), though other methods for defining this set are possible [34].

An obvious method for increasing the domain of attraction using these methods is to compute an affine feedback control law *on-line* at each sample time — a non-convex problem which has until recently been thought to be intractable. However, in [20, 26] an alternative convex parameterisation based on disturbance feedback was proposed for the full state information case, and was later shown to be equivalent to one based on affine state feedback in [19]. In the present paper, an analogous reparameterisation for *output* feedback is presented, together with techniques for synthesising robust time-invariant RHC laws from this parameterisation that guarantee constraint satisfaction for all time, and for which the control input at each time instant can be solved via a single tractable convex optimisation problem.

The proposed method has its origins in the recent work on robust optimisation of [6, 20], which developed a novel method for the solution of adjustable robust counterpart (ARC) optimisation problems, in which a subset of the decision variables may be selected after some or all of the uncertain problem parameters are realized. It was shown that if these decision parameters are restricted to be affine functions of the system uncertainty, rather than arbitrary nonlinear functions, then the resulting optimisation problem is convex and tractable under certain conditions.

The convex control parameterisation presented here was originally proposed for robust control of linear systems in [5, 41]. We employ the parameterisation in conjunction with a fixed linear state observer and a corresponding bound on the state estimation error, and show that RHC laws synthesised from the parameterisation can guarantee constraint satisfaction for all time. When the state estimation error bound matches the minimal robust positively invariant (mRPI) set for the system error dynamics, we show that the control law is actually time-invariant, but its calculation requires the solution of an infinite-dimensional optimisation problem when the mRPI set is not finitely determined. Finally, by employing an invariant outer approximation to the mRPI error set [33], we develop a time-invariant control law that can be computed by solving a finite-dimensional tractable optimisation problem at each time step.

The paper is organised as follows. Section 2 discusses the class of systems considered and defines a number of standing assumptions. Section 3 defines the affine output feedback policies considered throughout, and, in a manner similar to [5] but taking explicit account of the state estimate and observer error dynamics, demonstrates that one can define an *equivalent* but *convex* reparameterisation based on output error feedback. This equivalence is then exploited in Section 4 which is concerned with results concerning invariance and constraint satisfaction, and which contains the main contributions of the paper. Section 5 demonstrates how the proposed control law may be implemented via the solution of a single linear program (LP) at each time when all of the relevant constraints are polytopic, and provides a short numerical example. Some concluding remarks are made in Section 6.

Notation: $\mathbb{Z} := \{0, 1, \dots\}$ is the set of non-negative integers and $\mathbb{Z}_{[k,l]}$ represents the set of integers $\{k, k+1, \dots, l\}$. $\mathcal{B}_p^n(r) := \{x \in \mathbb{R}^n \mid \|x\|_p \leq r\}$ is the p-norm unit ball in $\mathbb{R}^n$, where $r \geq 0$. Given sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$, the Minkowski sum is defined as $X \oplus Y := \{x + y \mid x \in X, y \in Y\}$. Given a sequence of sets $\{X_i \subset \mathbb{R}^n\}_{i=a}^b$, define $\oplus_{i=a}^b X_i := X_a \oplus \cdots \oplus X_b$.

2 Problem Description

Throughout, we consider the following discrete-time linear time-invariant system:

$$x^+ = Ax + Bu + w \tag{1}$$

$$y = Cx + \eta \tag{2}$$

where $x \in \mathbb{R}^n$ is the system state at the current time instant, x^+ is the state at the next time instant, $u \in \mathbb{R}^m$ is the system input, $w \in \mathbb{R}^n$ is a disturbance, $y \in \mathbb{R}^p$ is the system output and $\eta \in \mathbb{R}^p$ is the measurement error. We assume that the pairs (A, B) and (C, A) are stabilisable and detectable respectively, and that there exist a controller gain K and Luenberger type observer gain L such that the matrices $A_K := (A + BK)$ and $A_L := (A - LC)$ are strictly stable. We define the estimated state $s \in \mathbb{R}^n$ at the current time instant such that

$$s^+ = As + Bu + L(y - Cs), \tag{3}$$

and define the state estimation error $e \in \mathbb{R}^n$ as $e := x - s$, such that

$$e^+ = (A - LC)e - L\eta + w, \tag{4}$$

where s^+ and e^+ represent the state estimate and estimation error at the next time instant. The actual values of the state, state estimate, estimation error, input and output at time instant k are denoted $x(k)$, $s(k)$, $e(k)$, $u(k)$ and $y(k)$ respectively. We assume that the system is subject to mixed constraints on the states and inputs, so that the system must satisfy

$$(x, u) \in Z, \tag{5}$$

where $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ is a closed and convex set containing the origin in its interior, and note that such a constraint may include constraints on the output y in (2). We further define a closed and convex target/terminal set $X_f \subset \mathbb{R}^n \times \mathbb{R}^n$ for the state estimate and error, such that $(s, e) \in X_f$. We assume that the disturbances w are unknown but contained in a compact set W containing the origin, and that the measurement errors η are unknown but contained in a compact set H, also containing the origin.

Before proceeding, we define some additional notation. In the sequel, predictions of the system's evolution over a finite control/planning horizon will be used to define a number of suitable control policies. Let the length N of this planning horizon be a positive integer and define stacked versions of the state estimate, estimation error, input, output, disturbance, and measurement error vectors $\mathbf{s} \in \mathbb{R}^{n(N+1)}$, $\mathbf{e} \in \mathbb{R}^{n(N+1)}$, $\mathbf{u} \in \mathbb{R}^{mN}$, $\mathbf{y} \in \mathbb{R}^{pN}$, $\mathbf{w} \in \mathbb{R}^{nN}$, and $\boldsymbol{\eta} \in \mathbb{R}^{pN}$ respectively, as

$$\mathbf{s} := [s_0', \dots, s_N']', \qquad \mathbf{e} := [e_0', \dots, e_N']' \tag{6a}$$

$$\mathbf{u} := [u_0', \dots, u_{N-1}']', \qquad \mathbf{y} := [y_0', \dots, y_{N-1}']' \tag{6b}$$

$$\mathbf{w} := [w_0', \dots, w_{N-1}']', \qquad \boldsymbol{\eta} := [\eta_0', \dots, \eta_{N-1}']' \tag{6c}$$

where $s_0 := s$ and $e_0 := e$ denote the current values of the state estimate and estimation error respectively, and $s_{i+1} := A_L s_i + Bu_i + Ly_i$ and $e_{i+1} = A_L e_i - L\eta_i + w_i$, $\forall i \in \mathbb{Z}_{[0,N-1]}$, denote the predictions of the state estimate and estimation error after i time instants. The predicted measurements after i time instants are $y_i = C(s_i + e_i) + \eta_i$, $\forall i \in \mathbb{Z}_{[0,N-1]}$. We define E to be the set of all convex and compact subsets of $\mathbb{R}^n$ containing the origin. We assume that the true initial state x is such that $x = s_0 + e_0$, where $e_0 \in \mathcal{E}$ is the initial state estimation error for some given $\mathcal{E} \in \mathsf{E}$. We let $\mathcal{W} := W^N := W \times \cdots \times W$ and $\mathcal{H} := H^N := H \times \cdots \times H$, so that $\mathbf{w} \in \mathcal{W}$ and $\boldsymbol{\eta} \in \mathcal{H}$.

We define a closed and convex set $\mathcal{Z}$, appropriately constructed from Z and X_f, such that the constraints to be satisfied are equivalent to $(\mathbf{s}, \mathbf{e}, \mathbf{u}) \in \mathcal{Z}$, i.e.

$$\mathcal{Z} := \left\{ (\mathbf{s}, \mathbf{e}, \mathbf{u}) \;\middle|\; \begin{array}{c} (s_i + e_i, u_i) \in Z,\ \forall i \in \mathbb{Z}_{[0,N-1]} \\ (s_N, e_N) \in X_f \end{array} \right\}. \tag{7}$$

Finally, we define the matrices $\mathbf{A}, \mathbf{B}, \mathbf{E}, \mathcal{L}, \mathcal{B}, \Phi$ and Γ (given in the Appendix) and affine functions f_e and f_s such that the vectors $\mathbf{s}$ and $\mathbf{e}$ can be written as

$$\mathbf{s} = f_s(s_0, e_0, \mathbf{u}, \mathbf{w}, \boldsymbol{\eta}) := \mathbf{A}s_0 + \mathbf{B}\mathbf{u} + \mathbf{E}\mathcal{L}(\mathbf{C}\mathbf{e} + \boldsymbol{\eta}) \tag{8}$$

$$\mathbf{e} = f_e(e_0, \mathbf{w}, \boldsymbol{\eta}) := \Phi e_0 - \Gamma\mathcal{L}\boldsymbol{\eta} + \Gamma\mathbf{w}, \tag{9}$$

and such that $\mathbf{s}$ may alternatively be expressed directly as an affine function of $\mathbf{y}$, i.e.

$$\mathbf{s} = \Phi s_0 + \Gamma\mathcal{B}\mathbf{u} + \Gamma\mathcal{L}\mathbf{y}. \tag{10}$$

3 Affine Feedback Parameterisations

3.1 Output Feedback

As a means of controlling the system (1) while ensuring the satisfaction of the constraints (7) for all possible realizations of the system uncertainty, we wish to construct a control policy such that each control input u_i is affine in the measurements $\{y_0, \ldots, y_{i-1}\}$, i.e.

$$u_i = g_i + \sum_{j=0}^{i-1} K_{i,j} y_j \tag{11}$$

where each $K_{i,j} \in \mathbb{R}^{m\times p}$ and $g_i \in \mathbb{R}^m$. For notational convenience we define the vector $\mathbf{g} \in \mathbb{R}^{mN}$ and matrix $\mathbf{K} \in \mathbb{R}^{mN\times pN}$ as

$$\mathbf{K} := \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ K_{1,0} & \ddots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ K_{N-1,0} & \cdots & K_{N-1,N-2} & 0 \end{bmatrix}, \mathbf{g} := \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{N-1} \end{bmatrix} \tag{12}$$

so that the control input sequence can be written as $\mathbf{u} = \mathbf{K}\mathbf{y} + \mathbf{g}$.

For a given initial state estimate s and estimation error set $\mathcal{E} \in \mathsf{E}$, the set of feasible output feedback policies which are guaranteed to satisfy the constraints $\mathcal{Z}$ for all possible uncertainty realizations is

$$\Pi_N^{of}(s, \mathcal{E}) := \left\{ (\mathbf{K}, \mathbf{g}) \;\middle|\; \begin{array}{r} (\mathbf{K}, \mathbf{g}) \text{ satisfies (12)} \\ \mathbf{s} = f_s(s, e, \mathbf{u}, \mathbf{w}, \boldsymbol{\eta}) \\ \mathbf{e} = f_e(e, \mathbf{w}, \boldsymbol{\eta}) \\ \mathbf{y} = \mathbf{C}(\mathbf{s} + \mathbf{e}) + \boldsymbol{\eta} \\ \mathbf{u} = \mathbf{K}\mathbf{y} + \mathbf{g},\; (\mathbf{s}, \mathbf{e}, \mathbf{u}) \in \mathcal{Z} \\ \forall e \in \mathcal{E},\; \forall \mathbf{w} \in \mathcal{W},\; \forall \boldsymbol{\eta} \in \mathcal{H} \end{array} \right\}. \tag{13}$$

Given an initial estimation error set $\mathcal{E}$, we define the set of all initial state estimates for which a constraint admissible policy exists as

$$\mathcal{S}_N^{of}(\mathcal{E}) := \left\{ s \;\middle|\; \Pi_N^{of}(s, \mathcal{E}) \neq \emptyset \right\}.$$

Remark 1. *The feedback policy* (11) *subsumes the class of "pre-stabilising" control policies in which the the control is based on perturbations* $\{c_i\}_{i=0}^{N-1}$ *to a* fixed *linear state feedback gain* K*, so that* $u_i = Ks_i + c_i$*, since the estimated state* s_i *may be expressed as an affine function of the measurements* $\{y_0, \ldots, y_{i-1}\}$ *(cf.* (10)*). Such a scheme is commonly employed for robust control of constrained systems under state feedback [2, 13, 24], or employed in conjunction with a stabilising linear observer gain* L *for output feedback [25, 35, 44]. It can also be shown to subsume tube-based schemes such as [28, 30] when the invariant sets defining the tube are based on static linear feedback, though these methods also confer additional stability properties which we do not address here.*

Remark 2. *As in the full state information case considered in [19], the set* $\Pi_N^{of}(s, \mathcal{E})$ *is non-convex, in general, due to the nonlinear relationship between the estimated states* $\mathbf{s}$ *and feedback gains* $\mathbf{K}$ *in* (13)*.*

3.2 Output Error Feedback

As an alternative to the parameterisation (11), we consider a control policy parameterised as an affine function of the uncertain parameters $\mathbf{w}$, $\boldsymbol{\eta}$ and e_0; a related parameterisation was first suggested as a means for finding solutions to a general class of robust optimisation problems, called affinely adjustable robust counterpart (AARC) problems [6,20], and recently as a means for robust control of systems with full state feedback [19,26] and output feedback [5,41]. The control policy is parameterized as

$$u_i = v_i + \sum_{j=0}^{i-1} M_{i,j}(y_j - Cs_j) \tag{14}$$

where each $M_{i,j} \in \mathbb{R}^{m \times p}$ and $v_i \in \mathbb{R}^m$, and note that $(y_i - Cs_i) = (Ce_i + \eta_i)$ for all $i \in \mathbb{Z}_{[0,N-1]}$. We further define matrices $\mathbf{M} \in \mathbb{R}^{mN \times nN}$ and vector $\mathbf{v} \in \mathbb{R}^{mN}$ as

$$\mathbf{M} := \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-2} & 0 \end{bmatrix}, \mathbf{v} := \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix} \tag{15}$$

so that the control input sequence can be written as

$$\mathbf{u} = \mathbf{M}(\mathbf{y} - \mathbf{C}\mathbf{s}) + \mathbf{v} \tag{16}$$

$$= \mathbf{M}(\mathbf{C}\mathbf{e} + \boldsymbol{\eta}) + \mathbf{v}. \tag{17}$$

By virtue of the relation (9), this control parameterization is *affine* in the unknown parameters e_0, $\mathbf{w}$, $\boldsymbol{\eta}$. For a given initial state estimate s and error set $\mathcal{E}$, the set of feasible feedback policies that are guaranteed to satisfy the system constraints for all possible uncertainty realizations is

$$\Pi_N^{ef}(s, \mathcal{E}) := \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies } (15) \\ \mathbf{s} = f_s(s, e, \mathbf{u}, \mathbf{w}, \boldsymbol{\eta}) \\ \mathbf{e} = f_e(e, \mathbf{w}, \boldsymbol{\eta}) \\ \mathbf{y} = \mathbf{C}(\mathbf{s} + \mathbf{e}) + \boldsymbol{\eta} \\ \mathbf{u} = \mathbf{M}(\mathbf{C}\mathbf{e} + \boldsymbol{\eta}) + \mathbf{v},\ (\mathbf{s}, \mathbf{e}, \mathbf{u}) \in \mathcal{Z} \\ \forall e \in \mathcal{E},\ \forall \mathbf{w} \in \mathcal{W},\ \forall \boldsymbol{\eta} \in \mathcal{H} \end{array} \right\}. \tag{18}$$

For a given error set $\mathcal{E}$, define the set of all constraint admissible initial state estimates to be

$$\mathcal{S}_N^{ef}(\mathcal{E}) := \left\{ s \;\middle|\; \Pi_N^{ef}(s,\mathcal{E}) \neq \emptyset \right\}.$$

We next characterise two critical properties of the parameterisation (14), which make it attractive in application to control of the system (1), and which parallel the results in [19] for the full state feedback case.

3.2.1 Convexity

Theorem 1. *The set of constraint admissible feedback policies $\Pi_N^{ef}(s,\mathcal{E})$ is convex and closed, and the set of feasible initial states $\mathcal{S}_N^{ef}(\mathcal{E})$ is convex.*

Proof. Define the set

$$\mathcal{C}_N(\mathcal{E}) := \bigcap_{\substack{\mathbf{w}\in\mathcal{W},\\ \boldsymbol{\eta}\in\mathcal{H},\ e\in\mathcal{E}}} \left\{ (\mathbf{M},\mathbf{v},s) \;\middle|\; \begin{aligned} (\mathbf{M},\mathbf{v}) \text{ satisfies (15)} \\ \mathbf{s} = f_s(s,e,\mathbf{u},\mathbf{w},\boldsymbol{\eta}) \\ \mathbf{e} = f_e(e,\mathbf{w},\boldsymbol{\eta}) \\ \mathbf{y} = \mathbf{C}(\mathbf{s}+\mathbf{e})+\boldsymbol{\eta} \\ \mathbf{u} = \mathbf{M}(\mathbf{Ce}+\boldsymbol{\eta})+\mathbf{v},\ (\mathbf{s},\mathbf{e},\mathbf{u})\in\mathcal{Z} \end{aligned} \right\} \tag{19}$$

which is closed and convex, since it is the intersection of closed and convex sets. The set $\mathcal{S}_N^{ef}(\mathcal{E})$ is a projection of this set, and is thus convex. The set $\Pi_N^{ef}(s,\mathcal{E})$ can similarly be written as an intersection of closed and convex sets, so is also closed and convex. □

Remark 3. *In certain cases it is possible to find a feasible policy $(\mathbf{M},\mathbf{v}) \in \Pi_N^{ef}(s,\mathcal{E})$ given an initial state estimate s using standard techniques in convex optimisation similar to those required in the case of robust control with state feedback [19]. For example, if the constraint set $\mathcal{Z}$ and uncertainty sets W, H and $\mathcal{E}$ are polytopes, a constraint admissible policy $(\mathbf{M},\mathbf{v}) \in \Pi_N^{ef}(s,\mathcal{E})$ can be found by solving a single tractable linear program (LP). If $\mathcal{Z}$ is a polytope and the sets $\mathcal{E}$, $\mathcal{H}$ and $\mathcal{W}$ are ellipsoids, then a constraint admissible policy can be found by solving a single tractable second-order cone program (SOCP).*

3.2.2 Equivalence

Theorem 2. *Given an initial state estimation set $\mathcal{E}$, the sets $\mathcal{S}_N^{ef}(\mathcal{E})$ and $\mathcal{S}_N^{of}(\mathcal{E})$ are equal. For a given state estimate s, for every pair $(\mathbf{K},\mathbf{g}) \in \Pi_N^{of}(s,\mathcal{E})$, there exists a pair $(\mathbf{M},\mathbf{v}) \in \Pi_N^{ef}(s,\mathcal{E})$ giving the same sequence of inputs and states for all possible realizations of the system uncertainty, and vice-versa.*

Proof. $\mathcal{S}_N^{of}(\mathcal{E}) \subseteq \mathcal{S}_N^{ef}(\mathcal{E})$: By definition, for any $s \in \mathcal{S}_N^{of}(\mathcal{E})$, there exists a pair $(\mathbf{K},\mathbf{g}) \in \Pi_N^{ef}(s,\mathcal{E})$. A bit of algebra shows that, given any uncertainty realization e, $\boldsymbol{\eta}$ and $\mathbf{w}$, the input sequence $\mathbf{u}$ can be written as

$$\mathbf{u} = \Delta^{-1}\mathbf{K}\left[\mathbf{C}(\mathbf{A}e+\mathbf{E}\mathbf{w})+\boldsymbol{\eta}\right] + \Delta^{-1}(\mathbf{KCA}s+g), \tag{20}$$

where $\Delta := (I-\mathbf{KCB})$, and the matrix Δ is always invertible since $\mathbf{KCB}$ is strictly lower triangular. Noting the identity $\mathbf{C}(\mathbf{A}e+\mathbf{E}\mathbf{w})+\boldsymbol{\eta} = (I+\mathbf{CE}\mathcal{L})(\mathbf{y}-\mathbf{C}s)$, the input sequence $\mathbf{u}$ can thus be written as

$$\mathbf{u} = \Delta^{-1}\mathbf{K}(I+\mathbf{CE}\mathcal{L})(\mathbf{y}-\mathbf{C}s) + \Delta^{-1}(\mathbf{KCA}s+g)$$

A constraint admissible policy $(\mathbf{M}, \mathbf{v}) \in \mathcal{S}_N^{ef}(\mathcal{E})$ can then be found by selecting

$$\mathbf{M} = \Delta^{-1}\mathbf{K}(I + \mathbf{C}\mathbf{E}\mathcal{L}), \;\; \mathbf{v} = \Delta^{-1}(\mathbf{K}\mathbf{C}\mathbf{A}s + g). \tag{21}$$

Thus, $s \in \mathcal{S}_N^{ef}(\mathcal{E})$ for all $s \in \mathcal{S}_N^{of}(\mathcal{E})$, so $\mathcal{S}_N^{of}(\mathcal{E}) \subseteq \mathcal{S}_N^{ef}(\mathcal{E})$.
$\mathcal{S}_N^{ef}(\mathcal{E}) \subseteq \mathcal{S}_N^{of}(\mathcal{E})$: By definition, for any $s \in \mathcal{S}_N^{ef}(\mathcal{E})$, there exists a pair $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{of}(s, \mathcal{E})$. Using the relation (10), the output error terms can be written as $\mathbf{y} - \mathbf{C}\mathbf{s} = (I - \mathbf{C}\Gamma\mathcal{L})\mathbf{y} - C\Phi s - C\Gamma\mathcal{B}\mathbf{u}$, and the control input sequence $\mathbf{u} = \mathbf{M}(\mathbf{y} - \mathbf{C}\mathbf{s}) + \mathbf{v}$ as

$$\mathbf{u} = \hat{\Delta}^{-1}\mathbf{M}(I - \mathbf{C}\Gamma\mathcal{L})\mathbf{y} + \hat{\Delta}^{-1}(\mathbf{v} - \mathbf{M}\mathbf{C}\Phi s),$$

where $\hat{\Delta} := (I + \mathbf{M}\mathbf{C}\Gamma\mathcal{B})$, and the matrix $\hat{\Delta}$ is always invertible since $\mathbf{M}\mathbf{C}\Gamma\mathcal{B}$ is strictly lower triangular. A constraint admissible policy $(\mathbf{K}, \mathbf{g}) \in \mathcal{S}_N^{of}(\mathcal{E})$ can then be found by selecting

$$\mathbf{K} = \hat{\Delta}^{-1}\mathbf{M}(I - \mathbf{C}\Gamma\mathcal{L}), \;\; \mathbf{g} = \hat{\Delta}^{-1}(\mathbf{v} - \mathbf{M}\mathbf{C}\Phi s) \tag{22}$$

Thus, $s \in \mathcal{S}_N^{of}(\mathcal{E})$ for all $s \in \mathcal{S}_N^{ef}(\mathcal{E})$, so $\mathcal{S}_N^{ef}(\mathcal{E}) \subseteq \mathcal{S}_N^{of}(\mathcal{E})$.

□

Remark 4. *A control policy based on the measurement prediction error terms* $(\mathbf{y} - \mathbf{C}\mathbf{s})$ *was proposed in [41], and independently in the context of robust optimization in [5], which gives an equivalence proof similar to that presented here but without the inclusion of a non-zero initial state estimate or observer dynamics. We make explicit use of these error dynamics to derive conditions under which receding horizon control (RHC) laws based on the parameterization* (14) *can be guaranteed to satisfy constraints for the resulting closed-loop system for all time.*

4 Geometric and Invariance Properties

In this section, we characterise some of the geometric and invariance properties associated with control laws synthesised from the feedback parameterisation (14). We first require the following assumption about the terminal constraint set X_f:

A1 (Terminal constraint) The state feedback gain matrix K and terminal constraint set X_f have been chosen such that:

- X_f is consistent with the set of states for which the constraints Z in (5) are satisfied under the control $u = Ks$, i.e. $(s, e) \in X_f$ implies $(s + e, Ks) \in Z$.
- X_f is robust positively invariant for the closed-loop system under the control $u = Ks$. Thus $(s, e) \in X_f$ guarantees $(s^+, e^+) \in X_f$ for all $w \in W$ and for all $\eta \in H$, where $s^+ = (A + BK)s + L(Ce + \eta)$ and $e^+ = A_L e - L\eta + w$.

Remark 5. *If the set* $W \times H$ *is a polytope or affine map of a p-norm ball and the constraints* Z *are polyhedral, then one can calculate an invariant set which satisfies the conditions* ***A1*** *by applying the techniques in [11, 22, 24] to the augmented system*

$$\begin{bmatrix} s^+ \\ e^+ \end{bmatrix} = \begin{bmatrix} (A + BK) & LC \\ 0 & (A - LC) \end{bmatrix} \begin{bmatrix} s \\ e \end{bmatrix} + \begin{bmatrix} 0 & L \\ I & -L \end{bmatrix} \begin{bmatrix} w \\ \eta \end{bmatrix} \tag{23}$$

Proposition 1 ((Monotonicity)). *If **A1** holds, then the following set inclusions hold:*

$$\mathcal{S}_1^{of}(\mathcal{E}) \subseteq \cdots \subseteq \mathcal{S}_{N-1}^{of}(\mathcal{E}) \subseteq \mathcal{S}_N^{of}(\mathcal{E}) \subseteq \mathcal{S}_{N+1}^{of}(\mathcal{E}) \ldots \tag{24}$$

$$\mathcal{S}_1^{ef}(\mathcal{E}) \subseteq \cdots \subseteq \mathcal{S}_{N-1}^{ef}(\mathcal{E}) \subseteq \mathcal{S}_N^{ef}(\mathcal{E}) \subseteq \mathcal{S}_{N+1}^{ef}(\mathcal{E}) \ldots \tag{25}$$

Proof. The proof of the first relation is by induction. Suppose that $s \in \mathcal{S}_N^{of}(\mathcal{E})$ and $(\mathbf{K}, \mathbf{g}) \in \Pi_N^{of}(s, \mathcal{E})$. Recalling the relation (10), the state estimates $\mathbf{s}$ can be found as an affine function of the measurements $\mathbf{y}$ using

$$\mathbf{s} = \Gamma(\mathcal{B}\mathbf{K} + \mathcal{L})\mathbf{y} + \Gamma\mathcal{B}\mathbf{g} + \Phi s_0 \tag{26}$$

One can thus find a pair $(\bar{\mathbf{K}}, \bar{\mathbf{g}}) \in \Pi_{N+1}^{of}(s, \mathcal{E})$, where $\bar{\mathbf{K}} := \left[\begin{smallmatrix} \mathbf{K} & 0 \\ \bar{K}_1 & \bar{K}_2 \end{smallmatrix}\right]$ and $\bar{\mathbf{g}} := \left[\begin{smallmatrix} \mathbf{g} \\ \bar{g} \end{smallmatrix}\right]$, by defining the matrices $\bar{K}_1 := \mathcal{A}_L(\mathcal{B}\mathbf{K}+\mathcal{L})$ and $\bar{K}_2 := 0$ and vector $\bar{g} := \big(\mathcal{A}_L(\mathcal{B}\mathbf{K} + \mathcal{L})\mathbf{g} + A_L^N s_0\big)$, where $\mathcal{A}_L := [\, A_L^{N-1} \cdots A_L \; I \,]$, such that the final stage input is $u_N = K s_N$. Since $s \in \mathcal{S}_N^{of}(\mathcal{E})$ implies $(s_N, e_N) \in X_f$ by definition, then it follows that $(s_N + e_N, u_N) \in Z$ and $(s_{N+1}, e_{N+1}) \in X_f$ for all $w \in W$ and all $\eta \in H$ if **A1** holds. Thus $(\bar{\mathbf{K}}, \bar{\mathbf{g}}) \in \Pi_{N+1}^{of}(s, \mathcal{E})$ and $s \in X_{N+1}(\mathcal{E})$. The second relation then follows from Theorem 2. □

4.1 Invariance Properties

We next consider some properties of receding horizon control (RHC) laws synthesised from the parameterization (11) (equivalently, (14)). In particular, we develop conditions under which such a RHC law can be guaranteed to be robust positively invariant for the resulting closed-loop system.

We define the set-valued map $\kappa_N : \mathcal{S}_N^{of} \times \mathsf{E} \to 2^{\mathbb{R}^m}$ as

$$\kappa_N(s, \mathcal{E}) := \left\{ u \;\middle|\; \exists (\mathbf{K}, \mathbf{g}) \in \Pi_N^{of}(s, \mathcal{E}) \text{ s.t. } u = g_0 \right\} \tag{27}$$

$$= \left\{ u \;\middle|\; \exists (\mathbf{M}, \mathbf{v}) \in \Pi_N^{ef}(s, \mathcal{E}) \text{ s.t. } u = v_0 \right\} \tag{28}$$

where $2^{\mathbb{R}^m}$ is the set of all subsets of $\mathbb{R}^m$, and (28) follows directly from Theorem 2. We define a function $\mu_N : \mathcal{S}_N^{of} \times \mathsf{E} \to \mathbb{R}^m$ as any selection from the set κ_N, i.e. given $\mathcal{E} \in \mathsf{E}$, $\mu_N(\cdot, \mathcal{E})$ must satisfy

$$\mu_N(s, \mathcal{E}) \in \kappa_N(s, \mathcal{E}), \; \forall s \in \mathcal{S}_N^{of}(\mathcal{E})$$

We wish to develop conditions under which time-varying or time-invariant control schemes based on the functions μ_N can be guaranteed to satisfy the system constraints Z for all time. We first introduce the following standard definition from the theory of invariant sets [22,33]:

Definition 1 (Minimal Robust Positively Invariant (mRPI) Error Set). *The set $\mathcal{E}_i$ is defined as*

$$\mathcal{E}_i := \bigoplus_{j=0}^{i} A_L^j (W \oplus L(-H)), \; \forall i \in \mathbb{Z}. \tag{29}$$

The minimal robust positively invariant (mRPI) set $\mathcal{E}_\infty$ is defined as the limit set of the sequence $\{\mathcal{E}_i : i \in \mathbb{Z}\}$, i.e. $\mathcal{E}_\infty := \lim_{i\to\infty} \mathcal{E}_i$.

Remark 6. *As noted in [22], unless the observer gain L is selected such that there exists a $k \in \mathbb{Z}$ and $0 \leq \alpha < 1$ such that $A_L^k = \alpha A_L$ (e.g. when L is a deadbeat observer so that A_L is nilpotent), then the set $\mathcal{E}_\infty$ may not be characterised by a finite number of inequalities, since it is a Minkowski sum with an infinite number of terms. However, in [33] it is shown how one can calculate a so-called ϵ-outer approximation $\mathcal{E}_I$ to the set $\mathcal{E}_\infty$ (which can be represented by a tractable number of inequalities if W and H are polytopes) such that $\mathcal{E}_\infty \subseteq \mathcal{E}_I \subseteq \mathcal{E}_\infty \oplus \mathcal{B}_p^n(\epsilon)$ and such that the set $\mathcal{E}_I$ is robust positively invariant. Further, it is shown in [33] that, if only the support function of the set $\mathcal{E}_I$ is required, then calculation of an* explicit *representation of $\mathcal{E}_I$ via Minkowski summation is* not *necessary, a fact which we exploit in the computational results of Section 5.*

4.1.1 Time-Varying and mRPI-based RHC Laws

We first consider the implementation of a time-varying receding horizon control (RHC) law based on the function $\mu_N(\cdot,\cdot)$. Taking the initial time to be 0 (note that this is always possible since the system (3)–(4) is time invariant), and given an initial state estimate $s(0)$ and initial state estimation error set $\mathcal{E}$, we define the *time-varying* RHC control law $\nu : \mathbb{R}^n \times \mathbb{Z} \times \mathsf{E} \to \mathbb{R}^m$ as

$$\nu(s(k), k, \mathcal{E}) := \begin{cases} \mu_N(s(k), \mathcal{E}), & \text{if } k = 0 \\ \mu_N(s(k), A_L^k \mathcal{E} \oplus \mathcal{E}_{k-1}), & \text{if } k > 0 \end{cases}. \tag{30}$$

Note that the error sets required in the calculation of $\nu(s(k), k, \mathcal{E})$ can be defined recursively, i.e. $A_L^{k+1}\mathcal{E} \oplus \mathcal{E}_k = A_L[A_L^k \mathcal{E} \oplus \mathcal{E}_{k-1}] \oplus \mathcal{E}_0$, though an explicit calculation of $\mathcal{E}$ via Minkowski summation is *not* required (cf. Section 5). The resulting closed-loop system can be written as:

$$x(k+1) = Ax(k) + B\nu(s(k), k, \mathcal{E}) + w \tag{31}$$

$$s(k+1) = As(k) + B\nu(s(k), k, \mathcal{E}) + L(y(k) - Cs(k)) \tag{32}$$

$$e(k+1) = (A - LC)e(k) - L\eta(k) + w(k) \tag{33}$$

$$y(k) = Cx(k) + \eta(k), \tag{34}$$

Note that given the estimation error set $\mathcal{E}$ at time 0, the estimation errors $\{e(i)\}_{i=0}^\infty$ in (33) are only known by the controller to satisfy $e(i) \in A_L^k \mathcal{E} \oplus \mathcal{E}_{k-1}$ for all $i \in \mathbb{Z}$. Our first invariance result follows immediately:

Proposition 2. *If **A1** holds and $s(0) \in \mathcal{S}_N^{of}(\mathcal{E})$, then the closed-loop system* (31)–(34) *satisfies the constraints* (5) *for all time and all possible uncertainty realizations if and only if the true initial state $x(0) \in \{s(0)\} \oplus \mathcal{E}$.*

Proof. If $s \in \mathcal{S}_N^{of}(\tilde{\mathcal{E}})$ for some $\tilde{\mathcal{E}} \in \mathsf{E}$, then there exists an output feedback policy pair $(\mathbf{K}, \mathbf{g}) \in \Pi_N^{of}(s, \tilde{\mathcal{E}})$ for which $\mu_N(s, \tilde{\mathcal{E}}) = g_0$. It is then easy to show that

$$s^+ = As + B\mu_N(s, \tilde{\mathcal{E}}) + L(Ce + \eta) \in X_{N-1}^{of}(A_L\tilde{\mathcal{E}} \oplus \mathcal{E}_0),\ \forall e \in \tilde{\mathcal{E}},$$

since one can construct a feasible policy pair $(\tilde{\mathbf{K}}, \tilde{\mathbf{g}}) \in \Pi_{N-1}^{of}(s^+, A_L\tilde{\mathcal{E}} \oplus \mathcal{E}_0)$ from $(\mathbf{K}, \mathbf{g})$ by dropping the first element of $\mathbf{g}$ and the first block row and column of $\mathbf{K}$. If **A1** holds, then $s^+ \in X_{N-1}^{of}(A_L\tilde{\mathcal{E}} \oplus \mathcal{E}_0)$ implies $s^+ \in X_N^{of}(A_L\tilde{\mathcal{E}} \oplus \mathcal{E}_0)$ from Proposition 1, and the result follows. □

We note that if the state estimation error set $\mathcal{E} = \mathcal{E}_\infty$, then the control law $\nu(\cdot)$ defined in (30) is actually *time-invariant*, so that

$$\nu(s(k), k, \mathcal{E}_\infty) = \mu_N(s(k), \mathcal{E}_\infty), \ \ \forall k \in \mathbb{Z}. \tag{35}$$

The next result follows immediately:

Corollary 1. *The set $\mathcal{S}_N^{of}(\mathcal{E}_\infty)$ is robust positively invariant for the closed-loop system* (31)–(34) *under the time-invariant control law* (35)*, i.e. if $s(0) \in \mathcal{S}_N^{of}(\mathcal{E}_\infty)$, then $s(k) \in \mathcal{S}_N^{of}(\mathcal{E}_\infty)$ for all $k \in \mathbb{Z}$ and for all possible uncertainty realizations. The constraints* (5) *are satisfied for all time and all possible uncertainty realizations if and only if the true initial state $x(0) \in \{s(0)\} \oplus \mathcal{E}_\infty$.*

4.1.2 A Time-Invariant Finite-Dimensional RHC Law

The central difficulty with the control law defined in (35) is that, in general, the set $\mathcal{E}_\infty$ is not finitely determined (cf. Remark 6). The calculation of the control law $\nu(\cdot, \cdot, \mathcal{E})$ in (30) is thus of increasing complexity with increased time, and the calculation of the control law $\nu(\cdot, \cdot, \mathcal{E}_\infty)$ in (35) requires the solution of an infinite-dimensional optimisation problem. We thus seek a control law which is of fixed and finite complexity, while preserving the time-invariant nature of (35). To this end, we define a robust positively invariant (RPI) error set $\mathcal{E}_I \in \mathsf{E}$ which satisfies the following:

A2 (Invariant Error Set) The set $\mathcal{E}_I \in \mathsf{E}$ is chosen such that it is robust positively invariant for the system $e^+ = A_L e - L\eta + w$, so that $A_L e - L\eta + w \in \mathcal{E}_I$ for all $e \in \mathcal{E}_I$, for all $w \in W$ and for all $\eta \in H$. Furthermore, for some p-norm, $\mathcal{E}_I$ is an ϵ-outer approximation for $\mathcal{E}_\infty$, so that there exists some $\epsilon > 0$ such that $\mathcal{E}_\infty \subseteq \mathcal{E}_I \subseteq \mathcal{E}_\infty \oplus \mathcal{B}_p^n(\epsilon)$.

Remark 7. *If **A2** holds, then $A_L\mathcal{E}_I \oplus (W \oplus L(-H)) \subseteq \mathcal{E}_I$ and $\mathcal{E}_\infty \subseteq \mathcal{E}_I$. Such a set can be calculated efficiently using standard techniques for finding* maximal *RPI sets [22] (cf. Remark 6), or as an outer approximation to the mRPI set using results from [33]. In both cases, the resulting set is polytopic when all of the relevant constraints and uncertainty sets are polytopic, and the set $\mathcal{E}_I$ can be characterised by a finite number of linear inequalities, though an* explicit *representation of the set $\mathcal{E}_I$ is not required (cf. Remark 6 and the results of Section 5).*

We can now guarantee an invariance condition similar to that in Proposition 2 using the finitely determined set $\mathcal{E}_I$, by slightly enlarging the disturbance set W from which feedback policies of the form (14) are selected. We define

$$W_\epsilon := W \oplus \mathcal{B}_p^n(\epsilon) \tag{36}$$

where p and ϵ satisfy the conditions of **A2** for the set $\mathcal{E}_I$, and similarly define $\mathcal{W}_\epsilon := W_\epsilon^N$. Using this enlarged disturbance set, we define a modified target/terminal constraint set $X_{f,\epsilon} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ which is closed and convex and which satisfies the following condition:

A3 (Modified terminal constraint) The state feedback gain matrix K and modified terminal constraint set $X_{f,\epsilon}$ have been chosen such that:

- $X_{f,\epsilon} \subseteq X_f$ is consistent with the set of states for which the constraints Z in (5) are satisfied under the control $u = Ks$, i.e. $(s, e) \in X_{f,\epsilon}$ implies $(s + e, Ks) \in Z$.
- $X_{f,\epsilon}$ is robust positively invariant for the system $s^+ = (A + BK)s + L(Ce + \eta)$, $e^+ = A_L e - L\eta + w$ for all $s \in X_{f,\epsilon}$, for all $w \in W_\epsilon$ and for all $\eta \in H$.

Using this modified target set, we define the modified constraint set $\mathcal{Z}_\epsilon \subseteq \mathcal{Z}$ as

$$\mathcal{Z}_\epsilon := \left\{ (\mathbf{s}, \mathbf{e}, \mathbf{u}) \;\middle|\; \begin{array}{c} (s_i + e_i, u_i) \in Z,\ \forall i \in \mathbb{Z}_{[0,N-1]} \\ (s_N, e_N) \in X_{f,\epsilon} \end{array} \right\}. \tag{37}$$

We also use the enlarged disturbance set W_ϵ to define a new set of feasible feedback control policies

$$\Pi^{of}_{N,\epsilon}(s, \mathcal{E}) := \left\{ (\mathbf{K}, \mathbf{g}) \;\middle|\; \begin{array}{r} (\mathbf{K}, \mathbf{g}) \text{ satisfies (12)} \\ \mathbf{s} = f_s(s, e, \mathbf{u}, \mathbf{w}, \boldsymbol{\eta}) \\ \mathbf{e} = f_e(e, \mathbf{w}, \boldsymbol{\eta}) \\ \mathbf{y} = \mathbf{C}(\mathbf{s} + \mathbf{e}) + \boldsymbol{\eta} \\ \mathbf{u} = \mathbf{K}\mathbf{y} + \mathbf{g},\ (\mathbf{s}, \mathbf{e}, \mathbf{u}) \in \mathcal{Z}_\epsilon \\ \forall e \in \mathcal{E},\ \forall \mathbf{w} \in \mathcal{W}_\epsilon,\ \forall \boldsymbol{\eta} \in \mathcal{H} \end{array} \right\}, \tag{38}$$

and feasible set

$$\mathcal{S}^{of}_{N,\epsilon}(\mathcal{E}) := \left\{ s \;\middle|\; \Pi^{of}_{N,\epsilon}(s, \mathcal{E}) \neq \emptyset \right\}. \tag{39}$$

In the sequel, we will choose an invariant set $\mathcal{E} = \mathcal{E}_I$ satisfying the conditions of **A3** in (38) such that a time-invariant control law constructed from $\Pi^{of}_{N,\epsilon}(s, \mathcal{E}_I)$ can be guaranteed to satisfy the system constraints for all time.

Remark 8. *An equivalent convex parameterisation can similarly be defined using the feedback parameterisation* (14)*, so that an admissible pair* $(\mathbf{K}, \mathbf{g}) \in \Pi^{of}_{N,\epsilon}(s, \mathcal{E}_I)$ *can be calculated using standard convex optimisation techniques (cf. Remark 3 and Theorem 2), where the optimisation problem to be solved is* finite-dimensional*, since the set* $\mathcal{E}_I$ *can be implicitly characterised by a* finite *number of inequalities (cf. Remark 6). We show in Section 5 that if all of the relevant constraint sets are polytopic, then such a policy can be found via the solution of a single, tractable linear program.*

We define the set-valued mapping $\kappa_{N,\epsilon} : \mathcal{S}^{of}_{N,\epsilon} \times \mathsf{E} \to 2^{\mathbb{R}^m}$ as

$$\kappa_{N,\epsilon}(s, \mathcal{E}_I) := \left\{ u \;\middle|\; \exists (\mathbf{K}, \mathbf{g}) \in \Pi^{of}_{N,\epsilon}(s, \mathcal{E}_I) \text{ s.t. } u = g_0 \right\}, \tag{40}$$

and define the *time-invariant* control law $\nu_\epsilon : \mathcal{S}^{of}_{N,\epsilon} \to \mathbb{R}^m$ as any selection from this set:

$$\nu_\epsilon(s) \in \kappa_{N,\epsilon}(s, \mathcal{E}_I). \tag{41}$$

When applied to the control of the system (1), the closed-loop system dynamics become

$$x^+ = Ax + B\nu_\epsilon(s) + w \tag{42}$$
$$s^+ = As + B\nu_\epsilon(s) + L(y - Cs). \tag{43}$$
$$e^+ = A_L e - L\eta + w \tag{44}$$
$$y = Cx + \eta, \tag{45}$$

where $w \in W$ and $\eta \in H$. It is critical to note that, though the control law (41) is conservatively constructed using the *enlarged* disturbance set W_ϵ, the disturbances w in (42) are generated from the *original* disturbance set W. It is this conservativeness which will ensure that the time-invariant control law (41) can guarantee constraint satisfaction of the closed-loop system for all time. We can now state our final result:

Theorem 3. *If* ***A2*** *and* ***A3*** *hold, then the set $\mathcal{S}^{of}_{N,\epsilon}(\mathcal{E}_I)$ is robust positively invariant for the closed-loop system* (43)*, i.e. if $s \in \mathcal{S}^{of}_{N,\epsilon}(\mathcal{E}_I)$, then $s^+ \in \mathcal{S}^{of}_{N,\epsilon}(\mathcal{E}_I)$ for all $e \in \mathcal{E}_I$, for all $\eta \in H$ and for all $w \in W$. Furthermore, the closed-loop system* (42) *satisfies the constraints* (5) *for all time and all possible uncertainty realizations if and only if the true initial state $x(0) \in \{s(0)\} \oplus \mathcal{E}_I$.*

Proof. If **A3** holds then it can be shown, using arguments identical to those in the proof of Proposition 2, that $s \in \mathcal{S}^{of}_{N,\epsilon}(\mathcal{E}_I)$ implies that the successor state $s^+ \in \mathcal{S}^{of}_{N,\epsilon}(A_L\mathcal{E}_I \oplus W_\epsilon \oplus L(-H)) = \mathcal{S}^{of}_{N,\epsilon}(A_L\mathcal{E}_I \oplus \mathcal{E}_0 \oplus \mathcal{B}^n_p(\epsilon))$. If **A2** holds, then $\mathcal{E}_\infty \subseteq A_L\mathcal{E}_I \oplus \mathcal{E}_0 \subseteq \mathcal{E}_I$, and thus $\mathcal{E}_I \subseteq \mathcal{E}_\infty \oplus \mathcal{B}^n_p(\epsilon) \subseteq A_L\mathcal{E}_I \oplus \mathcal{E}_0 \oplus \mathcal{B}^n_p(\epsilon)$. Writing the set $\Pi^{of}_{N,\epsilon}(s, \mathcal{E})$ in terms of set intersections as in (19) it is easy to verify that, for any sets $\mathcal{E}' \in \mathsf{E}$ and $\mathcal{E}'' \in \mathsf{E}$, $\mathcal{E}' \subseteq \mathcal{E}''$ implies $\Pi^{of}_{N,\epsilon}(s, \mathcal{E}'') \subseteq \Pi^{of}_{N,\epsilon}(s, \mathcal{E}')$ for all $s \in \mathbb{R}^n$, and thus $\mathcal{S}^{of}_{N,\epsilon}(\mathcal{E}'') \subseteq \mathcal{S}^{of}_{N,\epsilon}(\mathcal{E}')$. It follows that

$$\mathcal{S}^{of}_{N,\epsilon}(A_L\mathcal{E}_I \oplus \mathcal{E}_0 \oplus \mathcal{B}^n_p(\epsilon)) \subseteq \mathcal{S}^{of}_{N,\epsilon}(\mathcal{E}_I)$$

and thus that $s^+ \in \mathcal{S}^{of}_{N,\epsilon}(\mathcal{E}_I)$ for all $e \in \mathcal{E}_I$, for all $\eta \in H$ and for all $w \in W$. Finally we verify that the closed-loop system (42)–(45) satisfies the constraints Z for all time; we again use set intersection arguments and the fact that $X_{f,\epsilon} \subseteq X_f$ in **A3** to confirm that $\Pi^{of}_{N,\epsilon}(s, \mathcal{E}_I) \subseteq \Pi^{of}_{N}(s, \mathcal{E}_I)$. This implies that $\kappa_{N,\epsilon}(s, \mathcal{E}_I) \in \kappa_N(s, \mathcal{E}_I)$, which guarantees that $(s + e, \nu_\epsilon(s)) \in Z$ for all $e \in \mathcal{E}_I$ if and only if $s \in \mathcal{S}^{of}_{N,\epsilon}(\mathcal{E}_I)$. □

We note that, when the initial estimation error set $\mathcal{E}$ is large, it may be undesirable to immediately employ a time-invariant policy of the form (41) with $\mathcal{E} \subseteq \mathcal{E}_I$. In this case, one may prefer to employ a time-varying policy of the form (30) for some period of time, and transition to the time-invariant policy (41) when the initial estimation error is sufficiently reduced.

5 Computation of Feedback Control Laws

We next demonstrate how one may actually calculate feedback policies of the form (18) for the implementation of the control law (30). We consider the particular case when the constraint sets Z and X_f and uncertainty sets W, H and $\mathcal{E}$ are polytopes, so that the set $\mathcal{Z}$ defined in (37) is also polytopic. In this case one can define matrices S, T and U and a vector b of appropriate dimensions such that $\mathcal{Z}$ can be expressed as

$$\mathcal{Z} = \{(\mathbf{s}, \mathbf{e}, \mathbf{u}) \mid S\mathbf{s} + T\mathbf{e} + U\mathbf{u} \le b\} \tag{46}$$

so that the set of feasible control policies can be expressed as

$$\Pi_N^{of}(s,\mathcal{E}) = \left\{ (\mathbf{M},\mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M},\mathbf{v}) \text{ satisfies (15)} \\ S\mathbf{A}s + P\mathbf{v} + \delta_e + \delta_w + \delta_\eta \leq b \\ \delta_e = (P\mathbf{M}Q_e + R_e)e \\ \delta_w = (P\mathbf{M}Q_w + R_w)\mathbf{w} \\ \delta_\eta = (P\mathbf{M}Q_\eta + R_\eta)\boldsymbol{\eta} \\ \forall e \in \mathcal{E},\ \forall \mathbf{w} \in \mathcal{W},\ \forall \boldsymbol{\eta} \in \mathcal{H} \end{array} \right\}. \tag{47}$$

Recall that $a^T\mathbf{y} \leq e$ for all $\mathbf{y} \in \mathcal{Y}$ if and only if $\sup\left\{a^T\mathbf{y} \mid \mathbf{y} \in \mathcal{Y}\right\} \leq e$, where a is a vector of appropriate length, e is a scalar and $\sup\left\{a^T\mathbf{y} \mid \mathbf{y} \in \mathcal{Y}\right\}$ is the value of the *support function* of the set $\mathcal{Y}$ evaluated at a [22]. Hence, one can eliminate the universal quantifiers in (47) to obtain the equivalent expression

$$\Pi_N^{of}(s,\mathcal{E}) = \left\{ (\mathbf{M},\mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M},\mathbf{v}) \text{ satisfies (15)} \\ F\mathbf{A}s + P\mathbf{v} + \delta_e + \delta_w + \delta_\eta \leq b \\ \delta_e = \max_{e\in\mathcal{E}} (P\mathbf{M}Q_e + R_e)e \\ \delta_w = \max_{\mathbf{w}\in\mathcal{W}} (P\mathbf{M}Q_w + R_w)\mathbf{w} \\ \delta_\eta = \max_{\boldsymbol{\eta}\in\mathcal{H}} (P\mathbf{M}Q_\eta + R_\eta)\boldsymbol{\eta} \end{array} \right\}, \tag{48}$$

where the matrices P, Q_e, Q_w, Q_η, R_e, R_w and R_η are defined in the Appendix, and the maximisations are performed row-wise. Note that all of the maxima in (48) are attained since the sets $\mathcal{E}$, $\mathcal{W}$ and $\mathcal{H}$ are assumed compact. A pair $(\mathbf{M},\mathbf{v}) \in \Pi_N^{of}(s,\mathcal{E})$ can thus be found by forming the dual optimisation problem associated with each element of the vectors δ_e, δ_w and δ_η and introducing slack variables to form a single linear program, whose size is polynomial in the number of constraints defining the sets $\mathcal{Z}$, $\mathcal{E}$, $\mathcal{W}$ and $\mathcal{H}$. A procedure for doing this can be found, for example, in [6, 19], and so is not repeated here.

In particular, it is important to note that it is *not* necessary to explicitly perform the Minkowski summation of error sets in the calculation of the time varying control law (30), since only the support functions of these sets is of interest. Given an initial error set $\mathcal{E}$ at time 0, one needs to calculate at each time k a feasible policy pair $(\mathbf{M},\mathbf{v}) \in \Pi_N^{of}(s(k), A_L^k\mathcal{E} \oplus \mathcal{E}_{k-1})$. In this case the vector δ_e in (48) can be written as

$$\delta_e = \max_{e\in A_L^k\mathcal{E}+\mathcal{E}_{k-1}} (P\mathbf{M}Q_e + R_e)e \tag{49}$$

$$= \max_{e\in\mathcal{E}} (P\mathbf{M}Q_e + R_e)A_L^k e + \max_{e\in\mathcal{E}_{k-1}} (P\mathbf{M}Q_e + R_e)e \tag{50}$$

$$= \max_{e\in\mathcal{E}} (P\mathbf{M}Q_e + R_e)A_L^k e + \sum_{i=0}^{k-1} \max_{e\in\mathcal{W}\oplus(-L\mathcal{H})} (P\mathbf{M}Q_e + R_e)A_L^i e, \tag{51}$$

and one may dualise each row of each component of this summation, forming a single linear program whose size increases polynomially with time k.

An identical procedure may be used in the computation of an element of the set $\Pi_N^{of}(s,\mathcal{E}_I)$ in the implementation of the time-invariant control law (41) (cf. Remark 8), resulting in a tractable LP of *fixed* and *finite* complexity, where once again it is not necessary to explicitly form the Minkowski sum in (36), and where the support function of $\mathcal{E}_I$ can be determined using an *implicit* representation of a Minkowski sum of a finite number of polytopes as in [33].

5.1 Numerical Example

We consider the discrete-time system

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.2 \\ 1 \end{bmatrix} u + w \tag{52}$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x + \eta \tag{53}$$

with stable feedback gain K and observer gain L chosen as $K = [-0.75 \;\; -1.85]$ and $L = [1.15 \;\; 0.65]'$. The sets $\mathcal{Z}$, $\mathcal{E}$, W and H are defined as

$$Z := \left\{ (x,u) \in \mathbb{R}^2 \times \mathbb{R} \;\middle|\; \begin{array}{r} -3 \leq x_1 \leq 25 \\ -3 \leq x_2 \leq 25 \\ |u| \leq 5 \end{array} \right\} \tag{54}$$

$$\mathcal{E} := \left\{ e \in \mathbb{R}^2 \mid \|e\|_\infty \leq 0.4 \right\} \tag{55}$$

$$W := \left\{ w \in \mathbb{R}^2 \mid \|w\|_\infty \leq 0.1 \right\} \tag{56}$$

$$H := \left\{ \eta \in \mathbb{R}^2 \mid \|\eta\|_\infty \leq 0.1 \right\} \tag{57}$$

where x_i is the i^{th} element of x. In order to obtain the set X_f, we calculate the maximal RPI set compatible with Z for the system (23) using the method of [22, Alg. 6.2]. We consider the set of feasible initial state estimates $\mathcal{S}_i^{of}$ (equivalently $\mathcal{S}_i^{ef}$) for this system. For comparison, we also consider the sets S_i^K for which a feasible control policy can be found when the policy is parameterized in terms of perturbations to a *fixed* state feedback gain, such that $u_j = Ks_j + c_j$. Recall that $S_i^K \subseteq S_i^{of}$ for all $i \in \mathbb{N}$ (cf. Remark 1). The resulting sets of feasible initial state estimates for this system are shown in Figure 1.

6 Conclusions

The main contribution of this paper is to propose a new class of time-invariant receding horizon output feedback control laws for control of linear systems subject to bounded disturbances that guarantee robust constraint satisfaction for the resulting closed-loop system for all time. The proposed method is based on a fixed linear state observer combined with optimisation over the class of feedback policies which are affine in the estimated system state; this problem is non-convex, but can be convexified using an appropriate reparameterisation. As a consequence, receding horizon control laws in the proposed class can be computed using standard techniques in convex optimisation, while providing a larger region of attraction than methods based on calculating control perturbations to a static linear feedback law.

We have only considered the problem of finding a *feasible* control policy at each time, without regard to optimality. It is possible to define a variety of cost functions to motivate the selection from amongst this feasible set of policies, and we have not addressed any stability results which may be derived based on this selection; see, however, [18, 19, 21] for related results in the state feedback case.

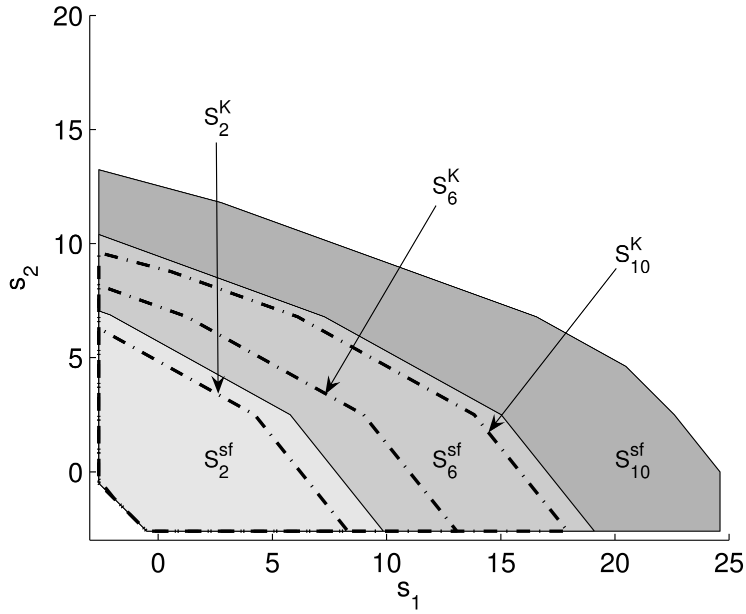

Figure 1: Feasible initial state estimate sets $\mathcal{S}_i^{of}$ and $\mathcal{S}_i^K$ for $i \in \{2, 6, 10\}$

A Matrix Definitions

Define $\mathfrak{A} \in \mathbb{R}^{n(N+1)\times n}$ and $\mathfrak{E} \in \mathbb{R}^{n(N+1)\times nN}$ as

$$\mathfrak{A}(L) := \begin{bmatrix} I_n \\ (A-LC) \\ (A-LC)^2 \\ \vdots \\ (A-LC)^N \end{bmatrix}, \quad \mathfrak{E}(L) := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ I_n & 0 & \cdots & 0 \\ (A-LC) & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (A-LC)^{N-1} & (A-LC)^{N-2} & \cdots & I_n \end{bmatrix}$$

so that $\mathbf{A} := \mathfrak{A}(0)$, $\mathbf{E} := \mathfrak{E}(0)$, $\Phi := \mathfrak{A}(L)$ and $\Gamma := \mathfrak{E}(L)$ The matrices $\mathcal{B} \in \mathbb{R}^{nN\times mN}$, $\mathcal{L} \in \mathbb{R}^{nN\times pN}$, $\mathbf{B} \in \mathbb{R}^{n(N+1)\times mN}$, and $\mathbf{C} \in \mathbb{R}^{pN\times n(N+1)}$ are defined as $\mathcal{B} := (I_N \otimes B)$, $\mathcal{L} := (I_N \otimes L)$, $\mathbf{B} := \mathbf{E}\mathcal{B}$ and $\mathbf{C} := [(I_N \otimes C)\ 0]$ respectively.

If the constraint set $\mathcal{Z}$ is polytopic and defined as in (46), then we define the matrices in (48) as $P := (S\mathbf{B} + U)$, $Q_e := \mathbf{C}\Phi$, $R_e := (S\mathbf{E}\mathcal{L}\mathbf{C} + T)\Phi$, $Q_w := \mathbf{C}\Gamma$, $R_w := (S\mathbf{E}\mathcal{L}\mathbf{C} + T)\Gamma$, $Q_\eta := (I - \mathbf{C}\Gamma\mathcal{L})$ and $R_\eta := (S\mathbf{E}(I - \mathcal{L}\mathbf{C}\Gamma) - T\Gamma)\,\mathcal{L}$.

A bit of algebra confirms that the matrix identities $\mathbf{E} = (I + \mathbf{E}\mathcal{L}\mathbf{C})\Gamma$ and $\mathbf{A} = (I + \mathbf{E}\mathcal{L}\mathbf{C})\Phi$ hold, so that one may also use the equivalent matrix definitions $R_e := S\mathbf{A} - (S-T)\Phi$, $R_w := S\mathbf{E} - (S-T)\Gamma$ and $R_\eta := (S-T)\Gamma\mathcal{L}$ above.

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