Quadratic Surface Lyapunov Functions in Global Stability Analysis of Saturation Systems

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Abstract

This paper considers quadratic surface Lyapunov functions in the study of global asymptotic stability of saturation systems (SAT), including those with unstable nonlinearity sectors. Quadratic surface Lyapunov functions were first introduced and successfully used to globally analyze asymptotic stability of limit cycles of relay feedback systems and later equilibrium points of on/off systems. Here, we show that quadratic surface Lyapunov functions can also be applied to analyze piecewise linear systems (PLS) with more than one switching surface. For that, we consider SAT. We present conditions in the form of LMIs that, when satisfied, guarantee *global* asymptotic stability of equilibrium points. A large number of examples was successfully proven globally stable, including systems systems of high dimension and systems with unstable nonlinearity sectors, for which classical methods like small gain theorem, Popov criterion, Zames-Falb criterion, IQCs, fail to analyze. In fact, existence of an example of a SAT with a globally stable equilibrium point that cannot be successfully analyzed with this new methodology is still an open problem. The results from this work suggests that other, more complex classes of PLS can be systematically globally analyzed using quadratic surface Lyapunov functions.

1 Introduction

Piecewise linear systems (PLS) are characterized by a finite number of linear dynamical models together with a set of rules for switching among these models. This captures discontinuity actions in the dynamics from either the controller or system nonlinearities. Although widely used and intuitively simple, PLS are computationally hard and very few results are available to analyze most PLS. In the analysis of equilibrium points of PLS, recent research has been concentrating on developing LMI based tools to construct piecewise quadratic Lyapunov functions. These ideas have been proposed and developed by [7], [10], and [6]. There are, however, several drawbacks with this approach. First, it fails to analyze limit cycles. Second, in general a refinement of partitions of the state space, in addition to the already existent natural partition, is necessary to analyze the system. The analysis method is efficient only when the number of partitions required to prove stability is small. As illustrated in an example in [2], however, even for second order systems, the method can become computationally intractable. This is even more relevant for high order systems where the geometric interpretation of the system is lost. Finding refinement of partitions to efficiently analyze high order systems is extremely hard. Finally, it fails to analyze PLS that are asymptotically, but not exponentially, stable.

The ideas introduced in [4], and used again in [3], were very successful in proving global asymptotic stability of limit cycles and equilibrium points of certain classes of PLS. On switching surfaces, we efficiently constructed quadratic Lyapunov functions that were used to show that impact maps, i.e., maps from one switching surface to the next switching surface, were contracting in some sense. The main difference between this and previous work [7, 10, 6] is that we look for quadratic Lyapunov functions on the switching surfaces instead of quadratic Lyapunov functions on the state space.

In [3], we analyzed equilibrium points of on/off systems using quadratic surface Lyapunov functions. In the state space, on/off systems are divided in two partitions by a switching surface. Therefore, the analysis was focused on studying a single switching surface. In the present work, using saturation systems (SAT), we show that quadratic surface Lyapunov functions can also be used to globally analyze PLS with more than two partitions and more than one switching surface.

The study of SAT is motivated by the possibility of actuator saturation or constraints on the actuators, reflected sometimes in bounds on available power supply or rate limits. These cannot be naturally dealt within the context of standard (algebraic) linear control theory, but are ubiquitous in control applications. The fact that linear feedback laws when saturated can lead to instability has motived a large amount of research. The well known result which states that a controllable lin-

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ear system is globally state feedback stabilizable, holds as long as the control does not saturate. In many applications, more often than not, the control is restricted to take values within certain bounds which may be met under closed-loop operation. Because feedback is cut, control saturation induces a nonlinear behavior on the closed-loop system. The problem of stabilizing linear systems with bounded controls has been studied extensively. See, for example, [13, 11, 14] and references therein.

In this paper, we focus on global asymptotic stability analysis of SAT. We are interested in those SAT where the origin is locally stable and is the only equilibrium point. Then, the question is if the origin is also globally asymptotically stable. Rigorous stability analysis for SAT is rarely done. The Zames-Falb criterion [15] can be used when the nonlinearity's slope is restricted, like in this case, but the method is difficult to implement. The Popov criterion can be used as a simplified approach to the analysis, but it is expected to be very conservative for systems of order greater than three. IQC-based analysis [8, 1, 9] gives conditions in the form of LMIs that, when satisfied, guarantee stability of SAT. However, none of these analysis tools can be used when a SAT has an unstable nonlinearity sector.

Here, we propose to construct quadratic Lyapunov functions on the switching surfaces of SAT to show that impact maps, i.e., maps from one switching surface to the next switching surface, associated with the system are contracting in some sense. This, in turn, proves the origin of a SAT is globally asymptotically stable. The search for these quadratic surface Lyapunov functions is done by solving a set of LMIs, which can be efficiently done using available computational tools.

2 Problem formulation

A SAT is defined as follows. Consider a SISO LTI system satisfying the following linear dynamic equations

$$\left\{ \begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Cx \end{array} \right. \tag{1}$$

where $x \in \mathbb{R}^n$, in feedback with a saturation controller (see figure 1) defined as

$$u(t) = \left\{ \begin{array}{ll} -d & \text{if } y(t) < -d \\ y(t) & \text{if } |y(t)| \leq d \\ d & \text{if } y(t) > d \end{array} \right. \tag{2}$$

where $d > 0$ (if $d = 0$ then the system is simply linear). By a solution of (1)-(2) we mean functions (x, y, u) satisfying (1)-(2). Since u is continuous and globally Lipschitz, $Ax + Bu$ is also globally Lipschitz. Thus, the SAT has a unique solution for any initial state.

In the state space, the saturation controller introduces two switching surfaces composed of hyperplanes of dimension $n - 1$

$$S = \{x \in \mathbb{R}^n : \; Cx = d\}$$

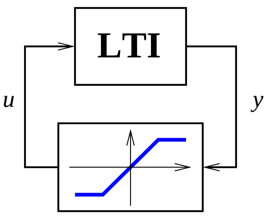

Figure 1: Saturation system

and

$$\underline{S} = \{x \in \mathbb{R}^n : \; Cx = -d\}$$

On one side of the switching surface S ($Cx > d$), the system is governed by $\dot{x} = Ax + Bd$. In between the two switching surfaces ($|Cx| \leq d$), the system is given by $\dot{x} = Ax + BCx = A_1 x$, where $A_1 = A + BC$. Finally, on the other side of $\underline{S}$ ($Cx < -d$) the system is governed by $\dot{x} = Ax - Bd$. Note that the vector field (1)-(2) is continuous along the switching surfaces since, for any $x \in S$, $A_1 x = (A + BC)x = Ax + Bd$, and for any $x \in \underline{S}$, $A_1 x = Ax - Bd$.

We are interested in those SAT with a unique locally stable equilibrium point. Only here can a SAT be globally stable. For that, it is necessary that $A + BC$ is Hurwitz in order to guarantee the origin is locally stable, and, if A is invertible, that $-CA^{-1}B < 1$, so the origin is the only equilibrium point. It is also necessary that A has no eigenvalues with positive real part, or otherwise there are initial conditions for which the system will grow unbounded (see, for example, [12]).

Consider $S_+ \subset S$ given by $S_+ = \{x \in S : \; CA_1 x \geq 0\}$. This set is important since it tells us which points in S correspond to the first switch of trajectories starting at any x_0 such that $Cx_0 < d$ (see figure 2). In other words, S_+ is the set of points in S that can be reached by trajectories of (1)-(2) when governed by the subsystem $\dot{x} = A_1 x$. In a similar way, define $S_- \subset S$ as $S_- = \{x \in S : \; CA_1 x \leq 0\}$. Note that $S = S_+ \bigcup S_-$ and $S_+ \bigcap S_- = \{x \in S : \; CAx = 0\}$. Define also $\underline{S}_+ = -S_+$ and $\underline{S}_- = -S_-$.

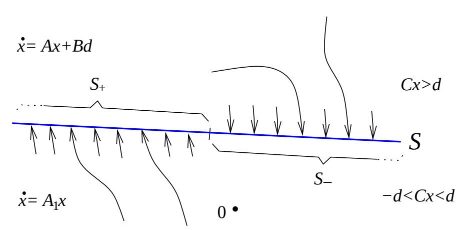

Figure 2: Both sets S_+ and S_- in S

Since A_1 must be Hurwitz, there is a set of points in S_- such that any trajectory starting in that set will not switch again and will converge asymptotically to the origin. In other words, let $S^* \subset S_-$ be the set of points x_0 such that the equations $Ce^{A_1 t} x_0 = \pm d$ do not have a solution for any $t > 0$. Note that this set S^* is not empty. To see this, let $P > 0$ satisfy $PA_1 + A_1' P = -I$. Then, an obvious point in S^* is the point x_1^* obtained from the intersection of S with the level set $x'Px = k$, where $k \geq 0$ is chosen such that the ellipse $x'Px = k$ is tangent to both S and $\underline{S}$ (see figure 3).

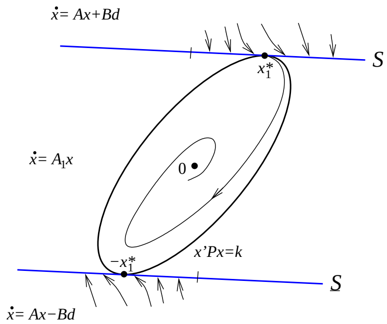

Figure 3: How to obtain x_1^*

The problem we propose to solve is to give sufficient conditions that, when satisfied, prove the origin of a SAT is globally asymptotically stable. The strategy of the proof is a follows. Consider a trajectory starting at some point $x_0 \in S_+$ (see figure 4). If all necessary conditions are met, the trajectory $x(t)$ will eventually switch at some time $t_1 > 0$, i.e., $Cx(t_1) = d$ and $Cx(t) \geq d$ for $t \in [0, t_1]$. Let $x_1 = x(t_1) \in S_-$. If $x_1 \in S^*$ then the trajectory will not switch again and converges asymptotically to the origin. Since we already know S^* is a stable set, we need to concentrate on those points in $S_- \backslash S^*$ since those are the ones that may lead to potentially unstable trajectories. Here, two scenarios can occur: either the trajectory switches at some point in S or it switches at some point in $\underline{S}$. Let $S_d \subset (S_- \backslash S^*)$ $(S_{-d} \subset (S_- \backslash S^*))$ be the set of points that will eventually switch in S ($\underline{S}$). If $x_1 \in S_d$ ($x_1 \in S_{-d}$) the trajectory switches in finite time $t_{2a} > t_1$ ($t_{2b} > t_1$) at $x_{2a} = x(t_{2a}) \in S_+$ ($x_{2b} = x(t_{2b}) \in \underline{S}_+$). Again, it would switch at $x_{3a} = x(t_{3a})$ ($x_{3b} = x(t_{3b})$) and so on.

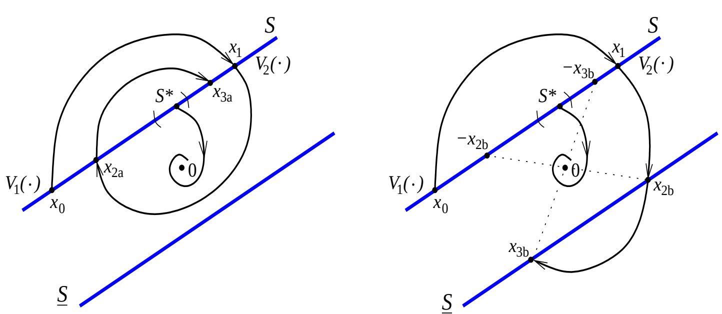

Figure 4: Possible state-space trajectories for a SAT

An interesting property of SAT is their symmetry around the origin. In other words, if $x(t)$ is a trajectory of (1)-(2) with initial condition x_0, then $-x(t)$ is a trajectory of (1)-(2) with initial condition $-x_0$. This means that it is equivalent to analyze the trajectory starting at x_0 or the trajectory starting at $-x_0$. This property is due to the fact that the vector field is symmetric around the origin. Due to this symmetry, whenever a trajectory intersects $\underline{S}$ (like, for example, at x_{2b} in figure 4), for purposes of analysis, it is equivalent to consider the trajectory continuing from the symmetric point around the origin in S ($-x_{2b}$ in figure 4).

As in [3], the idea is to check if x_{3a} or $-x_{3b}$ are closer in some sense to S^* than x_1. If so, this would mean that eventually $x(t_N) \in S^*$, for some N, and prove that the origin is globally asymptotically stable. This is the idea behind the results in the next section.

Before presenting the main results, it is convenient to notice that $x_0, x_1, x_{2a} \in S$ and $x_{2b} \in \underline{S}$ can be parametrized. Let $x_0 = x_0^* + \Delta_0$, $x_1 = x_1^* + \Delta_1$, $x_{2a} = x_0^* + \Delta_{2a}$ and $x_{2b} = -x_0^* + \Delta_{2b}$, where $x_0^*, x_1^* \in S$ and $C\Delta_0 = C\Delta_1 = C\Delta_{2a} = C\Delta_{2b} = 0$. Also, define $x_0^*(t)$ ($x_1^*(t)$) as the trajectory of $\dot{x} = Ax + Bd$ ($\dot{x} = A_1 x$), starting at x_0^* (x_1^*), for all $t > 0$. Since x_i^* are any points in S, we chose them to be such that $Cx_i^*(t) < d$ for all $t > 0$. The reason for this particular choice of x_0^* and x_1^* is so that $Cx_i^*(t) - d \neq 0$ for all $t > 0$. This will be necessary in proposition 3.1.

This choice of x_0^* and x_1^* is always possible. x_1^* is found as explained above (see figure 3). In this case, x_1^* is given by

$$x_1^* = \frac{P_d^{-1}C'}{CP_d^{-1}C'}d$$

where $P_d > 0$ satisfies $P_d A_1 + A_1' P_d = -I$. In a similar way, x_0^* is given by

$$x_0^* = (d + cA^{-1}Bd)\frac{P_u^{-1}C'}{CP_u^{-1}C'} - A^{-1}Bd$$

where $P_u > 0$ satisfies $P_u A + A' P_u = -I$.

3 Main results

There are three impact maps of interest associated with a SAT. The fist impact map (impact map 1) takes points from S_+ and maps them in S_-. The second impact map (impact map $2a$) takes points from $S_d \subset S_-$ and maps them back to S_+. Finally, the third impact map (impact map $2b$) takes points from from $S_{-d} \subset S_-$ and maps them in $\underline{S}_+$. Note that the impact maps associated with SAT are, in general, multivalued. Define the sets of expected switching times $\mathcal{T}_1$, $\mathcal{T}_{2a}$, and $\mathcal{T}_{2b}$ as the sets of all possible switching times associated with the respective impact map. In [2], we show how to get bounds on these sets.

Before presenting the main result, we show that each impact map associated with a SAT can be represented as a linear transformation analytically parametrized by the correspondent switching time.

Proposition 3.1 *Define*

$$w_1(t) = \frac{Ce^{At}}{d - Cx_0^*(t)}\ , \quad w_{2a}(t) = \frac{Ce^{A_1 t}}{d - Cx_1^*(t)}\ ,$$

$$\textit{and} \quad w_{2b}(t) = \frac{Ce^{A_1 t}}{-d - Cx_1^*(t)}$$

Let $H_1(t) = e^{At} + (x_0^*(t) - x_1^*)w_1(t)$, $H_{2a}(t) = e^{A_1 t} + (x_1^*(t) - x_0^*)w_{2a}(t)$, and $H_{2b}(t) = e^{A_1 t} + (x_1^*(t) + x_0^*)w_{2b}(t)$. Then, for any $\Delta_0 \in S_+ - x_0^*$ there exists a $t_1 \in \mathcal{T}_1$ such that

$$\Delta_1 = H_1(t_1)\Delta_0$$

Such t_1 is the switching time associated with Δ_1. Similarly, for any $\Delta_1 \in S_d - x_1^*$ there exists a $t_{2a} \in \mathcal{T}_{2a}$ such that

$$\Delta_{2a} = H_{2a}(t_{2a})\Delta_1$$

Such t_{2a} is the switching time associated with Δ_{2a}. Finally, for any $\Delta_1 \in S_{-d} - x_1^*$ there exists a $t_{2b} \in \mathcal{T}_{2b}$ such that

$$\Delta_{2b} = H_{2b}(t_{2b})\Delta_1$$

Such t_{2b} is the switching time associated with Δ_{2b}.

We need to show that these three impact maps are contracting in some sense. For that, define two quadratic Lyapunov functions on the switching surface S. Let V_1 and V_2 be given by

$$V_i(x) = x'P_i x - 2x'g_i + \alpha_i \tag{3}$$

where $P_i > 0$, for $i = 1, 2$. Global asymptotically stability of the origin follows if there exist $P_i > 0$, g_i, α_i such that

$$\begin{array}{rcll} V_2(\Delta_1) & < & V_1(\Delta_0) & \text{for all } \Delta_0 \in S_+ - x_0^* \\ V_1(\Delta_{2a}) & < & V_2(\Delta_1) & \text{for all } \Delta_1 \in S_d - x_1^* \\ V_1(-\Delta_{2b}) & < & V_2(\Delta_1) & \text{for all } \Delta_1 \in S_{-d} - x_1^* \end{array} \tag{4}$$

Note that in (4) we have mapped the point $\Delta_{2b} \in \underline{S}_+ + x_0^*$ into $S_+ - x_0^*$, taking advantage of the symmetry of the system.

Next, we define subsets in the domains of each impact map. Considering first impact map 1, for each $t_1 \in \mathcal{T}_1$ define S_{t_1} as the set of all initial conditions $x_0 \in S_+$ such that $y(t) \geq d$ on $[0, t_1]$, and $y(t_1) = d$. Analogously, for each $t_{2a} \in \mathcal{T}_{2a}$ ($t_{2b} \in \mathcal{T}_{2b}$) define $S_{t_{2a}}$ ($S_{t_{2b}}$) as the set of all initial conditions $x_{1a} \in S_d$ ($x_{1b} \in S_{-d}$) such that $-d \leq y(t) \leq d$ on $[0, t_{2a}]$ ($[0, t_{2b}]$), and $y(t_{2a}) = d$ ($y(t_{2b}) = -d$). Note that each of these subsets is a convex subset of a linear manifold of dimension $n-2$. Also, for a given $t_i \in \mathcal{T}_i$, impact map i is a linear map in S_{t_i}, $i = 1, 2a, 2b$. This is the main idea behind the next result. In this result, as a short hand, let $H_{it} = H_i(t)$, $w_{it} = w_i(t)$, and $P > 0$ on S stand for $x'Px > 0$ for all $x \in S$.

Theorem 3.1 *Define*

$$R_1(t) = P_1 - H_{1t}' P_2 H_{1t} - 2\left(g_1 - H_{1t}' g_2\right) w_{1t} + w_{1t}' \alpha w_{1t}$$

$$R_{2a}(t) = P_2 - H_{2at}' P_1 H_{2at} - 2\left(g_2 - H_{2at}' g_1\right) w_{2at} - w_{2at}' \alpha w_{2at}$$

$$R_{2b}(t) = P_2 - H_{2bt}' P_1 H_{2bt} - 2\left(g_2 + H_{2bt}' g_1\right) w_{2bt} - w_{2bt}' \alpha w_{2bt}$$

where $\alpha = \alpha_1 - \alpha_2$. The origin of the SAT is globally asymptotically stable if there exist $P_1, P_2 > 0$ and g_1, g_2, α such that

$$\left\{ \begin{array}{ll} R_1(t_1) > 0 & \text{on } S_{t_1} - x_0^* \\ R_{2a}(t_{2a}) > 0 & \text{on } S_{t_{2a}} - x_1^* \\ R_{2b}(t_{2b}) > 0 & \text{on } S_{t_{2b}} - x_1^* \end{array} \right. \tag{5}$$

for all expected switching times $t_1 \in \mathcal{T}_1$, $t_{2a} \in \mathcal{T}_{2a}$, and $t_{2b} \in \mathcal{T}_{2b}$.

A relaxation of the constraints on Δ_0 and Δ_1 in the previous theorem results in computationally efficient conditions.

Corollary 3.1 *The origin of the SAT is globally asymptotically stable if there exist $P_1, P_2 > 0$ and g_1, g_2, α such that*

$$\left\{ \begin{array}{ll} R_1(t_1) > 0 & \text{on } S - x_0^* \\ R_{2a}(t_{2a}) > 0 & \text{on } S - x_1^* \\ R_{2b}(t_{2b}) > 0 & \text{on } S - x_1^* \end{array} \right. \tag{6}$$

for all expected switching times $t_1 \in \mathcal{T}_1$, $t_{2a} \in \mathcal{T}_{2a}$, and $t_{2b} \in \mathcal{T}_{2b}$.

For each t_1, t_{2a}, t_{2b}, these conditions are LMIs which can be efficiently solved for $P_1, P_2 > 0$ and g_1, g_2, α using efficient available software. As we will see in the next section, although these conditions are more conservative than conditions (5), they are already enough to prove global asymptotic stability of many important SAT. Other ways to approximate (5) with less conservative sets of LMIs than (5) are explored in [2, 4].

The proofs of the above results are similar to the ones in [3] and are omitted here.

Note that in many cases, bounds on the expected switching times can be obtained, meaning that conditions (6) and (5) do not need to be satisfied for all expected switching times. Basically, since $|u| \leq d$ is a bounded input, and when A is Hurwitz, there exists a bounded set such that any trajectory will eventually enter and stay there. This will lead to bounds on the difference between any two consecutive switching times. Let t_{i-} and t_{i+}, $i = 1, 2a, 2b$, be bounds on the minimum and maximum switching times of the associated impact maps. The expected switching times $\mathcal{T}_i$ can, in general, be reduced to a smaller set (t_{-i}, t_{i+}). Conditions (6) and (5) can then be relaxed to be satisfied only on (t_{i-}, t_{i+}) instead on all $t_i \in \mathcal{T}_i$. Due to lack of space, these details have been omitted from this paper. They can, however, be found in [2].

Each condition in (6) depends only on a single scalar parameter. For instance, R_1 depends only on t_1 and not on t_{2a} or t_{2b}. Computationally, this means that when we grid each set of expected switching times, this will only affect one of the conditions in (6). Thus, if we need m_1 samples of $\mathcal{T}_1$, m_{2a} samples of $\mathcal{T}_{2a}$, and m_{2b} samples of $\mathcal{T}_{2b}$, we end up with a total of $m_1+m_{2a}+m_{2b}$ LMIs. Note that less conservative conditions than those in theorem 3.1 could be obtained. Such conditions, of the form $\bar{R}_1(t_1,t_{2a})>0$ and $\bar{R}_2(t_1,t_{2b})>0$, would, however, lead to $m_1\times m_{2a}+m_1\times m_{2b}$ LMIs, and the analysis problem would easily become computationally intractable. This difference in complexity is even more obvious in the analysis of other, more complex classes of PLS that may require simultaneous analysis of a large number of impact maps.

4 Examples

The following examples were processed in `matlab` code developed by the author. The latest version of this software is available at [5]. The input to the `matlab` function is a transfer function of an LTI system together with a parameter $d>0$. If the SAT is proven globally stable, the `matlab` function returns the parameters of the two quadratic surface Lyapunov functions (3). We then confirm conditions (6) are satisfied by plotting the minimum eigenvalues of each $R_i(t)$ on (t_{i-},t_{i+}), and showing that these are indeed positive in those intervals.

Before moving into the examples, it is important to explain how the bounds (t_{i-},t_{i+}) on the expected switching times are found. First, notice that $t_{1-}=t_{2a-}=0$. Zero switching time for the first impact map $\Delta_0\to\Delta_1$ and the second impact map $\Delta_1\to\Delta_{2a}$ correspond to points in S such that $CA_1x=0$. At those points, the Lyapunov functions (3) must be continuous since this is the only way

$$\begin{cases} V_2(\Delta_1)\le V_1(\Delta_0) \\ V_1(\Delta_{2a})\le V_2(\Delta_1) \end{cases}$$

can be satisfied simultaneously, for all $\Delta_0,\Delta_1,\Delta_{2a}=\Delta_0$ such that $x_0^*+\Delta_0=x_1^*+\Delta_1=x$ and $CAx=0$. Therefore, for those points we need $V_1(\Delta_0)=V_2(\Delta_1)$. This imposes certain restrictions on $P_1,P_2>0$, g_1,g_2, and α. The analysis of zero switching time for these two impact maps is similar to the case of on/off systems. See [2, section 6.7.2] for details.

As for the map $\Delta_1\to\Delta_{2b}$, zero switching never occurs since there is a "gap" between S and $\underline{S}$, resulting in a nonzero switching time for every trajectory starting in S_{-d}. For certain large values of $\|\Delta_1\|$, however, the switching times can be made arbitrarily small. But, when A is Hurwitz, all system trajectories eventually enter an invariant bounded set, as explained above. In this invariant bounded set, switching times for the impact map $\Delta_1\to\Delta_{2b}$ cannot be made arbitrarily small, and a lower bound can be found. Using the same ideas, upper bounds on expected switching times for all impact maps can be found. All the details can be found in [2]. The case when A has imaginary eigenvalues is currently under investigation.

Example 4.1 Consider the SAT on the left of figure 5 with $d=1$. It is easy to see the origin of this system is locally stable. The question is if the the origin is also globally asymptotically stable.

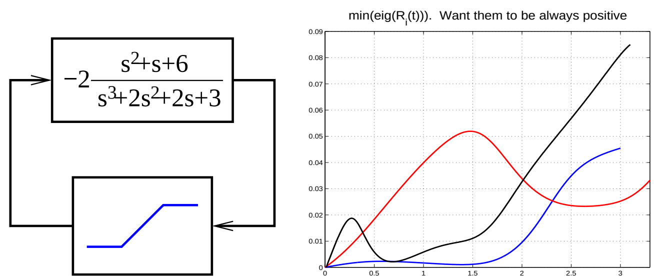

Figure 5: 3^{rd}-order system with unstable nonlinearity sector

Using conditions (6), we show that the origin is in fact asymptotically globally stable. The right side of figure 5 illustrates this fact: the minimum eigenvalue of each condition (6) is positive on its respective set of expected switching times. The expected switching times in this example are approximately $\mathcal{T}_1=(0,3)$, $\mathcal{T}_{2a}=(0,3.3)$, and $\mathcal{T}_{2b}=(0,3.1)$. For instance, if $t_1\ge 3$, there is no point in S_+ with switching time equal to t_1.

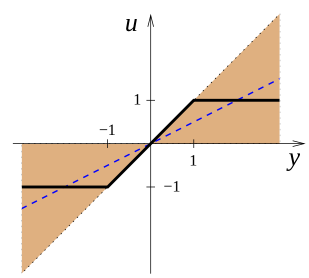

Figure 6: Saturation controller versus constant gain of 1/2 (dashed)

Note that this system has an unstable nonlinearity sector. If the saturation is replaced by a linear constant gain of 1/2, the system becomes unstable (see figure 6). This is very interesting since it tells us that classical analysis tolls like small gain theorem, Popov criterion, Zames-Falb criterion, and integral quadratic constraints, fail to analyze SAT of this nature. ∎

Example 4.2 Consider the SAT in figure 7 with $d=1$ and $k>0$. The origin of the SAT is locally stable for any $k>0$. Note that $\|Ce^{At}B\|_{\mathcal{L}_1}=k$, which means the small gain theorem can only be applied when $k<1$.

Let $k=2$. The right side of figure 7 shows how conditions (6) are satisfied in some intervals (t_{i-},t_{i+}), $i=1,2a,2b$. The intervals (t_{i-},t_{i+}) are bounds on the expected switching times. Such bounds are such that if conditions (6) are satisfied on (t_{i-},t_{i+}), then the system is globally asymptotically stable. For details on how to find these bounds see [2]. ∎

Example 4.3 Consider the SAT in figure 8 with $d=1$. This system is globally asymptotically stable. This

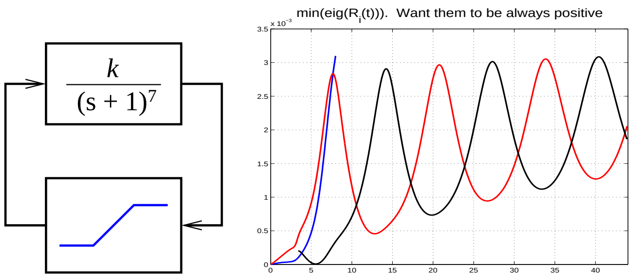

Figure 7: System with relative degree 7 (left); global stability analysis when $k = 2$ (right)

can be proven by direct application of the Popov criterion. What is interesting about this example is that the system is not exponential stable. Thus, the method of analysis using piecewise quadratic Lyapunov functions [10, 7, 6] fails to analyze the system.

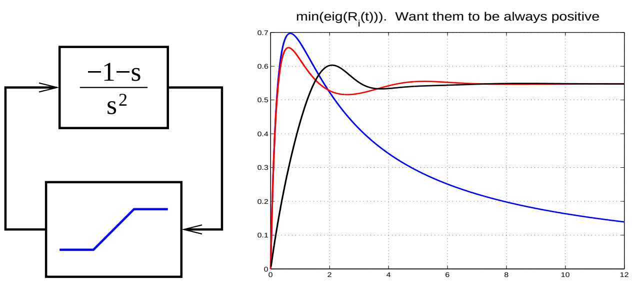

Figure 8: Second order system not exponentially stable

As seen on the right side of figure 8, quadratic surface Lyapunov functions can, however, be used to analyze and prove global asymptotic stability of the system. ■

5 Conclusion

This paper confirms the idea that global stability analysis of equilibrium points and limit cycles of certain classes of piecewise linear systems can be done using impact maps and quadratic surface Lyapunov functions. In particular, this paper showed the success of this methodology in global stability analysis of PLS with more than one switching surface.

In [4] and [3] we demonstrated how this approach is powerful in globally analyzing limit cycles of relay feedback systems and equilibrium points of on/off systems, respectively. Here, we demonstrated that similar ideas can be used to check if equilibrium points of saturation systems are globally asymptotically stable. Impact maps can be proven quadratically stable by constructing quadratic Lyapunov functions on switching surfaces. The search for quadratic surface Lyapunov functions is efficiently done by solving a set of LMIs. A large number of examples was successfully proven globally stable. These include systems of relative degree larger than one and of high dimension, and systems with unstable nonlinearity sectors, for which all classical fail to analyze. In fact, existence of an example with a globally stable equilibrium point that could not be successfully analyzed with this new methodology is still an open problem.

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