Offset-free Receding Horizon Control of Constrained Linear Systems

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Abstract:

This paper addresses the design of a dynamic state feedback receding horizon controller, which guarantees robust constraint satisfaction, robust stability and offset-free control of constrained linear systems in the presence of time-varying setpoints and unmeasured disturbances. This objective is obtained by first designing a dynamic linear offset-free controller and computing an appropriate domain of attraction for this controller. The linear (unconstrained) controller is then modified by adding a perturbation term, which is computed by a (constrained) robust receding horizon controller. The receding horizon controller has the property that its domain of attraction contains that of the linear controller. In order to ensure robust constraint satisfaction, in addition to offset-free control, the transient as well as the limiting behavior of the disturbance and setpoint need to be taken into account in the design of the receding horizon controller. The fundamental difference between the results in this paper and the existing literature on receding horizon control is that, in this paper, the transient effect of the disturbance and setpoint sequences on the so-called "target calculator" is explicitly incorporated in the formulation of the receding horizon controller. An example of the control of a continuous stirred tank reactor is presented.

Topical Heading: Process system engineering

Key Words: Model predictive control, constraints, offset-free control, tracking, robust control.

Introduction

The control of systems in the presence of constraints is an important task in many application fields because constraints “always” arise from physical limitations and quality or safety reasons. Moreover, in practical applications, disturbances are usually present and often they are not measurable or predictable. For example, in the chemical industries disturbances arise from interactions between different plant units, from changes in the raw materials and in the operating conditions (such as ambient temperature, humidity, etc.).

It is well-known that with an unmeasured persistent disturbance offset-free control is, in general, not possible. However, if the disturbance has the additional property that it is integrating or periodic, then offset-free control may be an achievable goal. In many practical applications, especially in the process industries, disturbances are often integrating and reach, after some transient, a constant value. Hence, one basic objective of an effective control algorithm is that it guarantees offset-free control whenever this is possible. Moreover, an effective control algorithm is also applicable to cases in which the setpoints of the controlled variables are allowed to vary over time.

The design of control algorithms able to stabilize constrained linear plants with input and state constraints subject to unknown, but bounded disturbances, has been the subject of several works over the last few decades; a number of excellent surveys are available [1–5], which discuss how the important goal of guaranteeing closed-loop stability and constraint satisfaction can be obtained. Existing control algorithms, which address the problem of robust control of constrained systems, are usually based on ideas from set invariance [1, 6, 7], reference governors [8–12] or receding horizon control [13–22]. It is interesting to note that, despite the practical importance of guaranteeing offset-free control in the presence of integrating disturbances, none of the existing receding horizon control algorithms with robust stability *and* robust constraint satisfaction guarantees are able to guarantee offset-free control.

Compared to linear (unconstrained) control, the rigorous study of designing controllers that guarantee offset-free control has received very little attention in the constrained control community, until relatively recently [23–30]. However, though the receding horizon control algorithms presented in [23–30] guarantee offset-free control around a neighborhood of the steady-state, they do not guarantee robust constraint satisfaction for all initial states over which the controller is defined (in other words, they do not guarantee feasibility of the optimization problem for all time and for all allowable disturbance and setpoint sequences). Furthermore, with the exception of [8–10, 14, 16] (which do not guarantee offset-free control for disturbances that decay to non-zero values), none of the existing receding horizon control algorithms that guarantee robust stability and robust constraint satisfaction, address the problem of tracking arbitrary setpoints.

In this paper, we present a novel receding horizon control algorithm for controlling constrained linear systems subject to unmeasured, bounded disturbances. The proposed algorithm is guaranteed to remove steady-state offset in the controlled variables whenever the disturbances reach an (unknown) steady-state value. Moreover, the setpoints of the controlled variables are allowed to vary arbitrarily with time within a pre-specified set, provided they also converge to some limit. Importantly, the algorithm also guarantees to satisfy input and state constraints for all allowable disturbance sequences. None of the existing receding horizon control algorithms are able to provide similar guarantees.

The difficulty with guaranteeing robust constraint satisfaction, *in addition* to offset-free control, is that the transient as well as the limiting behavior of the disturbance and setpoint need to be taken into account during the computation of the control input. The fundamental difference between the algorithm proposed in this paper and those available in the literature, is that we consider both the transient and limiting effect of all allowable future disturbance and setpoint sequences when computing the receding horizon control action. This is the key idea that allows us to guarantee offset-free control *and* robust constraint satisfaction. Existing approaches either neglect the transient response or they neglect the ultimate behavior of the disturbance and setpoint, hence why they are unable to offer the same kind of guarantee.

This paper is organized as follows. First, the problem definition is given, followed by the design of a linear offset-free controller. Following this, we show how a receding horizon controller can be used to improve on the linear controller by enlarging the region of attraction. The main characteristics of the proposed receding horizon controller are illustrated on an example of a continuous stirred tank reactor. Finally, the main contributions of this work are summarized and some possible extensions are discussed. To simplify the presentation and reading of this paper, the proofs of results that can be derived using well-known methods have been omitted. The only proofs that have been included are for those cases where we were unable to find equivalent and sufficiently detailed proofs, where we have used unconventional proof techniques or where we feel that a particular detail of the proof is important.

Notation: If a and b are vectors, then the column vector $(a,b) := (a^T \; b^T)^T$. Given an arbitrary set $\mathscr{X}$, $\mathscr{X}^N$ denotes the Cartesian product $\mathscr{X} \times \cdots \times \mathscr{X}$.

Problem Description and Preliminary Results

In this paper we consider a discrete-time linear time-invariant plant:

$$x(k+1) = Ax(k) + Bu(k) + Ed(k)\,, \tag{1a}$$

$$z(k) = C_z x(k)\,, \tag{1b}$$

in which k is the sample instant, $x \in \mathbb{R}^n$ is the plant state, $u \in \mathbb{R}^m$ is the control input (manipulated variable), $d \in \mathbb{R}^r$ is an unmeasured disturbance and $z \in \mathbb{R}^p$ is the controlled variable, i.e. the variable to be controlled to a given (time-varying) setpoint s. Affine inequality constraints are given on the state and input[1], i.e.

$$x \in \mathscr{X}\,, \quad u \in \mathscr{U}\,, \tag{2}$$

where $\mathscr{X}$ is a polyhedron (i.e. a closed and convex set that can be described by a finite number of affine inequality constraints) and $\mathscr{U}$ is a polytope (i.e. a bounded polyhedron); the interior of $\mathscr{X} \times \mathscr{U}$ contains the origin. We also make the following standard assumption:

Assumption 1 (General). A measurement of the plant state is available at each sample instant, (A,B) is stabilizable, (A,C_z) is detectable and (see e.g. [28])

$$\operatorname{rank}\begin{bmatrix} I-A & -B \\ C_z & 0 \end{bmatrix} = n+p\,. \tag{3}$$

Remark 1. The rank condition (3) is used to guarantee the existence of an offset-free steady-state for any constant setpoint and disturbance pair. In general, the steady-state is not necessarily unique for a given setpoint and disturbance. Note also that this condition implies that the number of controlled variables cannot exceed the number of control inputs or the number of states, i.e. $p \leq \min\{m,n\}$. It is easy to find examples for which offset-free control is not possible if (3) is not satisfied.

We also consider the following assumptions on the setpoint and disturbance sequences:

Assumption 2 (Setpoint). At each time instant, the current setpoint is known but future setpoint values are unknown. The setpoint sequence $s(\cdot)$ takes on values in a polytope $\mathscr{S} \subset \mathbb{R}^p$ containing the origin and asymptotically reaches an unknown steady-state value, i.e. $s(k) \in \mathscr{S}$ for all $k \in \mathbb{N}$ and there exists an $\bar{s} \in \mathscr{S}$ such that $\lim_{k\to\infty} s(k) = \bar{s}$.

Assumption 3 (Disturbance). At each time instant, current *and* future disturbances are *unknown*. The disturbance sequence $d(\cdot)$ takes on values in a polytope $\mathscr{D} \subset \mathbb{R}^r$ containing the origin and asymptotically reaches an *unknown* steady-state value $\bar{d}$, i.e. $d(k) \in \mathscr{D}$ for all $k \in \mathbb{N}$ and there exists a $\bar{d} \in \mathscr{D}$ such that $\lim_{k\to\infty} d(k) = \bar{d}$.

Remark 2. Note that, unlike many existing results, we do not assume that the disturbance or setpoint is constant. Furthermore, unlike [25], we do not assume that the setpoint and/or disturbance are generated by a known finite-dimensional exogenous system. The lack of these assumptions in this paper complicates the design of the controller.

Under the above assumptions we present a novel method for designing a dynamic, state feedback receding horizon controller that, for any allowable disturbance and setpoint sequences (i.e. any infinite disturbance and setpoint sequences that satisfy Assumptions 2 and 3), accomplishes the goal of asymptotically driving the controlled variable

[1]The results in this paper can easily be extended to the case with mixed constraints on the state and input.

to any given allowable asymptotic setpoint, while respecting the state and input constraints, i.e.

$$\lim_{k\to\infty} z(k) = \bar{s} \tag{4a}$$

$$x(k) \in \mathscr{X}\ ,\ u(k) \in \mathscr{U}\ ,\ \forall k \in \mathbb{N}\,. \tag{4b}$$

Before proceeding, we present here the following well-known result [6, 15, 18, 20, 31]:

Proposition 1. *Let the polyhedron*

$$\mathscr{P} := \left\{ v \in \mathbb{R}^t \ \mid Fv \leq g + Hw \textit{ for all } w \in \mathscr{W} \right\}$$

where $F \in \mathbb{R}^{q\times t}$ *and* $H \in \mathbb{R}^{q\times s}$ *are matrices,* $g \in \mathbb{R}^q$ *is a vector and* $\mathscr{W}$ *is a compact (i.e. closed and bounded) subset of* $\mathbb{R}^s$*, then*

$$\mathscr{P} = \left\{ v \in \mathbb{R}^t \ \middle|\ Fv \leq g + \min_{w\in\mathscr{W}} Hw \right\}$$

where the minimization is performed row-wise, i.e. if H_i *denotes the* i^{th} *row of* H*, then*

$$\min_{w\in\mathscr{W}} Hw := [\min_{w\in\mathscr{W}} H_1 w\ \cdots\ \min_{w\in\mathscr{W}} H_q w]^T\,.$$

Since our system, in closed-loop with the receding horizon controller, is *nonlinear* and we are interested in robust stability results, we review the following definitions and results, adapted from [32], for a generic nonlinear perturbed discrete-time system:

$$\zeta(k+1) = F(\zeta(k)) + w(k)\,, \tag{5}$$

in which $F : \mathbb{R}^\ell \to \mathbb{R}^\ell$ and $F(0) = 0$. Let $\mathbb{B}_r := \left\{ \xi \in \mathbb{R}^\ell \ \mid \|\xi\| \leq r \right\}$ if $r > 0$.

Definition 1. The origin is a robustly asymptotically stable fixed point of (5) if for all $\varepsilon > 0$, there exist a $\delta > 0$ and a $\mu > 0$ such that for all initial conditions $\zeta(0) \in \mathbb{B}_\delta$ and perturbation sequences $w(\cdot)$ satisfying $w(k) \in \mathbb{B}_\mu$ for all $k \in \mathbb{N}$, the following two conditions are satisfied:

1. (*Robust stability*) The solution of (5) satisfies $\zeta(k) \in \mathbb{B}_\varepsilon$ for all $k \in \mathbb{N}$;

2. (*Robust convergence*) The solution of (5) satisfies $\lim_{k\to\infty} \zeta(k) = 0$ if $\lim_{k\to\infty} w(k) = 0$.

Definition 2. If $\bar{w} := \lim_{k\to\infty} w(k)$ is the limit point of the perturbation sequence $w(\cdot)$, then a vector $\bar{\zeta}$ satisfying $\bar{\zeta} = F(\bar{\zeta}) + \bar{w}$ is a robustly asymptotically stable fixed point of (5) if the origin is a robustly asymptotically stable fixed point of the system $\chi(k+1) = G(\chi(k)) + \omega(k)$, in which $\chi := \zeta - \bar{\zeta}$, $\omega := w - \bar{w}$ and $G(\chi) := F(\bar{\zeta} + \chi) - F(\bar{\zeta})$.

Theorem 1. *[32, Thm. 3] Let $F : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be a Lipschitz continuous function in a neighborhood of the origin with $F(0) = 0$. If the origin is an exponentially stable fixed point of the unperturbed system $\zeta(k+1) = F(\zeta(k))$, then it is a robustly asymptotically stable fixed point of the perturbed system $\zeta(k+1) = F(\zeta(k)) + w(k)$.*

Linear Controller Design

In order to guarantee robust constraint satisfaction and stability in receding horizon control, it is by now standard practice to compute a suitable terminal constraint and terminal cost based on a stabilizing linear controller [2–5]. This section shows how one can compute such a linear controller and an appropriate terminal constraint. The difference between this paper and standard results is that the controller in this paper is dynamic, rather than static, hence the terminal constraint is computed in the joint plant-controller state-space, rather than just the plant state-space.

In order to guarantee offset-free control when disturbances are asymptotically constant and non-zero, it is standard practice to augment the plant model with a disturbance model and use this combined model to estimate the size of the disturbance. However, as pointed out in [27, 28], this is not necessarily as straightforward as is commonly thought. Care has to be taken in constructing the controller, since offset-free control is guaranteed only if the combined plant-disturbance system is detectable *and* the closed-loop system (feedback gain + observer + target calculator + plant) satisfy a couple of additional, technical assumptions (see [27, 28] for details). This section will therefore show in detail how one can design a dynamic controller (observer and static state feedback gain) for (1) that guarantees offset-free control. The general structure of the proposed controller is depicted in Figure 1 and each block is described next.

The Augmented System

We will make use of the following auxiliary system in order to define the controller dynamics:

$$\hat{x}(k+1) = Ax(k) + Bu(k) + (\hat{d}(k) + x(k) - \hat{x}(k)), \tag{6a}$$

$$\hat{d}(k+1) = \hat{d}(k) + x(k) - \hat{x}(k). \tag{6b}$$

Note that the system (6) corresponds to using a dead-beat observer for the following system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + \tilde{d}(k), \\ \tilde{d}(k+1) &= \tilde{d}(k), \\ y(k) &= x(k). \end{aligned}$$

in which it is clear that $\tilde{d} \in \mathbb{R}^n$ is an integrated (step) disturbance acting on the state $x \in \mathbb{R}^n$. The role of $\tilde{d}$ is essential in removing steady-state offset in the presence of an unknown persistent disturbance [27, 28]. As will be seen later, the dimensions of $\tilde{d}$ and d need not be the same in order to guarantee offset-free control.

By combining the plant dynamics (1) and the auxiliary system (6), we obtain the following augmented system:

$$\xi(k+1) = \mathscr{A}\xi(k) + \mathscr{B}u(k) + \mathscr{E}d(k)\,, \tag{7a}$$

$$z(k) = \mathscr{C}\xi(k)\,, \tag{7b}$$

in which $\xi := (x, \hat{x}, \hat{d})$ and $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{E}$ are suitably-defined.

Target Calculation

When a non-zero disturbance affects a system (and/or the current setpoint s is different from zero), the steady-state 'target' value of the state and input may need to be shifted in order to cancel the effect of such a disturbance on the controlled variable [33, 34]. To this aim, at each sample instant we use the estimate of the future disturbance and compute the steady-state target $(\bar{x}, \bar{u})$ such that one can drive the controlled variable to the current setpoint, by solving the following least-squares problem [34] in which $\bar{R} \in \mathbb{R}^{m\times m}$ is a positive definite matrix:

$$\left(\bar{x}^*(\xi,s), \bar{u}^*(\xi,s)\right) := \arg\min_{(\bar{x},\bar{u})} \frac{1}{2}\bar{u}^T\bar{R}\bar{u}, \tag{8a}$$

subject to

$$\begin{bmatrix} I-A & -B \\ C_z & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} I & -I & I \\ 0 & 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ I \end{bmatrix} s, \tag{8b}$$

where it can be seen that the current estimate of the limiting value of the disturbance is given by $\begin{bmatrix} I & -I & I \end{bmatrix}\xi(k) = x(k) - \hat{x}(k) + \hat{d}(k) = \hat{d}(k+1)$.

For a given augmented state ξ and a given setpoint s, one can think of $(\bar{x}^*(\xi,s), \bar{u}^*(\xi,s))$ as the new 'origin' around which the system should be regulated. Solving for $(\bar{x}^*(\xi,s), \bar{u}^*(\xi,s))$ is trivial:

Lemma 1 (Target calculation). *If Assumption 1 holds, then the target calculation* (8) *has a unique solution that is linear with respect to the augmented state ξ and the setpoint s, and has the form*

$$\begin{bmatrix} \bar{x}^*(\xi,s) \\ \bar{u}^*(\xi,s) \end{bmatrix} = \begin{bmatrix} \Pi_{x,\xi} & -\Pi_{x,\xi} & \Pi_{x,\xi} \\ \Pi_{u,\xi} & -\Pi_{u,\xi} & \Pi_{u,\xi} \end{bmatrix} \xi + \begin{bmatrix} \Pi_{x,s} \\ \Pi_{u,s} \end{bmatrix} s, \tag{9}$$

for suitably-defined constant matrices $\Pi_{x,\xi} \in \mathbb{R}^{n\times n}$, $\Pi_{u,\xi} \in \mathbb{R}^{m\times n}$, $\Pi_{x,s} \in \mathbb{R}^{n\times p}$ *and* $\Pi_{u,s} \in \mathbb{R}^{m\times p}$.

Proof. The statement follows immediately from the Lagrangian/KKT conditions for (8) [35, Sect. 16.1]. It is possible to verify that the matrix, which is to be inverted when obtaining the expression for the stationary point of the Lagrangian, is non-singular because of the rank condition in (3) *and* the fact that $\bar{R}$ is positive definite. Hence, the target calculation (8) has a unique minimizer [36]. □

Remark 3. Recall that (3) on its own is not sufficient to guarantee that the minimizer of (8) is unique. However, if $\bar{R}$ is positive definite, then we do not need to include a penalty on $\bar{x}$ in the cost (8a) in order to guarantee that the minimizer is unique.

Remark 4. Note that, unlike [27, 28], the constraints on the state and input are not included in (8). In order for the receding horizon controller to take account of the transient of the disturbance and setpoint in its predictions, it needs to calculate how $(\bar{x}^*(\xi,s), \bar{u}^*(\xi,s))$ will vary over the control horizon. As will be shown later, this is easy if the mapping $(\xi,s) \mapsto (\bar{x}^*(\xi,s), \bar{u}^*(\xi,s))$ is linear, as above. If inequality constraints had been included in (8), then the mapping $(\xi,s) \mapsto (\bar{x}^*(\xi,s), \bar{u}^*(\xi,s))$ would have been nonlinear, hence the computation of $(\bar{x}^*(\xi,s), \bar{u}^*(\xi,s))$ at each time instant in the prediction horizon, and hence the computation of the receding horizon control input, would be considerably more complicated.

Unconstrained Offset-free Controller Design

We now consider what would happen if one were to choose a gain matrix K such that $A+BK$ is strictly stable and, as is common practice, let the control input in the augmented system (7) be given by

$$u = \bar{u}^*(\xi,s) + K(x - \bar{x}^*(\xi,s)). \tag{10}$$

Clearly, as can be seen in Figure 1, there is a feedback path around the target calculator and therefore it is important to verify that the closed-loop system (observer + target calculator + state feedback gain + plant) is stable. We therefore present the following intermediate result:

Lemma 2 (Stability). *Suppose that Assumption 1 holds and* $K \in \mathbb{R}^{m\times n}$ *is such that* $A+BK$ *is strictly stable. If* $\Gamma \in \mathbb{R}^{m\times n}$ *is any constant matrix and*

$$\mathscr{K} := \begin{bmatrix} K+\Gamma & -\Gamma & \Gamma \end{bmatrix}, \tag{11}$$

then

$$\mathscr{A}_{\mathscr{K}} := \mathscr{A} + \mathscr{B}\mathscr{K} \tag{12}$$

is strictly stable.

Proof. The statement follows from straightforward block matrix manipulations [36], where one can show that $2n$ of the eigenvalues of $\mathscr{A}_{\mathscr{K}}$ are at zero and the rest are the eigenvalues of $A+BK$. $\square$

By defining

$$\Gamma := \Pi_{u,\xi} - K\Pi_{x,\xi}\,, \quad \mathscr{L} := \Pi_{u,s} - K\Pi_{x,s}\,, \tag{13}$$

and substituting (9) into (10) it easily follows that

$$u = \mathscr{K}\xi + \mathscr{L}s\,, \tag{14}$$

where $\mathscr{K}$ is defined as in (11).

After substituting (14) into (7), one can write an expression for the augmented system (7) under the linear control $u = \mathscr{K}\xi + \mathscr{L}s$ as

$$\xi(k+1) = \mathscr{A}_{\mathscr{K}}\xi(k) + \mathscr{E}d(k) + \mathscr{F}s(k)\,, \tag{15a}$$

$$z(k) = \mathscr{C}\xi(k)\,, \tag{15b}$$

where $\mathscr{F}$ is suitably-defined.

It immediately follows from Lemma 2 that the closed-system (15) is stable if the disturbance and setpoint are zero (or asymptotically constant). As a consequence, we introduce the following standing assumption:

Assumption 4 (Stabilizing gain). The matrix $K \in \mathbb{R}^{m\times n}$ is chosen such that $A+BK$ is strictly stable, $\mathscr{K}$ is given by (11), Γ and $\mathscr{L}$ given by (13) and $\mathscr{A}_{\mathscr{K}} := \mathscr{A} + \mathscr{B}\mathscr{K}$.

The following result states that if the control is given by $u = \mathscr{K}\xi + \mathscr{L}s$, then the value of the controlled variable for (15) is guaranteed to converge to the asymptotic setpoint $\bar{s}$, given any allowable infinite setpoint and disturbance sequence:

Lemma 3 (Offset-free control). *If Assumptions 1–4 hold, then the solution of the closed-loop system* (15) *satisfies* $\lim_{k\to\infty} z(k) = \bar{s}$ *for all* $\xi(0) \in \mathbb{R}^{3n}$.

Proof. See the Appendix. $\square$

Remark 5. In this paper, we have assumed that there is no mismatch between the plant model (A,B,E) and the actual plant dynamics (A_p,B_p,E_p). However, it is important to point out that, as in [24, 28], it is possible to verify that the offset-free property in Lemma 3 holds even if there is a mismatch between the plant model (A,B,E) and the actual plant dynamics (A_p, B_p, E_p), provided the augmented closed-loop system is stable. This fact is not surprising. Recall that the internal model principle [37] (roughly) states that a controller that guarantees offset-free control even when the system parameters are perturbed, must incorporate a model of the dynamic structure of the disturbance and the reference signals; in our case, the observer contains a model of the disturbance and the target calculator contains a model of the disturbance and setpoint. This point will be further illustrated in the example section.

The Maximal Constraint-admissible Robust Positively Invariant Set

As mentioned earlier, since the constraints on the state and input (2) are not included in the target calculation (8), there is no guarantee that the steady-state target will satisfy constraints. Furthermore, there is also no guarantee that the linear controller defined above will guarantee the satisfaction of the constraints during the transient. We therefore proceed along similar lines as in conventional receding horizon control [2–5] and compute a constraint-admissible, robust invariant set that could be used as a terminal constraint in our receding horizon controller (to be defined in the next section). However, since our linear controller is dynamic and not static, we depart from convention and compute an invariant set in the space of the augmented (plant + controller) state $\xi := (x, \hat{x}, \hat{d})$ of the closed-loop system (15), rather than in the plant state-space (as is usually done [2–5]).

Let the *constraint-admissible set* Ξ be defined as all augmented states for which the constraints on the plant state and plant input are satisfied, for any choice of setpoint $s \in \mathscr{S}$, if the control is given by $u = \mathscr{K}\xi + \mathscr{L}s$:

$$\Xi := \left\{ \xi \in \mathbb{R}^{3n} \mid x \in \mathscr{X} \text{ and } \mathscr{K}\xi + \mathscr{L}s \in \mathscr{U} \text{ for all } s \in \mathscr{S} \right\}. \tag{16}$$

The *maximal constraint-admissible robust positively invariant set* $\mathscr{O}_\infty$ for the closed-loop system (15) is defined as all initial states in Ξ for which the evolution of the system remains in Ξ for all allowable infinite setpoint and disturbance sequences, i.e.

$$\mathscr{O}_\infty := \{ \xi(0) \in \Xi \mid \xi(k+1) = \mathscr{A}_{\mathscr{K}}\xi(k) + \mathscr{E}d(k) + \mathscr{F}s(k) \in \Xi, \ \forall s(k) \in \mathscr{S}, \ d(k) \in \mathscr{D}, \ k \in \mathbb{N} \}. \tag{17}$$

Assumption 5 (Maximal invariant set). The set $\mathscr{O}_\infty$ is non-empty, contains the origin in its interior and is finitely determined (i.e. $\mathscr{O}_\infty$ can be described by a finite number of affine inequality constraints).

Note that since $\mathscr{X}$ and $\mathscr{U}$ are polyhedra given by affine inequalities, Ξ is easily computed by applying Proposition 1 to the above definitions. Since (15) is linear and time-invariant and Ξ is given by a finite number of affine inequality constraints, $\mathscr{O}_\infty$ (or an inner approximation to it) is easily computed by solving a finite number of LPs [31].

We are now in a position to collect all of the results in this section into a single statement. The following result states that, provided the state of the augmented system (controller + plant) is in $\mathscr{O}_\infty$ at time $k = 0$, then the evolution of the augmented system under the linear control $u = \mathscr{K}\xi + \mathscr{L}s$ is such that offset-free control is guaranteed, the state and input constraints are satisfied for all allowable setpoint and disturbance sequences and robust stability is guaranteed:

Theorem 2 (Linear controller). *Suppose that Assumptions 1–5 hold. The solution of the closed-loop system* (15) *satisfies* (4) *if and only if the initial augmented (controller + plant) state* $\xi(0) \in \mathscr{O}_\infty$*. Furthermore,* $\bar{\xi} := (I - \mathscr{A}_{\mathscr{K}})^{-1}(\mathscr{E}\bar{d} + \mathscr{F}\bar{s})$ *is the robustly asymptotically stable fixed point of* (15)*.*

Proof. See the Appendix. □

Before proceeding, we also define X_0, the set of plant states for which there exists a controller state such that the augmented state is in $\mathscr{O}_\infty$, as

$$X_0 := \left\{ x \in \mathbb{R}^n \;\middle|\; \exists(\hat{x}, \hat{d}) \in \mathbb{R}^{2n} \text{ such that } \xi \in \mathscr{O}_\infty \right\} . \tag{18}$$

Receding Horizon Controller Design

The set X_0, defined in (18), is the set of initial plant states for which one can initialize the controller state such that the controlled variable will ultimately be driven by the linear controller to the asymptotic setpoint $\bar{s}$. Clearly one would like to enlarge this set, if possible. It is well-known that receding horizon control allows one to achieve this aim [2–5]. This section therefore presents an approach for computing a (dynamic) receding horizon controller, which enlarges the set of initial plant states for which the controlled variable can ultimately be driven to the asymptotic setpoint.

As mentioned earlier, in order to guarantee robust constraint satisfaction and stability in receding horizon control, it is by now standard practice to compute a suitable terminal constraint and terminal cost based on a stabilizing linear controller [2–5]. However, all existing results on receding horizon control do not include the effect of all allowable disturbance and setpoint sequences on the target calculator in their predictions. This is an important difference that, coupled with the fact that the controller in this paper is dynamic and not static (as in [2–5]), makes it necessary to show in detail how the results from the previous section can be used to define an appropriate receding horizon controller.

Definition of the Receding Horizon Controller

For the sake of simplicity of exposition and implementation, we follow the approach of [14, 16–18, 20] by "pre-stabilizing" the plant and letting the linear control in (14) be modified with a perturbation term as follows:

$$u = \mathscr{K}\xi + \mathscr{L}s + v\,, \tag{19}$$

where $v \in \mathbb{R}^m$ is the perturbation term. The solution to the finite horizon optimal control problem (defined below) is then a finite sequence of input perturbations that guarantees robust constraint satisfaction over the horizon and optimizes some cost function.

Under the control in (19), the augmented state dynamics in (7) become

$$\xi(k+1) = \mathscr{A}_{\mathscr{K}}\xi(k) + \mathscr{B}v(k) + \mathscr{E}d(k) + \mathscr{F}s(k). \tag{20}$$

Remark 6. At this point, it is useful to recall that $\mathscr{K}$, $\mathscr{L}$, $\mathscr{A}_{\mathscr{K}}$ and $\mathscr{F}$ in (19)–(20) implicitly contain the solution of the target calculation (8)–(9). Since these equations are linear, it is easy to compute the set of admissible input perturbations, as detailed below.

Remark 7. Obviously, there exist many alternative robust receding horizon control formulations that can also be generalized to incorporate the results in this paper. For example, one could adopt a less conservative framework, such as the ones proposed in [15, 19, 22, 38]. However, this will come at a greater cost in off-line and/or on-line implementation. We believe that the pre-stabilizing framework adopted here is a practical approach that can be used to illustrate the main ideas of this paper without introducing unnecessary detail. The results in this paper are easily extended using the ideas in [15,19,22,38].

Before proceeding, we need to define some notation. Let the horizon length N be a positive integer and the block vectors $\mathbf{v} \in \mathbb{R}^{mN}$, $\mathbf{s} \in \mathbb{R}^{p(N-1)}$, $\mathbf{d} \in \mathbb{R}^{rN}$ be defined as $\mathbf{v} := (v_0, v_1, \cdots, v_{N-1})$, $\mathbf{s} := (s_1, s_2, \cdots, s_{N-1})$ and $\mathbf{d} := (d_0, d_1, \cdots, d_{N-1})$. Note that $\mathbf{s}$ and all related terms are present only if $N > 1$. Let ξ_i denote the predicted solution to (20) i time steps into the future (time $k+i$), given the augmented state $\xi := \xi(k)$ at the current time k, a finite sequence of control perturbations $\mathbf{v}$, the current setpoint $s_0 := s := s(k)$, a finite sequence of future setpoints $\mathbf{s}$ and a finite sequence of future disturbances $\mathbf{d}$. The corresponding predicted plant state $x_i := [I_n \; 0]\xi_i$ and input $u_i := \mathscr{K}\xi_i + \mathscr{L}s_i + v_i$ are similarly defined.

Given the above, we can now define the set of admissible input perturbations $\mathscr{V}_N(\xi, s)$ as the set of input perturbations of length N such that for all allowable future setpoint sequences of length $N-1$ and disturbance sequences of length N, the input constraints $\mathscr{U}$ are satisfied over the horizon $i = 0, \ldots, N-1$, the state constraints $\mathscr{X}$ are satisfied over the horizon $i = 0, \ldots, N-1$ and the augmented state at the end of the horizon is in $\mathscr{O}_\infty$ (hence the predicted plant state at the end of the horizon is also in $\mathscr{X}$), i.e.

$$\mathscr{V}_N(\xi,s) := \left\{ \mathbf{v} \in \mathbb{R}^{mN} \;\middle|\; \begin{array}{r} \xi_0 = \xi,\ s_0 = s, \\ \xi_{i+1} = \mathscr{A}_{\mathscr{K}}\xi_i + \mathscr{B}v_i + \mathscr{E}d_i + \mathscr{F}s_i,\ i = 0,\ldots,N-1, \\ x_i = \begin{bmatrix} I_n & 0 \end{bmatrix} \xi_i \in \mathscr{X},\ i = 0,\ldots,N-1,\ \xi_N \in \mathscr{O}_\infty, \\ u_i = \mathscr{K}\xi_i + \mathscr{L}s_i + v_i \in \mathscr{U},\ i = 0,\ldots,N-1 \\ \text{for all } \mathbf{s} \in \mathscr{S}^{N-1} \text{ and all } \mathbf{d} \in \mathscr{D}^N \end{array} \right\}. \tag{21}$$

Remark 8. Note that the predictions in (21) take into account the solution of the target calculation in (8)–(9) on the control and state trajectories over the horizon $i = 0,\ldots,N$. This is the fundamental difference between this paper and existing results in receding horizon control and therefore allows one to provide a guarantee of robust constraint satisfaction during transients.

In order to compute the receding horizon controller, we need to define a cost and set up an appropriate finite horizon optimal control problem (FHOCP). We choose to define the FHOCP to be solved for the current augmented state ξ and setpoint s, as

$$V_N^*(\xi,s) := \min_{\mathbf{v} \in \mathscr{V}_N(\xi,s)} V_N(\xi,s,\mathbf{v}), \tag{22}$$

where the cost to be minimized is

$$\begin{aligned} V_N(\xi,s,\mathbf{v}) := \sum_{i=0}^{N-1} (\tilde{x}_i - \overline{x}^*(\xi,s))^T Q (\tilde{x}_i - \overline{x}^*(\xi,s)) + (\tilde{u}_i - \overline{u}^*(\xi,s))^T R (\tilde{u}_i - \overline{u}^*(\xi,s)) \\ + (\tilde{x}_N - \overline{x}^*(\xi,s))^T P (\tilde{x}_N - \overline{x}^*(\xi,s)), \end{aligned} \tag{23}$$

with the matrices $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ and $P \in \mathbb{R}^{n\times n}$ positive definite. The vectors $\tilde{x}_i \in \mathbb{R}^n$ and $\tilde{u}_i \in \mathbb{R}^m$ are defined as

$$\tilde{x}_0 = x = \begin{bmatrix} I_n & 0 \end{bmatrix} \xi, \tag{24a}$$

$$\tilde{x}_{i+1} = A\tilde{x}_i + B\tilde{u}_i + (x - \hat{x} + \hat{d}), \quad i = 0,\ldots,N-1, \tag{24b}$$

$$\tilde{u}_i = \overline{u}^*(\xi,s) + K(\tilde{x}_i - \overline{x}^*(\xi,s)) + v_i, \quad i = 0,\ldots,N-1. \tag{24c}$$

The target state $\overline{x}^*(\xi,s)$ and target input $\overline{u}^*(\xi,s)$ are given by (8)–(9).

Remark 9. Note that the choice of cost (23) corresponds to assuming the disturbance sequence is constant over the control horizon and equal to the current estimate, and similarly with the setpoint. However, note that in the definition of the constraints on the input perturbations (21) it is assumed that the disturbance and setpoint may change over the prediction horizon. As will be shown below, these two facts can be combined to guarantee, respectively, offset-free control in the limit and robust constraint satisfaction during transients.

The minimizer of the FHOCP (22) is defined as

$$\mathbf{v}^*(\xi,s) := \left(v_0^*(\xi,s),\dots,v_{N-1}^*(\xi,s)\right) := \operatorname*{arg\,min}_{\mathbf{v}\in\mathscr{V}_N(\xi,s)} V_N(\xi,s,\mathbf{v})\,. \tag{25}$$

As is standard in receding horizon control, for the current augmented state ξ and setpoint s, we only keep the first element $v_0^*(\xi,s)$ of the solution to the FHOCP. Using this receding horizon principle, we define our receding horizon control input as

$$u(k) = \mathscr{K}\xi(k) + \mathscr{L}s(k) + v_0^*(\xi(k),s(k))\,. \tag{26}$$

Properties of the Receding Horizon Controller

It is well-known that if there are disturbances present and the receding horizon control action is computed by optimizing over *open-loop input sequences*, rather than *feedback policies*, then the optimization problem may become infeasible for large control horizons [3, 22]. Since we are not optimizing over feedback policies, but over a sequence of *perturbations* to a stabilizing control law (which is equivalent to optimizing over open-loop input sequences if the system is open-loop stable and the chosen control law is zero), it is important to be able to characterize the properties of the set of plant states for which one can guarantee that the FHOCP (22) has a solution.

Fortunately, optimizing over perturbations to a stabilizing control law does not suffer the same drawbacks as optimizing over open-loop input sequences [14, 16–18, 20]. However, since we are including the solution of the target calculator in our control input and predictions, it is still important to verify that feasibility of the optimization problem is not lost in the formulation presented in this paper. We will therefore proceed to show that if the FHOCP (22) is feasible at time $k = 0$, then the FHOCP (22) is feasible at all future time instants and offset-free control is guaranteed in the limit.

The set of plant states $X_N^{\mathbf{v}}$ for which one can initialize the controller state $(\hat{x},\hat{d})$ such that the set of admissible input perturbations $\mathscr{V}_N(\xi,s)$ is non-empty for all $s\in\mathscr{S}$ (and hence the FHOCP (22) has a solution), is given by

$$X_N^{\mathbf{v}} := \left\{x\in\mathscr{X}\;\middle|\;\exists(\hat{x},\hat{d})\in\mathbb{R}^{2n} \text{ such that } \mathscr{V}_N(\xi,s)\neq\emptyset \text{ for all } s\in\mathscr{S}\right\}. \tag{27}$$

As will be shown below, $X_N^{\mathbf{v}}$ is the set of plant states in $\mathscr{X}$ for which the controlled variable will ultimately be driven to the setpoint $\bar{s}$ by the receding horizon controller.

We can now give our first main result:

Theorem 3 (Domain of RHC law). *Suppose that Assumptions 1–5 hold. If X_0 is defined as in* (18) *and each $X_j^{\mathbf{v}}$, $j \in \{1,\dots,N\}$, is defined as in* (27) *with $N = j$, then all the sets in $\{X_0, X_1^{\mathbf{v}}, \dots, X_N^{\mathbf{v}}\}$ contain the origin in their interior and satisfy*

$$X_0 \subseteq X_1^{\mathbf{v}} \subseteq \cdots \subseteq X_{N-1}^{\mathbf{v}} \subseteq X_N^{\mathbf{v}}. \tag{28}$$

Proof. See the Appendix. □

Theorem 3 is very important because it shows that, under the above assumptions, the set of states for which the FHOCP (22) has a solution is non-empty and an increase in the horizon length does not decrease the size of the set of initial plant states for which the controlled variable can be driven to the setpoint. Furthermore, it also implies that the domain of attraction of the receding horizon controller contains the domain of attraction of the linear (unconstrained) controller defined in the previous section.

We now proceed to give conditions under which one can guarantee offset-free control in addition to robust constraint satisfaction. For this purpose, we introduce the following assumption:

Assumption 6. The matrices Q and R are chosen to be positive definite, the matrix P is the positive definite solution of the discrete algebraic Riccati equation $P = Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA$, and the matrix K is the corresponding gain $K = -(R + B^T PB)^{-1} B^T PA$. $\mathscr{K}$ is given by (11) with Γ given by (13), $\mathscr{A}_{\mathscr{K}} := \mathscr{A} + \mathscr{B}\mathscr{K}$ and $\mathscr{L}$ is given by (13).

Before giving our second main result, we need the following two lemmas, which are generalizations of results in [17]:

Lemma 4 (FHOCP equivalence). *Suppose that Assumptions 1 and 6 hold. If the matrix $W \in \mathbb{R}^{m\times m}$ is given by $W := R + B^T PB$, then $\mathbf{v}^*(\xi, s) = \operatorname{argmin}_{\mathbf{v}} \left\{ \sum_{i=0}^{N-1} v_i^T W v_i \mid \mathbf{v} \in \mathscr{V}_N(\xi, s) \right\}$.*

Proof. A similar result, for robust receding horizon controllers that do not provide offset-free control, is well-known [17, Rem. 3]. A detailed proof for the offset-free receding horizon controller proposed in this paper is reported in [36]. □

Lemma 5 (Robust feasibility and perturbation sequence). *Suppose that Assumptions 1–3 and 5–6 hold. If the set $\mathscr{V}_N(\xi(0), s(0))$ is non-empty, then the set $\mathscr{V}_N(\xi(k), s(k))$ is non-empty for all $k \in \mathbb{N}$ and*

$$\lim_{k\to\infty} v_0^*(\xi(k), s(k)) = 0.$$

Proof. The proof follows fairly standard arguments in receding horizon control and is reported in [36]. □

Theorem 4 (Offset-free control, robust constraint satisfaction and stability of RHC). *Let Assumptions 1–3 and 5–6 hold. One can choose the initial controller state $(\hat{x}(0), \hat{d}(0))$ such that the FHOCP* (22) *has a solution and the* *evolution of the augmented system* (7) *in closed-loop with the receding horizon control* (26) *satisfies* (4) *if and only if the initial plant state* $x(0) \in X_N^{\mathbf{v}}$. *Furthermore,* $\bar{\xi} := (I - \mathscr{A}_{\mathscr{K}})^{-1}(\mathscr{E}\bar{d} + \mathscr{F}\bar{s})$ *is the robustly asymptotically stable fixed point of* (20) *if it is in the interior of* $\mathscr{O}_\infty$.

Proof. See the Appendix. □

Implementation of the Receding Horizon Controller

Recall from (21) that the constraints on the input perturbations have to hold for all allowable setpoint and disturbance sequences. In this section, we point out the fact that the (infinite) set of constraints in (21) is easily rewritten in terms of a *finite* and *tractable* set of linear inequality constraints. This then allows one to compute the receding horizon control action by setting up and solving a tractable, strictly convex quadratic program (QP) at each sample instant.

Since $\mathscr{X}$, $\mathscr{U}$ and $\mathscr{O}_\infty$ are polyhedral sets with non-empty interiors, they are given by a finite number of affine inequality constraints. As a consequence, one can obtain an expression for the set of admissible input perturbations $\mathscr{V}_N(\xi, s)$ as

$$\mathscr{V}_N(\xi,s) = \left\{ \mathbf{v} \in \mathbb{R}^{mN} \;\middle|\; F\mathbf{v} \le b + G^d\mathbf{d} + G^s\mathbf{s} + H^\xi\xi + H^s s,\ \forall \mathbf{s} \in \mathscr{S}^{N-1}, \mathbf{d} \in \mathscr{D}^N \right\}, \tag{29}$$

where the matrices $F \in \mathbb{R}^{q\times mN}$, $G^d \in \mathbb{R}^{q\times rN}$, $G^s \in \mathbb{R}^{q\times p(N-1)}$, $H^\xi \in \mathbb{R}^{q\times 3n}$, $H^s \in \mathbb{R}^{q\times p}$ and the vector $b \in \mathbb{R}^q$ depend on the augmented system dynamics (20) (see [36] for the exact expressions).

By applying Proposition 1 one can compute an equivalent expression for $\mathscr{V}_N(\xi,s)$ in terms of a finite number of affine inequality constraints:

$$\mathscr{V}_N(\xi,s) = \left\{ \mathbf{v} \in \mathbb{R}^{mN} \;\middle|\; F\mathbf{v} \le c + H^\xi\xi + H^s s \right\}, \tag{30}$$

where

$$c := b + \min_{\mathbf{d}\in\mathscr{D}^N} G^d\mathbf{d} + \min_{\mathbf{s}\in\mathscr{S}^{N-1}} G^s\mathbf{s}. \tag{31}$$

Since $\mathscr{D}$ and $\mathscr{S}$ (and hence $\mathscr{D}^N$ and $\mathscr{S}^{N-1}$) are polytopes and can therefore be described by a finite number of affine inequality constraints, c can be computed efficiently by solving q LPs. However, if $\mathscr{D}$ and $\mathscr{S}$ are given only by upper and lower bounds on the components of d and s, respectively, then it is not necessary to solve LPs in order to compute c; computing the absolute values of the the components of G^d and G^s is sufficient. For example, let the disturbance and the setpoint take on values in the hypercubes $\mathscr{D} := \{ d \in \mathbb{R}^r \mid \|d\|_\infty \le \beta \}$ and $\mathscr{S} := \{ s \in \mathbb{R}^p \mid \|s\|_\infty \le \eta \}$. Recall now that the ∞-norm is the dual norm of the 1-norm [39], i.e. $\|a\|_1 = \max\left\{ a^T x \mid \|x\|_\infty \le 1 \right\}$ for any vector a. Hence, it is easy to show that

$$c = b - \beta\,\mathrm{abs}(G^d)\mathbf{1} - \eta\,\mathrm{abs}(G^s)\mathbf{1}, \tag{32}$$

where $\mathrm{abs}(M)$ is the matrix of the absolute values of the corresponding components of the matrix M and $\mathbf{1}$ is a column vector of ones of appropriate length.

It is also important to note that the number of constraints q in (30) is not dependent on the description for $\mathscr{S}$ and $\mathscr{D}$, but only dependent on N and the number of constraints that describe $\mathscr{X}$, $\mathscr{U}$ and $\mathscr{O}_\infty$. As a matter of fact, it is easy to show that the number of constraints $q = O(N)$.

Given all of the above, it is now clear that the solution to the FHOCP (22) exists if and only if $\mathscr{V}_N(\xi, s) \neq \emptyset$. The solution of the FHOCP (22) is the solution to the following tractable, strictly convex QP:

$$\mathbf{v}^*(\xi, s) = \arg\min_{\mathbf{v}} \left\{ \sum_{i=0}^{N-1} v_i^T W v_i \;\middle|\; F\mathbf{v} \leq c + H^\xi \xi + H^s s \right\}, \tag{33}$$

where $W := R + B^T B$ as in Lemma 4.

Illustrative example

Process and constraints

As an example, we consider a jacketed continuous stirred tank reactor (CSTR) studied by Henson and Seborg [40] in which an irreversible liquid-phase reaction occurs. A detailed nonlinear model has two states (reactant concentration and reactor temperature), one input (cooling liquid temperature) and two disturbances (feed temperature and feed reactant concentration). This CSTR shows three steady states, two of which are open-loop unstable, and for quality and safety reasons the middle conversion open-loop unstable steady-state is chosen as a nominal operating setpoint. Using a sampling time of $t_s = 0.1$ min and introducing deviation variables (from the corresponding steady state) a linearized model is as follows:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.7776 & -0.0045 \\ 26.6185 & 1.8555 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} -0.0004 \\ 0.2907 \end{bmatrix} u(k) + \begin{bmatrix} -0.0002 & 0.0893 \\ 0.1390 & 1.2267 \end{bmatrix} \begin{bmatrix} d_1(k) \\ d_2(k) \end{bmatrix}$$

$$z(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix},$$

in which x_1 and x_2 represent the reactant concentration and the reactor temperature, respectively; u represents the coolant temperature; d_1 and d_2 represent the feed temperature and the feed reactant concentration, respectively. Notice from the structure of C_z that the controlled variable is the reactor temperature, for which offset-free control to the setpoint s is required. The following constraints on the plant states and input and on the admissible disturbances and setpoint are considered:

$$\begin{bmatrix} -0.5 \\ -5 \end{bmatrix} \le \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 0.5 \\ 5 \end{bmatrix}, \qquad -15 \le u \le 15, \qquad \begin{bmatrix} -2 \\ -0.1 \end{bmatrix} \le \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \le \begin{bmatrix} 2 \\ 0.1 \end{bmatrix}, \qquad -1 \le s \le 1\,.$$

Results and comments

We present in Figure 2 the domain of attraction (i.e. $X_N^{\mathbf{v}}$) of four receding horizon controllers using different fixed horizons (specified in the figure) and the same penalty matrices: $Q = I_2$ and $R = 0.2$. Notice that X_0 is the domain of attraction of the linear controller. As expected from Theorem 3 we have that an increase in the fixed horizon length results in a larger feasible region and also that the domain of attraction of the linear controller is included in that of the receding horizon controllers.

We present in Figure 3 the closed-loop simulation results (controlled variable and input) obtained with four receding horizon controllers using the same fixed horizon, $N = 10$, different penalty matrices ($Q = I_2$ for all controllers and R specified in the figure). The initial plant state is $x(0) = \begin{bmatrix} -0.258 & 5 \end{bmatrix}^T$, the disturbances and the setpoint vary during the simulation time as reported in Table 1. The initial controller state $(\hat{x}(0), \hat{d}(0))$ (as well as the initial perturbation term) is computed from the following strictly convex QP:

$$\big(\hat{x}(0), \hat{d}(0), \mathbf{v}^*(\xi(0), s(0))\big) := \arg\min_{\hat{x},\hat{d},\mathbf{v}} \left\{ \lambda\left((\hat{x} - x(0))^T(\hat{x} - x(0)) + \hat{d}^T\hat{d}\right) + \sum_{i=0}^{N-1} v_i^T W v_i \;\middle|\; F\mathbf{v} \le c + H^{\xi}\xi(0) + H^s s(0) \right\},$$

in which $\lambda = 1000$.

For the receding horizon controller based on $Q = I_2$ and $R = 0.2$ the plant state sequence, $x(\cdot)$, is also shown in Figure 2. Notice that the state sequence $x(\cdot)$ initially starts at the boundary of the domain of attraction $X_{10}^{\mathbf{v}}$ and enters the domain of attraction of the linear controller X_0 in finite time. As expected from Theorem 4 the proposed controllers asymptotically drive the controlled variable to the asymptotic setpoint despite the presence of persistent unmeasured disturbances. Also, when the setpoint is changed the controllers drive the controlled variable to the new setpoint. Moreover, it is interesting to notice that the choice of penalty matrices has a direct impact on the closed-loop performance. As expected, when a lower input penalty R is chosen, the disturbance is rejected (and the setpoint is reached) more quickly and a larger control input is used.

In Figure 4 we present the closed-loop simulation results obtained with the same four controllers as in Figure 3 when the plant is described by the "original" nonlinear system [40]. From these results it is clear that the proposed controllers are able to achieve offset-free control even when there is a mismatch between the plant model and the actual plant dynamics. As mentioned after Lemma 3, as long as the closed-loop system is stable, it is possible to show that offset-free control holds independently of the actual plant dynamics.

We finally present in Figure 5 a comparison of the proposed receding horizon controller with a “standard” (i.e. non offset-free) robust receding horizon controller. As an example we chose the approach in [17], which is similar to the one proposed in this paper, in the sense that a pre-stabilizing gain matrix is used and the plant state prediction at the end of the horizon in restricted to be in the maximal disturbance invariant set $\mathscr{O}_\infty$. Both controllers are based on the same stabilizing gain matrix K, which is the optimal LQR gain with $Q = I_2$ and $R = 0.2$. The fixed horizon used for both controllers is $N = 10$ and the perturbation penalty for the “standard” controller is chosen as $W = R + B^T PB$ with P the solution to the corresponding steady-state Riccati equation. The initial plant state is $x(0) = \begin{bmatrix} -0.258 & 5 \end{bmatrix}^T$ and the disturbance varies as specified in Table 1. In this comparison the setpoint is the origin since the method in [17] does not apply to setpoints different from the origin (an extension of [17] to the setpoint tracking problem has been proposed in [16]; however, the controller proposed in [16] still does not guarantee offset-free control). As expected, the goal of offset-free control is achieved by the proposed method whereas the controller of [17] leaves a significant and undesired steady-state offset.

Conclusions

This paper has shown how one can design a dynamic receding horizon controller that guarantees robust constraint satisfaction, robust stability and offset-free control in the presence of asymptotically constant disturbances and setpoints. The design of the controller was split into two parts:

- *The design of a dynamic linear time-invariant controller*. A deadbeat observer is used to estimate the disturbance, the new steady state is given as a linear function of the current plant and observer states and of the current setpoint, and the controller aims to regulate the plant state and input to the new target steady-state.

- *The design of a dynamic nonlinear time-invariant receding horizon controller*. In order to increase the region of attraction of the linear controller a robust receding horizon controller, which computes perturbations to the linear control law, was proposed. The receding horizon controller includes the state and input constraints explicitly in its computations, as well as the transient effect of the unknown disturbance and time-varying setpoint on the target calculator and closed-loop response, thereby guaranteeing robust constraint satisfaction. It was shown that the specific formulation of the proposed receding horizon controller improves on the linear controller in terms of the domain of attraction. The proposed controller is computationally tractable since one has to solve, at each sampling time, a QP whose dimension increases linearly with an increase in the horizon length.

The paper also demonstrated the effectiveness of using the results in this paper in designing a controller for guaranteeing offset-free control of a continuous stirred tank reactor with respect to existing non offset-free algorithms. The simulation results were shown to be in agreement with the theory.

We conclude this paper with some recommendations on how the results in this paper may be extended:

- The choice of auxiliary system (i.e. observer) has an impact on the region of attraction and closed-loop performance of the system. A more detailed investigation into this topic could be undertaken.

- The constraints on the state and input were not included in the target calculation in (8). If the constraints are included in the target calculation, then the optimal steady-state target is no longer a linear function of the augmented state and setpoint. Clearly, this complicates the receding horizon controller design. However, the inclusion of constraints in the target calculation will enlarge the domain of attraction and increase the size of the disturbance and setpoints that can be handled by the controller. An extension of this paper, which includes constraints in the target calculation, could combine the results in [28] with those in [16].

- Due to the requirement that all setpoint sequences need to be handled, the approach presented in this paper is potentially conservative, since the maximal constraint-admissible robust positively invariant set $\mathscr{O}_\infty$ may be small or empty for the given range of setpoints. Further work could involve removing this source of conservativeness to allow a larger range of setpoints to be tracked without offset. Once again, this may be possible by combining results in [28] with those in [16].

- Clearly, the rank condition in (3) is not always satisfied. If this assumption is violated, then one might have to relax the requirement that offset-free control be achieved on all controlled variables. One possible approach to resolving this problem is to prioritize the controlled variables when performing the target calculation. The framework proposed in [41] may be useful in this context.

- Rather than optimizing over perturbations to a pre-stabilizing control law, one could consider optimizing over feedback policies [2, 3, 15, 19, 22]. This will enlarge the region of attraction of the receding horizon controller at the expense of an increase in computational complexity.

- The important problem of guaranteeing robust stability, performance, constraint satisfaction and offset-free control when output feedback (rather than state feedback) is used, remains to be addressed.

References

[1] F Blanchini. Set invariance in control. *Automatica*, 35:1747–1767, 1999. Survey paper.

[2] D Q Mayne. Control of constrained dynamic systems. *European Journal of Control*, 7:87–99, 2001. Survey paper.

[3] D Q Mayne, J B Rawlings, C V Rao, and P O M Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36:789–814, 2000. Survey paper.

[4] R Findeisen, L Imsland, F Allgöwer, and B A Foss. State and output feedback nonlinear model predictive control: An overview. *European Journal of Control*, 9(2–3):190–206, 2003. Survey paper.

[5] M Morari and J H Lee. Model predictive control: Past, present and future. *Computers & Chemical Engineering*, 23(4):667–682, 1999. Survey paper.

[6] F Blanchini. Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions. *IEEE Trans. Automatic Control*, 39(2):428–433, 1994.

[7] D Q Mayne and W R Schroeder. Robust time-optimal control of constrained linear systems. *Automatica*, 33:2103–2118, 1997.

[8] A Bemporad, A Casavola, and E Mosca. Nonlinear control of constrained linear systems via predictive reference management. *IEEE Trans. Automatic Control*, 42:340–349, 1997.

[9] A Bemporad. Reference governor for constrained nonlinear systems. *IEEE Trans. Automatic Control*, 43(3):415–419, March 1998.

[10] A Bemporad and E Mosca. Fulfilling hard constraints in uncertain linear systems by reference managing. *Automatica*, 34(4):451–461, 1998.

[11] E G Gilbert and I Kolmanovsky. Nonlinear tracking control in the presence of state and control constraints: A generalized reference governor. *Automatica*, 38:2063–2073, 2002.

[12] E G Gilbert, I Kolmanovsky, and K T Tan. Discrete-time reference governors and the nonlinear control of systems with state and control constraints. *International Journal of Robust and Nonlinear Control*, 5:487–504, 1995.

[13] T A Badgwell. Robust model predictive control of stable linear systems. *International Journal of Control*, 68(4):797–818, 1997.

[14] A Bemporad. Reducing conservativeness in predictive control of constrained systems with disturbances. In *Proc. 37th IEEE Conference on Decision and Control*, 1998.

[15] A Bemporad, F Borrelli, and M Morari. Min-max control of constrained uncertain discrete-time linear systems. *IEEE Trans. Automatic Control*, 48(9):1600–1606, September 2003.

[16] L Chisci and G Zappa. Dual mode predictive tracking of piecewise constant references for constrained linear systems. *Int. J. Control*, 76(1):61–72, 2003.

[17] L Chisci, J A Rossiter, and G. Zappa. Systems with persistent disturbances: predictive control with restricted constraints. *Automatica*, 37:1019–1028, 2001.

[18] E C Kerrigan and J M Maciejowski. On robust optimization and the optimal control of constrained linear systems with bounded state disturbances. In *Proc. European Control Conference*, Cambridge, UK, September 2003.

[19] E C Kerrigan and J M Maciejowski. Feedback min-max model predictive control using a single linear program: Robust stability and the explicit solution. *International Journal of Robust and Nonlinear Control*, 14:395–413, 2004.

[20] Y I Lee and B Kouvaritakis. Constrained receding horizon predictive control for systems with disturbances. *Int. J. Control*, 72(11):1027–1032, 1999.

[21] D Q Mayne and W Langson. Robustifying model predictive control of constrained linear systems. *Electronics Letters*, 37:1422–1423, 2001.

[22] P O M Scokaert and D Q Mayne. Min-max feedback model predictive control for constrained linear systems. *IEEE Trans. Automatic Control*, 43:1136–1142, 1998.

[23] J B Rawlings, E S Meadows, and K R Muske. Nonlinear model predictive control: a tutorial and survey. In *ADCHEM Conference*, pages 203–214, Kyoto, Japan, 1994.

[24] J M Maciejowski. The implicit daisy-chaining property of constrained predictive control. *Appl. Math. and Comp. Sci.*, 8(4):695–711, 1998.

[25] L Magni, G De Nicolao, and R Scattolini. Output feedback and tracking of nonlinear systems with model predictive control. *Automatica*, pages 1601–1607, 2001.

[26] L Magni. On robust tracking with non-linear model predictive control. *Int. J. Control*, 75(6):399–407, 2002.

[27] K R Muske and T A Badgwell. Disturbance modeling for offset-free linear model predictive control. *J. Proc. Cont.*, 12:617–632, 2002.

[28] G Pannocchia and J B Rawlings. Disturbance models for offset-free model predictive control. *AIChE J.*, 49:426–437, 2003.

[29] V Sakizlis, V Dua, J D Perkins, and E N Pistikopoulos. Robust model-based tracking control using parametric programming. *Computers and Chemical Engineering*, 28:195–207, 2004.

[30] V Sakizlis, N M P Kakalis, V Dua, J D Perkins, and E N Pistikopoulos. Design of robust model-based controllers via parametric programming. *Automatica*, 40:189–201, 2004.

[31] I Kolmanovsky and E G Gilbert. Theory and computation of disturbance invariance sets for discrete-time linear systems. *Mathematical Problems in Engineering: Theory, Methods and Applications*, 4:317–367, 1998.

[32] P O M Scokaert, J B Rawlings, and E S Meadows. Discrete-time stability with perturbations: Application to model predictive control. *Automatica*, 33:463–470, 1997.

[33] H Kwakernaak and R Sivan. *Linear Optimal Control Systems*. John Wiley & Sons, 1972.

[34] K R Muske and J B Rawlings. Model predictive control with linear models. *AIChE J.*, 39:262–287, 1993.

[35] J Nocedal and S J Wright. *Numerical Optimization*. Springer, 1999.

[36] G Pannocchia and E C Kerrigan. Offset-free receding horizon control of constrained linear systems subject to time-varying setpoints and persistent unmeasured disturbances. Technical Report CUED/F-INFENG/TR.468, Department of Engineering, University of Cambridge, Trumpington Street, CB2 1PZ Cambridge, UK, November 2003.

[37] B A Francis and W M Wonham. The internal model principle of control theory. *Automatica*, 12:457–465, 1976.

[38] E C Kerrigan and J M Maciejowski. Properties of a new parameterization for the control of constrained systems with disturbances. In *Proc. American Control Conference*, Boston, MA, USA, June 2004.

[39] R A Horn and C R Johnson. *Matrix Analysis*. Cambridge University Press, UK, 1985.

[40] M A Henson and D E Seborg. *Nonlinear Process Control*. Prentice Hall PTR, Upper Saddle River, New Jersey, USA, 1997.

[41] E C Kerrigan and J M Maciejowski. Designing model predictive controllers with prioritised constraints and objectives. In *Proc. IEEE Conference on Computer Aided Control System Design*, Glasgow, Scotland, September 2002.

Appendix: Selected Proofs

Proof of Lemma 3

Since $\lim_{k\to\infty} s(k) = \bar{s}$ and $\lim_{k\to\infty} d(k) = \bar{d}$ we have from (14)–(15) and from Lemma 2 that

$$\xi_\infty := \lim_{k\to\infty} \xi(k) = \mathscr{A}_{\mathscr{K}} \xi_\infty + \mathscr{E}\bar{d} + \mathscr{F}\bar{s} = \mathscr{A}\xi_\infty + \mathscr{B}u_\infty + \mathscr{E}\bar{d}, \tag{34}$$

in which $u_\infty = \mathscr{K}\xi_\infty + \mathscr{L}\bar{s}$. Let ξ_∞ be partitioned as follows: $\xi_\infty = (x_\infty, \hat{x}_\infty, \hat{d}_\infty)$ in which each block is a column vector of length n. We can rewrite (34) explicitly as follows:

$$x_\infty = Ax_\infty + Bu_\infty + E\bar{d} \tag{35a}$$

$$\hat{x}_\infty = Ax_\infty + Bu_\infty + (x_\infty - \hat{x}_\infty + \hat{d}_\infty) \tag{35b}$$

$$\hat{d}_\infty = x_\infty - \hat{x}_\infty + \hat{d}_\infty \,. \tag{35c}$$

From (35c) we immediately obtain that $x_\infty = \hat{x}_\infty$ (note that $x_\infty = \hat{x}_\infty$ and hence the rest of the proof holds independently of (35a), i.e. independently of the actual plant dynamics and disturbance), which combined with (35b) leads to

$$x_\infty = Ax_\infty + Bu_\infty + (x_\infty - \hat{x}_\infty + \hat{d}_\infty) \,. \tag{36}$$

Let $(\bar{x}_\infty, \bar{u}_\infty)$ denote the solution to the target calculation problem (8) for the augmented state ξ_∞ and the setpoint $\bar{s}$. From (8b) and (36) we can write

$$x_\infty - \bar{x}_\infty = A(x_\infty - \bar{x}_\infty) + B(u_\infty - \bar{u}_\infty) = (A + BK)(x_\infty - \bar{x}_\infty) \,, \tag{37}$$

where the last step comes from (10). It is important to notice that (37) and Assumption 4 implies that

$$x_\infty = \bar{x}_\infty \,. \tag{38}$$

Finally, from (38) and from (8b) we obtain:

$$\begin{aligned} \bar{s} = C_z\bar{x}_\infty = C_z x_\infty = \mathscr{C}\xi_\infty \\ = \lim_{k\to\infty} \mathscr{C}\xi(k) \,. \end{aligned}$$

Proof of Theorem 2

Robust constraint satisfaction follows immediately from the fact that $\mathscr{O}_\infty$ is robust positively invariant for the closed-loop system (15) and the fact that $\mathscr{O}_\infty$ is constraint-admissible.

Since $\mathscr{A}_\mathscr{K}$ is strictly stable, $(I-\mathscr{A}_\mathscr{K})^{-1}$ exists and hence $\bar{\xi}$ is well-defined and unique. Note also from the proof of Lemma 3 that $\bar{\xi} = \xi_\infty := \lim_{k\to\infty}\xi(k)$. Robust asymptotic stability follows from Theorem 1 by defining

$$\zeta := \xi - \bar{\xi}, \qquad w := \mathscr{E}(d-\bar{d}) + \mathscr{F}(s-\bar{s}).$$

Hence, we can write the closed-loop system dynamics in terms of the "shifted" variables as $\zeta(k+1) = \mathscr{A}_\mathscr{K}\zeta(k) + w(k)$. The proof is completed by noting that $\lim_{k\to\infty} w(k) = 0$.

Proof of Theorem 3

Though a result, similar to the one stated here, appears to be known [17, Sect. 4.2], we have been unable to find a detailed proof in the literature. Classical robust "open-loop" receding horizon control [3, Sect. 4.5] is well-known to exhibit infeasibility problems if the plant is open-loop unstable and no pre-stabilizing policy is used in the predictions [22]. However, it is a remarkable fact that one can remove this problem by optimizing over a sequence of perturbations to a pre-stabilizing control law. To show that this is indeed still true for the control algorithm proposed in this paper, we present a detailed proof.

It follows trivially from Assumption 5 that X_0 contains the origin in its interior. The rest of the proof is by induction.

Let the plant state $x \in X_j^{\mathbf{v}}$, where $j \in \{1,\ldots,N-1\}$, the controller state $(\hat{x},\hat{d})$ be such that $\mathscr{V}_j(\xi,s)$ is non-empty and $\mathbf{v}_j := (v_0,\ldots,v_{j-1}) \in \mathscr{V}_j(\xi,s)$ be an admissible perturbation sequence of length j. Also, let $\mathbf{s}_{j-1} := (s_1,\ldots,s_{j-1}) \in \mathscr{S}^{j-1}$ and $\mathbf{d}_j := (d_0,\ldots,d_{j-1}) \in \mathscr{D}^j$ be allowable setpoint and disturbance sequences of length $j-1$ and j, respectively.

From the definition of $\mathscr{V}_j(\xi,s)$, it follows that $\xi_j \in \mathscr{O}_\infty$ for all $\mathbf{s}_{j-1} \in \mathscr{S}^{j-1}$ and all $\mathbf{d}_j \in \mathscr{D}^i$. Recall that $\mathscr{O}_\infty$ is disturbance invariant and constraint-admissible for the closed-loop system (15), hence $\mathscr{O}_\infty$ is disturbance invariant and constraint-admissible for system (20) under the infinite perturbation sequence $\{v(k)\}_{k=0}^\infty := \{0,0,\ldots\}$.

It follows that if $\xi_j \in \mathscr{O}_\infty$ for all $\mathbf{s}_{j-1} \in \mathscr{S}^{j-1}$ and all $\mathbf{d}_j \in \mathscr{D}^i$, then $\xi_{j+1} \in \mathscr{O}_\infty$ for all $\mathbf{s}_i \in \mathscr{S}^i$ and all $\mathbf{d}_{i+1} \in \mathscr{D}^{i+1}$.

This implies that if $\mathbf{v}_j \in \mathscr{V}_j(\xi,s)$, then $(\mathbf{v}_j,0) \in \mathscr{V}_{j+1}(\xi,s)$. Hence, if $\mathscr{V}_j(\xi,s)$ is non-empty, then $\mathscr{V}_{j+1}(\xi,s)$ is non-empty. It follows from the definition of $X_j^{\mathbf{v}}$ that if $x \in X_j^{\mathbf{v}}$, then $x \in X_{j+1}^{\mathbf{v}}$, hence $X_j^{\mathbf{v}} \subseteq X_{j+1}^{\mathbf{v}}$.

Using similar arguments as above, the result is completed by noticing that $X_0 \subseteq X_1^{\mathbf{v}}$.

Proof of Theorem 4

Sufficiency. Suppose that $x(0) \in X_N^{\mathbf{v}}$, then it immediately follows from (27) that for any initial setpoint $s(0) \in \mathscr{S}$ one can choose a controller state $(\hat{x}(0), \hat{d}(0))$ such that $\mathscr{V}_N(\xi(0), s(0)) \neq \emptyset$ and hence the FHOCP (22) has a solution. This implies from Lemma 5 we have that $\mathscr{V}_N(\xi(k)) \neq \emptyset$ for all $k \in \mathbb{N}$ and also that

$$v_\infty := \lim_{k\to\infty} v(k) := \lim_{k\to\infty} v_0^*(\xi(k), s(k)) = 0\,. \tag{39}$$

The fact that (4a) holds can now be shown exactly as in the proof of Lemma 3, since from (20) and (39) it follows that

$$\xi_\infty = \lim_{k\to\infty} \xi(k) = \mathscr{A}_{\mathscr{K}}\xi_\infty + \mathscr{B}v_\infty + \mathscr{E}\bar{d} + \mathscr{F}\bar{s} = \mathscr{A}_{\mathscr{K}}\xi_\infty + \mathscr{E}\bar{d} + \mathscr{F}\bar{s} = \mathscr{A}\xi_\infty + \mathscr{B}u_\infty + \mathscr{E}\bar{d},$$

in which $u_\infty = \mathscr{K}\xi_\infty + \mathscr{L}\bar{s} + v_\infty = \mathscr{K}\xi_\infty + \mathscr{L}\bar{s}$.

The fact that (4b) holds follows trivially from Lemma 5 and the definition of $\mathscr{V}_N(\cdot)$.

Necessity. This is obvious because if $x(0) \notin X_N^{\mathbf{v}}$, then we either have that $x(0) \notin \mathscr{X}$ or that there exists an $s(0) \in \mathscr{S}$ such that for all $(\hat{x}(0), \hat{d}(0)) \in \mathbb{R}^{2n}$, $\mathscr{V}_N(\xi(0), s(0)) = \emptyset$ and hence the control input is undefined at time 0.

Finally, robust asymptotic stability follows from Theorem 1 and can be shown in a similar fashion as in the proof of Theorem 2. This is because it is easy to show that for any $\xi \in \mathscr{O}_\infty$, the optimal perturbation $v_0^*(\xi, s) = 0$ for all $s \in \mathscr{S}$. Hence, we can write the closed-loop system dynamics in a neighborhood of $\bar{\xi}$, in terms of the "shifted" variables, as $\zeta(k+1) = \mathscr{A}_{\mathscr{K}}\zeta(k) + w(k)$. Again, the proof is completed by noting that $\lim_{k\to\infty} w(k) = 0$.

Table 1: Disturbances and setpoint

t (min)	$[0,4)$	$[4,8)$	$[8,12)$	$[12,16)$	$[16,24)$	$[20,24]$
d	$[2\ 0.1]^T$	$[2\ -0.1]^T$	$[-2\ -0.1]^T$	$[-2\ 0.1]^T$	$[2\ 0.1]^T$	$[-2\ 0.1]^T$
s	0	0	1	1	-1	-1

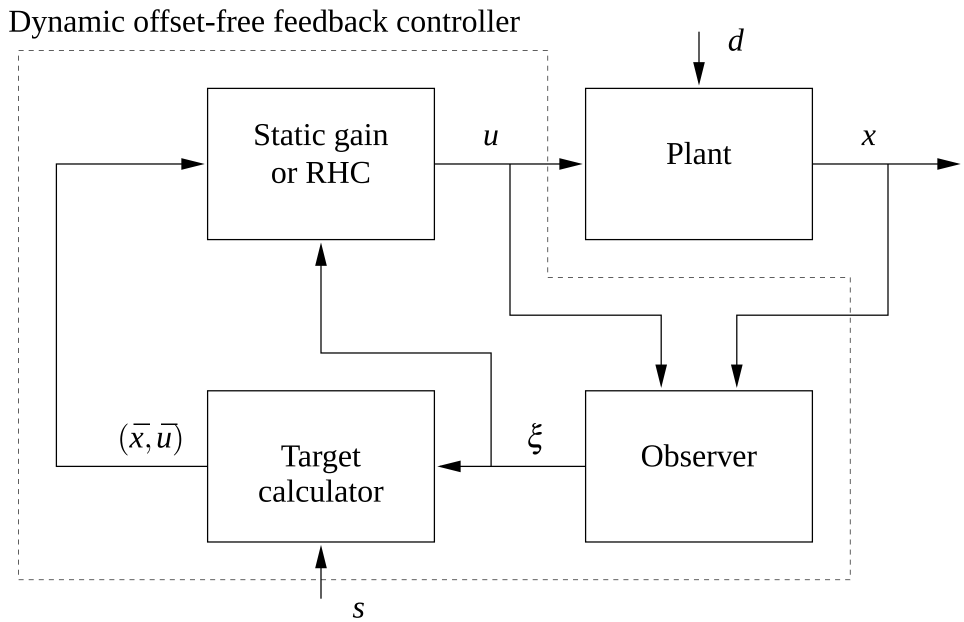


Figure 1: Graphical illustration of the proposed dynamic offset-free feedback controller

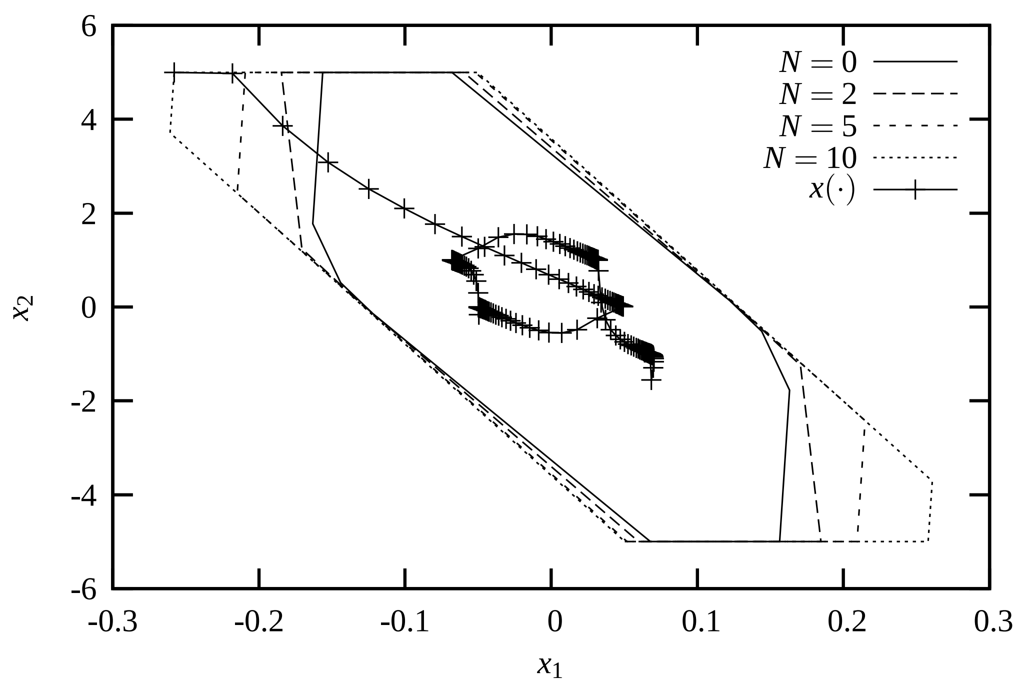


Figure 2: Domain of attraction ($X_N^{\mathbf{v}}$) for different fixed horizons

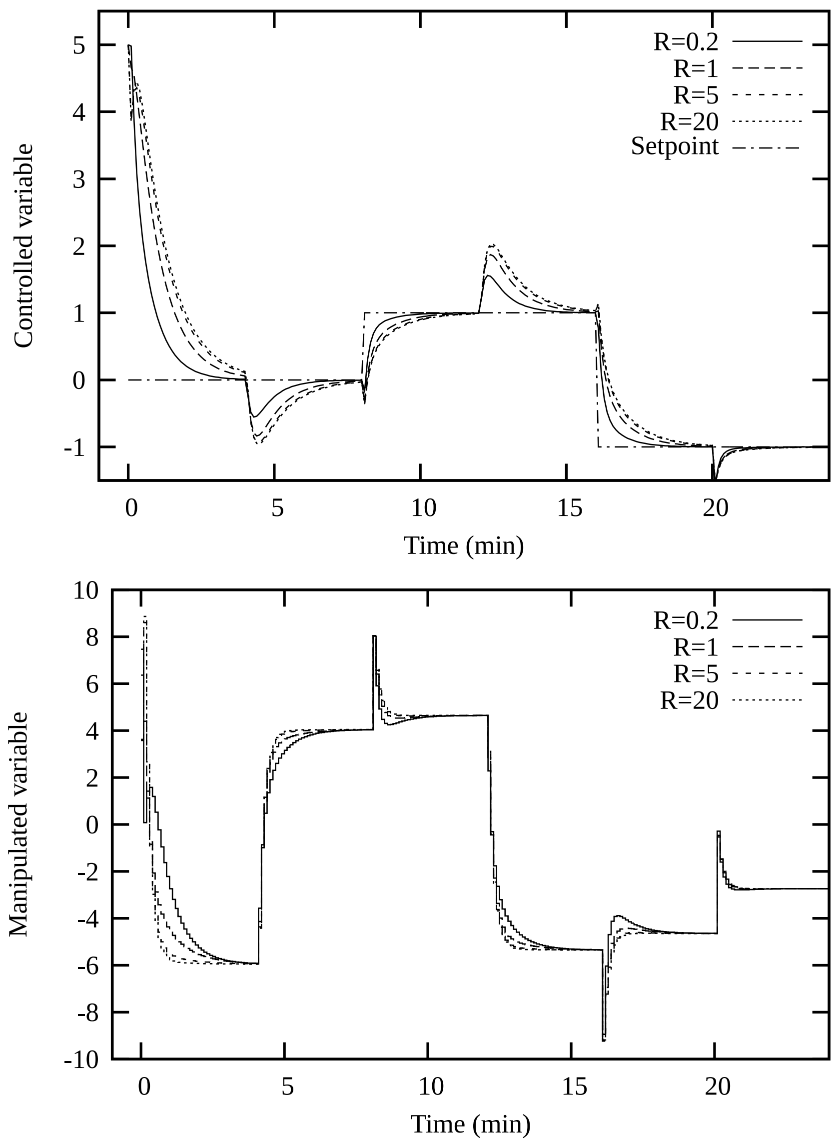


Figure 3: Closed-loop comparison of different receding horizon controllers (linear plant): controlled variable (top) and input (bottom)

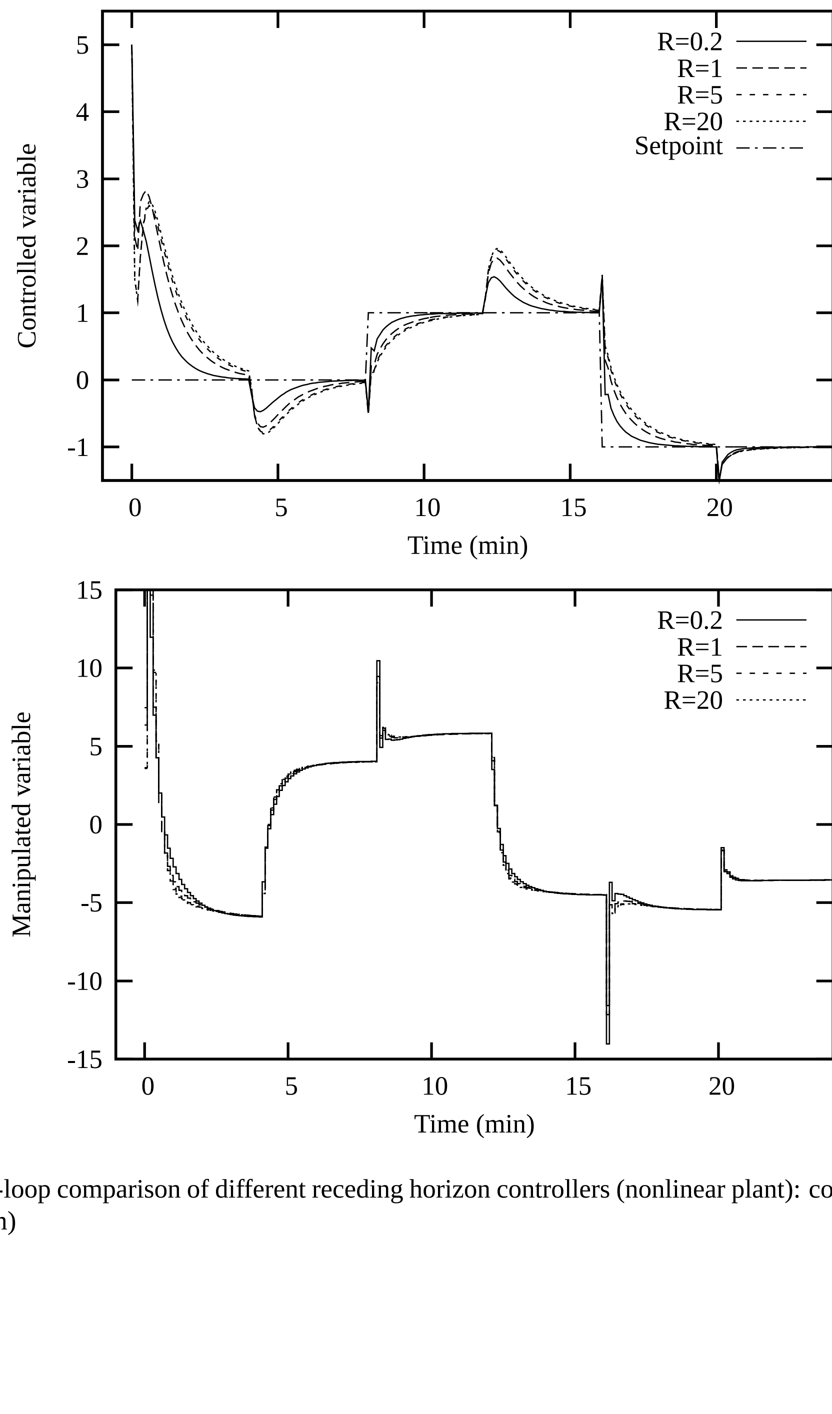
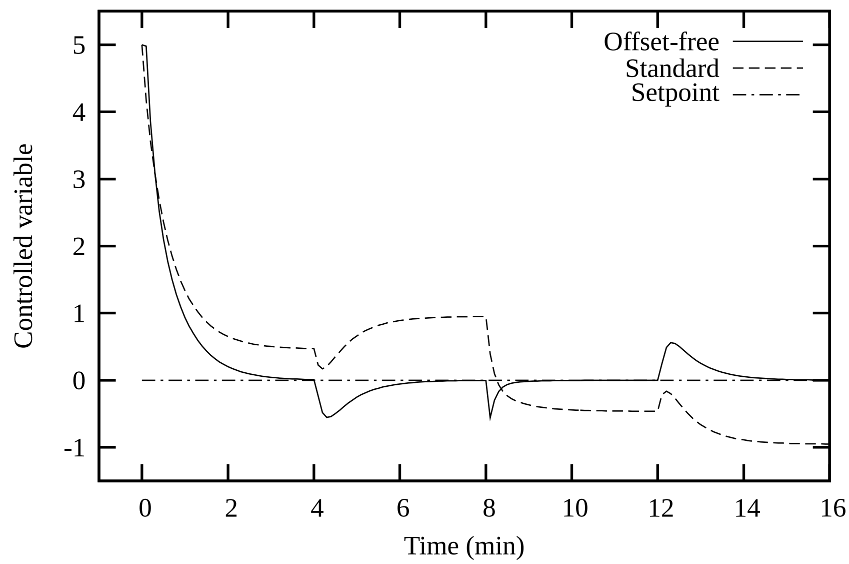


Figure 4: Closed-loop comparison of different receding horizon controllers (nonlinear plant): controlled variable (top) and input (bottom)



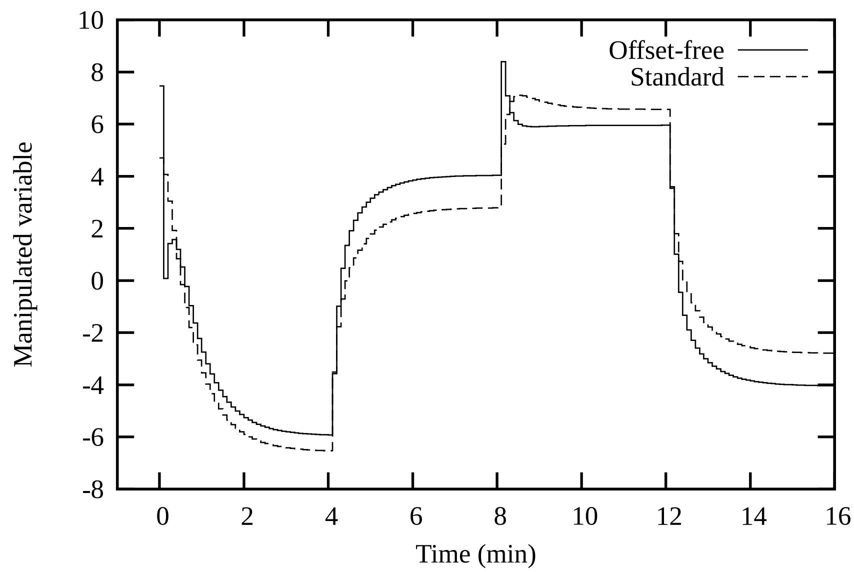


Figure 5: Closed-loop comparison of offset-free and standard robust receding horizon controllers: controlled variable (top) and input (bottom).