Characterization of the solution to a constrained H_∞ optimal control problem [1]

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Abstract

This paper obtains an explicit solution to a finite horizon min-max optimal control problem where the system is linear and discrete-time with control and state constraints, and the cost quadratic; the disturbance is negatively costed, as in the standard H_∞ problem, and is constrained. The cost is minimized over control policies and maximized over disturbance sequences so that the solution yields a feedback control. It is shown that, under certain conditions, the value function is piecewise quadratic and the optimal control policy piecewise affine, being quadratic and affine, respectively, in polyhedra that partition the domain of the value function.

Key words: min-max, constrained, H_∞, parametric optimization, optimal control.

1 Introduction

1.1 Background

Explicit solutions to constrained optimal control problems appeared in the papers [1–3] that deal with the constrained linear-quadratic problem, in the papers [3–7] and thesis [8] that deal with hybrid or piecewise affine systems, and in papers that deal with min-max optimal control problems [9–13]. In these papers it is shown that the value function is piecewise affine or piecewise quadratic (depending on the nature of the cost function in the optimal control problem) and the control law is piecewise affine, being quadratic or affine in polytopes that constitute a polytopic partition of the domain of the value function. When disturbances are present, it is necessary to compute the solution sequentially using dynamic programming as in [14, 15]; papers [10, 11, 13, 16] (that deal with state constraints) give recursions for the domains of the value functions. In this paper, which is motivated by recent research on H_∞ model predictive control [17–25], we characterize the solution to a constrained, min-max optimal control problem (in the sense of determining its most important properties) and discuss its use in receding horizon control. The term H_∞ is used somewhat loosely since we consider the min-max problem with fixed ρ (see (1.4) below). We consider, therefore, the problem of controlling a linear, discrete-time system described by

$$x^+ = Ax + Bu + Gw, \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the control and $w \in \mathbb{R}^p$ an additive disturbance (the 'adversary'); x^+ is the successor state. We frequently write the system dynamics in (1.1) in the form

$$x^+ = f(x, u, w)$$

where $f(x, u, w) \triangleq Ax+Bu+Gw$. The system is subject to hard control and state constraints $u \in U$, $x \in X$ where $U \subset \mathbb{R}^m$ and $X \subset \mathbb{R}^n$ are polytopes; each set contains the origin in its interior (the assumption that X is a polytope rather than a polyhedron [3] is made for simplicity). The disturbance w is constrained to lie in the polytope $W \subset \mathbb{R}^p$ that contains the origin.

Let $\pi \triangleq \{\mu_0(\cdot), \mu_1(\cdot), \dots, \mu_{N-1}(\cdot)\}$ denote a control policy (sequence of control *laws*) over horizon N and let

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[3] In this paper, a polyhedron is a closed set described by a finite set of linear inequalities; a polytope is a bounded polyhedron and is, therefore, compact.

$\mathbf{w} \triangleq \{w_0, w_1, \ldots, w_{N-1}\}$ denote a sequence of disturbances. Also, let $\phi(i; x, \pi, \mathbf{w})$ denote the solution of (1.1) when the initial state is x at time 0, the control policy is π and the disturbance sequence is $\mathbf{w}$, so that $\phi(i; x, \pi, \mathbf{w})$ is the solution, at time i of

$$x_{i+1} = Ax_i + B\mu_i(x_i) + Gw_i, \; x_0 = x \tag{1.2}$$

The cost $V_N(x, \pi, \mathbf{w})$, if the initial state is x, the control policy π and the disturbance sequence $\mathbf{w}$, is

$$V_N(x, \pi, \mathbf{w}) \triangleq \sum_{i=0}^{N-1} \ell(x_i, u_i, w_i) + V_f(x_N) \tag{1.3}$$

where, for all i, $x_i \triangleq \phi(i; x, \pi, \mathbf{w})$ and $u_i \triangleq \mu_i(x_i)$; $V_f(\cdot)$ is a terminal cost that may be chosen, together with a terminal constraint set X_f defined below, to ensure stability of the resultant receding horizon controller (see §7). The stage cost $\ell(\cdot)$ is a quadratic function, positive definite in x and u, and negative definite in w:

$$\ell(x, u, w) \triangleq (1/2)|x|_Q^2 + (1/2)|u|_R^2 - (\rho^2/2)|w|^2 \tag{1.4}$$

where $\rho > 0$, $|z|^2 \triangleq z'z$, $|z|_Z^2 \triangleq z'Zz$, and Q and R are positive definite. The stage cost may be expressed as

$$\ell(x, u, w) \triangleq (1/2)|y|^2 - (\rho^2/2)|w|^2, \; y \triangleq Hz \tag{1.5}$$

where $z \triangleq (x, u)$ and H is a suitably chosen matrix ((x, u) should be interpreted as a column vector $(x', u')'$ in matrix expressions). The terminal cost $V_f(\cdot)$ is either a quadratic function $V_f(x) \triangleq (1/2)|x|_{P_f}^2$ in which P_f is positive definite, or a strictly convex *piecewise quadratic* function (see Definition 2). The optimal control problem $\mathbb{P}_N(x)$ that we consider is

$$\mathbb{P}_N(x): \qquad V_N^0(x) = \inf_{\pi \in \Pi_N(x)} \max_{\mathbf{w} \in \mathcal{W}} V_N(x, \pi, \mathbf{w}) \tag{1.6}$$

where $\mathcal{W} \triangleq W^N$, is the set of admissible disturbance sequences, and $\Pi_N(x)$ is the set of admissible policies, i.e. those policies that satisfy, for all $\mathbf{w} \in \mathcal{W} \triangleq W^N$, the state and control constraints, and the terminal constraint $x_N \in X_f$. Inclusion of the hard disturbance constraint $\mathbf{w} \in \mathcal{W}$ is required when state constraints are present since, otherwise, we can expect that, for any policy π chosen by the controller, there exists a disturbance sequence $\mathbf{w}$ that transgresses the state constraint. The terminal constraint set X_f is a polytope, containing the origin in its interior, that satisfies $X_f \subseteq X$, ensuring satisfaction of the state constraint at time N. Hence the set of admissible policies is

$$\Pi_N(x) \triangleq \{\pi \mid \phi(i; x, \pi, \mathbf{w}) \in X, \; \mu_i(\phi(i; x, \pi, \mathbf{w})) \in U, \; \forall i \in \{0\} \cup \mathcal{I}_{N-1}, \phi(N; x, \pi, \mathbf{w}) \in X_f, \; \forall \mathbf{w} \in \mathcal{W}\} \tag{1.7}$$

where, for each integer J, $\mathcal{I}_J \triangleq \{1, \ldots, J\}$. Let $\mathcal{X}_N$ denote the set of initial states for which a solution to $\mathbb{P}_N(x)$ exists (the domain of $V_N^0(\cdot)$, the controllability set), i.e. $\mathcal{X}_N \triangleq \{x \mid \Pi_N(x) \neq \emptyset\}$.

1.2 Outline of the paper

Because of uncertainty (in the form of the additive disturbance w), the solution to the problem must be obtained by dynamic programming. In §2, we present the dynamic programming equations for the constrained min-max problem. At each time-to-go j (j ranges from 0 to N), a min-max problem must be solved; this may be decomposed into a max problem $\mathbb{P}_{\max}(z)$ followed by a min problem $\mathbb{P}_{\min}(x)$. Also in §2 we define two operators $\Gamma_{\mathcal{Z}}$ and Ψ mapping value functions into value functions; showing that the value functions of the H_∞ problem all have a certain property may be done by showing that this property is invariant under these operators. In §3 we show that certain properties, such as continuity, are invariant under these two operators. In §4 we present an improved algorithm for parametric *piecewise quadratic* programming and show that the *piecewise quadratic* property is invariant under these two operators if certain conditions are satisfied. The main obstacle to our program is that the operator Ψ requires the function it operates on to be continuously differentiable for invariance of the piecewise quadratic property; we show that the continuously differentiable property is invariant under $\Gamma_{\mathcal{Z}}$ if state constraints are absent or a certain assumption **A1** is satisfied. We obtain the solution to the H_∞ problem in §5 when state constraints are absent, and in §6 when assumption **A1** is satisfied. We show in §7 how our results may be used in receding horizon control. Some conclusions are drawn in §8. Lengthier proofs of results are given in the appendix.

2 Dynamic Programming for Constrained Problems

The solution to $\mathbb{P}_N(x)$ may be obtained as follows. For all $j \in \mathbb{N}_+ \triangleq \{1, 2, \ldots\}$, let the partial return function $V_j^0(\cdot)$ be defined as in (1.6) with j replacing N and let $\mathcal{X}_j$ (the controllability set) denote the domain of $V_j^0(\cdot)$; here j is "time-to-go". Then the sequences $\{V_j^0(\cdot), \kappa_j(\cdot), \mathcal{X}_j\}$, where $\kappa_j(\cdot)$ denotes the optimal control law $\mu_{N-j}^0(\cdot)$ at time $i = N - j$, may be calculated recursively as follows [10,16]:

$$V_j^0(x) = \min_{u \in U} \max_{w \in W} \{\ell(x, u, w) + V_{j-1}^0(f(x, u, w)) \mid f(x, u, W) \subseteq \mathcal{X}_{j-1}\} \tag{2.1}$$

$$\kappa_j(x) = \arg \min_{u \in U} \max_{w \in W} \{\ell(x, u, w) + V_{j-1}^0(f(x, u, w)) \mid f(x, u, W) \subseteq \mathcal{X}_{j-1}\} \tag{2.2}$$

$$\mathcal{X}_j = \{x \in X \mid \exists u \in U \text{ such that } f(x, u, W) \subseteq \{\mathcal{X}_{j-1}\} \tag{2.3}$$

with boundary conditions $V_0^0(x) = V_f(x), \mathcal{X}_0 = X_f \subseteq X$; $f(x,u,W) \triangleq \{f(x,u,w) \mid w \in W\}$; $\kappa_j(\cdot)$ is point valued due to strict convexity. For each integer j let $\mathcal{Z}_j \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be defined by

$$\mathcal{Z}_j \triangleq \{(x,u) \in X \times U \mid f(x,u,W) \subseteq \mathcal{X}_{j-1}\} \tag{2.4}$$

so that, from (2.3), $\mathcal{X}_j = \mathrm{Proj}_X \mathcal{Z}_j$ where, for all $\mathcal{Z} \subset X \times U$, Proj_X denotes the projection operator defined by $\mathrm{Proj}_X \mathcal{Z} = \{x \in X \mid \exists u \in U \text{ such that } (x,u) \in \mathcal{Z}\}$. Similarly, if Φ is a set in the product space $X \times U \times W$, $\mathrm{Proj}_Z \Phi$ denotes the set $\{z \in X \times U \mid \exists w \in W \text{ such that } (z,w) \in \Phi\}$. To analyze $\mathbb{P}_N(x)$ it is convenient to introduce the functions $J_j^0(\cdot)$, $j = 1, 2, \ldots$, defined by

$$J_j^0(x,u) \triangleq \max_{w \in W}\{\ell(x,u,w) + V_{j-1}^0(f(x,u,w))\} \tag{2.5}$$

The recursive equations (2.1)-(2.3) may therefore be rewritten as

$$V_j^0(x) = \min_{u \in U}\{J_j^0(x,u) \mid (x,u) \in \mathcal{Z}_j\}, \tag{2.6}$$

$$J_j^0(x,u) \triangleq \max_{w \in W}\{\ell(x,u,w) + V_{j-1}^0(f(x,u,w))\}, \tag{2.7}$$

$$\kappa_j(x) = \arg\min_{u \in U}\{J_j^0(x,u) \mid (x,u) \in \mathcal{Z}_j\}, \tag{2.8}$$

$$\nu_j(x,u) = \arg\max_{w \in W}\{\ell(x,u,w) + V_{j-1}^0(f(x,u,w))\}, \tag{2.9}$$

with endpoint conditions $V_0(\cdot) = V_f(\cdot)$, $\mathcal{X}_0 = X_f$. Under our assumptions the sets $\mathcal{X}_j$ and $\mathcal{Z}_j$ are compact. If $\mathcal{X}_0 = X_f$ is robust control invariant [4], the sets $\mathcal{X}_j$ are nested ($\mathcal{X}_j \supseteq \mathcal{X}_{j-1}$ for all $j \geq 1$) and robust control invariant.

To aid exposition of our results, we define two operators mapping value functions into value functions together with their associated minimizer or maximizer operators. For each polyhedral subset $\mathcal{Z}$ of $\mathbb{R}^n \times \mathbb{R}^m$, the first operator $\Gamma_{\mathcal{Z}}$, and its associated minimizer operator $\gamma_{\mathcal{Z}}$ are defined by:

$$\Gamma_{\mathcal{Z}}(J(\cdot))(x) \triangleq \min_{u \in U}\{J(x,u) \mid (x,u) \in \mathcal{Z}\}, \tag{2.10}$$

$$\gamma_{\mathcal{Z}}(J(\cdot))(x) \triangleq \arg\min_{u \in U}\{J(x,u) \mid (x,u) \in \mathcal{Z}\}. \tag{2.11}$$

Let Γ and γ denote the operators $\Gamma_{\mathcal{Z}}$ and $\gamma_{\mathcal{Z}}$ defined by (2.10) and (2.11), when $\mathcal{Z} = \mathbb{R}^n \times U$, so that $\Gamma(J(\cdot))(x) \triangleq \min_{u \in U} J(x,u)$ and $\gamma(J(\cdot))(x) \triangleq \arg\min_{u \in U} J(x,u)$. The second operator Ψ and its associated maximizer ψ are defined by

$$\Psi(V(\cdot))(z) \triangleq \max_{w \in W}\{\ell(z,w) + V(f(z,w))\}, \tag{2.12}$$

$$\psi(V(\cdot))(z) \triangleq \arg\max_{w \in W}\{\ell(z,w) + V(f(z,w))\}. \tag{2.13}$$

[4] See definition 5 in §7.

where $z \triangleq (x,u)$. The operators $\Gamma_{\mathcal{Z}}$ and $\gamma_{\mathcal{Z}}$ define the value function $V(\cdot)$ and minimizer $\kappa(\cdot)$ of the prototype minimization problem $\mathbb{P}_{\min}(x)$ defined by:

$$V(x) = \min_u\{J(x,u) \mid (x,u) \in \mathcal{Z}\} \tag{2.14}$$

$$\kappa(x) = \arg\min_u\{J(x,u) \mid (x,u) \in \mathcal{Z}\} \tag{2.15}$$

in the sense that $V(\cdot) = \Gamma_{\mathcal{Z}}(J(\cdot)), \kappa(\cdot) = \gamma_{\mathcal{Z}}(J(\cdot))$.

Similarly, the operators Ψ and ψ define the value function $J(\cdot)$ and maximizer $\nu(\cdot)$ of the prototype maximization problem $\mathbb{P}_{\max}(z)$ defined by:

$$J(z) = \max_w\{\ell(z,w) + V(f(z,w)) \mid w \in W\} \tag{2.16}$$

$$\nu(z) = \arg\max_w\{\ell(z,w) + V(f(z,w)) \mid w \in W\} \tag{2.17}$$

in the sense that $J(\cdot) = \Psi(V(\cdot)), \nu(\cdot) = \psi(V(\cdot))$. In terms of these operators, the dynamic programming equations (2.6)– (2.9) may be expressed as

$$V_j^0(\cdot) = \Gamma_{\mathcal{Z}_j}(J_j^0(\cdot)), \quad \kappa_j(\cdot) = \gamma_{\mathcal{Z}_j}(J_j^0(\cdot)), \tag{2.18}$$

$$J_j^0(\cdot) = \Psi(V_{j-1}^0(\cdot)), \quad \nu_j(\cdot) = \psi(V_{j-1}^0(\cdot)), \tag{2.19}$$

and the dynamic programming recursion (2.1) may be written

$$V_j^0(\cdot) = (\Gamma_{\mathcal{Z}_j} \circ \Psi)(V_{j-1}^0(\cdot)) \tag{2.20}$$

where $\circ$ denotes composition. Showing that property A is possessed by *all* the value functions $V_j^0(\cdot)$ is equivalent to showing that this property is *invariant* under the operator $\Gamma_{\mathcal{Z}} \circ \Psi$. Then, if $V_{j-1}^0(\cdot)$ possesses property A, so does $V_j^0(\cdot) = (\Gamma_{\mathcal{Z}_j} \circ \Psi)(V_{j-1}^0(\cdot))$; by induction, $V_1^0(\cdot)$, $V_2^0(\cdot), \ldots, V_N^0(\cdot)$ all have property A if the terminal cost $V_f(\cdot)$ does.

3 Invariance properties of $\Gamma_{\mathcal{Z}}$ and Ψ

In the following section, we show that basic properties (such as continuity, differentiability and convexity) are invariant under $\Gamma_{\mathcal{Z}}$ and Ψ and, hence under the dynamic programming recursion $\Gamma_{\mathcal{Z}} \circ \Psi$. Then, in §4, we establish the key property, the *piecewise quadratic* property, is invariant (under certain conditions) using a new technique for parametric piecewise quadratic programming. We require the following definitions in the sequel:

Definition 1 *A set $\mathcal{P}^{\mathcal{Z}} = \{P_i^{\mathcal{Z}} \mid i \in \mathcal{J}^{\mathcal{Z}}\}$, for some index set $\mathcal{J}^{\mathcal{Z}}$, is called a polyhedral (polytopic) partition of a polyhedral (polytopic) set $\mathcal{Z}$ if $\mathcal{Z} = \cup_{i \in \mathcal{J}^{\mathcal{Z}}} P_i^{\mathcal{Z}}$, and the sets $P_i^{\mathcal{Z}}$, $i \in \mathcal{J}^{\mathcal{Z}}$ are polyhedra (polytopes) with non-empty interiors relative to $\mathcal{Z}$ that are non-intersecting.*

Definition 2 *A function $J : \mathcal{Z} \to \mathbb{R}$ is said to be piecewise quadratic on a polyhedral (polytopic) partition $\mathcal{P}^{\mathcal{Z}} =$*

$\{P_i^{\mathcal{Z}} \mid i \in \mathcal{J}^{\mathcal{Z}}\}$ *of* $\mathcal{Z}$ *if it satisfies*

$$J(z) = J_i(z) \triangleq (1/2)|z|^2_{Q_i} + q_i'z + s_i, \qquad \forall z \in P_i^{\mathcal{Z}},\ i \in \mathcal{J}^{\mathcal{Z}}$$

for some Q_i, q_i, s_i, $i \in \mathcal{J}^{\mathcal{Z}}$*. Similarly, a function* $\kappa : \mathcal{Z} \to U$ *is said to be piecewise affine on a polyhedral partition* $\mathcal{P} = \{P_i^{\mathcal{Z}} \mid i \in \mathcal{J}^{\mathcal{Z}}\}$ *of* $\mathcal{Z}$ *if it satisfies*

$$\kappa(z) = K_i z + k_i, \qquad \forall z \in P_i^{\mathcal{Z}},\ i \in \mathcal{J}^{\mathcal{Z}},$$

for some K_i, k_i, $i \in \mathcal{J}^{\mathcal{Z}}$.

3.1 Basic properties of the operator $\Gamma_{\mathcal{Z}}$

We first show that the property of strict convexity is invariant under $\Gamma_{\mathcal{Z}}$.

Proposition 1 *Suppose* $\mathcal{Z}$ *is a convex and closed subset of* $\mathbb{R}^n \times U$*, and that* $J : \mathcal{Z} \to \mathbb{R}$ *is strictly convex. Then the value function* $V(\cdot) = \Gamma_{\mathcal{Z}}(J(\cdot))$ *is strictly convex with domain* $\mathcal{X}$*. If, in addition,* $J(\cdot)$ *is continuous, then, for all* $x \in \mathcal{X} = \mathrm{Proj}_X \mathcal{Z}$*, the solution* $\kappa(x)$ *to* $\mathbb{P}_{\min}(x)$ *exists and is unique.*

Proof: Convexity of $V(\cdot) = \Gamma_{\mathcal{Z}}(J(\cdot))$ given $J(\cdot)$ is convex is established in [26], §3.2.5; the extension to the result that $V(\cdot) = \Gamma_{\mathcal{Z}}(J(\cdot))$ is strictly convex if $J(\cdot)$ is strictly convex is trivial. ■

Our next result has simpler hypotheses than previous versions of this result (for example, $\mathcal{Z}$ is not required to have an interior and non-degeneracy conditions on multipliers are not required) and makes a stronger assertion: continuity is invariant under $\Gamma_{\mathcal{Z}}$ in $\mathcal{X}$ (rather than in the interior of $\mathcal{X}$). The existence and continuity of $\kappa(\cdot)$ follows.

Theorem 1 *Suppose* $\mathcal{Z}$ *is a polyhedron in* $\mathbb{R}^n \times U$ *and that* $J : \mathcal{Z} \to \mathbb{R}$ *is continuous. Then, for all* $x \in \mathcal{X} = \mathrm{Proj}_X \mathcal{Z}$*, the solution* $\kappa(x)$ *to* $\mathbb{P}_{\min}(x)$ *exists and the value function* $V(\cdot) = \Gamma_{\mathcal{Z}}(J(\cdot))$ *is continuous with domain* $\mathcal{X}$*. If, in addition,* $\kappa(x)$ *is unique for each* $x \in \mathcal{X}$*, then* $\kappa(\cdot) = \gamma_{\mathcal{Z}}(J(\cdot))$ *is continuous on* $\mathcal{X}$*.*

The requirement that the piecewise quadratic property is invariant under the operator Ψ (that arises in the max problem $\mathbb{P}_{\max}(z)$) forces us to make the strong demand (as we show later) that continuous differentiability is invariant under $\Gamma_{\mathcal{Z}}$. We can meet this demand in two ways.

Firstly: assume $X = X_f = \mathbb{R}^n$ so that $\mathcal{Z} = \mathbb{R}^n \times U$. Under this assumption, $\mathbb{P}_{\min}(x)$ becomes the simpler problem $V(x) = \min_{u \in U} J(x, u)$. The dynamic programming recursion then yields $\mathcal{X}_j = \mathbb{R}^n$, $\mathcal{Z}_j = \mathbb{R}^n \times U$, $\Gamma_{\mathcal{Z}_j} = \Gamma$ for all $j \in \mathbb{N}_+$. That continuous differentiability is invariant under Γ as shown in:

Theorem 2 *Suppose that* $X = X_f = \mathbb{R}^n$ *and that* $J : \mathbb{R}^n \times U \to \mathbb{R}$ *is continuously differentiable and strictly convex. Then the value function* $V(\cdot) = \Gamma(J(\cdot))$ *of* $\mathbb{P}_{\min}(x)$ *is continuously differentiable and strictly convex.*

Secondly, it is also possible to obtain invariance of continuous differentiability under $\Gamma_{\mathcal{Z}}$ (when state constraints are present) by requiring satisfaction of a certain condition. We delay proving this result until §4.

3.2 Basic properties of the operator Ψ

Our first result is that the property strict convexity and continuity is invariant under Ψ.

Proposition 2 *Suppose* $V : \mathcal{X} \to \mathbb{R}$ *where* $\mathcal{X} \subseteq X$ *and that, for each* $w \in W$*, the function* $z \mapsto V(z, w)$ *is strictly convex and continuous. Then, the value function* $J(\cdot) = \Psi(V(\cdot))$ *is strictly convex and continuous with domain* $\mathcal{Z} = \{z \in X \times U \mid f(z, W) \subseteq \mathcal{X}\}$*.*

Proof: Since $J(\cdot)$ is the maximum (over w in W) of a set of strictly convex functions $z \mapsto V'(z, w) \triangleq \ell(z, w) + V(f(z, w))$, $w \in W$, it is convex; strict convexity follows from the fact that $z \mapsto \ell(z, w)$ is uniformly strictly convex over $w \in W$. Since W is constant, the continuity of $J(\cdot) = \Psi(V(\cdot))$ follows from the continuity of $V'(\cdot)$ and the maximum theorem (Theorem 5.4.3 in [27]). ■

In order to show (in §4) that the *piecewise-quadratic property* is invariant under Ψ, we require that the function $w \mapsto V'(z, w) \triangleq \ell(z, w) + V(f(z, w))$ is strictly concave for suitably large ρ when $V : \mathcal{X} \to \mathbb{R}$ is piecewise quadratic on a polyhedral partition of the polyhedron $\mathcal{X}$, in which case the maximizer $\nu(\cdot) = \psi(V(\cdot))$ is unique and $J(\cdot) = \Psi(V(\cdot))$ is piecewise quadratic on a polyhedral partition of $\Phi \triangleq \{(z, w) \in (\mathbb{R}^n \times U) \times W \mid f(z, w) \in \mathcal{X}\}$. Generally $w \mapsto V'(z, w)$ (being the sum of a concave and a convex function) is not strictly concave so $J(\cdot) = \Psi(V(\cdot))$ is not necessarily piecewise quadratic. However, for suitably large ρ, the function $V'(\cdot)$ is strictly concave, if $V : \mathcal{X} \to \mathbb{R}$ is continuously differentiable and strictly convex, as we now establish.

Proposition 3 *Let* $\mathcal{X}$ *be a polyhedron in* $\mathbb{R}^n$ *containing the origin in its relative interior and suppose* $V : \mathcal{X} \to \mathbb{R}$ *is continuously differentiable, piecewise quadratic, and strictly convex. Then* $V'(\cdot)$ $(V'(z, w) \triangleq \ell(z, w) + V(Fz + Gw))$ *is continuously differentiable, strictly convex in* z *for each* $w \in W$*, and there exists a* $\rho^* > 0$ *such that, for all* $\rho \geq \rho^*$*,* $V'(\cdot)$ *is strictly concave in* w *for each* z *in* $\mathcal{Z} \triangleq \mathrm{Proj}_Z \Phi$*,* $\Phi \triangleq \{(z, w) \in (\mathbb{R}^n \times U) \times W \mid f(z, w) \in \mathcal{X}\}$*.*

The proof (given in the appendix) is constructive and yields a value for ρ^*. We can exploit Proposition 3 to show that the property of continuous differentiability is invariant under Ψ:

Proposition 4 *Suppose that $V(\cdot)$ in $\mathbb{P}_{\max}(z)$ is continuously differentiable, and strictly convex. Suppose also that $\rho \geq \rho^*$ where ρ^* is defined in Proposition 3. Then the value function $J(\cdot) = \Psi(V(\cdot))$ of $\mathbb{P}_{\max}$ is continuously differentiable.*

The proof of this result is almost identical to the proof of Theorem 2.

4 Parametric programming and invariance of the piecewise quadratic property

An important objective is to show that the *piecewise (pw) quadratic property* is invariant under the dynamic programming recursion. This result permits determination of explicit control using dynamic programming coupled with parametric programming, which is the tool we need.

4.1 The operator $\Gamma_{\mathcal{Z}}$ and problem $\mathbb{P}_{\min}(x)$

We first look at the parametric problem $\mathbb{P}_{\min}(x)$ defined in (2.14) under the assumption that $J(\cdot)$ is continuous piecewise quadratic on a polyhedral partition $\mathcal{P}^{\mathcal{Z}} = \{P_i^{\mathcal{Z}} \mid i \in \mathcal{J}^{\mathcal{Z}}\}$ of a polyhedron $\mathcal{Z}$; $J(z) = J_i(z) = (1/2)|z|^2_{Q_i} + q_i'z + s_i$ for all $z \in P_i^{\mathcal{Z}}$, all $i \in \mathcal{J}^{\mathcal{Z}}$ (here, the subscript i in $J_i(\cdot)$ is used to specify the cost in a particular polyhedron and not the cost at a particular time-to-go as in the dynamic programming recursion (2.7)).

4.1.1 Invariance of the pw quadratic property under $\Gamma_{\mathcal{Z}}$

Definition 3 *A polyhedron $P_i^{\mathcal{Z}}$ in a polyhedral partition $\mathcal{P}^{\mathcal{Z}} = \{P_i^{\mathcal{Z}} \mid i \in \mathcal{J}^{\mathcal{Z}}\}$ of a polyhedron $\mathcal{Z}$ is said to be* active *at $z \in \mathcal{Z}$ if $z = (x, u) \in P_i^{\mathcal{Z}}$. The set of polyhedra active at $z \in \mathcal{Z}$ is*

$$S(z) \triangleq \{i \in \mathcal{J}^{\mathcal{Z}} \mid z \in P_i^{\mathcal{Z}}\}.$$

A polyhedron $P_i^{\mathcal{Z}}$ in a polyhedral partition $\mathcal{P}^{\mathcal{Z}} = \{P_i^{\mathcal{Z}} \mid i \in \mathcal{J}^{\mathcal{Z}}\}$ of a polyhedron $\mathcal{Z}$ is said to be active *for Problem $\mathbb{P}_{\min}(x)$ if $(x, \kappa(x)) \in P_i^{\mathcal{Z}}$. The set of active polyhedra for $\mathbb{P}_{\min}(x)$ is $S^0(x)$ defined by*

$$S^0(x) \triangleq \mathcal{S}(x, \kappa(x)) = \{i \in \mathcal{J}^{\mathcal{Z}} \mid (x, \kappa(x)) \in P_i^{\mathcal{Z}}\}$$

where $\kappa(x)$, the solution of $\mathbb{P}_{\min}(x)$, is unique if $J(\cdot)$ is strictly convex.

Definition 4 *For each $\bar{x} \in \mathcal{X}$, the polyhedron $P_{\bar{x}}$ is defined by $P_{\bar{x}} \triangleq \bigcap_{i \in S^0(\bar{x})} P_i^{\mathcal{Z}}$.*

Because $J(\cdot)$ is continuous, for each $\bar{x} \in \mathcal{X}$, $J_i(z) = J(z)$ for all $i \in \mathcal{S}^0(\bar{x})$, all $z \in P_{\bar{x}}$. We now define, for each $\bar{x} \in \mathcal{X}$, the simpler problem $\mathbb{P}_{\bar{x}}(x)$, parameterized by $\bar{x}$ and defined by

$$V_{\bar{x}}(x) = \min_u \{J(x, u) \mid (x, u) \in P_{\bar{x}}\} \tag{4.1}$$

$$\kappa_{\bar{x}}(x) = \arg\min_u \{J(x, u) \mid (x, u) \in P_{\bar{x}}\}. \tag{4.2}$$

so that $V_{\bar{x}}(\cdot)$ is the value function, and $\kappa_{\bar{x}}(\cdot)$ the minimizer, for problem $\mathbb{P}_{\bar{x}}(x)$. By construction, $S(x, \kappa_{\bar{x}}(x)) = S^0(\bar{x})$ at all x such that $(x, \kappa_{\bar{x}}(x)) \in P_{\bar{x}}$. The question arises: how is the solution $\kappa(x)$ of the original problem $\mathbb{P}_{\min}(x)$ related to the solution $\kappa_{\bar{x}}(x)$ of the derived problem $\mathbb{P}_{\bar{x}}(x)$? This is answered by:

Proposition 5 *Suppose $J : \mathcal{Z} \to \mathbb{R}$ is continuous piecewise quadratic and strictly convex and that $\bar{x} \in \mathcal{X}$ is given. For all x such that $\mathcal{S}^0(x) = \mathcal{S}^0(\bar{x})$ (i.e. $(x, \kappa(x)) \in P_{\bar{x}}$) the following three statements are equivalent:*
(i) u is optimal for the original problem $\mathbb{P}_{\min}(x)$ ($u = \kappa(x)$).
(ii) u is optimal for the derived problem $\mathbb{P}_{\bar{x}}(x)$ ($u = \kappa_{\bar{x}}(x)$).
(iii) u is optimal for problems $\mathbb{P}_i(x)$: $\min_u\{J(x, u) \mid (x, u) \in P_i^{\mathcal{Z}}\}$ for all $i \in S^0(x)$.

Consider an arbitrary point $\bar{x} \in \mathcal{X}$. Then $\mathcal{S}^0(\bar{x})$ is the set of active polytopes at $(\bar{x}, \kappa(\bar{x}))$, and $P_{\bar{x}} = \cap\{P_i^{\mathcal{Z}} \mid i \in \mathcal{S}^0(\bar{x})\}$. The polyhedron $P_{\bar{x}}$ admits the representation

$$P_{\bar{x}} = \left\{ z \,\middle|\, \begin{array}{l} \tilde{M}_{\bar{x}} u = \tilde{N}_{\bar{x}} x + \tilde{p}_{\bar{x}} \\ \bar{M}_{\bar{x}} u \leq \bar{N}_{\bar{x}} x + \bar{p}_{\bar{x}} \end{array} \right\} \tag{4.3}$$

where, because of the definition of $P_{\bar{x}}$, equality constraints $\tilde{M}_{\bar{x}} u = \tilde{N}_{\bar{x}} x + \tilde{p}_{\bar{x}}$ that define the boundary common to $P_i^{\mathcal{Z}}, i \in \mathcal{S}^0(\bar{x})$ arise. Let $\bar{M}_{\bar{x}}^j$, $\bar{N}_{\bar{x}}^j$ and $\bar{p}_{\bar{x}}^j$ denote, respectively, the jth row of $\bar{M}_{\bar{x}}$, $\bar{N}_{\bar{x}}$ and $\bar{p}_{\bar{x}}$ and let $I^0(\bar{x})$ index those constraints in the *second* set of constraints in (4.3) that are active at $(\bar{x}, \kappa(\bar{x}))$; the set $I^0(\bar{x})$ is therefore defined by

$$I^0(\bar{x}) \triangleq \{j \mid \bar{M}_{\bar{x}}^j \kappa(\bar{x}) = \bar{N}_{\bar{x}}^j \bar{x} + \bar{p}_{\bar{x}}^j\}. \tag{4.4}$$

Let $M_{\bar{x}}$, $N_{\bar{x}}$ and $p_{\bar{x}}$ denote, respectively, the matrices with rows $\bar{M}_{\bar{x}}^j$, $\bar{N}_{\bar{x}}^j$ and $\bar{p}_{\bar{x}}^j$, $j \in I^0(\bar{x})$ and let $r_{\bar{x}}$ denote the row dimension of these matrices. Because $J(z) = J_i(z)$ for all $i \in \mathcal{S}^0(\bar{x})$, all $z \in P_{\bar{x}}$, we may now define, for each $\bar{x} \in \mathcal{X}$, the equality constrained problem $\mathbb{P}_{\bar{x}}^e(x)$ by

$$V_{\bar{x}}^e(x) = \min_u \{J(x, u) \mid \tilde{M}_{\bar{x}} u = \tilde{N}_{\bar{x}} x + \tilde{p}_{\bar{x}}, \; M_{\bar{x}} u = N_{\bar{x}} x + p_{\bar{x}}\}, \tag{4.5}$$

$$\kappa_{\bar{x}}^e(x) = \arg\min_u \{J(x, u) \mid \tilde{M}_{\bar{x}} u = \tilde{N}_{\bar{x}} x + \tilde{p}_{\bar{x}}, \; M_{\bar{x}} u = N_{\bar{x}} x + p_{\bar{x}}\}. \tag{4.6}$$

where $J_i(\cdot)$, for any $i \in \mathcal{S}^0(\bar{x})$, may be used in place of $J(\cdot)$. Hence $\mathbb{P}_{\bar{x}}^e(x)$ is a simple quadratic optimization problem with affine equality constraints; the solution to this problem is, as is well known [28]:

$$V^e_{\bar{x}}(x) = (1/2)x'Q_{\bar{x}}x + q'_{\bar{x}}x + s_{\bar{x}}, \tag{4.7}$$
$$\kappa^e_{\bar{x}}(x) = K_{\bar{x}}x + k_{\bar{x}} \tag{4.8}$$

for some $Q_{\bar{x}}$, $q_{\bar{x}}$, $s_{\bar{x}}$, $K_{\bar{x}}$ and $k_{\bar{x}}$. Since the control law $\kappa^e_{\bar{x}}(\cdot)$ ensures, by construction, that $M_{\bar{x}}\kappa^e_{\bar{x}}(x) = N_{\bar{x}}x+p_{\bar{x}}$ for all x such that $(x, \kappa^e_{\bar{x}}(x)) \in P_{\bar{x}}$, the cone of feasible directions $h \in \mathbb{R}^q$ at $u = \kappa^e_{\bar{x}}(x)$ is $\mathcal{F}_{\bar{x}} = \{h \mid \tilde{M}_{\bar{x}}h = 0,\ M_{\bar{x}}h \leq 0\}$. Then

$$PC_{\bar{x}} \triangleq \{[\tilde{M}'_{\bar{x}},\ -\tilde{M}'_{\bar{x}},\ M'_{\bar{x}}]\lambda \mid \lambda \geq 0\}, \tag{4.9}$$

is the polar cone [5] at 0 of the cone $\mathcal{F}_{\bar{x}}$. Because the gradient of the cost function $J(\cdot)$ is not necessarily continuous in $P_{\bar{x}}$, we define the polytope $X_{\bar{x}}$ as follows:

$$X_{\bar{x}} \triangleq \left\{ x \left| \begin{array}{l} \bar{M}^j_{\bar{x}}\kappa^e_{\bar{x}}(x) \leq \bar{N}^j_{\bar{x}}x + \bar{p}^j_{\bar{x}},\ \forall j \in \mathcal{I}_{r_{\bar{x}}} \setminus I^0(\bar{x}) \\ -\nabla_u J_i(x, \kappa^e_{\bar{x}}(x)) \in PC_{\bar{x}}\ \forall i \in \mathcal{S}^0(\bar{x}) \end{array} \right. \right\} \tag{4.10}$$

where $\mathcal{I}_{r_{\bar{x}}} \triangleq \{1, 2, \ldots, r_{\bar{x}}\}$. The convex hull of $\{\nabla_u J_i(x, \kappa^e_{\bar{x}}(x)),\ i \in \mathcal{S}^0(\bar{x})\}$ is the subgradient $\delta J(\cdot)$ of the convex function $J(\cdot)$ (that is not necessarily differentiable) at $(x, \kappa^e_{\bar{x}}(x))$. The first condition in (4.10) (that holds with equality for $j \in I^0(\bar{x})$) ensures satisfaction of the constraints defining $P_{\bar{x}}$ everywhere in $X_{\bar{x}}$ and the second (that may be expressed as $-\delta J(x, \kappa^e_{\bar{x}}(x)) \subset PC_{\bar{x}}$) is a necessary and sufficient condition for the optimality of $\kappa^e_{\bar{x}}(\cdot)$ in $X_{\bar{x}}$ (see, e.g., [29], §12.6 and [26], §4.2.3). Hence, in $X_{\bar{x}}$, the value function $V(\cdot)$ (for $\mathbb{P}_{\min}(x)$) is equal to the quadratic function $V^e_{\bar{x}}(\cdot)$ and the associated optimal control law $\kappa(\cdot)$ is equal to the affine function $\kappa^e_{\bar{x}}(\cdot)$. As before, the second constraint in the definition of $X_{\bar{x}}$ is a set of easily determined linear inequalities so that $X_{\bar{x}}$ is a polytope (polyhedral if $\mathcal{Z}$ is polyhedral). We have proven:

Theorem 3 *Suppose $J : \mathcal{Z} \to \mathbb{R}$ is continuous piecewise quadratic and strictly convex on a polyhedral (polytopic) partition $\mathcal{P}^{\mathcal{Z}} = \{P^{\mathcal{Z}}_i \mid i \in \mathcal{J}^{\mathcal{Z}}\}$ of the polyhedron (polytope) $\mathcal{Z}$. Then the value function $V(\cdot) = \Gamma_{\mathcal{Z}}(J(\cdot))$ is continuous piecewise quadratic and strictly convex and the optimal control law $\kappa(\cdot) = \gamma_{\mathcal{Z}}(J(\cdot))$ is piecewise affine on a polyhedral (polytopic) partition $\mathcal{P}^{\mathcal{X}} = \{P^{\mathcal{X}}_i \mid i \in \mathcal{J}^{\mathcal{X}}\}$ of $\mathcal{X} = \mathrm{Proj}_X \mathcal{Z}$ satisfying, respectively, $V(x) = V_{\bar{x}}(x)$ and $\kappa(x) = \kappa_{\bar{x}}(x)$ for all x in $P^{\mathcal{X}}_i$ where each $P^{\mathcal{X}}_i = X_{\bar{x}}$ for some $\bar{x} \in \mathcal{X}$.*

When $X = X_f = \mathbb{R}^n$, $\mathcal{Z} = \mathbb{R}^n \times U$ is polyhedral rather than polytopic; a piecewise quadratic (affine) characterization of $V(\cdot)$ ($\kappa(\cdot)$) on a polytopic subset S of $\mathcal{X} = \mathbb{R}^n$ may be obtained (see §4.2).

[5] The polar cone of a convex set C is the set $C^* = \{y \mid y'x \leq 0,\ \forall x \in C\}$.

4.1.2 *Invariance of continuous differentiability under $\Gamma_{\mathcal{Z}}$*

This has been established for the case $X = X_f = \mathbb{R}^n$ in Theorem 2. When state constraints are present, continuous differentiability is invariant under $\Gamma_{\mathcal{Z}}$ under a rather strong condition that we now state:

A1: The pair $(J(\cdot), \mathcal{Z})$ is such that any two adjacent polyhedra $P^{\mathcal{X}}_i = X_{\bar{x}_i}$ and $P^{\mathcal{X}}_j = X_{\bar{x}_j}$ in the polyhedral partition $\mathcal{P}^{\mathcal{X}}$ of $\mathcal{X}$ (see Theorem 3) satisfy $I^0_i \subset I^0_j$ or $I^0_j \subset I^0_i$ where, for each i, I^0_i indexes *all* the constraints in the definition of $P^{\mathcal{X}}_{\bar{x}_i}$ (see (4.3)) that are active at $(\bar{x}_i, \kappa(\bar{x}_i))$.

Theorem 4 *Suppose $J : \mathcal{Z} \to \mathbb{R}$ is continuously differentiable, piecewise quadratic and strictly convex on a polyhedral partition $\mathcal{P}^{\mathcal{Z}}$ of a polyhedron $\mathcal{Z}$. If $(J(\cdot), \mathcal{Z})$ satisfies assumption **A1**, then the value function $V(\cdot) = \Gamma_{\mathcal{Z}}(J(\cdot))$ is continuously differentiable, piecewise quadratic and strictly convex, and the optimal control law $\kappa(\cdot) = \gamma_{\mathcal{Z}}(J(\cdot))$ is continuous and piecewise affine on a polyhedral partition $\mathcal{P}^{\mathcal{X}} = \{P^{\mathcal{X}}_i \mid i \in \mathcal{J}^{\mathcal{X}}\}$ of $\mathcal{X} = \mathrm{Proj}_X \mathcal{Z}$ (each $P^{\mathcal{X}}_i = X_{\bar{x}}$ for some $\bar{x} \in \mathcal{X}$).*

Proof: An outline of the proof to this result is given in [16]; a full proof appears in [30]. A related result is given in [8] where continuous differentiability is established under a different hypothesis (a non-degeneracy condition that includes linear independence of the active constraints). ■

Assumption **A1** cannot be verified a priori and is, therefore, of limited use. However, the assumption suggests that continuous differentiability is possible and is satisfied in an illustrative example with state constraints provided in [30].

4.2 *The operator Ψ and problem $\mathbb{P}_{\max}(z)$*

We look now at the parametric problem $\mathbb{P}_{\max}(z)$ defined in (2.16) under the assumption that $V : \mathcal{X} \to \mathbb{R}$ is piecewise quadratic on a polytopic partition $\mathcal{P}^{\mathcal{X}} = \{P^{\mathcal{X}}_i \mid i \in \mathcal{J}^{\mathcal{X}}\}$ of a polytope $\mathcal{X}$; $V(x) = V_i(x) = (1/2)|x|^2_{Q_i} + q'_i x + s_i$ for some Q_i, q_i and s_i, for all $x \in P^{\mathcal{X}}_i$, all $i \in \mathcal{J}^{\mathcal{X}}$ and, in addition, is continuously differentiable and strictly convex. The following result, an analog of Theorem 3, shows that the *pw quadratic property* is invariant under Ψ if the operand is continuously differentiable:

Theorem 5 *Suppose $V : \mathcal{X} \to \mathbb{R}$ is continuously differentiable, piecewise quadratic and strictly convex on a polyhedral partition $\mathcal{P}^{\mathcal{X}} = \{P^{\mathcal{X}}_i \mid i \in \mathcal{J}^{\mathcal{X}}\}$ of the polyhedron $\mathcal{X}$. Then $V' : \Phi \to \mathbb{R}$, where $V'(z, w) \triangleq \ell(z, w) + V(f(z, w))$ and $\Phi \triangleq \{(z, w) \in (\mathbb{R}^n \times U) \times W \mid f(z, w) \in \mathcal{X}\}$, is continuously differentiable, piecewise quadratic and strictly convex in z on a polyhedral partition $\mathcal{P}^{\Phi}$ of* *the polyhedron Φ. Moreover, there exists a $\rho^* > 0$ such that $V'(\cdot)$ is strictly concave in w for all $\rho \geq \rho^*$; for all $\rho \geq \rho^*$, the value function $J(\cdot) = \Psi(V(\cdot))$ is continuous, piecewise quadratic and strictly convex, and the optimal control law $\nu(\cdot) = \psi(V(\cdot))$ continuous and piecewise affine, on a polyhedral partition $\mathcal{P}^{\mathcal{Z}} = \{P_i^{\mathcal{Z}} \mid i \in \mathcal{J}^{\mathcal{Z}}\}$ of $\mathcal{Z} = \mathrm{Proj}_Z \Phi$ (each $P_i^{\mathcal{Z}} = Z_{\bar{z}}$ for some $\bar{z} \in \mathcal{Z}$ where $Z_{\bar{z}}$ is defined similarly to* (4.10)*).*

Proof: The proof of this result is almost identical to the proof of Theorem 2 since $P_{\max}(x)$ may be expressed as $-\min_{w \in W}\{-V'(z,w)\}$. ∎

Reverse Transformation Algorithm for $\mathbb{P}_{min}(x)$

Proposition 5 motivates the following improved reverse transformation algorithm for the *piecewise quadratic* parametric problem $\min_u\{J(x,u) \mid (x,u) \in \mathcal{Z}\}$:

1. *Initialize:* Set $\mathcal{R} = \emptyset$.
2. *Update:* Select $\bar{x} \in \mathcal{X} \setminus \mathcal{R}$ ($\mathcal{X} = \mathrm{Proj}_X \mathcal{Z}$), solve $\mathbb{P}_{\min}(\bar{x})$ and determine $P_{\bar{x}}$. Determine the affine minimizer $\kappa_{\bar{x}}(\cdot)$, the quadratic value function $V_{\bar{x}}(\cdot)$, and the polytope $X_{\bar{x}}$. Set $V(\cdot) = V_{\bar{x}}(\cdot)$ and $\kappa(\cdot) = \kappa_{\bar{x}}(\cdot)$ on $X_{\bar{x}}$. Set $\mathcal{R} = X_{\bar{x}} \cup \mathcal{R}$.
3. *Iterate:* While $\mathcal{R} \neq \mathcal{X}$, repeat Step 2.

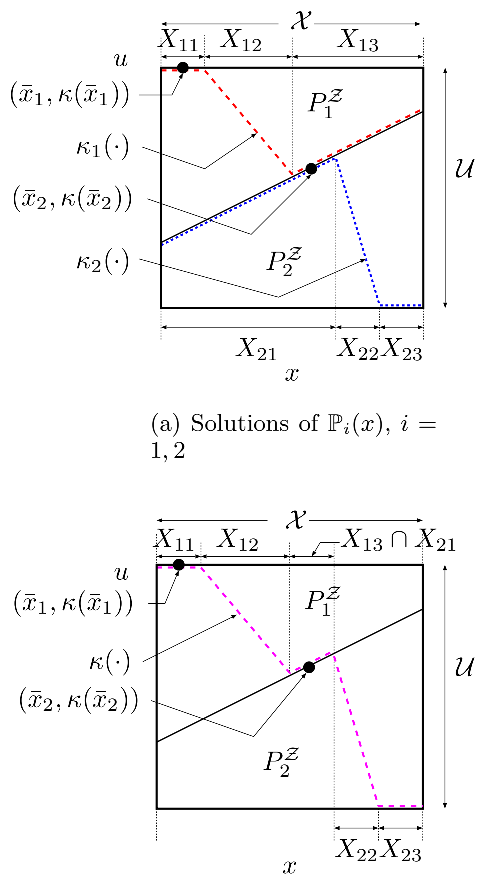
Consider the example shown in Figure 1. For $i = 1, 2$, the solution of problem $\mathbb{P}_i(x)$: $\min_u\{J(x,u) \mid (x,u) \in P_i^{\mathcal{Z}}\}$, is the piecewise affine function $\kappa_i(\cdot)$ defined on the polytopic partition $\{X_{i1}, X_{i2}, X_{i3}\}$ of $\mathcal{X}$. The original version of the algorithm [3, 8, 16] determines $\kappa_1(\cdot)$ and $\kappa_2(\cdot)$ shown in Figure 1(a) where two representative values $\bar{x}_1$ and $\bar{x}_2$ for $\bar{x}$ are indicated; these solutions overlap (at each $x \in \mathcal{X}$, both solutions exist) and further investigation is needed to choose the appropriate minimizer. The improved algorithm, in contrast, yields the single solution $\kappa(\cdot)$ shown in Figure 1(b). When $\mathcal{X} = \mathbb{R}^n$, the restriction of $V(\cdot)$ and $\kappa(\cdot)$ to any polytopic subset of $\mathcal{X}$ may be obtained by replacing $\mathcal{X}$ by S and $X_{\bar{x}}$ by $X_{\bar{x}} \cap S$ in the above algorithm.

5 H_∞ control; no state constraints

In this case, $X = X_f = \mathbb{R}^n$ so the only constraints are $u \in U$ and $w \in W$. Consequently $\mathcal{X}_j = \mathbb{R}^n$ for all j and the dynamic programming equations (2.1) - (2.3) are replaced by the conventional dynamic programming equations in which $\mathcal{X}_{j-1}$ is replaced by $\mathbb{R}^n$; equivalently, the new dynamic programming equations are given by (2.6)-(2.9) with the constraint $(x,u) \in \mathcal{Z}_j$ omitted. The prototype problems in the dynamic programming recursion become:

$$\mathbb{P}_{\min}(x): \qquad V(x) = \min_{u \in U} J(x,u) \tag{5.1}$$

$$\mathbb{P}_{\max}(z): \qquad J(z) = \max_{w \in W} \ell(z,w) + V(f(z,w)) \tag{5.2}$$



(b) Solution of $\mathbb{P}(x)$

Fig. 1. Reverse transformation algorithm

in which $V(\cdot)$ and $J(\cdot)$ in (5.1) represent, respectively, $V_j^0(\cdot)$ and $J_j^0(\cdot)$, and $J(\cdot)$ and $V(\cdot)$ in (5.2) represent, respectively, $J_j^0(\cdot)$ and $V_{j-1}^0(\cdot)$. Since $J(\cdot)$ is convex in u and $V'(\cdot)$ ($V'(z,w) \triangleq \ell(z,w) + V(f(z,w))$ is (under appropriate conditions) concave in w, the respective value functions have identical properties ($\max_w\{V'(z,w) \mid w \in W\} = -\min_w\{-V'(z,w) \mid w \in W\}$).

To establish that continuous differentiability, strict convexity and the piecewise quadratic property are all invariant under the dynamic programming recursion operator $\Gamma \circ \Psi$, we require:

Proposition 6 *There exists a $\rho^* > 0$ such that $V_j'(\cdot)$ is strictly concave in w for all $\rho \geq \rho^*$, all $j \in \{1, \ldots, N-1\}$.*

We can now obtain the properties of the solution to the H_∞ problem when there are no state or terminal constraints ($X = X_f = \mathbb{R}^n$).

Theorem 6 *Suppose the H_∞ problem $\mathbb{P}_N(x)$ has no state and terminal constraints, that $\rho \geq \rho^*$, and that $V_f(\cdot)$ is continuously differentiable, strictly convex, and* *piecewise quadratic on a polyhedral partition $\mathcal{P}^{X_f}$ of $X_f = \mathbb{R}^n$. Then, for each $j \in \{0, 1, \ldots, N\}$, the value functions $V_j^0(\cdot)$ of $\mathbb{P}_N(x)$ is continuously differentiable, strictly convex, and piecewise quadratic on a polyhedral partition of $\mathcal{X}_j = \mathbb{R}^n$.*

Proof: That continuous differentiability, strict convexity and the piecewise quadratic property are invariant under Γ is established in Theorem 2, Proposition 1, and Theorem 3 respectively. That these properties are invariant under Ψ is established in Proposition 4, Proposition 2 and Theorem 5 respectively. The invariance of these properties under $\Gamma \circ \Psi$ follows. It follows that, for each $j \in \{0, 1, \ldots, N\}$ the value function $V_j^0(\cdot)$ of $\mathbb{P}_j(x)$ is continuously differentiable, strictly convex, and piecewise quadratic on a polyhedral partition of $\mathcal{X}_j = \mathbb{R}^n$. ∎

Restrictions of the functions $V_j^0(\cdot)$ and $\kappa_j(\cdot)$ to polytopic subsets S_j of $\mathbb{R}^n$ satisfying $f(S_j, U, W) \subseteq S_{j-1}$, $j = 1, \ldots, N$ may be obtained using the modification to the reverse transformation algorithm described in §4; the condition imposed on the sets S_j (that may be arbitrarily large) ensures that the value functions and control laws are all well defined.

6 H_∞ control; state constraints

In this case, we require that **A1** (see §4.1.2) is satisfied by $(J_j^0(\cdot), \mathcal{Z}_j)$ for each $j \in \{1, \ldots, N\}$. Since the sequence $\{(J_j^0(\cdot), \mathcal{Z}_j)\}$ is affected by ρ, we must make the additional strong assumption:

A2: There exists a $\rho^* > 0$ such that, for each $j \in \{1, \ldots, N\}$, $V_j'(\cdot) = \ell(\cdot) + V_{j-1}^0 \circ f(\cdot)$ is strictly concave in w and **A1** is satisfied by $(J_j^0(\cdot), \mathcal{Z}_j)$ for all $\rho \geq \rho^*$.

We then have:

Theorem 7 *Suppose that $V_f(\cdot)$ in the H_∞ problem is continuously differentiable, strictly convex, and continuous piecewise quadratic on a polytopic partition $\mathcal{P}^{\mathcal{X}_0}$ of a polytope $\mathcal{X}_0 \triangleq X_f \subset \mathbb{R}^n$. Suppose also that assumption **A2** is satisfied. Then, with $\rho = \rho^* > 0$, for each $j \in \{1, \ldots, N\}$, the value function $V_j^0(\cdot)$ is continuously differentiable, strictly convex, and continuous piecewise quadratic on a polytopic partition $\mathcal{P}^{\mathcal{X}_j}$ of $\mathcal{X}_j$, and the optimal control law $\kappa_j(\cdot)$ is continuous and piecewise affine on the same polytopic partition $\mathcal{P}^{\mathcal{X}_j}$ of $\mathcal{X}_j$.*

Proof: Invariance of continuous differentiability under $\Gamma_{\mathcal{Z}}$ is established in Theorem 3, invariance of strict convexity in Proposition 1 and invariance of the piecewise quadratic property in 3. The remainder of the proof is the same as the proof of Theorem 6. ∎

An illustrative example with state constraints is provided in [30].

7 H_∞ receding horizon control

7.1 Introduction

Since we make use, in this section, of the solution for the infinite horizon, linear unconstrained H_∞ problem, we assume, in the sequel, that (A, B) is stabilizable and that (C, A, B) has no zeros on the unit circle where $Q = C'C$. Since Q is assumed to be positive definite, (C, A) is detectable. These conditions, and the fact that R is assumed positive definite, ensure that the conditions assumed in [31], Appendix B, are satisfied for the full information case. Hence there exists a $\tilde{\rho} > 0$ such that a positive definite solution P_f to the associated (generalized) H_∞ algebraic Riccati equation exists for all $\rho > \tilde{\rho}$; suppose $\rho_f > \tilde{\rho}$, that P_f is the solution of the H_∞ algebraic Riccati equation with $\rho = \rho_f$, and that the associated optimal control and disturbance laws are $u = K_u x$ and $w = K_w x$ respectively. It is shown in [31] that, under these assumptions, the state matrices $A_f \triangleq A + BK_u$ and $A_c \triangleq A + BK_u + GK_w$ are both stable. The terminal cost function $V_f(\cdot)$ for the constrained H_∞ control problem is the infinite horizon value function defined (globally in $\mathbb{R}^n$) by $V_f(x) = (1/2)|x|^2_{P_f}$ and satisfies $V_f(x) = \max_w\{\ell_f(x, K_u x, w) + V_f(f(x, K_u x, w))\}$ so that, with $\Delta\phi(x, u, w) \triangleq \phi(f(x, u, w)) - \phi(x)$, we have

$$[\Delta V_f + \ell_f](x, K_u x, w) \leq 0 \tag{7.1}$$

for all (x, w) where $\ell_f(\cdot)$, defined by,

$$\ell_f(x, u, w) \triangleq (1/2)|x|^2_Q + (1/2)|u|^2_R - (\rho_f^2/2)|w|^2 \tag{7.2}$$

is the stage cost when $\rho = \rho_f$. The stage cost for the constrained problem $\mathbb{P}_N(x)$ is, as before, $\ell(x, u, w) \triangleq (1/2)|x|^2_Q + (1/2)|u|^2_R - (\rho^2/2)|w|^2$ where $\rho \geq \rho^* \geq \rho_f \geq \tilde{\rho}$. It follows that

$$\ell(x, u, w) \leq \ell_f(x, u, w) \quad \forall (x, u, w). \tag{7.3}$$

An important consequence that we use later is that

$$\begin{aligned} &\max_{w \in W}[\Delta V_f + \ell](x, K_u x, w) \leq \\ &\max_{w \in W}[\Delta V_f + \ell_f](x, K_u x, w) \\ &\leq \max_{w \in \mathbb{R}^n}[\Delta V_f + \ell_f](x, K_u x, w) \\ &= [\Delta V_f + \ell_f](x, K_u, K_w x) = 0. \end{aligned}$$

so that

$$[\Delta V_f + \ell](x, K_u x, w) \leq 0 \tag{7.4}$$

for all (x, w). We now require [32, 33]:

Definition 5 *A set Ω is robust positively invariant for $x^+ = f(x, w)$ if, for every $x \in \Omega$, $f(x, W) \subseteq \Omega$. A set Ω is robust control invariant for $x^+ = f(x, u, w)$ if, for every $x \in \Omega$, there exists a $u \in U$ such that $f(x, u, W) \subseteq \Omega$.*

Algorithms for the construction of these sets are given in [32, 34]. The terminal constraint set X_f is chosen to be a robust positively invariant set containing the origin in its interior for the system $x^+ = A_f x + Gw$, $A_f \triangleq A + BK_u$. Any robust positively invariant set X_f for $x^+ = A_f x + Gw$ satisfies

$$f(x, K_u x, W) \subseteq X_f \ \forall\, x \in X_f \tag{7.5}$$

We assume that the set W is sufficiently small to ensure the existence of a robust positively invariant set X_f (for the system $x^+ = f(x, K_u x, w) = A_f x + Gw$) that satisfies

$$X_f \subseteq X, \quad K_u X_f \subseteq U. \tag{7.6}$$

The controller $u = K_u x$ maintains the state of $x^+ = f(x, u, w)$ in X_f if the initial state is in X_f.

7.2 No state constraints

Here *both* $X = \mathbb{R}^n$ and $X_f = \mathbb{R}^n$. Let X_f^* now denote any robust positively invariant set for $x^+ = f(x, K_u x, w) = A_f x + Gw$ containing the origin in its interior and satisfying $f(x, K_u x, W) \subseteq X_f^* \ \forall\, x \in X_f^*$ and $K_u X_f^* \subseteq U$. Standard stability results [35, 36] (that require a terminal constraint) cannot be employed. The dynamic programming recursions for the value functions $V_j^0(\cdot)$ and the associated control laws $\kappa_j(\cdot)$ are given by (2.6)-(2.9) with the constraint $(x, u) \in \mathcal{Z}_j$ omitted and employing the *global* terminal cost function $V_f(x) = (1/2)|x|^2_{P_f}$ defined above. The value functions $V_j^0(\cdot)$ have domain $\mathbb{R}^n$ but the resultant receding horizon control law $\kappa_N(\cdot)$ (that also has domain $\mathbb{R}^n$) is not necessarily stabilizing in $\mathbb{R}^n$ because the terminal constraint $x_N \in X_f^*$ is not enforced. Assume that the value functions $V_j^0(\cdot)$ and control laws $\kappa_j(\cdot)$ have been determined (by solving (2.6)-(2.9) with the constraints $(x, u) \in \mathcal{Z}_j$ omitted) on sufficiently large polytopic subsets S_j of $\mathbb{R}^n$ (satisfying $\mathcal{X}_j^* \subseteq S_j$ and $f(S_j, U, W) \subseteq S_{j-1}$ for all j); it can be shown that $V_j^0(0) = 0$ and $\kappa_j(0) = 0$ for each j. The sets $\mathcal{X}_j^*$ are computed using

$$\mathcal{X}_j^* = \{x \in S_j \mid f(x, \kappa_j(x), W) \subseteq \mathcal{X}_{j-1}^*\} \tag{7.7}$$

$$\mathcal{X}_0^* = X_f^* \tag{7.8}$$

Each set $\mathcal{X}_j^*$ contains the origin. Let the control law $\kappa_0(\cdot)$ be defined by $\kappa_0(x) \triangleq K_u x$ and let $V_0^0(\cdot) \triangleq V_f(\cdot)$. We define the value function $V^* : \mathcal{X}^* \to \mathbb{R}$, the control law $\kappa^* : \mathcal{X}^* \to U$ and the set $\mathcal{X}^*$ by:

$$\begin{aligned} V^*(x) &\triangleq \min_j \{V_j^0(x) \mid x \in \mathcal{X}_j^*,\ j \in \mathcal{J}\}, \\ j^*(x) &\triangleq \arg\min_j \{V_j^0(x) \mid x \in \mathcal{X}_j^*,\ j \in \mathcal{J}\}, \\ \kappa^*(x) &\triangleq \kappa_{j^*(x)}(x), \\ \mathcal{X}^* &\triangleq \cup\{\mathcal{X}_j^* \mid j \in \mathcal{J}\} \end{aligned} \tag{7.9}$$

where $\mathcal{J} \triangleq \{0, 1, \ldots, N\}$. The function $V^*(\cdot)$ is continuous at the origin and satisfies $V^*(0) = 0$.

Proposition 7 *The ℓ_2 gain from the disturbance w to the costed output $y = Hz$, $z = (x, u)$ is finite. If the disturbance sequence $\mathbf{w}$ is identically zero, the origin is asymptotically stable with a domain of attraction $\mathcal{X}^*$.*

Proof: Suppose that $x \in \mathcal{X}^*$ which implies $x \in \mathcal{X}_{j^*(x)}^*$, $V^*(x) = V_{j^*(x)}^0(x)$ and $\kappa^*(x) = \kappa_{j^*(x)}(x)$. Then $f(x, \kappa^*(x), w) \in \mathcal{X}_{j^*(x)-1}^*$ for all $w \in W$ so that, from the dynamic programming equations (2.6)-(2.9):

$$\begin{aligned} V^*(x) &= \max_{w \in W} \{\ell(x, \kappa_{j^*(x)}, w)) + \\ &\qquad V_{j^*(x)-1}^0(f(x, \kappa_{j^*(x)}, w))\} \\ &\geq \max_{w \in W} \{\ell(x, \kappa^*(x), w)) + V^*(f(x, \kappa^*(x), w))\}. \end{aligned}$$

Hence

$$[\Delta V^* + \ell](x, \kappa^*(x), w) \leq 0 \tag{7.10}$$

for all $x \in \mathcal{X}^*$, all $w \in W$. It follows, by standard calculations, that, for any $x \in \mathcal{X}^*$, any integer $M > 0$

$$\sum_{k=0}^{M} |y(k)|^2 \leq \rho^2 \sum_{k=0}^{M} |w(k)|^2 + 2V^*(x) \tag{7.11}$$

where $y = Hz$, $z \triangleq (x, u)$ is the costed output of the system $x^+ = f(x, \kappa_N(x), w)$ and $V^*(\cdot)$ is positive definite. Hence the ℓ_2 gain from the disturbance w to the costed output y is finite. Asymptotic stability may be proved in the usual way using (7.10), the positive definiteness of $(x, u) \mapsto \ell(x, u, 0)$ (this provides a lower bound for $V^*(\cdot)$), and the continuity of $V^*(\cdot)$ at the origin. ■

A disadvantage of this approach is that the sets $\mathcal{X}_j^*$ are subsets, possibly small subsets, of $\mathbb{R}^n$, and are polygons (unions of polyhedra) rather than polyhedra.

7.3 H_∞ receding horizon control: state constraints

Standard results [17,19,36] may be employed. The conditions (i) $X_f \subseteq X$ is robust positively invariant for $x^+ = f(x, K_u x, w)$, $K_u X_f \subseteq U$ and, (ii) $V_f(\cdot)$ is a local Control Lyapunov function satisfying (7.4) for all $x \in X_f$ and all $w \in W$ are the 'stabilizing conditions' for the min-max optimal control problem and ensure [16] that $\mathcal{X}_N$ is robust positively invariant for $x^+ = f(x, \kappa_N(x), w)$ and that

$$[\Delta V_N^0 + \ell](x, \kappa_N(x), w) \leq 0 \tag{7.12}$$

for all $(x, w) \in \mathcal{X}_N \times W$. Finite ℓ_2 gain from the disturbance w to the costed output y follows as shown above. Also, if the disturbance is identically zero, the origin is exponentially stable with a domain of attraction $\mathcal{X}_N$.

8 Conclusion

The most important results of this paper are Theorem 1 and Theorems 6 and 7 that establish that the value functions are piecewise quadratic and the optimal control laws are piecewise affine for min-max optimal control problems for, respectively, the two cases: (i) control constraints only, and (ii) control, state and terminal constraints. The results for case (i) require few assumptions. Case (ii) requires a strong assumption that may not be satisfied. Both cases require the solution to a parametric program in which the constraints are polyhedral and the cost *piecewise quadratic* (rather than quadratic) is required. A novel solution to this problem is presented in §4 and should prove useful in the determination of 'explicit' control for other problems. Implementation of the results for control is briefly discussed in §7.

Appendices

To prove Theorem 1, we make use of a special case of a theorem of Clarke et al (Theorem 3.1, page 126 in [37]), namely:

Theorem 8 *Take a non-negative valued, convex function $\psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. Let $\mathcal{U}(x) \triangleq \{u \in \mathbb{R}^m \mid \psi(x,u) \leq 0\}$ and $\mathcal{X} \triangleq \{x \in \mathbb{R}^n \mid \mathcal{U}(x) \neq \emptyset\}$. Assume there exists a $\delta > 0$ such that*

$$u \in \mathbb{R}^m,\ x \in \mathcal{X} \text{ and } g \in \partial_u \psi(x,u) \implies |g| > \delta$$

(here $\partial_u \psi(x,u)$ denotes the convex subdifferential of ψ with respect to the variable u). Then, for each $x \in \mathcal{X}$, we have $d(u, \mathcal{U}(x)) \leq \psi(x,u)/\delta$ for all $u \in \mathbb{R}^m$.

This regularity theorem has a role in the proof of Theorem 1 via the following Corollary:

Corollary 1 *Suppose $\mathcal{Z}$ is a polyhedron in $\mathbb{R}^n \times \mathbb{R}^m$ and let $\mathcal{X}$ denote its projection on $\mathbb{R}^n$ ($\mathcal{X} = \{x \mid \exists u \in \mathbb{R}^m$ such that $(x,u) \in \mathcal{Z}\}$). Let $\mathcal{U}(x) \triangleq \{u \mid (x,u) \in \mathcal{Z}\}$. Then there exists a $K > 0$ such that, for all $x, x' \in \mathcal{X}$, for all $u \in \mathcal{U}(x)$, there exists a $u' \in \mathcal{U}(x')$ such that $|u' - u| \leq K|x' - x|$.*

Proof: The polyhedron $\mathcal{Z}$ admits the representation $\mathcal{Z} = \{(x,u) \mid (m^j)'u - (n^j)'x - p^j \leq 0,\ j \in \mathcal{I}_J\}$ for some m^j, n^j and p^j, $j \in \mathcal{I}_J \triangleq \{1, \ldots, J\}$. We recall $\mathcal{X} = \{x \mid (x,u) \in \mathcal{Z}$ for some $u \in U\}$ and $\mathcal{U}(x) \triangleq \{u \mid (x,u) \in \mathcal{Z}\}$. Define $\mathcal{D}$ to be the collection of all index sets $I \subseteq \mathcal{I}_J$ such that $\sum_{j \in I} \lambda^j m^j \neq 0,\ \forall \lambda \in \Lambda_I$ in which, for a particular index set I, Λ_I is defined to be $\Lambda_I \triangleq \{\lambda \mid \lambda^j \geq 0,\ \sum_{j \in I} \lambda^j = 1\}$. Because $\mathcal{D}$ is a finite set, there exists a $\delta > 0$ such that for all $I \in \mathcal{D}$, all $\lambda \in \Lambda_I$, $|\sum_{j \in I} \lambda^j m^j| > \delta$. Let $\psi(\cdot)$ be defined by $\psi(x,u) \triangleq \max\{(m^j)'u - (n^j)'x - p^j, 0 \mid j \in \mathcal{I}_J\}$. We now claim that, for every $(x,u) \in \mathcal{X} \times \mathbb{R}^m$ such that $\psi(x,u) > 0$ and every $g \in \partial_u \psi(x,u)$ (the subgradient of ψ at (x,u)) we have $|g| > \delta$. Assuming for the moment that the claim is true, the proof of the Corollary may be completed with the aid of Theorem 8. Assume, as stated in the Corollary, that $x, x' \in \mathcal{X}$ and $u \in \mathcal{U}(x)$; the theorem asserts

$$d(u, \mathcal{U}(x')) \leq (1/\delta)\psi(x', u),\ \forall x' \in \mathcal{X}.$$

But $\psi(x,u) = 0$ (since $u \in \mathcal{U}(x)$) so that

$$d(u, \mathcal{U}(x')) \leq (1/\delta)[\psi(x', u) - \psi(x,u)] \leq (c/\delta)|x' - x|$$

where c is the Lipschitz constant for $x \mapsto \psi(x,u)$ ($\psi(\cdot)$ is piecewise affine and continuous). This proves the Corollary with $K = c/\delta$.

It remains to confirm the claim. Take any $(x,u) \in \mathcal{X} \times \mathbb{R}^m$ such that $\psi(x,u) > 0$. Then $\max_j\{(m^j)'u - (n^j)'x - p^j, 0 \mid j \in \mathcal{I}_J\} > 0$. Let $I^0(x,u)$ denote the active constraints (those at which the maximum is achieved). Then

$$(m^j)'u - (n^j)'x - p^j > 0,\ \forall j \in I^0(x,u).$$

Since $x \in \mathcal{X}$, there exists a $\bar{u} \in \mathcal{U}(x)$ so that

$$(m^j)'\bar{u} - (n^j)'x - p^j \leq 0,\ \forall j \in I^0(x,u).$$

Subtracting these two inequalities yields

$$(m^j)'(u - \bar{u}) > 0,\ \forall j \in I^0(x,u).$$

But then, for all $\lambda \in \Lambda_{I^0(x,u)}$, $|\sum_{j \in I^0(x,u)} \lambda^j m^j (u - \bar{u})| > 0$, so that $\sum_{j \in I^0(x,u)} \lambda^j m^j \neq 0$. It follows that $I^0(x,u) \in \mathcal{D}$, so that

$$|\sum_{j \in I^0(x,u)} \lambda^j m^j| > \delta,\ \forall \lambda \in \Lambda_{I^0(x,u)}. \qquad (*)$$

Now take any $g \in \partial_u f(x,u) = \mathrm{co}\{m^j \mid j \in I^0(x,u)\}$ (co denotes 'convex hull'). There exists a $\lambda \in \Lambda_{I^0(x,u)}$ such that $g = \sum_{j \in I^0(x,u)} \lambda^j m_j$. But then $|g| > \delta$ by equation $(*)$ above. This proves the claim. ∎

We can now proceed with the proof of Theorem 1.

Proof of Theorem 1: Continuity of $V(\cdot) = \Gamma_{\mathcal{Z}} J(\cdot)$

The constraint $(x,u) \in \mathcal{Z}$ imposes an implicit state-dependent constraint $u \in \mathcal{U}(x)$ on u where the set-valued function $\mathcal{U}(\cdot)$ is defined by

$$\mathcal{U}(x) \triangleq \{u \mid (x,u) \in \mathcal{Z}\}.$$

We claim that $\mathcal{U}(\cdot)$ is continuous (being both outer and inner semi-continuous) on $\mathcal{X} = \text{Proj}_X \mathcal{Z}$, the domain of $\mathcal{U}(\cdot)$. By definition [27], the set-valued map $\mathcal{U}(\cdot)$ is outer semi-continuous at $x \in \mathcal{X}$ if $\mathcal{U}(x)$ is closed and if, for any compact set G such that $\mathcal{U}(x) \cap G = \emptyset$ there exists an $\varepsilon > 0$ such that $\mathcal{U}(x) \cap G = \emptyset$ for all $x' \in B(x, \varepsilon) \cap \mathcal{X}$, $B(x, \varepsilon) \triangleq \{z \mid |z - x| \leq \varepsilon\}$. The set-valued map $\mathcal{U}(\cdot)$ is inner semi-continuous at $x \in \mathcal{X}$ if, for any open set $G \subseteq \mathbb{R}^m$ such that $G \cap \mathcal{U}(x) \neq \emptyset$, there exists an $\varepsilon > 0$ such that $G \cap \mathcal{U}(x') \neq \emptyset$ for all $x' \in B(x, \varepsilon) \cap \mathcal{X}$. The set-valued map $\mathcal{U}(\cdot)$ is outer semi-continuous because its graph, $\mathcal{Z}$, is closed so that, given any sequence $\{(x_i, u_i)\}$ in $\mathcal{Z}$ ($u_i \in \mathcal{U}(x_i)$ for all i) such that $(x_i, u_i) \to (\bar{x}, \bar{u})$, we have $(\bar{x}, \bar{u}) \in \mathcal{Z}$ so that $\bar{u} \in \mathcal{U}(\bar{x})$. Hence $\mathcal{U}(\cdot)$ is outer semi-continuous [27].

We now establish inner semi-continuity using Corollary 1 above. Let x, x' be arbitrary points in $\mathcal{X}$ and $\mathcal{U}(x)$ and $\mathcal{U}(x')$ the associated control constraint sets. Let G be an open set such that $\mathcal{U}(x) \cap G \neq \emptyset$ and let u be an arbitrary point in $\mathcal{U}(x) \cap G$. Because G is open, there exist an $\varepsilon > 0$ such that $B(u, \varepsilon) \triangleq \{v \mid |v - u| \leq \varepsilon\} \subset G$. Let $\varepsilon' \triangleq \varepsilon / K$ where K is defined in Corollary 1. From Corollary 1, there exists a $u' \in \mathcal{U}(x') \cap G$ for all $x' \in B(x, \varepsilon') \cap \mathcal{X}$. This implies $\mathcal{U}(x') \cap G \neq \emptyset$ for all $x' \in B(x, \varepsilon') \cap \mathcal{X}$, so that $\mathcal{U}(\cdot)$ is inner semi-continuous. Since $J(\cdot)$ is continuous and $\mathcal{U}(\cdot)$ takes values in the compact set U, condition (2b) in Theorem 5.4.1 of [27] holds so $V(\cdot)$ is continuous by Corollary 5.4.2 of [27].

Proof of Theorem 2: Continuous differentiability of $V(\cdot) = \Gamma(J(\cdot))$

Since U, being constant, is continuous in z, the continuity of the value function $V(\cdot) = \Gamma(J(\cdot))$ follows from the maximum theorem (e.g. Theorem 5.4.3 in [27]). Since the function $u \mapsto J(x, u)$ is strictly convex for all x, the optimizer $\kappa(\cdot) = \gamma(J(\cdot))$ is unique (a singleton) for each x; by the same maximum theorem, $\kappa(\cdot)$ is continuous. Since $J(\cdot)$ is continuously differentiable and U is compact, and the optimizer $\kappa(x)$ is unique and continuous, it follows from the proof of Theorem 5.4.7 in [27] that the directional derivative of $V(\cdot)$ satisfies

$$dV(x; h) = (\partial/\partial x) J(x, \kappa(x)) h$$

at any x, any direction h. Hence $V(\cdot)$ is Gateau differentiable at any x with Gateau derivative $G(x) = (\partial/\partial x) J(x, \kappa(x))$. Since $G(\cdot)$ is continuous, $V(\cdot)$ is continuously (Frechet) differentiable in $\mathcal{Z}$ with derivative $(\partial/\partial x) V(x) = (\partial/\partial x) J(x, \kappa(x))$ [38]. ■

Proof of Proposition 3: Strict concavity of $w \mapsto V'(z, w)$

Continuous differentiability of $V'(\cdot)$ ($V'(z, w) \triangleq \ell(z, w) + V(f(z, w))$) follows from continuous differentiability of $\ell(\cdot)$ and $V(\cdot)$. Take any two points w_1, w_2 in W. For all $\lambda \in [0, 1]$, let $w_\lambda \triangleq w_1 + \lambda(w_2 - w_1)$, and, for each $z \in \mathcal{Z}$, let the real valued function $\phi(\cdot)$ be defined on $[0, 1]$ by $\phi(\lambda) \triangleq V(z, w_\lambda)$. Suppose that $V(x) = (1/2) x' Q_i x + q_i' x + s_i$ in $P_i^{\mathcal{X}}$ (for each $i \in \mathcal{J}^{\mathcal{X}}$). Then

$$\begin{aligned} V'(z,w) &= (1/2)(Fz+Gw)'Q_i(Fz+Gw)+q_i'(Fz+Gw) \\ &+ s_i + \ell(z,w) = -(1/2)w'(\rho^2 I - G'Q_iG)w + b_i'w + c_i \end{aligned}$$

on the polyhedron $P_i^\Phi = \{(z, w) \in \mathbb{R}^n \times U \times W \mid Fz + Gw \in P_i^{\mathcal{X}}\}$, where b_i and c_i depend on z. For any $\varepsilon > 0$, there exists a $\rho^* > 0$ such that $C_i \triangleq \rho^2 I - G' Q_i G \geq \varepsilon I$ for all $\rho \geq \rho^*$, all $i \in \mathcal{J}^{\mathcal{X}}$. The function $\phi(\cdot)$ is continuously differentiable and satisfies:

$$\begin{aligned} \phi(\lambda) &= -(1/2)(h'C_ih)\lambda^2 + b_i\lambda + c_i \\ \phi'(\lambda) &= -(h'C_ih)\lambda + b_i \end{aligned}$$

for all $\lambda \in [0, 1]$ such that $Fz + Gw_\lambda \in P_i^\Phi$. Since $\phi'(\cdot)$ is continuous, $\phi'(\cdot)$ is strictly decreasing if $\rho \geq \rho^*$. It follows, by a trivial modification to the proof of Theorem 4.4 in [39], that $\phi(\lambda) > \phi(0) + \lambda(\phi(1) - \phi(0))$ for all $\lambda \in (0, 1)$ which establishes the strict concavity of $\phi(\cdot)$ and, hence, of $w \mapsto V'(z, w)$ if $\rho \geq \rho^*$. ■

Proof of Proposition 5: equivalence of solutions to $\mathbb{P}_{\min}(x)$ and $\mathbb{P}_{\bar{x}}(x)$

Suppose that $u = \kappa(x)$ is optimal for the original problem $\mathbb{P}_{\min}(x)$; this implies that $V^0(x) = V(x, u) \leq V(x, u')$ for any u' satisfying $(x, u') \in P_{\bar{x}}$. But $(x, u) \in P_{\bar{x}}$ since, by assumption, $\mathcal{S}^0(x) = \mathcal{S}^0(\bar{x})$; hence u is also optimal for $\mathbb{P}_{\bar{x}}(x)$. On the other hand, suppose u is optimal for $\mathbb{P}_{\bar{x}}(x)$ so that $(x, u) \in P_{\bar{x}}$ and $\mathcal{S}(x, u) = \mathcal{S}^0(\bar{x})$. By assumption, $\mathcal{S}^0(x) = \mathcal{S}^0(\bar{x})$ so that $(x, \kappa(x)) \in P_{\bar{x}}$; consequently $V(x, u) \leq V(x, \kappa(x)) = V^0(x)$. But $V^0(x) \leq V(x, u)$ (by optimality of $\kappa(x)$ for problem $\mathbb{P}_{\min}(x)$) so that $V(x, u) = V^0(x)$ and u is also optimal for $\mathbb{P}_{\min}(x)$. This proves equivalence of (i) and (ii); equivalance of (iii) to (i) and (ii) follows easily. ■

Proof of Proposition 6: Existence of ρ^*

The proof is by induction. Suppose, for any j in $\{1, \ldots, N-1\}$, there exists a ρ_j^* such that $V_i'(z, w) = \ell(z, w) + V_{i-1}^0(f(z, w))$ is strictly concave in w for all $\rho \geq \rho_j^*$, all $z \in \mathcal{Z}_i$, all $i \in \{1, \ldots, j\}$. It follows from Theorem 6 (with N replaced by j), that $V_i^0(\cdot)$ is continuously differentiable, strictly convex, and piecewise quadratic for all $i \in \{1, \ldots, j\}$. By Proposition 3, there exists a $\rho_{j+1}^* \geq \rho_j^*$ such that $V_{j+1}'(z, w) = \ell(z, w) + V_j^0(f(z, w))$ is strictly concave in w for all $\rho \geq \rho_{j+1}^*$, and all $z \in \mathcal{Z}_{j+1}$. Hence $V_i'(z, w) = \ell(z, w) + V_j^0(f(z, w))$ is strictly concave in w for all $\rho \geq \rho^*_{j+1}$, all $z \in \mathcal{Z}_i$, all $i \in \{1, \ldots, j+1\}$. But, by Proposition 3 and our assumptions on $V^0_0(\cdot) \triangleq V_f(\cdot)$, there exists a ρ^*_1 such that $V'_1(z,w) = \ell(z,w) + V^0_0(f(z,w))$ is strictly concave in w for all $\rho \geq \rho^*_1$, all $z \in \mathcal{Z}_1$. By induction, there exists a $\rho^* = \rho^*_N$ such that $V'_i(z,w) = \ell(z,w) + V^0_{j-1}(f(z,w))$ is strictly concave in w for all $\rho \geq \rho^*$, all $z \in \mathcal{Z}_i$, all $i \in \{1, \ldots, N\}$.

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