H_∞ Sampled-Data Synthesis and Related Numerical Issues

M. W. Cantoni* K. Glover

Department of Engineering, University of Cambridge, Cambridge CP2 1PZ, United Kingdom.

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Abstract

In this paper we present a new, compact derivation of state-space formulae for the so-called discretisation-based solution of the H_∞ sampled-data control problem. Our approach is based on the established technique of continuous time-lifting, which is used to isometrically map the continuous-time, linear, periodically time-varying, sampled-data problem to a discrete-time, linear, time-invariant problem. State-space formulae are derived for the equivalent, discrete-time problem by solving a set of two-point, boundary-value problems. The formulae accommodate a direct feed-through term from the disturbance inputs to the controlled outputs of the original plant and are simple, requiring the computation of only a single matrix exponential. It is also shown that the resultant formulae can be easily re-structured to give a numerically robust algorithm for computing the state-space matrices.

KEY WORDS : Sampled-Data Systems, H_∞-Synthesis, Numerical Issues.

1 Notation

The non-negative real numbers and the non-negative integers are denoted by $\mathbb{R}_+$ and $\mathbb{Z}_+$, respectively. $\mathrm{L}_2(\mathbb{R}_+, \mathbb{R}^n)$ denotes the Hilbert space of continuous-time signals, $f : \mathbb{R}_+ \to \mathbb{R}^n$, square (Lebesgue) integrable over the interval $[0, \infty)$. $\ell_2(\mathbb{Z}_+, \mathbb{R}^n)$ denotes the Hilbert space of discrete-time signals, $f : \mathbb{Z}_+ \to \mathbb{R}^n$, square summable over the interval $[0, \infty)$. $\mathcal{K}^n := \mathrm{L}_2([0, h), \mathbb{R}^n)$, denotes the Hilbert space of functions $f : [0, h) \to \mathbb{R}^n$, square (Lebesgue) integrable over the interval $[0, h)$. $\ell_2(\mathbb{Z}_+, \mathcal{K}^n)$ denotes the Hilbert space of discrete-time signals, $f : \mathbb{Z}_+ \to \mathcal{K}^n$. For notational convenience we may occasionally drop explicit reference to the vector space to which the signals are mapping (for example, $\mathrm{L}_2(\mathbb{R}_+, \mathbb{R}^n)$ may be written $\mathrm{L}_2(\mathbb{R}_+)$.)

Given a matrix M, M' denotes the transpose matrix and M^H the complex conjugate transpose matrix. $\lambda(M)$ denotes the set of eigen-values of M. For a bounded linear operator B, B^* denotes the adjoint operator and $\|B\|$ the operator induced-norm. The lower, linear, fractional transformation $\mathcal{F}_\ell(\cdot, \cdot)$, is defined to be $\mathcal{F}_\ell(G, K) := G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$, where G is partitioned appropriately. $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is used to denote a state-space realisation.

*All correspondence, fax : +44 1223 332 662 and e-mail : mwc@eng.cam.ac.uk

2 Introduction

In general, sampled-data (SD) systems are hybrid. That is, at some points in the system signals evolve in continuous time and at others in discrete time. Typically the plant evolves in continuous time and the controller in discrete time. It is well known that provided the sampling frequency is not pathological, the stability of an SD, control system can be determined by its behaviour at the discrete sampling instants (Chen and Francis, 1991). However, since the plant evolves in continuous time, performance criteria are most readily formulated in the continuous-time domain. With this in mind, a framework which captures both the inter-sample behaviour and the behaviour at the sampling instants is clearly sensible for the development of useful SD controllers. Given a linear, time-invariant (LTI) plant, such a framework has been developed for H_∞-based, SD synthesis and is based on the technique of continuous time-lifting (Bamieh et al., 1991).

In this paper we present a new, compact derivation of state-space formulae for the synthesis of SD controllers that achieve a specified bound on the closed-loop, $\mathrm{L}_2(\mathbb{R}_+)$-induced norm. Our approach is based on the established technique of continuous time-lifting (Bamieh et al., 1991). This technique is used to isometrically transform the continuous-time, linear, periodically, time-varying (LPTV), SD problem to an equivalent discrete-time, LTI problem. Formally, this transformation is achieved by introducing the time-lifting operator which is an isomorphism between the continuous-time signal-space $\mathrm{L}_2(\mathbb{R}_+)$ and the discrete-time signal-space $\ell_2(\mathbb{Z}_+, \mathcal{K})$. Time-Lifting all continuous-time signals in the SD closed-loop results in a system that is LTI and evolves in discrete-time. Although handling this discrete-time equivalent is complicated by the fact that the input and output spaces are infinite-dimensional, there exist various methods for determining a finite-dimensional equivalent (Bamieh and Pearson, 1992; Toivonen, 1992; Chen and Francis, 1995; Kabamba and Hara, 1993; Toivonen, 1993).

Perhaps the most simple explicit formulae for the state-space matrices of the finite-dimensional, discrete-time, equivalent problem are given in (Toivonen, 1993) and involve the computation of only a single matrix exponential. Toivonen (1993) derives a finite-dimensional, discrete-time, equivalent problem without using time-lifting. Instead he uses classical, linear-quadratic, optimal-control theory to give a closed-loop characterisation of the worst-case, inter-sample disturbance. Toivonen and Sagfors (1997) clarify the relationship between the solutions obtained via time-lifting and the linear-quadratic, optimal control approach. They point out that the time-lifting based solution can be interpreted as consisting of a lifting transformation followed by an open-loop characterisation of the worst-case, inter-sample disturbance. In this way the two methods are seen to be equivalent, although expressions for the finite-dimensional, discrete-time, equivalent problem were different. In this paper we derive, using only the time-lifting technique, essentially identical (except for the accommodation of a direct feed-through term D_{11}) state-space formulae to those obtained by Toivonen (1993), which indicates directly the equivalence of the two approaches. Our formulation involves solving of a set of two-point, boundary value problems (TPBVP) which characterise the behaviour of certain state-space operators that describe the infinite-dimensional, time-lifted, discrete-time equivalent problem and leads to state-space formulae that involve the computation of only a single matrix-exponential.

The state-space formulae derived involve the exponential of a Hamiltonian structured matrix expressible explicitly in terms of the state-space matrices of the original plant. Since the eigen-values of a Hamiltonian matrix are symmetric about the imaginary axis, direct computation of its exponential can be dominated by large terms if any of the eigen-values have large real part. These large terms can cancel or become small in the formulae used to determine the state-space matrices of the finite-dimensional equivalent. Hence, without care the formulae derived can be numerically sensitive. However, we show that the state-space formulae derived can be easily re-structured to give a numerically robust formulae for computation.

The structure of the paper is as follows: In Section 3, we give a brief introduction to the continuous time-lifting technique and state some well known results concerning the stability and closed-loop, induced norm of SD, control systems. With this foundation, in Section 4 we derive new formulae for constructing a state-space realisation of the SD-equivalent, finite-dimensional, discrete-time, LTI plant. In Section 5, we show how to re-structure the new formulae for numerically robustness and in section 6, we discuss two assumptions required to make this possible. Finally, in Section 7 we make some concluding remarks.

3 Continuous Time-Lifting and SD Systems

In this section we give a brief introduction to continuous time-lifting and state some well known results concerning the stability and closed-loop, induced norm of SD, control systems (Chen and Francis, 1990, 1991; Francis and Georgiou, 1988; Bamieh and Pearson, 1992; Toivonen, 1992; Yamamoto, 1994; Kabamba and Hara, 1993).

Consider the SD, control system shown in Figure 1. G is a finite-dimensional, LTI plant with state-space realisation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t),\ x(0) = 0, \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t), \\ y(t) &= C_2 x(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^{m_1}$ is the disturbance input, $u(t) \in \mathbb{R}^{m_2}$ is the control input, $z(t) \in \mathbb{R}^{p_1}$ is the controlled output and $y(t) \in \mathbb{R}^{p_2}$ is the measured output. $\mathcal{S}$ is an ideal sampling operator and we assume that it is periodic and synchronised with the zero-order hold $\mathcal{H}$. K is a discrete-time, LTI controller. Note that there are no direct feed-through terms from the disturbance and control inputs to the measured output. This is required to ensure boundedness of the sampling operator and it also limits frequency domain aliasing. Forcing D_{21} and D_{22} to be zero can be achieved by placing any strictly-causal, stable LTI filter on the measured outputs before the sampler and absorbing this into the plant.

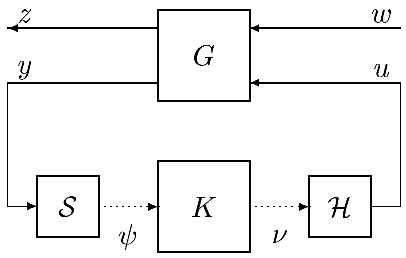

Figure 1: SD Control System

We introduce the isometric time-lifting operator $\mathcal{L}_h : \mathrm{L}_2(\mathbb{R}_+, \mathbb{R}^n) \to \ell_2(\mathbb{Z}_+, \mathcal{K}^n)$, defined by

$$\tilde{f}[k] = (\mathcal{L}_h f)[k], \;\; \tilde{f}[k] := f(kh+t), \;\; \text{for } \; 0 \leq t < h.$$

Note that $\mathcal{L}_h$ is a linear operator, that it is bijective and therefore invertible. Let $\tilde{w}[k] = (\mathcal{L}_h w)[k]$, $\tilde{z}[k] = (\mathcal{L}_h z)[k]$ and $x[k] = x(kh)$. Absorbing the sampling and hold operators into G and expressing the evolution of the system in terms of the state at the sampling instants kh, we obtain the discrete-time, system of state-space equations

$$\begin{aligned} x[k+1] &= A_d x[k] + \tilde{B}_1 \tilde{w}[k] + B_{2d}\nu[k], \; x[0] = 0, \\ \tilde{z}[k] &= \tilde{C}_1 x[k] + \tilde{D}_{11}\tilde{w}[k] + \tilde{D}_{12}\nu[k], \\ \psi[k] &= C_2 x[k], \end{aligned}$$

where

$$A_d : \mathbb{R}^n \to \mathbb{R}^n, \qquad A_d x = e^{hA}x, \tag{2}$$

$$B_{2d} : \mathbb{R}^{m_2} \to \mathbb{R}^n, \qquad B_{2d} = \int_0^h e^{(h-t)A} dt B_2, \tag{3}$$

$$\tilde{B}_1 : \mathcal{K}^{m_1} \to \mathbb{R}^n, \qquad \tilde{B}_1 \tilde{w} = \int_0^h e^{(h-\tau)A} B_1 \tilde{w}(\tau)\; d\tau, \tag{4}$$

$$\tilde{C}_1 : \mathbb{R}^n \to \mathcal{K}^{p_1}, \qquad (\tilde{C}_1 x)(t) = C_1 e^{tA} x, \tag{5}$$

$$\tilde{D}_{11} : \mathcal{K}^{m_1} \to \mathcal{K}^{p_1}, \qquad (\tilde{D}_{11}\tilde{w})(t) = D_{11}\tilde{w}(t) + \int_0^t C_1 e^{(t-\tau)A} B_1 \tilde{w}(\tau)\; d\tau \tag{6}$$

and

$$\tilde{D}_{12} : \mathbb{R}^{m_2} \to \mathcal{K}^{p_1}, \qquad (\tilde{D}_{12}\nu)(t) = D_{12}\nu + \int_0^t C_1 e^{(t-\tau)A}\; d\tau B_2 \nu. \tag{7}$$

These state-space equations describe the behaviour of the discrete-time, LTI operator

$$\tilde{G} = \begin{bmatrix} \mathcal{L}_h G_{11} \mathcal{L}_h^{-1} & \mathcal{L}_h G_{12} \mathcal{H} \\ \mathcal{S} G_{21} \mathcal{L}_h^{-1} & \mathcal{S} G_{22} \mathcal{H} \end{bmatrix}.$$

Provided the sampling is not pathological and since $\tilde{G}_{22}$ characterises the behaviour of G_{22} at the sampling instants, K will be internally stabilising for G iff it is internally stabilising for $\tilde{G}$ (Bamieh and Pearson, 1992; Chen and Francis, 1991). Furthermore, since the time-lifting operator is an isometry, it is clear that given a internally stabilising controller K, $\|\mathcal{F}_\ell(\tilde{G}, K)\| < \gamma$ iff $\|\mathcal{F}_\ell(G, \mathcal{H}K\mathcal{S})\| < \gamma$.

The synthesis of K to achieve a specified bound on $\|\mathcal{F}_\ell(\tilde{G}, K)\|$ is is complicated by the fact that the input and output spaces of the closed-loop are infinite-dimensional. Fortunately, there exist methods for transforming the problem into an equivalent finite-dimensional one (Bamieh and Pearson, 1992; Toivonen, 1992; Chen and Francis, 1995; Toivonen, 1993). Close inspection of the state-space operators $\tilde{B}_1 : \mathcal{K}^{m_1} \to \mathbb{R}^n$, $\tilde{C}_1 : \mathbb{R}^n \to \mathcal{K}^{p_1}$ and $\tilde{D}_{12} : \mathbb{R}^{m_2} \to \mathcal{K}^{p_1}$, reveals that they all have finite-rank and therefore finite-dimensional matrix representations. However, $\tilde{D}_{11} : \mathcal{K}^{m_1} \to \mathcal{K}^{p_1}$ does not have finite-rank and as such we would like to remove it. This can be achieved using an operator version of the loop-shifting principle.

Lemma 3.1 *(Bamieh and Pearson, 1992) Construct the LTI plant*

$$\begin{aligned}
\underline{G} &= \left[\begin{array}{c|cc} \underline{A} & \underline{B}_1 & \underline{B}_2 \\ \hline \underline{C}_1 & 0 & \underline{D}_{12} \\ C_2 & 0 & 0 \end{array}\right] && (8)\\
\underline{A} &= A_d + \gamma^{-2}\tilde{B}_1\tilde{D}_{11}^*\tilde{S}^{-1}\tilde{C}_1 && (9)\\
\underline{B}_1 &= \gamma^{-1}\tilde{B}_1\tilde{R}^{-\frac{1}{2}} && (10)\\
\underline{B}_2 &= \gamma^{-2}\tilde{B}_1\tilde{D}_{11}^*\tilde{S}^{-1}\tilde{D}_{12} + B_{2d} && (11)\\
\underline{C}_1 &= \gamma\tilde{S}^{-\frac{1}{2}}\tilde{C}_1 && (12)\\
\underline{D}_{12} &= \gamma\tilde{S}^{-\frac{1}{2}}\tilde{D}_{12} && (13)
\end{aligned}$$

where $\tilde{R} = (I - \gamma^{-2}\tilde{D}_{11}^*\tilde{D}_{11})$ and $\tilde{S} = (I - \gamma^{-2}\tilde{D}_{11}\tilde{D}_{11}^*)$. Then provided $\|\tilde{D}_{11}\| < \gamma$ and given a discrete-time, LTI controller K, the following are equivalent:

(i) $\mathcal{F}_\ell(\tilde{G}, K)$ is internally stable and $\|\mathcal{F}_\ell(\tilde{G}, K)\| < \gamma$

(ii) $\mathcal{F}_\ell(\underline{G}, K)$ is internally stable and $\|\mathcal{F}_\ell(\underline{G}, K)\| < \gamma$.

In analogy with the standard finite-dimensional H_∞ problem, we require the assumption $\gamma > \|\tilde{D}_{11}\|$. This ensures that $\tilde{R}$ and $\tilde{S}$ are well defined. Toivonen (1993) points out that this is equivalent to the condition that given any bounded $x[k]$ and $\nu[k]$,

$$\int_0^h \left(\tilde{z}[k]'\tilde{z}[k] - \gamma^2\tilde{w}[k]'\tilde{w}[k]\right) dt,$$

has bounded supremum over $\tilde{w}[k] \in \mathcal{K}^{m_1}$. To ensure that the assumption is not violated we need at the very least a way of bounding $\|\tilde{D}_{11}\|$. Since $\tilde{D}_{11} : \mathcal{K}^{m_1} \to \mathcal{K}^{p_1}$ is not a compact operator, we cannot simply look for the largest eigen-value of $\tilde{D}_{11}^*\tilde{D}_{11}$ to calculate $\|\tilde{D}_{11}\|$ (Kreyszig, 1989). However, using the principle of loop-shifting we can construct a compact operator $\check{D}_{11} : \mathcal{K}^{m_1} \to \mathcal{K}^{p_1}$ such that $\|\tilde{D}_{11}\| < \gamma \Leftrightarrow \|\check{D}_{11}\| < \gamma$. In this way we can evaluate a bound on $\|\tilde{D}_{11}\|$ in terms of the eigen-values of $\check{D}_{11}^*\check{D}_{11}$. The action of such an operator $g(t) = (\check{D}_{11}f)(t)$, can be expressed in terms of the system of equations

$$\begin{aligned} \dot{x} &= \check{A}x + \check{B}f,\ x(0) = 0, \\ g &= \check{C}x,\ \ 0 \le t \le h, \end{aligned} \tag{14}$$

where $R = (I - \gamma^{-2}D_{11}^{'}D_{11})$, $S = (I - \gamma^{-2}D_{11}D_{11}^{'})$, $\check{A} = A + \gamma^{-2}B_1R^{-1}D_{11}^{'}C_1$, $\check{B} = \gamma^{-1}B_1R^{-\frac{1}{2}}$, and $\check{C} = \gamma S^{-\frac{1}{2}}C_1$. Provided $\|D_{11}\| < \gamma$, $R^{-\frac{1}{2}}$ and $S^{-\frac{1}{2}}$ are well defined. Let

$$E = \begin{bmatrix} -A' - \gamma^{-2}C_1^{'}D_{11}R^{-1}B_1^{'} & -C_1^{'}S^{-1}C_1 \\ \gamma^{-2}B_1R^{-1}B_1^{'} & A + \gamma^{-2}B_1R^{-1}D_{11}^{'}C_1 \end{bmatrix}, \tag{15}$$

and

$$Q(t) = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{bmatrix} = e^{tE}. \tag{16}$$

Lemma 3.2 *$\gamma^2 > \|D_{11}\|^2$ is an eigen-value of $\check{D}_{11}^*\check{D}_{11}$ iff $\det[Q_{11}(h)] = 0$.*

Proof : The proof is essentially the same as that of Theorem 13.5.1 in (Chen and Francis, 1995). ∎

Remark 3.3 *As a result of Lemma 3.2, under the assumption that $\|\tilde{D}_{11}\| < \gamma$, $Q_{11}(h)$ is invertible.*

What follows is the important result of Bamieh and Pearson (1992), which implicitly characterises a finite-dimensional (in terms of input, output and state spaces) equivalent of $\underline{G}$.

Theorem 3.4 *(Bamieh and Pearson, 1992) Given the discrete-time, LTI plant $\underline{G}$ with infinite-dimensional input and output spaces defined in (8), obtain the following symmetric factorisations*

$$\underline{B}_1\underline{B}_1^* = T_{\underline{B}_1}^*\begin{bmatrix} \Sigma_{\underline{B}_1} & 0 \\ 0 & 0 \end{bmatrix}T_{\underline{B}_1},$$

and

$$\begin{bmatrix} \underline{C}_1^* \\ \underline{D}_{12}^* \end{bmatrix}\begin{bmatrix} \underline{C}_1 & \underline{D}_{12} \end{bmatrix} = T_{\underline{C}_1,\underline{D}_{12}}^*\begin{bmatrix} \Sigma_{\underline{C}_1,\underline{D}_{12}} & 0 \\ 0 & 0 \end{bmatrix}T_{\underline{C}_1,\underline{D}_{12}}.$$

Define the discrete-time, finite-dimensional, LTI plant

$$\grave{G} = \left[\begin{array}{c|cc} \underline{A} & \grave{B}_1 & \underline{B}_2 \\ \hline \grave{C}_1 & 0 & \grave{D}_{12} \\ C_2 & 0 & 0 \end{array}\right] \tag{17}$$

where

$$\grave{B}_1 = T_{\underline{B}_1}^*\begin{bmatrix} \Sigma_{\underline{B}_1}^{\frac{1}{2}} \\ 0 \end{bmatrix}, \tag{18}$$

and

$$\begin{bmatrix} \grave{C}_1 & \grave{D}_{12} \end{bmatrix} = \begin{bmatrix} \Sigma^{\frac{1}{2}}_{\underline{C}_1,\underline{D}_{12}} & 0 \end{bmatrix} T_{\underline{C}_1,\underline{D}_{12}}. \tag{19}$$

Then given a discrete-time, LTI controller K, the following are equivalent:

(i) $\mathcal{F}_\ell(\underline{G}, K)$ is internally stable and $\|\mathcal{F}_\ell(\underline{G}, K)\| < \gamma$

(ii) $\mathcal{F}_\ell(\grave{G}, K)$ is internally stable and $\|\mathcal{F}_\ell(\grave{G}, K)\| < \gamma$.

4 State-Space Formulae for the SD Equivalent, Discrete-Time, LTI Plant $\grave{G}$

In this section we present a new, compact derivation of formulae for the state-space matrices of the SD-equivalent, discrete-time, LTI plant $\grave{G}$. The formulae are expressible explicitly in terms of the state-space matrices of the original plant G and are similar to those presented in (Toivonen, 1993; Toivonen and Sagfors, 1997). Using Theorem 3.4, we construct a state-space realisation for $\grave{G}$ by solving a set of TPBVPs. The formulae derived accommodate a direct feed-through from the disturbance inputs w to the controlled outputs z of the original plant G, that is $D_{11} \neq 0$.

To calculate the state-space matrices for $\grave{G}$ defined in (17), we see from equations (8-19) that finite-dimensional expressions for the following operators are required:

$$\gamma^{-2}\tilde{B}_1(I - \gamma^{-2}\tilde{D}^*_{11}\tilde{D}_{11})^{-1}\tilde{B}^*_1, \tag{20}$$

$$\gamma^2 \begin{bmatrix} \tilde{C}^*_1 \\ \tilde{D}^*_{12} \end{bmatrix} (I - \gamma^{-2}\tilde{D}_{11}\tilde{D}^*_{11})^{-1} \begin{bmatrix} \tilde{C}_1 & \tilde{D}_{12} \end{bmatrix}, \tag{21}$$

$$\gamma^{-2}\tilde{B}_1\tilde{D}^*_{11}(I - \gamma^{-2}\tilde{D}_{11}\tilde{D}^*_{11})^{-1} \begin{bmatrix} \tilde{C}_1 & \tilde{D}_{12} \end{bmatrix}. \tag{22}$$

Given $\gamma > \|\tilde{D}_{11}\|$, calculate the matrix exponential

$$\hat{Q}(h) = e^{h\hat{E}} = \exp\left\{ h \begin{bmatrix} -\hat{A}' & -\hat{C}'\hat{C} \\ \hat{B}\hat{B}' & \hat{A} \end{bmatrix} \right\}, \tag{23}$$

where

$$\hat{A} = \bar{A} + \gamma^{-2} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} R^{-1} D'_{11} \begin{bmatrix} C_1 & D_{12} \end{bmatrix}, \tag{24}$$

$$\hat{C}'\hat{C} = \begin{bmatrix} C'_1 \\ D'_{12} \end{bmatrix} S^{-1} \begin{bmatrix} C_1 & D_{12} \end{bmatrix}, \tag{25}$$

$$\hat{B}\hat{B}' = \gamma^{-2} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} B'_1 & 0 \end{bmatrix} \tag{26}$$

and

$$\bar{A} = \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}. \tag{27}$$

Then partition $\hat{Q}(h)$ conformably with $\hat{E}$.

Theorem 4.1 *The state-space matrices of the discrete-time, finite-dimensional, LTI plant* $\grave{G}$ *defined in equation (17) are given by:*

$$\underline{A} = \begin{bmatrix} I_n & 0 \end{bmatrix} (\hat{Q}_{11}(h)')^{-1} \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \tag{28}$$

$$\underline{B}_2 = \begin{bmatrix} I_n & 0 \end{bmatrix} (\hat{Q}_{11}(h)')^{-1} \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}, \tag{29}$$

$\grave{B}_1$ *a matrix satisfying*

$$\grave{B}_1 \grave{B}_1' = \begin{bmatrix} I_n & 0 \end{bmatrix} \hat{Q}_{21} \hat{Q}_{11}(h)^{-1} \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \tag{30}$$

and $\grave{C}_1$ *and* $\grave{D}_{12}$ *matrices satisfying*

$$\begin{bmatrix} \grave{C}_1' \\ \grave{D}_{12}' \end{bmatrix} \begin{bmatrix} \grave{C}_1 & \grave{D}_{12} \end{bmatrix} = -\gamma^2 \hat{Q}_{11}(h)^{-1} \hat{Q}_{12}(h). \tag{31}$$

Proof : From Theorem 3.4, it is required to determine finite-dimensional expressions for the operators given in equations (20), (21) and (22). Consider the first operator $\gamma^{-2}\tilde{B}_1(I - \gamma^{-2}\tilde{D}_{11}^*\tilde{D}_{11})^{-1}\tilde{B}_1^*$. Define

$$y = \gamma^{-1}\tilde{B}_1^* z,\ z \in \mathbb{R}^n, \text{ and } u = (I - \gamma^{-2}\tilde{D}_{11}^*\tilde{D}_{11})^{-1} y. \tag{32}$$

Let $r = \gamma^{-1}\tilde{D}_{11}u$, then from (6),

$$\begin{aligned} \dot{x}_1 &= Ax_1 + \gamma^{-1}B_1 u,\ x_1(0) = 0, \\ r &= C_1 x_1 + \gamma^{-1} D_{11} u. \end{aligned} \tag{33}$$

Moreover, from (32), $u = \gamma^{-1}\tilde{D}_{11}^* r + \gamma^{-1}\tilde{B}_1^* z$. From (4) and (6), this can be expressed as

$$\begin{aligned} \dot{x}_2 &= -A' x_2 - C_1' r,\ x_2(h) = z, \\ u &= \gamma^{-1} B_1' x_2 + \gamma^{-1} D_{11}' r. \end{aligned} \tag{34}$$

Note that the zero final condition response gives $\gamma^{-1}\tilde{D}_{11}^* r$ and the final condition response gives $\gamma^{-1}\tilde{B}_1^* z$. The systems (33) and (34) combined, form a TPBVP with solution,

$$\begin{bmatrix} x_2(t) \\ x_1(t) \end{bmatrix} = Q(t) \begin{bmatrix} x_2(0) \\ 0 \end{bmatrix}, \tag{35}$$

where $Q(t)$ is defined in equation (16). Now note that from equations (4) and (33)

$$\gamma^{-1}\tilde{B}_1 u = x_1(h). \tag{36}$$

From (35), $x_1(h) = Q_{21}(h)Q_{11}(h)^{-1}z$. Combining this with (36), characterises a finite-dimensional expression for the first operator. Note that $Q_{11}(h)$ is invertible by Remark 3.3 and that by the structure of $\hat{E}$ that $Q_{21}(h)Q_{11}(h)^{-1} = \begin{bmatrix} I_n & 0 \end{bmatrix} \hat{Q}_{21}\hat{Q}_{11}(h)^{-1} \begin{bmatrix} I_n \\ 0 \end{bmatrix}$.

Before continuing, we need to state a result from matrix theory (Van Loan, 1978). Let M_{11} and M_{22} both be square, then,

$$\exp\left\{t\begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22}\end{bmatrix}\right\} = \begin{bmatrix} e^{tM_{11}} & \hat{M}_{12} \\ 0 & e^{tM_{22}}\end{bmatrix}, \tag{37}$$

where $\hat{M}_{12} = \int_0^t e^{(t-\tau)M_{11}} M_{12} e^{\tau M_{22}}\, d\tau$, for $t \geq 0$. Now consider the second and third operators

$$\gamma^2 \begin{bmatrix} \tilde{C}_1^* \\ \tilde{D}_{12}^* \end{bmatrix} (I - \gamma^{-2}\tilde{D}_{11}\tilde{D}_{11}^*)^{-1} \begin{bmatrix} \tilde{C}_1 & \tilde{D}_{12} \end{bmatrix}$$

and $\gamma^{-2}\tilde{B}_1\tilde{D}_{11}^*(I - \gamma^{-2}\tilde{D}_{11}\tilde{D}_{11}^*)^{-1}\begin{bmatrix} \tilde{C}_1 & \tilde{D}_{12}\end{bmatrix}$, respectively. Define $f = \begin{bmatrix} \tilde{C}_1 & \tilde{D}_{12}\end{bmatrix} z$, $z \in \mathbb{R}^{n+m_2}$, which by (5), (7) and (37) can be expressed as

$$f = \begin{bmatrix} C_1 & D_{12}\end{bmatrix} e^{t\bar{A}} z,$$

where $\bar{A}$ is defined in equation (27). Also define

$$g = (I - \gamma^{-2}\tilde{D}_{11}\tilde{D}_{11}^*)^{-1} f. \tag{38}$$

Let $w = \gamma^{-1}\tilde{D}_{11}^* g$, then from (6),

$$\begin{aligned} \dot{\xi}_1 &= -\bar{A}'\xi_1 - \begin{bmatrix} C_1' \\ D_{12}' \end{bmatrix} g,\ \xi_1(h) = 0, \\ w &= \begin{bmatrix} \gamma^{-1}B_1' & 0 \end{bmatrix}\xi_1 + \gamma^{-1}D_{11}' g. \end{aligned} \tag{39}$$

Moreover, from (38), $g = \begin{bmatrix} C_1 & D_{12}\end{bmatrix} e^{t\bar{A}} z + \gamma^{-1}\tilde{D}_{11} w$. Again from (6), this can be expressed as

$$\begin{aligned} \dot{\xi}_2 &= \bar{A}\xi_2 + \begin{bmatrix} \gamma^{-1}B_1 \\ 0 \end{bmatrix} w,\ \xi_2(0) = z, \\ g &= \begin{bmatrix} C_1 & D_{12}\end{bmatrix}\xi_2 + \gamma^{-1}D_{11} w. \end{aligned} \tag{40}$$

Note that the zero initial condition response gives $\gamma^{-1}\tilde{D}_{11}w$, and the initial condition response gives $\begin{bmatrix} C_1 & D_{12}\end{bmatrix} e^{t\bar{A}} z$. The systems (40) and (39) combined, form a TPBVP with solution

$$\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = \hat{Q}(t) \begin{bmatrix} \xi_1(0) \\ z \end{bmatrix}. \tag{41}$$

Now note that

$$\gamma^2 \begin{bmatrix} \tilde{C}_1^* \\ \tilde{D}_{12}^* \end{bmatrix} g = \gamma^2 \xi_1(0), \tag{42}$$

and

$$\gamma^{-2}\tilde{B}_1\tilde{D}_{11}^{*}g = \gamma^{-1}\tilde{B}_1 w = \begin{bmatrix} I_n & 0 \end{bmatrix}\{\xi_2(h) - e^{h\bar{A}}z\} \tag{43}$$

Recall the partitioning of $\hat{Q}(h)$. From (41), $\xi_1(0) = -\hat{Q}_{11}(h)^{-1}\hat{Q}_{12}(h)z$. Combining this with (42) characterises a finite-dimensional expression for the second operator. Similarly,

$$\xi_2(h) = \{\hat{Q}_{22}(h) - \hat{Q}_{21}(h)\hat{Q}_{11}(h)^{-1}\hat{Q}_{12}(h)\}z. \tag{44}$$

Define

$$J = \begin{bmatrix} 0 & -I_{n+m_2} \\ I_{n+m_2} & 0 \end{bmatrix}, \tag{45}$$

and note that $-\hat{E}' = J\hat{E}J^{-1}$. So,

$$(\hat{Q}(h)')^{-1} = e^{-h\hat{E}'} = J\hat{Q}(h)J^{-1}.$$

That is, $\hat{Q}(h)'J\hat{Q}(h)J^{-1} = J\hat{Q}(h)J^{-1}\hat{Q}(h)' = I$, which yields the relationship

$$\hat{Q}_{22}(h) - \hat{Q}_{21}(h)\hat{Q}_{11}(h)^{-1}\hat{Q}_{12}(h) = (\hat{Q}_{11}(h)')^{-1}. \tag{46}$$

Combining this with (43) and (44), characterises a finite-dimensional expression for the third operator. From (9),

$$\begin{aligned} \underline{A} &= e^{hA} + \begin{bmatrix} I_n & 0 \end{bmatrix}\{(\hat{Q}_{11}(h)')^{-1} - e^{h\bar{A}}\}\begin{bmatrix} I_n \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} I_n & 0 \end{bmatrix}(\hat{Q}_{11}(h)')^{-1}\begin{bmatrix} I_n \\ 0 \end{bmatrix}. \end{aligned}$$

Moreover, from (11),

$$\begin{aligned} \underline{B}_2 &= \int_0^h e^{(h-t)A}dtB_2 + \begin{bmatrix} I_n & 0 \end{bmatrix}\{(\hat{Q}_{11}(h)')^{-1} - e^{h\bar{A}}\}\begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix} \\ &= \begin{bmatrix} I_n & 0 \end{bmatrix}(\hat{Q}_{11}(h)')^{-1}\begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}. \end{aligned}$$

Now, to see that $\hat{Q}_{11}(h)$ is invertible, note that from (41) and (37),

$$\hat{Q}_{11}(h) = \begin{bmatrix} Q_{11}(h) & 0 \\ ? & I_{m_2} \end{bmatrix},$$

where $Q_{11}(h)$ is defined as before and "?" denotes an irrelevant sub-matrix. Thus, $\hat{Q}_{11}(h)$ is invertible iff $Q_{11}(h)$ is, which is the case by Remark 3.3. This completes the proof. ∎

In view of Theorems 3.4 and 4.1, with the assumptions that the sampling is not pathological and that (A, B_2) is stabilisable and (A, C_2) detectable, the synthesis of K to achieve a specified bound γ on $\|\mathcal{F}_\ell(G, \mathcal{H}K\mathcal{S})\|$ is equivalent to the synthesis of K to achieve the same bound on $\|\mathcal{F}_\ell(\grave{G}, K)\|$. [1] The solution to the later synthesis problem is well-known (Gahinet and Apkarian, 1994; Iglesias and Glover, 1991; Kondo et al., 1990). Theorems 3.4 and 4.1 can also be used to construct a bisection type algorithm to compute the closed-loop, $\mathrm{L}_2(\mathbb{R}_+)$-induced norm of a given SD, control system.

5 Re-Structuring for Numerical Robustness

In this section we re-structure the formulae presented in Theorem 4.1, to give a numerically robust algorithm for calculating the state-space matrices of the SD-equivalent, discrete-time, LTI plant $\grave{G}$.

As detailed in Theorem 4.1, computation of the state-space matrices for $\grave{G}$ involves the matrix exponential of the Hamiltonian structured matrix $\hat{E}$. Since $\hat{E}$ has Hamiltonian structure, its eigen-values are symmetric about the imaginary axis. Therefore, if $\hat{E}$ has eigen-values with large, positive real part with respect to the sampling frequency, the computation of $\hat{Q}(h) = e^{h\hat{E}}$ can be numerically sensitive, with the calculation being dominated by large terms. Such a situation can occur if the original plant G has (stable or unstable) poles that are fast compared to the sampling period. In the final computation of the state-space matrices for $\grave{G}$, the large terms that can dominate the calculation of $\hat{Q}(h)$ cancel or become very small (see equations (28-31)). This effect makes the computation of the SD-equivalent system numerically sensitive as a whole. In what follows, we re-structure the formulae given in Theorem 4.1 to avoid direct calculation of the full matrix exponential $e^{h\hat{E}}$. The re-structuring is concerned with restricting all matrix exponential calculations to involve only matrices with all eigen-values having real part less than or equal to some number of order 1. By similar argument, the following approach should also be used when computing bounds on $\|\tilde{D}_{11}\|$ (see Lemma 3.2).

Let $M \in \mathbb{C}^{2(n+m_2)\times 2(n+m_2)}$, be an invertible matrix such that

$$\hat{E} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{bmatrix}, \tag{47}$$

where $\tilde{M} = M^{-1}$, $\lambda(T_{11}) \subset \mathbb{C}_{>\alpha} := \{s : \mathrm{Re}(s) > \alpha\}$, $\lambda(T_{22}) \subset \bar{\mathbb{C}}_{\leq\alpha} := \{s : \mathrm{Re}(s) \leq \alpha\}$, $\alpha \in [0, l/h]$ (l of order 1) and the matrices M and $\tilde{M}$ are partitioned conformably with $\hat{E}$. That such a matrix exists follows from two well known results from matrix theory. Namely, Schur Decomposition and Block Triangularisation (Golub and Van Loan, 1989). Decomposing the matrix $\hat{E}$ into its spectral components involves first obtaining an ordered Schur decomposition. It is well known that the numerical accuracy of performing the ordered Schur Decomposition can be improved by balancing the original matrix (Wilkinson and Reinsch, 1971). This involves

[1] Note that this finite-dimensional, LTI, discrete-time, H_∞-synthesis problem is in fact singular in the sense that $\grave{D}_{21} = 0$.

applying similarity transformations to the matrix concerned so that corresponding rows and columns have approximately the same vector norms. As implemented in MATLAB[2], the similarity transformations can often be ill-conditioned, especially when there are both relatively very large and very small, non-zero elements in the original matrix (MATLAB, 1992). On examining the structure of the matrix $\hat{E}$, we note that "squared" terms like BB' and $C'C$ appear. This can further accentuate differences between large and small elements of B and C, possibly effecting a balancing algorithm used on $\hat{E}$ prior to Schur Decomposition. Hence, we recommended that the original system G be scaled prior to forming $\hat{E}$, so that corresponding rows and columns of the state-space matrices are of comparable size. Such an algorithm is implemented in the μ-tools[3] tool-box in MATLAB. Having performed the Schur Decomposition the next step required to obtain the decomposition in (47) is Block Triangularisation. This involves solving a Sylvester equation and can be numerical sensitive if the subsets $\lambda(T_{ii})$ are insufficiently separated (Golub et al., 1979). Algorithms exist which dynamically determine the partitioning to limit the loss of accuracy (Bavely and Stewart, 1979).

Before continuing we require a technical result.

Lemma 5.1 *M_{11} has full column rank iff (A, C_1) has no unobservable modes in $\mathbb{C}_{>\alpha}$. Similarly, $\tilde{M}_{11}$ has full row rank iff (A, B_1) has no uncontrollable modes in $\mathbb{C}_{>\alpha}$.*

Proof : From equation (47), we have

$$\hat{E}\begin{bmatrix} M_{11} \\ M_{12} \end{bmatrix} = \begin{bmatrix} -\hat{A}' & -\hat{C}'\hat{C} \\ \hat{B}\hat{B}' & \hat{A} \end{bmatrix}\begin{bmatrix} M_{11} \\ M_{12} \end{bmatrix} = \begin{bmatrix} M_{11} \\ M_{12} \end{bmatrix} T_{11}.$$

Since T_{11} has full rank and is square with all eigen-values having real part strictly greater than α, the proof that M_{11} has full column rank iff (A, C_1) has no unobservable modes in $\mathbb{C}_{>\alpha}$, is roughly the same as the proof that the Algebraic Riccati equation

$$A'X + XA + XRX + Q = 0,$$

has stabilising solution iff (A, R) is stabilisable (see (Zhou et al., 1995, Theorem 13.6)). The proof that $\tilde{M}_{11}$ has full row rank iff (A, B_1) has no uncontrollable modes in $\mathbb{C}_{>\alpha}$, follows similarly. For the full proof see Theorem 4.7 in (Cantoni and Glover, 1996). ∎

With this lemma in mind we make the following assumptions.

(A1) (A, B_1) has no uncontrollable modes in $\mathbb{C}_{>\alpha}$ and

(A2) (A, C_1) has no unobservable modes in $\mathbb{C}_{>\alpha}$.

For the sake of continuity, we postpone justification of these assumptions to the next section.

[2]Copyright Maths Works Inc.
[3]Copyright Musyn, 1993.

With the matrix $\hat{E}$ decomposed into its $\mathbb{C}_{>\alpha}$ and $\mathbb{C}_{\leq\alpha}$ spectral components, we now concentrate on re-structuring the formulae given in Theorem 4.1, to avoid direct computation of the matrix exponential $e^{hT_{11}}$. Let p denote the dimension of T_{11} and $r = n+m_2-p$. By Lemma 5.1, under assumptions (A1) and (A2), M_{11} has full column rank and $\tilde{M}_{11}$ has full row rank. So we can construct invertible matrices

$$\begin{bmatrix} M_{11}^L \\ M_{11}^{\perp} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{M}_{11}^R & \tilde{M}_{11}^{\perp} \end{bmatrix},$$

such that

$$\begin{bmatrix} M_{11}^L \\ M_{11}^{\perp} \end{bmatrix} M_{11} = \begin{bmatrix} I_p \\ 0_{r,p} \end{bmatrix} \quad \text{and} \quad \tilde{M}_{11}\begin{bmatrix} \tilde{M}_{11}^R & \tilde{M}_{11}^{\perp} \end{bmatrix} = \begin{bmatrix} I_p & 0_{p,r} \end{bmatrix},$$

where $0_{i,j}$ denotes an $i \times j$ matrix of zeros. Furthermore, we define the matrices

$$\begin{aligned}
\Omega_L &= \left(\begin{bmatrix} e^{-hT_{11}} M_{11}^L \\ M_{11}^{\perp} \end{bmatrix} M_{12} e^{hT_{22}} \tilde{M}_{21} + \begin{bmatrix} I_p \\ 0_{r,p} \end{bmatrix} \tilde{M}_{11} \right)^{-1}, \\
\Omega_R &= \left(M_{11} \begin{bmatrix} I_p & 0_{p,r} \end{bmatrix} + M_{12} e^{hT_{22}} \tilde{M}_{21} \begin{bmatrix} \tilde{M}_{11}^R e^{-hT_{11}} & \tilde{M}_{11}^{\perp} \end{bmatrix} \right)^{-1}, \\
\Psi &= \begin{bmatrix} \tilde{M}_{11}^R e^{-hT_{11}} & \tilde{M}_{11}^{\perp} \end{bmatrix} \Omega_R, && (48) \\
\Phi &= \left(M_{22} e^{hT_{22}} \tilde{M}_{21} \begin{bmatrix} \tilde{M}_{11}^R e^{-hT_{11}} & \tilde{M}_{11}^{\perp} \end{bmatrix} + M_{21} \begin{bmatrix} I_p & 0_{p,r} \end{bmatrix} \right) \Omega_R && (49)
\end{aligned}$$

and

$$\Upsilon = \Omega_L \left(\begin{bmatrix} e^{-hT_{11}} M_{11}^L \\ M_{11}^{\perp} \end{bmatrix} M_{12} e^{hT_{22}} \tilde{M}_{22} + \begin{bmatrix} I_p \\ 0_{r,p} \end{bmatrix} \tilde{M}_{12} \right). \tag{50}$$

Note that none of the matrices above involve the exponential of a matrix with eigen-values having real part greater than l, a number of order 1.

Theorem 5.2 *The state-space matrices of the SD-equivalent, discrete-time, LTI plant $\grave{G}$ are given by:*

$$\underline{A} = \begin{bmatrix} I_n & 0 \end{bmatrix} \Psi' \begin{bmatrix} I_n \\ 0 \end{bmatrix},$$

$$\underline{B}_2 = \begin{bmatrix} I_n & 0 \end{bmatrix} \Psi' \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix},$$

$\grave{B}_1$ *a matrix satisfying* $\grave{B}_1 \grave{B}_1' = \begin{bmatrix} I_n & 0 \end{bmatrix} \Phi \begin{bmatrix} I_n \\ 0 \end{bmatrix}$ *and*

$\grave{C}_1$ *and* $\grave{D}_{12}$ *matrices satisfying* $\begin{bmatrix} \grave{C}_1' \\ \grave{D}_{12}' \end{bmatrix} \begin{bmatrix} \grave{C}_1 & \grave{D}_{12} \end{bmatrix} = -\gamma^2 \Upsilon.$

Proof : Note from equation (47) that

$$\hat{Q}_{11}^{-1}(h)\left(M_{11}e^{hT_{11}}\tilde{M}_{11}+M_{12}e^{hT_{22}}\tilde{M}_{21}\right)=I.$$

Post-multiplying both sides by $\left[\begin{array}{cc}\tilde{M}_{11}^{R}e^{-hT_{11}} & \tilde{M}_{11}^{\perp}\end{array}\right]$ we obtain

$$\hat{Q}_{11}^{-1}(h)\Omega_R^{-1}=\left[\begin{array}{cc}\tilde{M}_{11}^{R}e^{-hT_{11}} & \tilde{M}_{11}^{\perp}\end{array}\right],$$

and hence from equation (48), $\Psi=\hat{Q}_{11}^{-1}(h)$.

Let $\Phi=\hat{Q}_{21}\hat{Q}_{11}^{-1}$, and note from equation (47) that

$$\Phi\left(M_{11}e^{hT_{11}}\tilde{M}_{11}+M_{12}e^{hT_{22}}\tilde{M}_{21}\right)=\left(M_{21}e^{hT_{11}}\tilde{M}_{11}+M_{22}e^{hT_{22}}\tilde{M}_{21}\right).$$

Post-multiplying both sides by $\left[\begin{array}{cc}\tilde{M}_{11}^{R}e^{-hT_{11}} & \tilde{M}_{11}^{\perp}\end{array}\right]$ we obtain

$$\Phi\Omega_R^{-1}=\left(M_{22}e^{hT_{22}}\tilde{M}_{21}\left[\begin{array}{cc}\tilde{M}_{11}^{R}e^{-hT_{11}} & \tilde{M}_{11}^{\perp}\end{array}\right]+M_{21}\left[\begin{array}{cc}I_p & 0_{p,r}\end{array}\right]\right),$$

which is equation (49).

Similarly, let $\Upsilon=\hat{Q}_{11}^{-1}(h)\hat{Q}_{12}(h)$, and note from equation (47) that

$$\left(M_{11}e^{hT_{11}}\tilde{M}_{11}+M_{12}e^{hT_{22}}\tilde{M}_{21}\right)\Upsilon=\left(M_{11}e^{hT_{11}}\tilde{M}_{12}+M_{12}e^{hT_{22}}\tilde{M}_{22}\right).$$

Pre-multiplying both sides by $\left[\begin{array}{c}e^{-hT_{11}}M_{11}^{L}\\ M_{11}^{\perp}\end{array}\right]$ we obtain

$$\Omega_L^{-1}\Upsilon=\left(\left[\begin{array}{c}e^{-hT_{11}}M_{11}^{L}\\ M_{11}^{\perp}\end{array}\right]M_{12}e^{hT_{22}}\tilde{M}_{22}+\left[\begin{array}{c}I_p\\ 0_{r,p}\end{array}\right]\tilde{M}_{12}\right),$$

which is equation (50). ∎

6 Comments Concerning Assumptions (A1) and (A2)

Under assumptions (A1) and (A2), the matrices M_{11} and $\tilde{M}_{11}$ have full column and full row rank respectively. This is required to enable re-structuring of the formulae given in Theorem 4.1. In this section, we will argue that the assumptions (A1) and (A2) are reasonable.

There are two cases in which the assumptions can be violated. The first case is when there is any unstable mode which is unobservable (and/or uncontrollable) from the controlled outputs (disturbance inputs) which has a time-constant greater than or equal to the sampling period divided by some number l of order 1. The second case is when any unstable mode which is unobservable (and/or uncontrollable) from the controlled outputs (disturbance inputs) has time constant less than the sampling period divided by some number l of order 1.

In the first case we can simply choose α to be larger than real part of the unobservable (and/or uncontrollable) mode. In this case such an α will be less than or equal to l/h and therefore lie in the required interval, $[0, l/h]$.

Remark 6.1 *With this in mind, a suitable choice for α is the smallest number in $[0, l/h]$, such that respectively, (A, C_1) and (A, B_1) have no unobservable and uncontrollable modes in the set $\{s : \alpha < \mathrm{Re}(s) \leq l/h\}$, where l is a number of order 1. If this cannot be determined exactly then setting $\alpha = l/h$ will suffice.*

The second case is more interesting and we illustrate it with a numerical example. Consider a problem for which the original plant has state-space realisation

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{array}\right],$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 50 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \text{ and } C_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

The sampling period is chosen to be $h = 1s$. This system has an unstable mode at $s = 50$ which is unobservable from the controlled output. Note that the unstable mode is fast (time constant $0.02s$) compared to the sampling period ($1s$). In this case

$$\hat{E} = \begin{bmatrix} 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & -50 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 50 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which has eigen-values $0, -1.7321, 1.7321, -50, 50,$ and 0. For this problem there are two eigen-values which are faster than the sampling frequency. Again for simplicity we choose $l = 1$. Since in this example the real part of the unobservable mode is greater than the sampling frequency we cannot select α, which must lie in $[0, l/h]$, to lie to the right of it. Therefore (A1) will be violated for all possible choices of α in the sensible interval $[0, l/h]$. However, we suggest from a synthesis point of view, that in this case it is the structure of the original problem that is not sensible. The unstable mode is excitable by the disturbance, yet this is not observed from the controlled output and hence is not "observed" in the optimisation problem of bounding the norm of the controlled output for worst case disturbances. Since this unstable mode has a time constant of 0.02 seconds, in one sampling period it can increase by a large amount ($e^{50} = 5.18E + 21$). It is reasonable then to assume that this will require a large control action in the next sampling period, which if not penalised at the controlled output (that is $D_{12} = 0$) will be allowed, which is not sensible. If penalised at controlled output (that is D_{12} full column rank) this would result in very poor performance. Further to this, one would intuitively expect that it be very difficult to stabilise a system with such a fast unstable pole using such a slow sampling frequency. Similar arguments hold for when (A, B_1) has an uncontrollable mode that is faster than the sampling frequency.

In view of the cases considered, we claim that the assumptions (A1) and (A2) are reasonable in the context of H_∞, SD synthesis.

7 Conclusions

In this paper we have presented a new, compact derivation of formulae for the synthesis of H_∞, SD controllers for continuous-time, LTI plants. It is shown that these formulae can be re-structured to give a numerically robust algorithm for computing the state-space matrices of a realisation for the SD-equivalent, discrete-time, LTI plant.

We note that the formulae derived are essentially identical (except for the accommodation of a direct feed-through term D_{11}) to those derived by Toivonen (1993) using classical, linear-quadratic, optimal-control theory. This indicates directly that the time-lifting approach and the classical, linear-quadratic, optimal-control theoretic approach are equivalent, as was also pointed out in (Toivonen and Sagfors, 1997). The formulae are simple, requiring the computation of only a single matrix exponential. It is the simplicity of the formulae that facilitates the re-structuring summarised in Theorem 5.2 for numerical robustness.

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