Frequency-Domain Analysis of Linear Periodic Operators with Application to Sampled-Data Control Design

M. W. Cantoni* K. Glover

Department of Engineering, University of Cambridge, Cambridge CP2 1PZ, United Kingdom.

Abstract

The development of controller usually involves two major steps: A design step in which a generalised plant is constructed, followed by a synthesis step in which an optimisation problem is solved for the generalised plant. Frequency-Domain analysis methods are commonly used in the design step; for example loop-shaping. In this paper we develop a new frequency-domain analysis method for periodic operators. Specifically, we define the *average-power gain matrix*, which for a periodic operator characterises the average power of the asymptotic ("steady-state") response to a sinusoidal input of a single frequency. To illustrate an application of the new analysis method we show how to use the average-power gain matrix in the H_∞ loop-shaping design of sampled-data, closed-loop systems.

Keywords : Frequency-Domain Analysis, Periodic Operators, Sampled-Data Control, H_∞ Loop-Shaping.

1 Introduction

In this paper we develop a new frequency-domain analysis method for periodic operators. Our motivation stems from the wide use of frequency-domain analysis tools in the important design step of optimal controller development. For example, in the H_∞ loop-shaping design procedure of McFarlane and Glover [12], the singular values of the frequency response matrix for the open-loop plant are shaped to trade-off desired performance and robustness, followed by a synthesis step based on H_∞ optimisation. To date, most of the work on frequency-domain analysis methods for periodic operators has been presented in the context of sampled-data (SD), closed-loop systems [1, 2, 15, 16, 11]. Much of this work concentrates on the definition of a frequency response operator to characterise the asymptotic response to an infinite series of sinusoidal functions of frequencies separated by integer multiples of the frequency of periodicity. Such a definition of the frequency response is necessary when studying robustness [8]. However, from the perspective of performance it is also useful to characterise the asymptotic response of an operator to a sinusoidal input of a single frequency. For example, in disturbance rejection problems it is useful to know if the "gain" of the system is small for signals from a specific band of frequencies. Furthermore, since robustness properties are to a large extent governed by the structure of the generalised plant and type of optimisation problem solved in the synthesis step, from the perspective of designing for performance a characterisation of the asymptotic response of performance-indicating, closed-loop operators (which may be periodic in the case of SD control) to sinusoidal inputs of a single frequency, is useful.

With this as motivation, we define the *average-power gain matrix* for periodic operators. The singular values of this finite-dimensional matrix characterise the average power of the asymptotic ("steady-state") response to sinusoidal inputs of a single frequency. State-space formulae for computing the average-power gain matrix are presented for the sampled-data case. To illustrate the value of this new frequency-domain analysis method we show how to use the average-power gain matrix in the H_∞ loop-shaping design of SD closed-loop systems.

2 Notation

We denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}_+$, $\mathbb{Z}_+$ and $\mathbb{R}_{[0,h)}$ the field of real numbers, the field of complex numbers, the non-negative reals, the non-negative integers and the interval $[0,h)$ of $\mathbb{R}_+$, respectively. Let $\mathscr{X}$ be an arbitrary Hilbert space. Then $L_2(\mathbb{R}_+,\mathscr{X})$ denotes the Hilbert space of functions $f:\mathbb{R}_+\to\mathscr{X}$, square (Lebesgue) integrable over $\mathbb{R}_+$, $\ell_2(\mathbb{Z}_+,\mathscr{X})$ the Hilbert space of functions $f:\mathbb{Z}_+\to\mathscr{X}$ square summable over $\mathbb{Z}_+$. $L_2^e(\mathbb{R}_+,\mathscr{X})$ and $\ell_2^e(\mathbb{Z}_+,\mathscr{X})$ denote the extended $\mathcal{L}_2$ and ℓ_2 spaces as defined in the usual manner. We also use the short hand $\mathscr{E}^m$ to denote the Hilbert space of functions $f:\mathbb{R}_{[0,h)}\to\mathbb{R}^m$, square (Lebesgue) integrable over the interval $\mathbb{R}_{[0,h)}$. The inner-product on a Hilbert space $\mathscr{X}$ is denoted by $\langle\cdot,\cdot\rangle_{\mathscr{X}}$ and the 2-norm on a vector space X by $\|\cdot\|_X$.

For a matrix $M\in\mathbb{C}^{m\times p}$, $M'\in\mathbb{C}^{p\times m}$ denotes the complex conjugate transpose and $\overline{\sigma}\{M\}$ and $\underline{\sigma}\{M\}$ the maximum and minimum singular values respectively. For arbitrary Hilbert spaces $\mathscr{X}_0$ and $\mathscr{X}_1$, $\mathfrak{B}_{\mathscr{X}_0\to\mathscr{X}_1}$ denotes the vector space of bounded linear operators mapping $\mathscr{X}_0$ to $\mathscr{X}_1$. For an operator $T\in\mathfrak{B}_{\mathscr{X}_0\to\mathscr{X}_1}$, $T^*\in\mathfrak{B}_{\mathscr{X}_1\to\mathscr{X}_0}$ denotes the adjoint operator and $\|T\|_{\mathscr{X}_0\to\mathscr{X}_1}$ the 2-norm induced operator norm.

The $\boldsymbol{\mathcal{Z}}$-Transform operator is defined by

$$\widehat{f}(\lambda)=(\boldsymbol{\mathcal{Z}}f)(\lambda)\triangleq\sum_{k=0}^{\infty}\lambda^k f_k,$$

*All correspondence, e-mail : mwc@eng.cam.ac.uk

where $f : \mathbb{Z}_+ \to \mathscr{X}$. The $\boldsymbol{\mathcal{Z}}$-Transform operator yields an isomorphism between $\ell_2(\mathbb{Z}_+, \mathscr{X})$ and $H_2(\mathscr{X})$, the Hardy space of functions $\hat{f} : \mathbb{C} \to \mathscr{X}$, analytic in the open unit disc ($\mathbb{D}$) and square (Lebesgue) integrable on the unit circle ($\partial\mathbb{D}$) of the complex plane.

$H_\infty(\mathfrak{B}_{\mathscr{X}_0 \to \mathscr{X}_1})$ denotes the Hardy space of $\mathfrak{B}_{\mathscr{X}_0 \to \mathscr{X}_1}$-valued functions of a complex variable that are analytic in $\mathbb{D}$ and essentially bounded on $\partial\mathbb{D}$. $\|\cdot\|_\infty$ denotes the norm on $H_\infty(\mathfrak{B}_{\mathscr{X}_0 \to \mathscr{X}_1})$ with $\|\cdot\|_\infty = \operatorname*{ess\,sup}_{\omega \in [0,2\pi)} \|\cdot(e^{j\omega})\|_{\mathscr{X}_0 \to \mathscr{X}_1}$.

$\boldsymbol{\mathcal{L}}_h : L_2^e(\mathbb{R}_+, \mathbb{R}^m) \to \ell_2^e(\mathbb{Z}_+, \mathscr{E}^m)$ denotes the time-lifting operator and is defined by

$$\mathsf{f}_k(\theta) = (\boldsymbol{\mathcal{L}}_h f)_k(\theta) = f(t) \quad kh \le t < (k+1)h.$$

Note that $\boldsymbol{\mathcal{L}}_h$ is bijective and therefore invertible.

3 The Average-Power Gain Matrix

Consider a bounded, causal, linear, periodically-time varying (LPTV) operator $H : L_2(\mathbb{R}_+, \mathbb{R}^m) \to L_2(\mathbb{R}_+, \mathbb{R}^p)$, with period h. $\mathsf{H} = \boldsymbol{\mathcal{L}}_h H \boldsymbol{\mathcal{L}}_h^{-1}$ is then a bounded, causal, linear, shift-invariant operator mapping $\ell_2(\mathbb{Z}_+, \mathscr{E}^m)$ to $\ell_2(\mathbb{Z}_+, \mathscr{E}^p)$ [3]. By causality and shift-invariance the behaviour of H can be expressed in terms of the lower-triangular, block-Toeplitz matrix

$$\mathscr{T}_{\mathsf{H}} = \begin{bmatrix} \mathsf{H}_0 & 0 & 0 & \cdots \\ \mathsf{H}_1 & \mathsf{H}_0 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$

where each $\mathsf{H}_k : \mathscr{E}^m \to \mathscr{E}^p$. $\mathscr{T}_{\mathsf{H}}$ is uniquely identifiable by the sequence $\{\mathsf{H}_k\}_{k=0}^\infty$. We associate with this sequence the function $\widehat{\mathsf{H}}$ of a complex variable λ, defined by

$$\widehat{\mathsf{H}}(\lambda) = \sum_{k=0}^{\infty} \mathsf{H}_k \lambda^k.$$

Note that since H is bounded and shift-invariant $\widehat{\mathsf{H}} \in H_\infty(\mathfrak{B}_{\mathscr{E}^{\mathbb{R}^m} \to \mathscr{E}^{\mathbb{R}^p}})$ and $\|\mathsf{H}\|_{\ell_2(\mathbb{Z}_+, \mathscr{E}^m) \to \ell_2(\mathbb{Z}_+, \mathscr{E}^p)} = \|\widehat{\mathsf{H}}\|_\infty$ [13, 9]. Moreover, $H = \boldsymbol{\mathcal{L}}_h^{-1} \boldsymbol{\mathcal{Z}}^{-1} \boldsymbol{\mathcal{M}}_{\widehat{\mathsf{H}}} \boldsymbol{\mathcal{Z}} \boldsymbol{\mathcal{L}}_h$ where $\boldsymbol{\mathcal{M}}_{\widehat{\mathsf{H}}}$ denotes the multiplication operator with frequency domain symbol $\widehat{\mathsf{H}}$. Since $\boldsymbol{\mathcal{Z}}$ and $\boldsymbol{\mathcal{L}}_h$ are isomorphisms it then follows that $\|H\|_{L_2(\mathbb{R}_+, \mathbb{R}^m) \to L_2(\mathbb{R}_+, \mathbb{R}^p)} = \|\widehat{\mathsf{H}}\|_\infty$. We refer to $\widehat{\mathsf{H}}$ as the transfer operator for H.

We now state an important result which characterises the asymptotic ("steady-state") response of H to a power function.

Lemma 3.1 *[14] Consider an input to H of the form*

$$u(kh+\theta) \;=\; \mu^k \mathsf{v}(\theta), \; \mathsf{v} \in \mathscr{E}^m, \; |\mu| \ge 1,$$

for $k \in \mathbb{Z}_+$ and $\theta \in \mathbb{R}_{[0,h)}$. Let $y(t) = (Hu)(t)$. Then $y(t)$ asymptotically approaches

$$y_{ss}(t) = y_{ss}(kh+\theta) = \mu^k (\widehat{\mathsf{H}}(\mu^{-1})\mathsf{v})(\theta)$$

as $t \to \infty$.

Observing that via time-lifting a sinusoidal function $u(t) = e^{j\omega t} u^d$ ($u^d \in \mathbb{R}^m$) can be expressed as a power function of the form considered in Lemma 3.1, we can say that $y(t) = (He^{j\omega t} u^d)(t)$ asymptotically approaches

$$y_{ss}^\omega(t) = y_{ss}^\omega(kh+\theta) = (e^{j\widehat{\omega} h})^k (\widehat{\mathsf{H}}(e^{-j\widehat{\omega} h})\mathsf{u}^d)(\theta), \qquad (1)$$

where $\mathsf{u}^d(\theta) = e^{j\omega\theta} u^d \in \mathscr{E}^m$, $\widehat{\omega} = \omega -]\frac{h\omega}{2\pi}[\ (\frac{2\pi}{h})$ and $]\cdot[$ denotes the integer part of a real number. Although strictly speaking $y_{ss}^\omega(t)$ is never in "steady-state" (unless ω is commensurate with the sampling frequency), this characterisation of the asymptotic behaviour of a periodic operator leads to the definition of $\widehat{\mathsf{H}}(e^{-j\widehat{\omega} h})$ ($\widehat{\omega} \in [0, \frac{2\pi}{h})$) as the frequency response operator for H and its induced norm to be the frequency response (cf. [16, 15]).

In the following lemma we characterise the average power of $y_{ss}^\omega(t)$.

Lemma 3.2 *Let $y_{ss}^\omega(t)$ be as defined in (1). Then,*

$$\begin{aligned} \|y_{ss}^\omega\|_{pow} &= \sqrt{\lim_{T\to\infty} \frac{1}{T} \int_0^T y_{ss}^\omega(t)' y_{ss}^\omega(t) dt} \\ &= \frac{1}{\sqrt{h}} \|(\widehat{\mathsf{H}}(e^{-j\widehat{\omega} h})\mathsf{u}^d)(\theta)\|_{\mathscr{E}^p}. \end{aligned}$$

Proof : Let $\mathsf{h}(\theta) = (\widehat{\mathsf{H}}(e^{-j\widehat{\omega} h})\mathsf{u}^d)(\theta)$. Then,

$$\begin{aligned} \|y_{ss}^\omega\|_{pow}^2 &= \lim_{T\to\infty} \frac{1}{T} \int_0^T y_{ss}^\omega(t)' y_{ss}^\omega(t) dt \\ &= \lim_{N\to\infty} \frac{1}{Nh} \sum_{k=0}^{N-1} \int_{kh}^{(k+1)h} y_{ss}^\omega(t)' y_{ss}^\omega(t) dt \\ &= \lim_{N\to\infty} \frac{1}{Nh} \sum_{k=0}^{N-1} \int_0^h \mathsf{h}(\theta)' \mathsf{h}(\theta) d\theta \\ &= \frac{1}{h} \|\mathsf{h}(\theta)\|_{\mathscr{E}^p}^2. \end{aligned}$$

This completes the proof. ∎

Let $\Phi_{y,u}(\omega) : \mathbb{R}^m \to \mathscr{E}^p$ denote the operator $\widehat{\mathsf{H}}(e^{-j\widehat{\omega} h})\Xi(\omega)$ where $\Xi(\omega) : \mathbb{R}^m \to \mathscr{E}^m$ denotes the operator $e^{(j\omega I_m)\theta}$ and I_m is the $m \times m$ identity. Then by Lemma 3.2,

$$\|y_{ss}^\omega\|_{pow} = \frac{1}{\sqrt{h}} \sqrt{(u^d)' \Phi_{y,u}(\omega)^* \Phi_{y,u}(\omega) u^d}. \qquad (2)$$

Note that $\Phi_{y,u}(\omega)^* \Phi_{y,u}(\omega)$ is a square, symmetric matrix and hence, that its square root exists. Let $\Psi_{y,u}(\omega) = \frac{1}{\sqrt{h}} (\Phi_{y,u}(\omega)^* \Phi_{y,u}(\omega))^{\frac{1}{2}}$. Then from equation (2), we see that the average power of the "steady-state" response can be characterised in terms of the singular values of the matrix $\Psi_{z,w}(\omega)$. This leads to the following definition :

Definition 3.3 *The frequency dependent $m \times m$ matrix $\Psi_{y,u}(\omega) = \frac{1}{\sqrt{h}} (\Phi_{y,u}(\omega)^* \Phi_{y,u}(\omega))^{\frac{1}{2}}$, where $\Phi_{y,u}(\omega) = \widehat{\mathsf{H}}(e^{-j\widehat{\omega} h})\Xi(\omega)$ and $\Xi(\omega) = e^{(j\omega I_m)\theta}$, is defined to be the the average-power gain matrix for the periodic operator H.*

Remark 3.4 *Note that linear, time-invariant (LTI) operators are included in the class of LPTV operators. So $\Psi_{\cdot,\cdot}$ is well defined for LTI operators. In fact, by the way we have defined the average-power gain matrix, for a bounded, causal, LTI operator F mapping u to y say, we have that $\Psi_{y,u}(\omega) = (\widehat{F}(j\omega)'\widehat{F}(j\omega))^{\frac{1}{2}}$. That is the non-zero singular values of $F(j\omega)$ are equal to the non-zero singular values of $\Psi_{y,u}(\omega)$. In forming $\Psi_{y,u}(\omega)$ the phase information of $F(j\omega)$ is lost but the gain information is retained.*

Before going on we present some technical results that will be used in the following sections. They are stated without proof due to constraints on length.

Lemma 3.5 *Given a bounded, causal, LTI operator F with frequency domain transfer matrix $\widehat{F}$,*

$$\|\widehat{\mathsf{F}}(e^{-j\hat{\omega}h})\|_{\mathscr{E}^m \to \mathscr{E}^p} = \overline{\sigma}^{\,\omega}_{sup}\{\widehat{F}\},$$

where $\widehat{\mathsf{F}}$ is the transfer operator for the time-lifted operator $\mathsf{F} = \boldsymbol{\mathcal{L}}_h F \boldsymbol{\mathcal{L}}_h^{-1}$ and

$$\overline{\sigma}^{\,\omega}_{sup}\{\cdot\} = \sup_{n\in\mathbb{Z}} \overline{\sigma}\{\cdot(j(\widehat{\omega} + n\frac{2\pi}{h}))\}, \tag{3}$$

$\widehat{\omega} = \omega -]\frac{h\omega}{2\pi}[\,(\frac{2\pi}{h})$.

Lemma 3.6 *Given a bounded, causal LTI operator F mapping u to y and a bounded, causal, LPTV operator P mapping y to x, consider the bounded, causal, LPTV operator PF mapping u to x. Let $u = e^{j\omega t}u^d$, where $\|u^d\|_{\mathbb{R}^m} = 1$. Then,*

$$\overline{\sigma}\{\Psi_{x,u}(\omega)\} \le \overline{\sigma}\{\Psi_{x,y}(\omega)\}\overline{\sigma}\{\widehat{F}(j\omega)\},$$

where $\Psi_{\cdot,\cdot}(\omega)$ is the average-power gain matrix as defined in Definition 3.3 and $\widehat{F}$ is the frequency domain transfer matrix for F.

Lemma 3.7 *Given a causal, bounded LPTV operator P mapping u to y and a causal, bounded, LTI operator F mapping y to x, consider the causal, bounded, LPTV operator FP mapping u to x. Let $u = e^{j\omega t}u^d$, where $\|u^d\|_{\mathbb{R}^m} = 1$. Then,*

$$\overline{\sigma}\{\Psi_{x,u}(\omega)\} \le \overline{\sigma}^{\,\omega}_{sup}\{\widehat{F}\}\overline{\sigma}\{\Psi_{y,u}(\omega)\}.$$

where $\Psi_{\cdot,\cdot}(\omega)$ is the average-power gain matrix as defined in Definition 3.3, $\widehat{F}$ is the frequency-domain transfer matrix for F and $\overline{\sigma}^{\,\omega}_{sup}\{\cdot\}$ is defined in equation (3).

4 State-Space Formulae for the SD Case

Consider the stable, SD, closed-loop system shown in Figure 1, where $\boldsymbol{\mathcal{S}}$ is an ideal, h-periodic sample operator synchronised with the zero-order hold operator $\boldsymbol{\mathcal{H}}$ and the causal, LTI sub-systems G and K_d have state-space realisations

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & 0 & 0 \end{array}\right] \text{ and } K_d = \left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right].$$

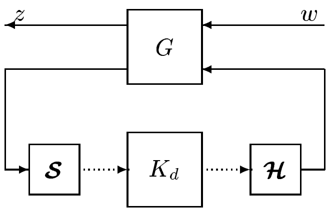

Figure 1: SD Closed-Loop

The closed-loop operator mapping w to z is an LPTV operator. In what follows we derive state-space formulae for the average-power gain matrix $\Psi_{z,w}(\omega)$ as defined in Definition 3.3. The state-space formulae derived are expressible explicitly in terms of the state-space matrices of the closed-loop, sub-systems G and K_d.

As described in the previous section, time-lifting the input and output signals and absorbing the ideal sample and hold operators into G, yields a discrete-time, linear, shift-invariant characterisation of the closed-loop operator mapping w to z. This can be expressed in terms of the system of equations

$$\begin{aligned} \begin{bmatrix} x_{k+1} \\ \xi_{k+1} \end{bmatrix} &= \mathsf{P}\begin{bmatrix} x_k \\ \xi_k \end{bmatrix} + \mathsf{Q}\ \mathsf{w}_k(\theta), \\ \mathsf{z}_k(\theta) &= \mathsf{R}\begin{bmatrix} x_k \\ \xi_k \end{bmatrix} + \mathsf{S}\ \mathsf{w}_k(\theta), \end{aligned} \tag{4}$$

where

$$\mathsf{P} = \begin{bmatrix} \mathsf{A} + \mathsf{B}_2 D_d C_2 & \mathsf{B}_2 C_d \\ B_d C_2 & A_d \end{bmatrix}, \tag{5}$$

$$\mathsf{Q} = \begin{bmatrix} \mathsf{B}_1 \\ 0 \end{bmatrix}, \tag{6}$$

$$\mathsf{R} = \begin{bmatrix} \mathsf{C}_1 + \mathsf{D}_{12} D_d C_2 & \mathsf{D}_{12} C_d \end{bmatrix}, \tag{7}$$

$$\mathsf{S} = \mathsf{D}_{11}, \tag{8}$$

$$\mathsf{A} : \mathbb{R}^n \to \mathbb{R}^n : x \to e^{hA}x, \tag{9}$$

$$\mathsf{B}_1 : \mathscr{E}^{m_1} \to \mathbb{R}^n : \mathsf{w}(\theta) \to \int_0^h e^{(h-\tau)A}B_1\mathsf{w}(\tau)d\tau, \tag{10}$$

$$\mathsf{B}_2 : \mathbb{R}^{m_2} \to \mathbb{R}^n : \nu \to \int_0^h e^{(h-\tau)A}d\tau B_2\nu, \tag{11}$$

$$\mathsf{C}_1 : \mathbb{R}^n \to \mathscr{E}^{p_1} : x \to C_1 e^{\theta A}x, \tag{12}$$

$$\mathsf{D}_{11} : \mathscr{E}^{m_1} \to \mathscr{E}^{p_1} : \mathsf{w}(\theta) \to D_{11}\mathsf{w}(\theta) + \int_0^t C_1 e^{(\theta-\tau)A}B_1\mathsf{w}(\tau)d\tau, \tag{13}$$

$$\mathsf{D}_{12} : \mathbb{R}^{m_2} \to \mathscr{E}^{p_1} : \nu \to D_{12}\nu + \int_0^t C_1 e^{(\theta-\tau)A}d\tau B_2\nu, \tag{14}$$

$\mathsf{w}_k(\theta) = w(\theta + kh)$ and $\mathsf{z}_k(\theta) = z(\theta + kh)$. As defined in Section 3, the transfer operator of the time-lifted, closed-loop is then

$$\widehat{\mathsf{H}}(\lambda) = \mathsf{R}\lambda(I - \lambda\mathsf{P})^{-1}\mathsf{Q} + \mathsf{S}.$$

Let $\mathsf{P}_{st}(\omega)$ denote the matrix $(e^{j\widehat{\omega}h}I - \mathsf{P})^{-1}$ and $\Phi_{z,w}(\omega)$ and $\Xi(\omega)$ be defined as in Section 3. Then $\Phi_{z,w}(\omega)^*\Phi_{z,w}(\omega)$ can be expressed as

$$\Xi(\omega)^*[\mathsf{Q}^*\mathsf{P}_{st}(\omega)'\mathsf{R}^* + \mathsf{S}^*][\mathsf{R}\ \mathsf{P}_{st}(\omega)\mathsf{Q} + \mathsf{S}]\Xi(\omega). \quad (15)$$

Expanding this and substituting expressions for the lifted, closed-loop, state-space operators given in equations (5–8), it can be seen that in order to compute $\Phi_{z,w}(\omega)^*\Phi_{z,w}(\omega)$ we need to first compute finite-dimensional expressions for the following matrices :

$$\mathsf{A},\quad \mathsf{B}_2,\quad \mathsf{B}_1\Xi(\omega),\quad \begin{bmatrix}\mathsf{C}_1^*\\ \mathsf{D}_{12}^*\end{bmatrix}\begin{bmatrix}\mathsf{C}_1 & \mathsf{D}_{12}\end{bmatrix},$$
$$\Xi(\omega)^*\mathsf{D}_{11}^*\mathsf{D}_{11}\Xi(\omega),\quad \Xi(\omega)^*\mathsf{D}_{11}^*\begin{bmatrix}\mathsf{C}_1 & \mathsf{D}_{12}\end{bmatrix}.$$

Subsequent computation of $\Phi_{z,w}(\omega)^*\Phi_{z,w}(\omega)$ and hence $\Psi_{z,w}(\omega) = \frac{1}{\sqrt{h}}(\Phi_{z,w}(\omega)^*\Phi_{z,w}(\omega))^{\frac{1}{2}}$, is then a simple matter of standard finite-dimensional matrix arithmetic and will not be explicitly presented here.

Define the square matrices $\breve{A} = \begin{bmatrix} A & B_2\\ 0 & 0\end{bmatrix}$,

$$\begin{aligned}
\underline{A} &= \begin{bmatrix} \breve{A} & \begin{bmatrix}B_1\\0\end{bmatrix}\\ 0 & (j\omega I_m)\end{bmatrix},\\
\underline{C} &= \begin{bmatrix} C_1 & D_{12} & D_{11}\end{bmatrix},\\
E &= \begin{bmatrix} \underline{A} & 0\\ -\underline{C}'\underline{C} & -\underline{A}'\end{bmatrix}
\end{aligned} \quad (16)$$

and

$$Q = \begin{bmatrix} Q_{11} & Q_{12}\\ Q_{21} & Q_{22}\end{bmatrix} = e^{hE}, \quad (17)$$

where the partitioning of Q conforms with that of E.

Theorem 4.1 *Let n and m denote the number of states and the number of inputs to the 11-block of G respectively and let s denote the number of outputs from K_d. For notational convenience, define $l = n+s$, $q = n+m$ and $r = m+s$. Then,*

$$\begin{aligned}
\Xi(\omega)^*\mathsf{D}_{11}^*\mathsf{D}_{11}\Xi(\omega) &= -\begin{bmatrix}0_{m,l} & I_m\end{bmatrix}Q_{22}^{-1}Q_{21}\begin{bmatrix}0_{l,m}\\ I_m\end{bmatrix},\\
\Xi(\omega)^*\mathsf{D}_{11}^*\begin{bmatrix}\mathsf{C}_1 & \mathsf{D}_{12}\end{bmatrix} &= -\begin{bmatrix}0_{m,l} & I_m\end{bmatrix}Q_{22}^{-1}Q_{21}\begin{bmatrix}I_q\\ 0_{s,q}\end{bmatrix},\\
\begin{bmatrix}\mathsf{C}_1^*\\ \mathsf{D}_{12}^*\end{bmatrix}\begin{bmatrix}\mathsf{C}_1 & \mathsf{D}_{12}\end{bmatrix} &= -\begin{bmatrix}I_q & 0_{q,s}\end{bmatrix}Q_{22}^{-1}Q_{21}\begin{bmatrix}I_q\\ 0_{s,q}\end{bmatrix},\\
\mathsf{B}_1\Xi(\omega) &= \begin{bmatrix}I_n & 0_{n,r}\end{bmatrix}Q_{11}\begin{bmatrix}0_{l,m}\\ I_m\end{bmatrix},\\
\mathsf{A} &= \begin{bmatrix}I_n & 0_{n,r}\end{bmatrix}Q_{11}\begin{bmatrix}I_n\\ 0_{r,n}\end{bmatrix},\\
\mathsf{B}_2 &= \begin{bmatrix}I_n & 0_{n,r}\end{bmatrix}Q_{11}\begin{bmatrix}0_{n,s}\\ I_s\\ 0_{m,s}\end{bmatrix},
\end{aligned}$$

where $0_{i,k}$ denotes an $i\times k$ matrix of zeros and I_k the $k\times k$ identity.

Proof : Consider $\hat{z} = \Xi(\omega)^*\mathsf{D}_{11}^*\mathsf{D}_{11}\Xi(\omega)z$ for some $z \in \mathbb{R}^m$ and let $u = \Xi(\omega)z$, $y = \mathsf{D}_{11}u$, $f = \mathsf{D}_{11}^*y$. By the definition of $\Xi(\omega)$ it is clear that $u = \Xi(\omega)z$ can be characterised by the system of equations

$$\begin{aligned}
\dot{\xi}(\theta) &= (j\omega I_m)\xi(\theta),\ \xi(0) = z,\\
u(\theta) &= \xi(\theta).
\end{aligned} \quad (18)$$

It follows that $\hat{z} = \Xi(\omega)^*f$ can be expressed as

$$\begin{aligned}
\dot{\zeta}(\theta) &= -(j\omega I_m)'\zeta(\theta) - f(\theta),\ \zeta(h) = 0,\\
\hat{z} &= \zeta(0).
\end{aligned} \quad (19)$$

Examining equation (13), we see that $y = \mathsf{D}_{11}u$, respectively $f = \mathsf{D}_{11}^*y$, can be characterised by the system of equations

$$\begin{aligned}
\dot{x}(\theta) &= \breve{A}x(\theta) + \begin{bmatrix}B_1\\0\end{bmatrix}u(\theta),\ x(0) = 0,\\
y(\theta) &= \begin{bmatrix}C_1 & D_{12}\end{bmatrix}x(\theta) + D_{11}u(\theta),
\end{aligned} \quad (20)$$

respectively

$$\begin{aligned}
\dot{\eta}(\theta) &= -\breve{A}'\eta(\theta) - \begin{bmatrix}C_1'\\ D_{12}'\end{bmatrix}y(\theta),\ \eta(h) = 0,\\
f(\theta) &= \begin{bmatrix}B_1' & 0\end{bmatrix}\eta(\theta) + D_{11}'y(\theta).
\end{aligned} \quad (21)$$

Combining equations (18–21) we obtain the two-point, boundary-value problem

$$\begin{bmatrix}\dot{x}\\ \dot{\xi}\\ \dot{\eta}\\ \dot{\zeta}\end{bmatrix}(\theta) = E\begin{bmatrix}x\\ \xi\\ \eta\\ \zeta\end{bmatrix}(\theta),\quad \begin{bmatrix}x(0)\\ \xi(0)\\ \eta(h)\\ \zeta(h)\end{bmatrix} = \begin{bmatrix}0\\ z\\ 0\\ 0\end{bmatrix}$$

where E is defined in equation (16). The solution satisfies

$$\begin{bmatrix}x(h)\\ \xi(h)\\ \dots\\ \eta(h)\\ \zeta(h)\end{bmatrix} = e^{tE}\begin{bmatrix}x(0)\\ \xi(0)\\ \dots\\ \eta(0)\\ \zeta(0)\end{bmatrix}.$$

Using the terminal condition $\begin{bmatrix}\eta(h)\\ \zeta(h)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ we find that

$$\begin{bmatrix}\eta(0)\\ \zeta(0)\end{bmatrix} = -Q_{22}^{-1}Q_{21}\begin{bmatrix}x(0)\\ \xi 0\end{bmatrix} = -Q_{22}^{-1}Q_{21}\begin{bmatrix}0\\ z\end{bmatrix}.$$

where Q is defined in equation (17). Hence, noting that $\hat{z} = \zeta(0) = \Xi(\omega)^*\mathsf{D}_{11}^*\mathsf{D}_{11}\Xi(\omega)z$ we conclude that

$$\Xi(\omega)^*\mathsf{D}_{11}^*\mathsf{D}_{11}\Xi(\omega) = -\begin{bmatrix}0_{m,l} & I_m\end{bmatrix}Q_{22}^{-1}Q_{21}\begin{bmatrix}0_{l,m}\\ I_m\end{bmatrix}.$$

The proof follows in a similar manner for the other operators and is omitted for the sake of brevity. ■

Remark 4.2 *Note that the formulae given in Theorem 4.1 require the exponentiation of a Hamiltonian-structured matrix E. This can become numerically sensitive if any of the eigen-values of E have large real part. However, by the nature of the formulae presented, they can be easily restructured to give numerically robust computational formulae (see [5] for details of a similar restructuring for numerical robustness.)*

5 H_∞ Loop-Shaping Design

A framework for the design of SD control systems is proposed in [4], motivated by the H_∞ loop-shaping design procedure of McFarlane of Glover [12]. In brief, given a causal, LTI, open-loop system G to be controlled, design in the proposed framework involves the selection of stable, minimum-phase, LTI weights W_1 and W_2, which in the full continuous-time, LTI case are used to shape the open-loop singular values to trade off desired performance and robustness. This design step is followed by the synthesis of a causal, discrete-time, LTI controller K_d^∞ to bound the induced norm of the operator mapping $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ in the closed-loop configuration shown in Figure 2, where F_b is a simple, strictly-causal, stable, LTI filter used to ensure boundedness of the sample-operator. The weight W_2 is then absorbed into the pre-sampling filter F_b, to give the final controller controller $K_{sd} = \boldsymbol{\mathcal{H}} K_d^\infty \boldsymbol{\mathcal{S}} F$, where $F = F_b W_2$.

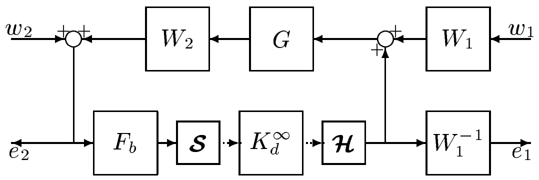

Figure 2: SD Design Framework

The operator mapping $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ is of the form

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} K_{sd}^\infty \\ I \end{bmatrix} (I - G_s K_{sd}^\infty)^{-1} \begin{bmatrix} G_s & I \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \qquad (22)$$

where $G_s = W_2 G W_1$ and $K_{sd}^\infty = W_1^{-1} \boldsymbol{\mathcal{H}} K_d^\infty \boldsymbol{\mathcal{S}} F_b$. Bounding the induced norm of this operator is appealing from the perspective of both performance and robustness. Suppose we know that the induced norm of this operator is bounded from above by say γ. It is shown in [4] that $K_{sd} = \boldsymbol{\mathcal{H}} K_d^\infty \boldsymbol{\mathcal{S}} F$ ($F = F_b W_2$) will stabilise all plants in the ball constructed by perturbing the left-coprime factors of the weighted plant G_s by bounded, operators of induced norm less than $\frac{1}{\gamma}$. Furthermore, using results in [6], it can be shown that K_{sd} stabilises all plants in the weighted, open H_2-gap ball of h-periodic operators centred at G_s and with radius $\frac{1}{\gamma}$ if and only if γ is greater than or equal to the induced norm of the closed-loop operator in equation (22).

We now turn our attention to performance. Using the identity [7, 10, 6],

$$\left\| \begin{bmatrix} K_{sd}^\infty \\ I \end{bmatrix} (I - G_s K_{sd}^\infty)^{-1} \begin{bmatrix} G_s & I \end{bmatrix} \right\|_{L_2 \to L_2} = \left\| \begin{bmatrix} G_s \\ I \end{bmatrix} (I - K_{sd}^\infty G_s)^{-1} \begin{bmatrix} K_{sd}^\infty & I \end{bmatrix} \right\|_{L_2 \to L_2}, \qquad (23)$$

we can obtain bounds on the behaviour of all six common closed-loop operators. For example, the operators characterising tracking, sensor noise rejection and plant-input disturbance rejection. In [12], for the full continuous-time, LTI case, frequency-by-frequency bounds are derived for the frequency response of such operators in terms of the bound γ obtained in the synthesis step, the weighted plant G_s (or its coprime factors) and the weights W_1 and W_2. These bounds aid in the selection of weights W_1 and W_2 in the design step. In what follows, we derive for the SD case similar frequency-by-frequency bounds on the singular values of the average-power gain matrix as defined in Definition 3.3, for a subset of the six closed-loop operators. The operators for which bounds are given have been chosen according to their significance from a performance perspective. Looking at frequency-by-frequency bounds for closed-loop operators that reflect robustness is meaningless in the SD setting. To determine robustness properties of periodic systems, it is necessary to consider the behaviour of the system in response to an infinite series of sinusoidal inputs of frequencies separated by integer multiples of the sampling frequency [8].

Theorem 5.1 *Denote the weighted tracking error, plant-input disturbance to plant-output error and the sensor noise to plant-input error in response to unit average-power sinusoidal inputs by*

$$\begin{aligned} e_r(t) &= W_2 (I - G K_{sd})^{-1} e^{j\omega t} r^d, \\ e_d(t) &= (I - K_{sd} G)^{-1} e^{j\omega t} d^d \end{aligned}$$

and

$$e_n(t) = (I - K_{sd} G)^{-1} K_{sd} e^{j\omega t} n^d,$$

respectively, where $\|r^d\|_{\mathbb{R}^p} = \|d^d\|_{\mathbb{R}^m} = \|n^d\|_{\mathbb{R}^p} = 1$. Let $e_{r,ss}^\omega$, $e_{d,ss}^\omega$ and $e_{n,ss}^\omega$ denote the functions which $e_r(t)$, $e_d(t)$ and $e_n(t)$ asymptotically approach as $t \to \infty$. Then if

$$\left\| \begin{bmatrix} K_{sd}^\infty \\ I \end{bmatrix} (I - G_s K_{sd}^\infty)^{-1} \begin{bmatrix} G_s & I \end{bmatrix} \right\|_{L_2 \to L_2} < \gamma, \qquad (24)$$

we have

$$\begin{aligned} \|e_{r,ss}^\omega\|_{pow} &\le \overline{\sigma}\{\Psi_{e_r,r}(\omega)\} < \gamma\, \overline{\sigma}\{\widehat{M}_s^l(j\omega)\} \overline{\sigma}\{\widehat{W}_2(j\omega)\}, \\ \|e_{d,ss}^\omega\|_{pow} &\le \overline{\sigma}\{\Psi_{e_d,d}(\omega)\} < \gamma\, \overline{\sigma}_{sup}^{\,\omega}\{\widehat{W}_1 \widehat{M}_s^r\} \frac{1}{\underline{\sigma}\{\widehat{W}_1(j\omega)\}}, \\ \|e_{n,ss}^\omega\|_{pow} &\le \overline{\sigma}\{\Psi_{e_n,n}(\omega)\} < \gamma\, \overline{\sigma}_{sup}^{\,\omega}\{\widehat{W}_1 \widehat{M}_s^r\} \overline{\sigma}\{\widehat{W}_2(j\omega)\}, \end{aligned}$$

where $\Psi_{\cdot,\cdot}(\omega)$ is the average-power gain matrix of an operator as defined in Definition 3.3, $\overline{\sigma}_{sup}^{\,\omega}\{\cdot\}$ is defined in

(3), $G_s = N_s^r (M_s^r)^{-1} = (M_s^l)^{-1} N_s^l$ are normalised, coprime factorisations and $\widehat{M}_s^r$, $\widehat{M}_s^l$, $\widehat{W}_1$ and $\widehat{W}_2$ are the frequency domain transfer matrices for the LTI operators M_s^r, M_s^l, W_1 and W_2 respectively.

Proof : We will prove the bound on $\overline{\sigma}\{\Psi_{e_n,n}(\omega)\}$. The other bounds follow in a similar manner. First note that by Lemma 3.1 in [4] and from equations (23) and (24) that

$$\left\| \begin{bmatrix} K_{sd}^{\infty} \\ I \end{bmatrix} (I - G_s K_{sd}^{\infty})^{-1} (M_s^l)^{-1} \right\|_{L_2 \to L_2} =$$

$$\left\| (M_s^r)^{-1} (I - K_{sd}^{\infty} G_s)^{-1} \begin{bmatrix} K_{sd}^{\infty} & I \end{bmatrix} \right\|_{L_2 \to L_2} < \gamma. \quad (25)$$

Let $H = (M_s^r)^{-1}(I - K_{sd}^{\infty} G_s)^{-1} K_{sd}^{\infty}$, $y = Hu$ and $u(t) = e^{j\omega t} u^d$, $\|u_d\|_{\mathbb{R}^p} = 1$. Define $\mathsf{u}^d(\theta) e^{j\omega\theta}$. Then by Lemma 3.2 and the definition of $\Psi_{y,u}(\omega)$ we have

$$\begin{aligned} \|\Psi_{y,u}(\omega) u^d\|_{\mathbb{R}^m} &= \frac{1}{\sqrt{h}} \|(\widehat{\mathsf{H}}(e^{-j\widehat{\omega} h}) \mathsf{u}^d)\|_{\mathscr{E}^m} \\ &\leq \|\widehat{\mathsf{H}}(e^{-j\widehat{\omega} h})\|_{\mathscr{E}^p \to \mathscr{E}^m}. \quad (26) \end{aligned}$$

From equation (25), $\|\widehat{\mathsf{H}}(e^{-j\widehat{\omega} h})\|_{\mathscr{E}^p \to \mathscr{E}^m} < \gamma$. So taking the supremum over $\|u^d\|_{\mathbb{R}^p} = 1$ in equation (26) gives

$$\overline{\sigma}\{\Psi_{y,u}(\omega)\} < \gamma.$$

Now note that $e_n(t) = W_1 M_s^r H W_2 e^{j\omega t} n^d$. Thus, by Lemmas 3.6 and 3.7, we have

$$\begin{aligned} \overline{\sigma}\{\Psi_{e_n,n}(\omega)\} &\leq \overline{\sigma}_{sup}^{\omega}\{\widehat{W}_1 \widehat{M}_s^r\} \overline{\sigma}\{\Psi_{y,u}(\omega)\} \overline{\sigma}\{\widehat{W}_2(j\omega)\} \\ &< \gamma\, \overline{\sigma}_{sup}^{\omega}\{\widehat{W}_1 \widehat{M}_s^r\} \overline{\sigma}\{\widehat{W}_2(j\omega)\}. \end{aligned}$$

This completes the proof. ∎

Remark 5.2 *These bounds are similar to those given in [12] for the full continuous-time, LTI case. We have given the "tracking-error bound" as a bound on the actual tracking-error weighted by $\widehat{W}_2$. This was necessary to obtain a reasonable bound. It can be argued that provided $\widehat{W}_2$ is selected to roll-off at a frequency sufficiently greater than the bandwidth of the unweighted open loop G, this bound will not be too far from that of the actual tracking error. Also, $\widehat{W}_1$ can be selected to give $\sup_{\omega} \overline{\sigma}\{\widehat{W}_1 \widehat{M}_s^r(j\omega)\} \approx 1$, so that the given bounds on input-disturbance and sensor noise rejection are sensible in the frequency ranges where they are important.*

Acknowledgements

M. W. Cantoni is supported financially by the Gledden Studentship, University of Western Australia, and the ORS Award, United Kingdom.

References

[1] Araki, M. and Ito, Y., "Frequency-response of Sampled-Data Systems I: Open-Loop Consideration." *Proceedings of the 12th IFAC World Congress*, 7:289-292, 1993.

[2] Araki, M. Hagiwara, T. and Ito, Y., "Frequency-response of Sampled-Data Systems I: Closed-Loop Consideration." *Proceedings of the 12th IFAC World Congress*, 7:293-296, 1993.

[3] Bamieh, B. A., Pearson, J. B., Francis, B. A. and Tannenbaum, A. R., "A Lifting Technique for Linear Periodic Systems With Applications to Sampled-Data Control." *Systems and Control Letters,* 17:79-88, 1991.

[4] Cantoni, M. W. and Glover, K., "A Design-Framework for Continuous-Time Systems Under Sampled-Data Control." *Proceedings of the 35th IEEE CDC*, pp. 458-463, 1996.

[5] Cantoni, M. W. and Glover, K., "H_∞ Sampled-Data Synthesis and Related Numerical Issues." To appear in Automatica, 1997.

[6] Cantoni, M. W. and Glover, K., "Linear Periodic Systems: Representations, The Gap Metric, Robust Stability and Sampled-Data Control." Paper in preparation, 1997.

[7] Doyle, J. C., Georgiou, T. T. and Smith, M. C., "The Parallel Projection Operators of A Nonlinear Feedback System." *Systems and Control Letters*, 20:79-85, 1993.

[8] Dullerud, G. E., *Control of Uncertain Sampled-Data Systems.* PhD Dissertation, University of Cambridge, U. K., 1994.

[9] Foiaş, C. and Frazho, A., *The Commutant Lifting Approach to Interpolation Problems.* Operator Theory : Advances and Applications, Vol. 44, Birkhäuser Verlag, Berlin, 1990.

[10] Foiaş, C., Georgiou, T. T. and Smith, M. C., "Geometric Techniques for Robust Stabilization of Linear Time-Varying Systems." *Proceedings of the 29th IEEE CDC*, 2868-2873, 1990.

[11] Goodwin, G. C. and Salgado, M., "Frequency Domain Sensitivity Functions For Continuous-Time Systems Under Sampled-Data Control." *Automatica*, 30:1263-1270, 1994.

[12] McFarlane, D. C. and Glover, K., *Robust controller Design Using Normalised Coprime Factorisation Plant Descriptions.* Lecture Notes in Control and Information Sciences, Springer-Verlag, Berlin, 1990.

[13] Sz-Nagy, B. and Foiaş, C., *Harmonic Analysis of Operators on Hilbert Space.* North-Holland Publishing Company, Amsterdam, 1970.

[14] Yamamoto, Y., "A Function Space Approach to Sampled-Data Control Systems and Tracking Problems." *IEEE Transactions on Automatic Control,* 39:703-713, 1994

[15] Yamamoto, Y. and Araki, M., "Frequency Responses for Sampled-Data Systems - Their Equivalence and Relationships." *Linear Algebra and its Applications,* 205-206:1319-1339, 1994.

[16] Yamamoto, Y. and Khargonekar, P. P., "Frequency Response of Sampled-Data Systems." *IEEE Transactions on Automatic Control*, 41:166-176, 1996.