Global Stability of Relay Feedback Systems

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Abstract

For a large class of relay feedback systems (RFS) there will be limit cycle oscillations. Conditions to check existence and *local* stability of limit cycles for these systems are well known. *Global* stability conditions, however, are practically non-existent. This paper presents conditions in the form of linear matrix inequalities (LMIs) that guarantee *global* asymptotic stability of a limit cycle induced by a relay with hysteresis in feedback with an LTI stable system. The analysis is based on finding global quadratic Lyapunov functions for a Poincaré map associated with the RFS. We found that a typical Poincaré map induced by an LTI flow between two hyperplanes can be represented as a linear transformation analytically parametrized by a scalar function of the state. Moreover, level sets of this function are convex. The search for globally quadratic Lyapunov functions is then done by solving a set of LMIs. Most examples of RFS analyzed by the authors were proven globally stable. Systems analyzed include minimum-phase systems, systems of relative degree larger than one, and of high dimension. This leads us to believe that quadratic stability of associated Poincaré maps is common in RFS.

1 Introduction

Although widely used [3, 4, 7, 18, 20], very few results are available to analyze most PLS. More precisely, one typically cannot guarantee stability, robustness, and performance properties of PLS designs. Rather, any such properties are inferred from extensive computer simulations. However, in the absence of rigorous analysis tools, PLS designs come with no guarantees. In other words, complete and systematic analysis and design methodologies have yet to emerge.

In this paper, a new methodology to analyze global asymptotic stability of PLS is proposed. This methodology is based in proving quadratic Lyapunov functions on the switching surfaces that can be used to prove the map from one switching surface to the next switching surface is contracting in some norm. The main difference between this and previous work, e.g. [16], is that we look for quadratic Lyapunov functions on the switching surfaces instead of quadratic Lyapunov functions on the state space. An immediate advantage is that this allows us to analyze not only equilibrium points (very recently we were able to prove global asymptotic stability of on/off systems [8] and saturation systems [9]) but also limit cycles.

To demonstrate the success of this methodology, we apply it to a simple yet very hard to analyze class of PLS known as relay feedback systems (RFS). Although the focus of this paper is on RFS, it is important to point out that most ideas behind the main results described here can be used in the analysis of more general PLS.

Analysis of RFS is a classic field. The early work was motivated by relays in electromechanical systems and simple models of dry friction. Applications of relay feedback range from stationary control of industrial processes to control of mobile objects as used, for example, in space research. A vast collection of applications of relay feedback can be found in the first chapter of [21]. More recent examples include the delta-sigma modulator (as an alternative to conventional A/D converters) and the automatic tuning of PID regulators. In the delta-sigma modulator, a relay produces a bit stream output whose pulse density depends on the applied input signal amplitude (see, for example, [1]). Various methods were applied to the analysis of delta-sigma modulators. In most situations, however, none allowed to verify global stability of nonlinear oscillations. As for the automatic tuning of PID regulators, implemented in many industrial controllers, the idea is to determine some points on the Nyquist curve of a stable open loop plant by measuring the frequency of oscillation induced by a relay feedback (see, for example, [4]). One problem that needs to be solved here is the characterization of those systems that have unique global attractive unimodal limit cycles. This problem is important because it gives the class of systems where relay tuning can be used.

Some important questions can be asked about RFS: do they have limit cycles? If so, are they locally stable or unstable? And if there exist a unique locally stable limit cycle, is it also globally stable? Over many years, researchers have been trying to answer these questions. [5] and [21] are references that survey a number of anal-

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ysis methods. Rigorous results on existence and *local* stability of limit cycles of RFS can be found in [2, 15]. In [2], necessary and sufficient conditions for local stability of limit cycles are presented. [15] emphasizes fast switches and their properties and also proves volume contraction of RFS. In [10], reasonably large regions of stability around limit cycles were characterized. For second-order systems, convergence analysis can be done in the phase-plane [19, 13]. Stable second-order non-minimum phase processes can in this way be shown to have a globally attractive limit cycle. In [17] it is proved that this also holds for processes having an impulse response sufficiently close, in a certain sense, to a second-order non-minimum phase process. Many important RFS, however, are not covered by this result. It is then clear that the problem of rigorous *global* analysis of relay-induced oscillations is still open.

In this paper we prove *global* stability of limit cycles of RFS by finding quadratic stability of associated Poincaré maps. These results are based on the discovery that most Poincaré maps can be represented as linear transformations parametrized by a scalar function of the state. Quadratic stability can then be easily checked by solving a set of linear matrix inequalities (LMIs), which can be efficiently solved using available computational tools. Using these ideas, most RFS analyzed by the authors were proven to be globally stable. Systems analyzed include minimum-phase systems, systems of relative degree larger than one, and of high dimension. This leads us to believe that quadratic stability of Poincaré maps is common in RFS.

This paper is organized as follows. Section 2 is dedicated to give some background on RFS followed by the main result of this paper (section 3). There, we show that most Poincaré maps can be decomposed in a such a way that it is possible to reduce the problem of quadratic stability of Poincaré maps to solving a set of LMIs. Section 4 contains some illustrative examples. Finally, conclusions and future work are discussed in section 5.

2 Background

We start by defining a RFS. Consider an LTI system

$$\left\{ \begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Cx \end{array} \right. \tag{1}$$

where $x \in \mathbb{R}^n$ and A is Hurwitz, in feedback with a relay (see figure 1) defined as

$$u(t) \in \begin{cases} \{-1\} & \text{if } y(t) > d, \text{ or } y(t) > -d \\ & \quad \text{and } u(t-0) = -1 \\ \{1\} & \text{if } y(t) < -d, \text{ or } y(t) < d \\ & \quad \text{and } u(t-0) = 1 \\ \{-1, 1\} & \text{if } y(t) = -d \text{ and } u(t-0) = -1 \\ & \quad \text{or } y(t) = d \text{ and } u(t-0) = 1 \end{cases} \tag{2}$$

where $d \geq 0$ is the hysteresis parameter. By a solution of (1)-(2) we mean functions (x, y, u) satisfying (1)-(2), where u is piecewise constant. Note that existence of a solution is always guaranteed if $d > 0$, or if $d = 0$ and $CB < 0$, which are the cases we consider in this paper. t is a switching time of a solution of (1)-(2) if u is discontinuous at t. We say a trajectory of (1)-(2) switches at some time t if t is a switching time.

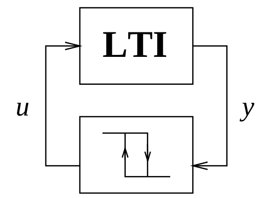

Figure 1: Relay Feedback System

The *switching surfaces* S and $\underline{S}$ of the RFS are the surfaces of dimension $n-1$ where y is equal to d and $-d$, respectively. More precisely

$$S = \{x \in \mathbb{R}^n : \; Cx = d\}$$

and $\underline{S} = -S$. Consider a subset S_a of S given by

$$S_a = \{x \in S : \; CAx + CB \geq 0\}$$

This set is important since it characterizes those points in S that can be reached by any trajectory starting at $\underline{S}$. We call it the arrival set in S (see figure 2). Similarly, define $\underline{S_a} = -S_a$

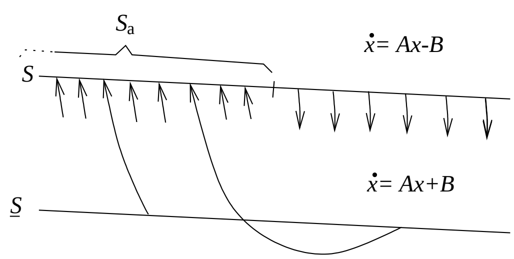

Figure 2: The arrival set S_a

Note that trajectories of $\dot{x} = Ax - B$ starting at any point $x_0 \in S$ will converge to the equilibrium point $A^{-1}B$. When connected in feedback with the relay, one of the following two possible scenarios will occur for a certain trajectory starting at x_0: it will either cross $\underline{S}$ at some time, or it will never cross $\underline{S}$. The last situation is not interesting to us since it does not lead to limit cycle trajectories. One way to ensure a switch is to have $CA^{-1}B + d < 0$, although this is not a necessary condition for the existence of limit cycles. However, if we are looking for globally stable limit cycles, it is in fact necessary to have $CA^{-1}B + d < 0$. Otherwise a trajectory starting at $A^{-1}B$ would not converge to the limit cycle. Throughout the paper, it is assumed $CA^{-1}B + d < 0$.

As we mentioned before, for a large class of processes, there will be limit cycle oscillations. Let $\xi(t)$ be a non-trivial periodic solution of (1)-(2) with period T, and let γ be the limit cycle defined by the trace of $\xi(t)$. The limit cycle γ is called *symmetric* if $\xi(t+T/2)=-\xi(t)$. It is called *unimodal* if it only switches twice per cycle. A class of limit cycles we are particularly interested in is the class of all symmetric unimodal limit cycles. [2] gives necessary and sufficient conditions for the existence and local stability of symmetric unimodal limit cycles.

An interesting property of linear systems in relay feedback is their symmetry around the origin, i.e., if $x(t)$ is a trajectory of $\dot{x}=Ax-B$ starting at $x_0 \in S$, then $-x(t)$ is a trajectory of $\dot{x}=Ax+B$ starting at $-x_0 \in \underline{S}$. This means that a limit cycle only needs to be analyzed over half of its period.

We are now ready to define what we mean by a Poincaré map for a RFS. Consider a symmetric unimodal limit cycle γ, with period $2t^*$, obtained with the initial condition $x^* \in S_a$. This means that a trajectory $x(t)$ starting at x^* crosses the switching surface $\underline{S}$ at $-x^*=x(t^*) \in \underline{S_a}$ (see figure 3).

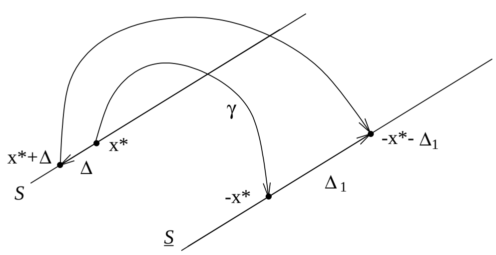

Figure 3: Definition of a Poincaré map for a RFS

To study the behavior of the system around the limit cycle we perturb x^* by Δ such that $x^*+\Delta \in S_a$. With the initial condition $x(0)=x^*+\Delta$, consider the solution of (1)-(2) until the system switches. Let $-x^*-\Delta_1$ be the switching point in $\underline{S}$. We are interested in studying the map from Δ to Δ_1 (see figure 3).

Note that, in general, Δ_1 is not unique, that is, the map from Δ to Δ_1 is, in general, a multivalued map. Define t_Δ as the set of all times $t_i \geq 0$ such that $y(t_i)=-d$ and $y(t) \geq -d$ on $[0,t_i]$. For example, in the case of figure 4, $t_\Delta=\{t_1,t_2,t_3\}$. This means that a switch can occur at $t=t_1$, $t=t_2$, or $t=t_3$. Obviously, if no switch occurred at $t=t_1$ or $t=t_2$, then a switch must occur at $t=t_3$.

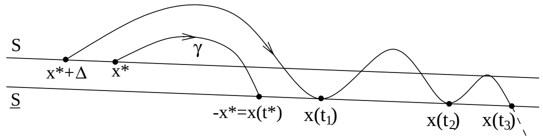

Figure 4: Poincaré map is in general multivalued

Let $-x^*-\Delta_1 \in x(t_\Delta)$. Since $-x^*-\Delta_1 \in \underline{S_a}$ then $x^*+\Delta_1 \in S_a$. Consider the multivalued Poincaré map $T_0 : S_a \to S_a$ defined by $x^*+\Delta_1 \in T_0(x^*+\Delta)$. Since x^* is fixed, the Poincaré map can be redefined as the map $T : S_a - x^* \to S_a - x^*$ given by $\Delta_1 \in T(\Delta)$, where $T(\Delta)=T_0(x^*+\Delta)-x^*$. In result, $\Delta=0$ is an equilibrium point of the discrete-time system

$$\Delta_{k+1} \in T(\Delta_k)$$

In this paper, we are interested in systems that have a unique locally stable unimodal limit cycle. For such systems, the idea is to find a global quadratic Lyapunov function for the associated Poincaré map. If this map is found to be quadratically stable then it follows that the limit cycle is globally asymptotically stable.

3 Decomposition and stability of Poincaré maps

This section contains the main result of this paper. We start by showing that most Poincaré maps induced by an LTI flow between the switching surfaces S and $\underline{S}$ can be represented as a linear transformation analytically parametrized by a scalar function of the state. The proof can be found in [11, 12].

Theorem 3.1 *Consider the Poincaré map T defined above. Let $v_t=\left(e^{At}-e^{At^*}\right)\left(x^*-A^{-1}B\right)$ and assume $|Cv_t| \geq K\|v_t\|$, for some $K>0$ and all $t>0$. Define*

$$H(t)=\left(\frac{v_t C}{C v_t}-I\right)e^{At}$$

for $t>0$ (for $t=t^*$, $H(t)$ is defined via continuation). Then, for any $\Delta \in S_a - x_0^*$ and $\Delta_1 \in T(\Delta)$ there exists a $t>0$ such that

$$\Delta_1 = H(t)\Delta \tag{3}$$

Such t is the switching time associated with Δ_1.

This theorem says that most Poincaré maps induced by an LTI flow between two hyperplanes can be represented as a linear transformation analytically parametrized by a scalar function of the state. The advantage of expressing the Poincaré map this way is to have all nonlinearities depending on only one parameter t. Although t depends on Δ, once t is fixed, the Poincaré map becomes linear in Δ. Note that $H(t)$ defined above is continuous in $t>0$.

As we will see next, based on this theorem, it is possible to reduce the problem of quadratic stability of Poincaré maps to the solution of a set of LMIs. The Poincaré map T defined above is quadratically stable if there exists a symmetric matrix $P>0$ such that

$$T^T(\Delta)PT(\Delta) < \Delta^T P \Delta\ , \quad \forall \Delta \in S_a - x^*,\ \Delta \neq 0 \tag{4}$$

A sufficient condition for the quadratic stability of the Poincaré map can easily be obtained by substituting (3) in (4). Therefore, the limit cycle is globally asymptotically stable if there exists a $P > 0$ such that

$$\Delta^T \left(P - H^T(t)PH(t)\right)\Delta > 0 \qquad (5)$$

for all $\Delta \in S_a$, with associated switching times $t \in t_\Delta$.

There are several alternatives to transform (5) into a set of LMIs. A simple sufficient condition is

$$P - H^T(t)PH(t) > 0 \quad \text{ on } S - x^* \qquad (6)$$

for some $P > 0$ and all $t > 0$, where "$D > 0$ on X" stands for $x^T Dx > 0$ for all nonzero $x \in X$. As we will see in the next section using some illustrative examples, although this condition is more conservative than (5), it can prove global asymptotic stability of many important RFS.

Other less conservative conditions are considered and discussed in [11, 12]. These are based on the fact that T is a map from S_a to S_a, and that the set of points in S_a with the same switching time t forms a convex set of dimension $n-2$.

Condition (6) can be relaxed by noticing that since A is Hurwitz and $u = \pm 1$ is a bounded input, there is a bounded set such that any trajectory will eventually enter and stay there. This will lead to bounds on the difference between any two consecutive switching times. Let t_- and t_+ be bounds on the minimum and maximum switching times of trajectories in that set. Call the elements of $[t_-, t_+]$ the *expected switching times*. Condition (6) can then be relaxed to be satisfied on $[t_-, t_+]$ instead of $t > 0$. We briefly explain how these bounds t_+ and t_- can be obtained. The details can be found in [11, 12].

By definition of S_a, $y(t) > -d$ at least in some interval $(0, \epsilon)$, where $\epsilon > 0$. However, since we are assuming $CA^{-1}B < -d$, and A Hurwitz, it is easy to see that $y(t)$ cannot remain larger than $-d$ for all $t > 0$. For any initial condition x_0, $Ce^{At}(x_0 - A^{-1}B) \to 0$ as $t \to \infty$. Hence, since for sufficiently large time t, $x(t)$ is bounded, an upper bound on t_+ can be obtained.

If $x_0 \in S_a$, $y(t)$ will be positive at least in some interval $(0, \epsilon)$. It can be shown [11, 12] that for t large enough, ϵ cannot be made arbitrarily small. So, since for sufficiently large time t, $x(t)$ is bounded, a lower bound on the time it takes between two consecutive switches can be obtained.

4 Examples

The following examples were processed in `matlab` code written by the authors. This code can be obtained at [14]. The input to the `matlab` function is a transfer function of an LTI system, together with the hysteresis parameter d. If the RFS is proven to be stable, the `matlab` functions return the matrix $P > 0$ in (6). For space limitation reasons, we have omitted several details in these examples. Please refer to [11, 12] for completeness and more examples.

Example 4.1 Consider the system in (1) with transfer function

$$H(s) = -\frac{s^2 + s - 4}{3(s+1)(s+2)(s+3)}$$

in feedback with an ideal relay ($d = 0$). This is possible since any state-space realization of H results in $CB < 0$. Although very simple, this system has never been proven to be globally stable. The closed-loop system has one unimodal symmetric limit cycle with period approximately equal to $2*1.4$. This corresponds to $x^* \approx [0.60 \text{ -}0.44 \;\; 0.32]^T \in S_a$. We analyzed this same RFS in [10]. There, we characterized a reasonably large region of stability around the limit cycle. Using the software described above we were able to find $P > 0$ satisfying (6) for all expected switching times. Therefore, the limit cycle is in fact globally asymptotically stable. ■

Example 4.2 Consider the following 3^{rd}-order minimum phase system in

$$H(s) = \frac{s^2 + 3s + 10}{(s^2 + 4s + 2)(s + 3)}$$

in feedback with an hysteresis with $d = 0.25$. The RFS has one unimodal symmetric limit cycle with period approximately equal to $2 * 0.94$. Moreover, there exists a $P > 0$ satisfying (6) for all expected switching times which means that the limit cycle is globally asymptotically stable. ■

Example 4.3 Consider the following LTI system with relative degree 7

$$H(s) = \frac{1}{(s+1)^7}$$

in feedback with an hysteresis with $d = 0.1$. The RFS has a symmetric unimodal limit cycle with period $2t^*$, where $t^* \approx 6.89$. Note how the period of the limit cycle is much larger than the hysteresis parameter d. Again, it is possible to find a $P > 0$ satisfying (6) for all expected switching times. We then conclude that the limit cycle is globally asymptotically stable ■

5 Conclusion

This paper introduces the idea that global stability analysis of certain trajectories like equilibrium points and limit cycles of piecewise linear systems can be done using surface Lyapunov functions. The development of stability conditions is based on expressing

Poincaré maps induced by LTI flow between a set and an hyperplane as linear transformations analytically parametrized by a scalar function of the state. Moreover, level sets of this function are convex. This way, quadratic constraints for Poincaré maps can expressed as sets of LMIs.

To show how this approach can be powerful in the analysis of piecewise linear systems, we applied it to a simple, yet very hard to analyze, class of PLS known as relay feedback systems. We addressed the problem of global quadratic stability analysis of limit cycles for RFS with hysteresis. This is, in fact, a hard problem since very few results existed until now. However, with this new results, most RFS analyzed by the authors were proven to be globally stable. Systems analyzed include minimum-phase systems, systems of relative degree larger than one, and of high dimension. This leads us to believe that quadratic stability is common in RFS.

Knowing that this methodology worked so well for this simple class of PLS, we pose the following question: will the same ideas work well for other, more complicated classes of PLS? Very recently we were able to prove global asymptotic stability of equilibrium points of on/off systems [8] and saturation systems [9] showing that this methodology can in fact be used to analyze many PLS. The ideas are similar: on the switching surfaces we find quadratic Lyapunov functions that are used to prove that the map from one switching surface to the next switching surface is contracting in some norm. These recent results together with the one in this paper opens the door to the possibility that more general PLS can be systematically analyzed using surface Lyapunov functions.

There are still many open problems following this work. An important extension of the results from this paper is to find conditions that do not depend on P but guarantee its existence. Another interesting direction is to study robustness and performance of RFS.

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