Necessary Conditions for Robust Stability of a Class of Nonlinear Systems

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Abstract

Input-output stability results for feedback systems are developed. Robust Stability conditions are presented for nonlinear systems with nonlinear uncertainty defined by some function (with argument equal to the norm of the input) that bounds its output norm. A *sufficient small gain theorem* for a class of these systems is known. Here, *necessary* conditions are presented for the vector space $(\ell_\infty, \|\cdot\|_\infty)$. These results capture the conservatism of the *small gain theorem* as it is applied to systems that do not have linear gain. The theory is also developed for the case of ℓ_2 signal norms, indicating some difficulties which make this case less natural than ℓ_∞.

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1 Introduction

This paper considers the development of *necessary* conditions (conditions that when not met imply that there exists a perturbation that destabilizes the system) for the robust stabilization of certain classes of nonlinear plants. The problem of robust stabilization may be stated as follows. Given a nominal plant model and a family of possible true plants, under what condition does a compensator which stabilizes the nominal plant also stabilize every plant in the given family?

The idea that a loop of less than unity gain ensures stability of a feedback loop has been appreciated since the early days of classical control. In mathematical terms, it is related to well-known ideas on invertibility of nonlinear operators of the form $I + G_1 G_2$ where I is the identity and G_1, G_2 are nonlinear operators on Banach spaces.

The usual form of the small gain theorem assumes gain properties of the form

$$\|(Mu)_T\|_p \leq \gamma \|u_T\|_p \tag{1}$$

for the operator M where u denotes the input signal, $\gamma = \sup_{x \neq 0} \frac{\|Mx\|_p}{\|x\|_p}$ ($\|\cdot\|_p$ denotes the usual *p-norm*), and u_T denotes the truncation of the signal u at time T. With this structure, it is shown, for $1 \leq p \leq \infty$, that, if M is linear and if $\|\Delta\|_{\ell_{p-ind}} < 1$ then the feedback system of Δ and M achieves robust stability if and only if $\|M\|_{\ell_{p-ind}} \leq 1$. For details, see Dahleh and Diaz-Bobillo (1995) and Young and Dahleh (1995). If M is nonlinear, the necessity part only holds if M is fading memory and $p = 2$ (see Shamma (1991) or Shamma and Zhao (1993)).

In Mareels and Hill (1992), a different notion of stability called monotone stability is used to obtain sufficient conditions for stability of feedback systems (where the systems in the loop can be nonlinear). This notion of stability is a generalization of (1). It allows more general bounding functions of the form

$$\|(Mu)_T\| \leq F(\|u_T\|)$$

where $F(\cdot)$ is a monotone function. Systems satisfying the last inequality are called monotone stable.

While sufficiency conditions for robust stability were shown with this new notion of nonlinear gain (also used by others like Sontag *et al.* (1994), Teel (1995), or Teel (1996)) no results on the necessity of such conditions are known. Such results are useful to understand the degree of conservatism that the small gain theorem has. Necessity conditions for linear gain exists (see Dahleh and Diaz-Bobillo (1995) or Khammash and Dahleh (1992)). There also exist necessity conditions for nonlinear systems that have their output norm bounded by a *linear* function of the input norm (see Shamma (1991) or Shamma and Zhao (1993)).

The main results of this paper are *necessary* conditions for the robust stability of nonlinear systems. For the vector space $(\ell_\infty, \|\cdot\|_\infty)$ we give necessary conditions on a system M for robust stability with either nonlinear time varying (NLTV) or nonlinear time invariant (NLTI) perturbations. These conditions are also studied for the vector space $(\ell_2, \|\cdot\|_2)$ but, here, in contrast to the $(\ell_\infty, \|\cdot\|_\infty)$ vector space, the results lead to fundamental questions. On one hand, the necessary conditions derived on M for robust stability are for non-causal perturbations. The construction of a causal perturbation is still under investigation. On the other hand, several other problems arise in this vector space which do not occur in ℓ_∞. These problems will be analyzed in section 5.

Using known sufficient results for robust stability (that we will recall here) we will derive an equivalent sufficient condition that is *close*[1] to the necessary conditions.

The remainder of this paper is organized as follows. Section 2 starts by establishing notation and giving some mathematical preliminaries. Section 3 deals with sufficiency of the small gain theorem.

[1] We will explain what we mean by *close* later.

Sections 4 and 5 present necessary conditions of the small gain theorem in $(\ell_\infty, \|\cdot\|_\infty)$ and $(\ell_2, \|\cdot\|_2)$ respectively. Section 6 shows, using an example, how the results given in the previous sections are important when analyzing the robust stability of a given closed loop system. Finally, concluding remarks are given in section 7.

2 Mathematical Preliminaries

We start by defining some standard concepts. The field of real numbers is denoted by $\Re$, the set of $n \times 1$ vectors with elements in $\Re$ is denoted by $\Re^n$, and the set of all $n \times m$ matrices with elements in $\Re$ is denoted by $\Re^{n\times m}$. The set of nonnegative reals (integers) is denoted by $\Re_+$ (Z_+). Superscript $(\cdot)^T$ denotes transpose.

The extended space of sequences in $\Re^n$ is denoted by ℓ_{pe} for every $1 \le p \le \infty$ or just by ℓ when it is obvious or when it just does not matter what $p - norm$ is being used. The restriction of f to the interval $[a, b]$ is denoted by $f|_{[a,b]}$. For every $f = \{f(0), f(1), f(2), \cdots\} \in \ell$ define $\|f\|_{p|_{[a,b]}}$ as

$$\|f\|_{p|_{[a,b]}} = \left(\sum_{n=a}^{b} |f(n)|^p\right)^{1/p}$$

The set of all $f \in \ell$ such that

$$\|f\|_p = \left(\sum_{n=0}^{\infty} |f(n)|^p\right)^{1/p} < \infty$$

is denoted by ℓ_p. The set of all $f \in \ell$ with $f \notin \ell_p$ is denoted by $\ell \setminus \ell_p$.

Given $f \in \ell$ define the support of $f \in \ell$ by $supp(f) = \{n : f(n) \neq 0\}$.

For $k \in Z_+$, S_k denotes the kth-shift (time-delay) operator on ℓ, and P_k the kth-truncation operator on ℓ. Let $H : \ell \to \ell$ be an operator. Then, H is called causal if $P_k H f = P_k H P_k f$, $\forall_{k\in Z_+}$, strictly causal if $P_k H f = P_k H P_{k-1} f$, $\forall_{k\in Z_+}$, and time invariant if $HS_1 = S_1 H$.

Let X_e and Y_e be two signal spaces. Then an operator $G : X_e \to Y_e$ provides an input-output system representation. We do not make explicit the role of initial conditions although this can be important in a complete stability analysis (see Hill (1991), Jiang *et al.* (1994), or Teel (1996)).

The following definition provides a concept of input-output stability.

Definition 2.1 *The system G is* monotone stable *if there exists a monotonic increasing homeomorphism*[2] $F : \Re_+ \to \Re_+$ *with* $F(0) = 0$ *and a constant* $\beta \in \Re_+$ *such that*

$$\|(Gu)_T\| \le F(\|u_T\|) + \beta \tag{2}$$

for all $u \in X_e$ *and* $T \ge 0$.

If G is linear, causal, and bounded then F in (2) can be written, for all $s \ge 0$, as $F(s) = \gamma s$, where $\gamma = \|G\| \ge 0$ (note that $\|G\|$ represents the induced norm of G and it is defined as $\sup_{f\neq 0} \frac{\|Gf\|}{\|f\|}$).

Consider now the feedback system in figure 1.

Assumption 2.1 *Let V_{1e} and V_{2e} be two signal spaces. The operators $G_1 : V_{1e} \to V_{2e}$ and $G_2 : V_{2e} \to V_{1e}$ are such that for all input signals $r_1 \in V_{1e}$ and $r_2 \in V_{2e}$ there exist unique signals $u_1, y_2 \in V_{1e}$ and $u_2, y_1 \in V_{2e}$.*

[2] A function F is an homeomorphism if it is continuous and has a continuous inverse.

Figure 1: Closed loop system

This assumption ensures the feedback system model is mathematically well-posed in the sense that unique signals exist in the chosen signal spaces. Sufficiency conditions to ensure this situation are available in the literature (see Vidyasagar and Desoer (1975)).

Definition 2.2 *The feedback system in figure 1 under assumption 2.1 is called* monotone stable *if there exist functions* $F_1, F_2 : \Re_+ \times \Re_+ \to \Re_+$ *and constants* $\beta_1, \beta_2 \in \Re_+$ *such that*

$$\begin{aligned} \|y_{1T}\| &\leq F_1(\|r_{1T}\|, \|r_{2T}\|) + \beta_1 \\ \|y_{2T}\| &\leq F_2(\|r_{1T}\|, \|r_{2T}\|) + \beta_2 \end{aligned} \quad (3)$$

$\forall_{T \geq 0}$, $\forall_{r_1} \in \ell_{p_1 e}$, $\forall_{r_2} \in \ell_{p_2 e}$, *and* $F_1(\sigma, \cdot)$, $F_1(\cdot, \sigma)$, $F_2(\sigma, \cdot)$, $F_2(\cdot, \sigma)$ *are monotonic increasing homeomorphisms of* $\Re_+$ *onto* $\Re_+$ *for any* $\sigma \in \Re_+$ *and with* $F_1(0,0) = F_2(0,0) = 0$.

Comment: The last definition of stability has some implications. First, we see that if, for a certain system, there exist bounded inputs ($r_1, r_2 \in \ell$) that produce unbounded outputs ($y_1, y_2 \notin \ell$) then there are no functions F_1 and F_2 that satisfy the definition and therefore the system is unstable.

There is another important implication which has to do with the β_i. There are systems that are stable when we allow $\beta_i \neq 0$ but they are unstable when we impose $\beta_i = 0$. To see this, assume for instance that β_1, β_2 can be different from zero (as in the definition). Then, we can actually have systems such that the input norm can be made arbitrarily small but the output norm remains the same. Although the ratio $\frac{\|y_i\|}{F_i(\|r\|)}$ goes to infinity (for any F_i as in definition 2.2), with stability defined this way, these kinds of systems are stable. This is due to the fact that we considered $\beta_i \neq 0$.

Example 2.1 *A very simple example is a relay (see figure 2). In* ℓ_∞*, this static nonlinear system is stable if we allow* $\beta \neq 0$ *(in this case,* $\beta = 1$*). If we impose* $\beta = 0$ *then there is no* F *satisfying definition 2.1 which means that the system is unstable*

Figure 2: Relay

Definition 2.3 *A nonlinear operator* G *is said to be* finite memory *if there exists an increasing integer function* $FM(\cdot; G) : Z_+ \to Z_+$ *with* $FM(t; G) \geq t$ *such that*

$$(I - P_{FM(t;G)})Gf = (I - P_{FM(t;G)})G(I - P_t)f \quad (4)$$

for all $f \in \ell_p$ *and* $t \in Z_+$*. The function* $FM(\cdot; G)$ *is called the finite memory function associated with* G*.*

Figure 3: G is finite-memory

The last definition states that (see figure 3) the effects of a finite-duration of the input eventually vanish completely and therefore the recent operator output depends only on the recent inputs and not on the extreme past inputs.

The following proposition is from Shamma (1991).

Proposition 2.1 *Let G, a nonlinear operator, have finite-memory with associated finite-memory function $FM(\cdot;G)$. Then for $f_1 \in \ell_2$ with $supp(f_1) \subset [0,n]$ and $f_2 \in \ell_2$ with $supp(f_2) \subset [FM(n;G)+1,\infty)$*

$$G(f_1+f_2) = Gf_1 + Gf_2$$

In the following definition, assume that G is some nonlinear operator and $\|\cdot\| = \|\cdot\|_p$ for some $1 \le p \le \infty$.

Definition 2.4 *Let $\eta_G(s) : \Re_+ \to \Re_+$ be a non-decreasing function defined, for all $s \ge 0$, as*

$$\eta_G(s) = \sup_{\|f\|=s} \|G(f)\| \tag{5}$$

Note that η_G may not exist. If there exist an f with $\|f\| < \infty$ such that $\|G(f)\| = \infty$ then $\eta_G(\|f\|)$ is not defined and therefore, for this given system, η_G does not exist.

Note also that G is monotone stable if and only if η_G exists. If G is monotone stable then there exists a monotone increasing homeomorphism F and a $\beta \in \Re_+$ satisfying (2). Since $\eta_G(s) \le F(s) + \beta < \infty$ it means that η_G exists. Conversely, make $\beta = \eta_G(0)$ and if η_G exists and it is a homeomorphism, then just make $F(s) = \eta_G(s) - \beta$. If η_G is not a homeomorphism, pick a homeomorphism F satisfying $F(s) \ge \eta_G(s) - \beta$. This means that G is monotone stable.

This last definition is the natural extension of the gain of a linear system defined as $\gamma = \sup_{f\neq 0} \frac{\|Gf\|}{\|f\|}$. Note that (5) can be written as

$$\sup_{\|f\|=s} \frac{\|Gf\|}{\eta_G(\|f\|)} = 1$$

In the case where G is a linear system, η_G is just a linear function, that is, $\eta_G(s) = \gamma s$. But, in general, if G is a nonlinear operator, η_G is some non-decreasing function. As in the linear case, (5) tells us that for any $s \ge 0$, there exists a signal f with $\|f\| = s$ such that $\|G(f)\|$ is equal or arbitrary close to $\eta_G(\|f\|)$, that is, given any $\epsilon > 0$ there exists a signal f with $\|f\| = s$ such that $\eta_G(\|f\|) - \|G(f)\| < \epsilon$. For all other $u \in \ell$, $\|G(u)\| \le \eta_G(\|u\|)$.

Note that this is not necessarily true for F in 2.1. The only information we have from F is that for every $u \in \ell$, $\|G(u)\| \le F(\|u\|) + \beta$. The relation between F and η_G is therefore $\eta_G(s) \le F(s) + \beta$ for all $s \ge 0$. In fact, there may exist $s \ge 0$ for which one cannot find any $f \in \ell$ with $\|f\| = s$ such that $\|G(f)\| = F(\|f\|) + \beta$.

Another important difference that follows from what was just discussed is that η_G does not need to be an homeomorphism. In fact, η_G does not need to have an inverse or to be continuous. Also, note that $\eta_G(0)$ does not need to be equal to zero.

Example 2.2 *For a relay (see figure 2), $\eta_G(s) = 1$ for all $s \geq 0$. In 2.1, we can choose $\beta = 1$ and F can be, for example, $F(s) = \epsilon s$, where $\epsilon \geq 0$.*

The properties of η_G will play a key role in section 4 and 5 when we talk about necessity of the small gain theorem.

Consider the system in figure 4.

Figure 4: Closed loop system

Let $\mathbf{\Delta}$ denote the class of allowable perturbations. We now define $C_{\mathbf{\Delta},p}$ as the subset of $\mathbf{\Delta}$ containing elements with $\eta_\Delta(s) < \Omega(s)$, where Ω is a monotonic increasing homeomorphism.

Definition 2.5 *Given a monotonic increasing homeomorphism Ω define*

$$C_{\mathbf{\Delta},p} = \{\Delta \in \mathbf{\Delta} : \ \eta_\Delta(s) < \Omega(s)\}$$

This is the same to say that, for every $f \in \ell_p$, $\|\Delta(f)\|_p \leq \eta_\Delta(\|f\|_p)$ where $\eta_\Delta(s) < \Omega(s)$. This means that $\|\Delta(f)\|_p < \Omega(\|f\|_p)$.

For perturbations $\Delta \in C_{\mathbf{\Delta},p}$, the problem will be to find necessary and sufficient conditions on M to guarantee robust stability.

3 Sufficiency of the Small-Gain Theorem

In this section we will present a sufficient condition to achieve robust stability when the perturbation belongs to $C_{\mathbf{\Delta},p}$. First we will present some known results that will be used to derive a sufficient condition on some system M, perturbed by $\Delta \in C_{\mathbf{\Delta},p}$, that guarantees the robust stability of the feedback system in figure 4. This condition is not equivalent to the one that will be presented in sections 4 and 5, where necessity will be discussed, but, as we will see, they are *close* in the sense that both *look in to* the composition of the gain functions of the system that is being analyzed and its perturbation.

Definition 3.1 *Define the following function classes:*

$$\begin{aligned}
Q &= \{F : \Re_+ \to \Re_+ |\ F \text{ is a monotonic increasing homeomorphism of } \Re_+ \text{ onto } \Re_+\} \\
N &= \{F \in Q |\ \exists_{\rho \in Q} \text{ s.t. } F(x) \leq x - \rho(x)\} \\
N_y &= \{F \in Q |\ \exists_{\rho \in Q} \text{ s.t. } F(x) \leq x - \rho(x) \text{ for all } x \geq y\} \text{ where } y \geq 0.
\end{aligned}$$

So, $N \subset N_y$. Define also $Q0 = Q \cup \{O_F\}$ and $N0_y = N_y \cup \{O_F\}$ where O_F denotes the zero function $F \equiv 0$.

Let i denote the identity function.

Fact 3.1 *If $F_1, F_2 \in Q$ then F_1^{-1}, $F_1 \circ F_2$, $F_1 + F_2 \in Q$.*

Proposition 3.1 *$F \in N_y$ if and only if $\exists_{\rho \in Q}$ such that $(i + \rho) \circ F(x) \leq x$ for all $x \geq y$.*

Proof: ($\Rightarrow$) Assume that $F \in N_y$. This implies that $\exists_{\alpha \in Q}$ such that $F(x) \leq x - \alpha(x)$, $\forall x \geq y$. This is the same to say that $\forall x \geq y$, $F(x) + \alpha(x) \leq x$ or $(i + \rho) \circ F(x) \leq x$ where $\rho(x) = \alpha \circ F^{-1} \in Q$ from fact 3.1.

($\Leftarrow$) $(i+\rho) \circ F(x) = F(x) + \rho(F(x)) \leq x$, $\forall x \geq y$. Let $\xi(x) = (\rho \circ F)(x) \in Q$ by fact 3.1. Then, $F(x) \leq x - \rho(F(x)) = x - \xi(x)$, $\forall x \geq y$, which means that $F \in N_y$. ■

Consider the feedback system in figure 1.

Let each system be monotone stable with gain functions F_1 and F_2, and β_1 and β_2 as in definition 2.1. This means that

$$\|y_{1T}\| \leq F_1(\|u_{1T}\|) + \beta_1 \quad (6)$$

$$\|y_{2T}\| \leq F_2(\|u_{2T}\|) + \beta_2 \quad (7)$$

The proof of the following result can be found in Mareels and Hill (1992).

Theorem 3.1 *Consider the system in figure 1. Suppose G_1 and G_2 are stable and satisfy (6,7). The feedback system is monotone stable if there exist $\rho \in Q$ and $s^* \geq 0$ such that*

$$F_2 \circ (i + \rho) \circ F_1 \in N_{s^*} \quad (8)$$

Corollary 3.1 *Consider the system in figure 1. Suppose G_1 and G_2 are stable and satisfy (6,7). The feedback system is monotone stable if there exist $\rho_1, \rho_2 \in Q$ and $s^* \geq 0$ such that*

$$(i + \rho_1) \circ F_2 \circ (i + \rho_2) \circ F_1(s) \leq s \quad for\ all\ s \geq s^* \quad (9)$$

Proof: The result follows from the last theorem and proposition 3.1. ■

Consider again the system in figure 4. Assume that there exists a $\beta \in (0, 1)$ such that $\Delta \in C^{\beta}_{\mathbf{\Delta},p} = \{\Delta \in \mathbf{\Delta} :\ \eta_\Delta(s) \leq (1 - \beta)\Omega(s)\}$. It is easy to see that $C^{\beta}_{\mathbf{\Delta},p} \subset C_{\mathbf{\Delta},p}$ ($C_{\mathbf{\Delta},p}$ is defined in definition 2.5). Assume also that M is monotone stable with gain function $m(s)$.

We will prove that it is sufficient to have $m(s) \leq (1 - \epsilon)\Omega^{-1}(s)$ for some $0 < \epsilon < 1$ and for all $s \geq s^*$, for some $s^* \geq 0$, in order to have robust stability. We will show that when M satisfies this condition we can always find monotonically increasing functions ρ_1 and ρ_2 satisfying $(i + \rho_1) \circ \eta_\Delta \circ (i + \rho_2) \circ m(s) \leq s$ and this way prove robust stability.

Corollary 3.2 *The system in figure 4 achieves robust stability for all $\Delta \in C^{\beta}_{\mathbf{\Delta},p}$ if there exist an $\epsilon \in (0, 1)$ and an $s^* \geq 0$ such that $m(s) \leq (1 - \epsilon)\Omega^{-1}(s)$ for all $s \geq s^*$.*

Proof: In equation (9), let $\rho_1(s) = \frac{\beta}{1-\beta}s$ and $\rho_2(s) = \frac{\epsilon}{1-\epsilon}s$. Using corollary 3.1 with $F_1 = m$ and $F_2 = \eta_\Delta$ the result follows. ■

Note: this is not a new result. This last corollary says the same as corollary 3.1. Here, ϵ and β represent the same as ρ_1 and ρ_2 in corollary 3.1. The reason why we have included this corollary is to give a sufficient condition that is *arbitrarily close* to the necessary condition we will talk about in the following sections. What we mean by *arbitrary close* is that when both ϵ and β approach zero, the sufficient condition approaches $(\Omega \circ \eta_M)(s) \leq s$ which is (as we will see in sections 4 and 5) the condition for necessity.

4 ℓ_∞ Stability Robustness Necessary Conditions

Consider the system in figure 4. In Dahleh and Diaz-Bobillo (1995), Dahleh and Khammash (1993), and Khammash and Pearson (1991) necessary conditions for stability robustness were presented for the case when M is linear time invariant. We will now extend those conditions to certain classes of nonlinear M. First, we will consider the case where the perturbation is NLTV. Then, we will prove that the necessity conditions still holds if the perturbation is NLTI.

Before we move to the next section, a remark is in the order to the effect of initial conditions. Since only input-output stability is considered, the effects of initial conditions is not addressed explicitly. For general NLTV systems, the initial condition can dramatically alter the resulting input-output behavior. However, since in the proofs of necessity in both ℓ_∞ and ℓ_2 cases we assume having finite memory for M, the effects of initial condition vanish after some finite time.

4.1 ℓ_∞ stability robustness with NLTV perturbations

Before we present the main theorem of the section we give a lemma that will be used during the proof of the theorem.

Lemma 4.1 *Let $A : \Re_+ \to \{true, false\}$ be some boolean function and $f : \Re_+ \to \Re_+$ an increasing homeomorphism. Saying that there exists a monotonic increasing sequence $\{s_n\}$ with $s_n \to \infty$ as $n \to \infty$ and $f^{-1}(s_{n+1}) - f^{-1}(s_n) \leq L$, for some $L > 0$, such that $A(s_n) = false$, $\forall_{n \in Z_+}$, is equivalent to say that $\exists_{N>0} : \forall_{s^*}, \exists_{s^* < s \leq f(f^{-1}(s^*)+N)}$ such that $A(s) = false$.*

Proof: ($\Rightarrow$) Given any $s^* \geq 0$, one can always find an $n \in Z_+$ such that $s_n \leq s^* < s_{n+1}$. Pick $s = s_{n+1}$. Then $A(s) = false$ and $f^{-1}(s) - f^{-1}(s^*) \leq f^{-1}(s_{n+1}) - f^{-1}(s_n) \leq L$ or $s \leq f(f^{-1}(s^*) + L)$. So, just take $N = L$ and the result follows.

($\Leftarrow$) In this case we need to construct a sequence $\{s_n\}$ according to the lemma. Let $s_0^* \geq 0$. Then, $\exists_{s_0^* < s_0 \leq f(f^{-1}(s_0^*)+N)}$ such that $A(s_0) = false$. Let $s_1^* = s_0$ and $S_1 = \{s_1^* < s \leq f(f^{-1}(s_1)^* + N) : A(s) = false\}$. By assumption, S_1 is non-empty. Let $s_1 = \max S_1$. Again, let $s_2^* = s_1$ and $S_2 = \{s_2^* < s \leq f(f^{-1}(s_2^*) + N) : A(s) = false\}$. Let $s_2 = \max S_2$. This means that $s_2 \geq f(f^{-1}(s_1^*) + N) = f(f^{-1}(s_0) + N) \geq f(N)$. Constructing $s_3, s_4, \cdots$ the same way and letting $L = N$ the result follows. ■

Next, assume that $C_{\mathbf{\Delta}_{TV},\infty}$ represents the set of all causal NLTV perturbations according to definition 2.5. The case of time invariant M is considered first.

Theorem 4.1 *Assume that M is finite memory, monotone stable, causal, and NLTI. The system in figure 4 achieves robust stability for all $\Delta \in C_{\mathbf{\Delta}_{TV},\infty}$ only if given any $N > 0$, there exists an $s^* \geq 0$ such that $(\Omega \circ \eta_M)(s) \leq s$ for all $s^* < s \leq \Omega(\Omega^{-1}(s^*) + N)$.*

This theorem tells us that if the system in figure 4 is robust stable it implies that for any $N > 0$, there is an interval on the real line (the interval is $(s^*, \Omega(\Omega^{-1}(s^*) + N))$ for some $s^* \geq 0$) such that the composition of Ω with η_M in less or equal than the identity function in that interval.

Proof: To simplify the proof, consider M and Δ SISO.

The approach we use is to show that a destabilizing perturbation $\Delta \in C_{\mathbf{\Delta}_{TV},\infty}$ can be constructed whenever the conditions of the theorem are not satisfied. So, assume that $\exists_{N>0} : \forall_{s^* \geq 0}$, $\exists_{s^* < s \leq \Omega(\Omega^{-1}(s^*)+N)}$: $\eta_M(s) > \Omega^{-1}(s)$. From lemma 4.1 this is equivalent to say that there exists a monotonic increasing sequence $\{s_n\}$ with $s_n \to \infty$ as $n \to \infty$ and $\Omega^{-1}(s_{n+1}) - \Omega^{-1}(s_n) \leq L$, for some $L > 0$, such that $\eta_M(s_n) > \Omega^{-1}(s_n)$, $\forall_{n \in Z_+}$.

Since M is *finite-memory*, from definition 2.3, this means that there exists an increasing integer function $FM(\cdot; M): Z_+ \to Z_+$ with $FM(t; M) \geq t$ satisfying equation (4).

As in Dahleh and Diaz-Bobillo (1995), Dahleh and Khammash (1993), and Khammash and Pearson (1991), the proof is divided in two parts: construction of an unbounded signal and construction of a destabilizing perturbation using that signal.

Construction of the unbounded signals

We need to construct ξ satisfying:

1. ξ is unbounded;

2. We want $\eta_\Delta(s) < \Omega(s)$ which means that we need to have $\|P_t\xi\|_\infty < \Omega(\|P_t y\|_\infty)$ for all $t \geq 0$.

Assume that $N_0 = 0$. The construction of ξ proceeds as follows (see figure 5).

Figure 5: Construction of ξ for $t = N_{n-1}, \cdots, N_n - 1$

For all $n = 1, 2, 3, \cdots$, let $N_{0n} = FM(N_{n-1}; M)$ (this will guarantee that $z(N_{0n}) = 0$). From assumption we know that $\eta_M(s_n) > \Omega^{-1}(s_n)$. Let $\epsilon_n = \eta_M(s_n) - \Omega^{-1}(s_n)$. Now, choose $N_n > N_{0n}$ and $|\xi(t)| \leq s_n$ for $t = N_{0n}, ..., N_n - 1$ such that $\|P_{N_n-1}\xi\|_\infty = s_n$ and $\eta_M(\|P_{N_n-1}\xi\|_\infty) - \|P_{N_n-1}z\|_\infty < \epsilon_n$. This way $\|P_{N_n-1}z\|_\infty > \Omega^{-1}(s_n)$. Since

$$\begin{aligned} y &= z + r \\ &= z + sgn(z)(\Omega^{-1}(s_{n+1}) - \Omega^{-1}(s_n)) \end{aligned}$$

we have

$$\begin{aligned} \|P_{N_n-1}y\|_\infty &= \|P_{N_n-1}z\|_\infty + (\Omega^{-1}(s_{n+1}) - \Omega^{-1}(s_n)) \\ &> \Omega^{-1}(s_n) + (\Omega^{-1}(s_{n+1}) - \Omega^{-1}(s_n)) \end{aligned}$$

or

$$\|P_{N_n-1}y\|_\infty > \Omega^{-1}(s_{n+1})$$

Since $|\xi(t)| \leq s_{n+1}$ for $t = N_{0n+1}, ..., N_{n+1} - 1$ we actually have

$$\|P_t y\|_\infty > \Omega^{-1}\left(\|P_t\xi\|_\infty\right) \tag{10}$$

for *all* t. Also, since $s_n \to \infty$ as $n \to \infty$ we have that $\|\xi(t)\| \to \infty$ and $\|y(t)\| \to \infty$ as $n \to \infty$ (or as $t \to \infty$). So, both requirements for ξ are met.

Note that, from assumption, we have $\Omega^{-1}(s_{n+1}) - \Omega^{-1}(s_n) \leq L$ for all $n \in Z_+$. Therefore, $\|r\|_\infty = \sup_n(\Omega^{-1}(s_{n+1}) - \Omega^{-1}(s_n)) \leq L$, i. e., $r \in \ell_\infty$.

Construction of the destabilizing perturbation

The idea now is to construct a destabilizing perturbation $\Delta \in C_{\mathbf{\Delta}_{TV},\infty}$ using the signals ξ and y (see figure 6).

We have $\xi = \{\xi(i)\}_{i=0}^\infty \in \ell$ and $y = \{y(i)\}_{i=0}^\infty \in \ell$ such that (10) is satisfied. We can rewrite (10) as $\|P_t\xi\|_\infty < \Omega(\|P_t y\|_\infty)$.

Figure 6: Construction of Δ

Now, Δ is trivial if $y = 0$: just pick Δ itself to be zero. So, assume that $y \neq 0$. Constructing $(y(i_1), y(i_2), ...)$ as in Dahleh and Diaz-Bobillo (1995), Dahleh and Khammash (1993), and Khammash and Pearson (1991) we can now construct our Δ.

So, Δ is constructed by having (see figure 7) $\xi = \Delta(y) = \bar{\Delta}\Omega(y)$. This can be seen as a series of two systems. The first is a static nonlinear system whose argument is $y(t)$ while the second ($\bar{\Delta}$) is just an LTV system.

Figure 7: Structure of Δ

$\bar{\Delta}$ is a matrix constructed as follows

$$\bar{\Delta} = \begin{pmatrix} \ddots & 0 & & & & & & & & \\ 0 & \frac{\xi(i_1)}{\Omega(y(i_1))} & 0 & & & & & & & \\ & \vdots & \vdots & \ddots & & & & & & \\ & \frac{\xi(i_2-1)}{\Omega(y(i_1))} & 0 & \cdots & 0 & & & & & \\ & & & & \frac{\xi(i_2)}{\Omega(y(i_2))} & 0 & & & & \\ & & & & \vdots & \vdots & \ddots & & & \\ & & & & \frac{\xi(i_3-1)}{\Omega(y(i_2))} & 0 & \cdots & 0 & & \\ & & & & & & & & \frac{\xi(i_3)}{\Omega(y(i_3))} & 0 \\ & & & & & & & & \vdots & \ddots \end{pmatrix}$$

where the first nonzero column is i_1th, the second is the i_2th, and so on.

It is easy to see that $\xi = \Delta(y)$.

Each row of the above matrix has at most one nonzero element which has absolute value less than 1. This means that $\|\bar{\Delta}\|_{\ell_{\infty-ind}} < 1$.

Now, let's see if Δ belongs to the set $C_{\mathbf{\Delta}_{TV},\infty}$. For every t we have $\|P_t\xi\|_\infty = \|P_t\Delta(y)\|_\infty = \|\bar{\Delta}P_t\Omega(y)\|_\infty \leq \|\bar{\Delta}\|_{\ell_{\infty-ind}}\|P_t\Omega(y)\|_\infty < \|P_t\Omega(y)\|_\infty = \Omega(\|P_t y\|_\infty)$ or just $\|P_t\xi\|_\infty < \Omega(\|P_t y\|_\infty)$ which means that $\eta_\Delta(s) \leq \|\bar{\Delta}\|_{\ell_{\infty-ind}}\Omega(s) < \Omega(s)$. So, $\Delta \in C_{\mathbf{\Delta}_{TV},\infty}$. Moreover, Δ is causal and NLTV.

So, we found a bounded input that produces an unbounded output. This means that in definition 2.2 there is no function F_1 such that (3) is satisfied because there exists a bounded $u_1 = r$

(with $u_2 = 0$) that produces an unbounded y_1. Therefore, we conclude that the closed loop system is unstable. ■

Corollary 4.1 *Consider the system in figure 4. If there exists an $s^* \geq 0$ such that $(\Omega \circ \eta_M)(s) > s$ for all $s > s^*$ then there exists a perturbation $\Delta \in C_{\mathbf{\Delta}_{TV},\infty}$ that makes the system unstable.*

Proof: The proof follows from the last theorem since this corollary is just a special case of it. ■

Remark 4.1 *For $\Omega(s) = s$, theorem 4.1 provides a necessity proof for $\ell_\infty -$ stability of finite memory systems that satisfies*

$$\sup_{\|f\|=s} \frac{\|M(f)\|}{\|f\|} = 1$$

Moreover, the destabilizing perturbation can be LTV.

Remark 4.2 *In the previous theorem we consider the system M to be time invariant. The results can actually be extended to the case where M is nonlinear time varying by replacing $(\Omega \circ \eta_M)(s) \leq s$ with $\inf_k(\Omega \circ \eta_{MS_k})(s) \leq s$ (note that the operator MS_k represents the original operator M restricted to inputs which start after time k). The proof, omitted here, is based on the same ideas of the proofs of the previous theorem and theorem 3.2 in Shamma (1991).*

4.2 ℓ_∞ stability robustness with NLTI perturbations

Assume here that $C_{\mathbf{\Delta}_{TI},\infty}$ represents the set of all NLTI perturbations according to definition 2.5. The proof of the following theorem is similar to the one done in Dahleh and Diaz-Bobillo (1995).

Theorem 4.2 *The system in figure 4 achieves robust stability for all $\Delta \in C_{\mathbf{\Delta}_{TI},\infty}$ only if given any $N > 0$, there exists an $s^* \geq 0$ such that $(\Omega \circ \eta_M)(s) \leq s$ for all $s^* < s \leq \Omega(\Omega^{-1}(s^*) + N)$.*

Proof: The proof of this theorem follows exactly as the proof of theorem 4.1 except for the construction of the destabilizing perturbation. Given the signals y and ξ, we show that a nonlinear time invariant perturbation can be constructed to destabilize the closed-loop system.

Let the signals y and ξ be given as before. Then Δ must be such that

$$\eta_\Delta(s) \leq \|\bar{\Delta}\|_{\ell_{\infty-ind}} \Omega(s) < \Omega(s) \tag{11}$$

and $\xi = \Delta(y)$. We just need to redefine $\bar{\Delta}$. So, let $\bar{\Delta}$ be defined as follows

$$(\bar{\Delta} f)(t) = \begin{cases} k\xi(t-j), & \text{if for some integer } \ j \geq 0,\ P_t f = P_t S_j \bar{y}, \\ 0, & \text{otherwise.} \end{cases}$$

where S_j is the shift operator by j steps. It is easy to see that the new Δ is a nonlinear, time invariant, and causal system. It satisfies (11) (because $\|\bar{\Delta}\|_{\ell_{\infty-ind}} < 1$) which means that $\Delta \in C_{\mathbf{\Delta}_{TI},\infty}$ and maps y to ξ. ■

5 ℓ_2 Stability Robustness Necessary Conditions

Once again, we will extend the conditions for stability robustness presented in Dahleh and Diaz-Bobillo (1995) to certain classes of nonlinear M.

5.1 ℓ_2 stability robustness with non-causal perturbations

The following theorem gives a necessary condition on the system in figure 4 in order to guarantee that the closed loop system is stable. Here, M is assumed to be some NLTI and finite memory system with its output $\ell_2 - norm$ bounded (to an input u) by $\eta_M(\|u\|_2)$ according to definition 2.4.

Define $C^{\beta}_{\mathbf{\Delta}_{NC},2,x}$ (with $x > 0$), for some given $k > 0$ and $\beta > 0$, as

$$C^{\beta}_{\mathbf{\Delta}_{NC},2,x} = \{\Delta \in \mathbf{\Delta} : \eta_{\Delta}(s) < (1+\beta)ks^x,\ \Delta \ \ non\ causal\}$$

This is a special case of definition 2.5 where $\Omega(s) = (1+\beta)ks^x$ and Δ is non-causal.

One of the assumptions in the next theorem is having $x \leq 1$. The reason why the theorem does not follow for $x > 1$ is because it is assumed that M is finite memory. In fact, if $x > 1$ and M is NLTI then it can not be finite memory. It has to be infinite memory. This can be shown by contradiction.

Assume $\eta_M(s) = (\frac{s}{k})^{\frac{1}{x}}$ and M is finite memory. Then, there exists an $f \in \ell_2$ and an integer $N \geq 0$ such that $supp(f) = [0, N-1]$ and $\|Mf\|_{2|_{[0,N-1]}} = \eta_M(\|f\|_{2|_{[0,N-1]}})$. This means that for this particular f we have

$$\|Mf\|_{2|_{[0,N-1]}} = \|Mf\|_2 = \left(\frac{1}{k}\|f\|_2\right)^{\frac{1}{x}} \tag{12}$$

Let $T = FM(N;M) + 2$. Define

$$\xi = \sum_{i=0}^{\infty} S_{iT} f = (f, 0's, f, 0's, f, 0's, \cdots) \tag{13}$$

where $0's$ denotes a string of zeros of length $T - N - 1$. From proposition 2.1 it follows that the response $M\xi = M\sum_n f_n = \sum_n Mf_n$ despite the nonlinearity of M. Given this decomposition, $M\xi$ may be block partitioned as follows

$$y = (Mf, x_1, Mf, x_2, Mf, \cdots) \tag{14}$$

where $x_i \in \ell_2$ are some signals. From (13) we see that

$$\begin{aligned} \|P_{nT-1}\xi\|_2 &= \sqrt{\|f\|_2^2 + \|f\|_2^2 + \cdots + \|f\|_2^2} \\ &= \sqrt{n}\|f\|_2 \end{aligned}$$

and from (14) we have

$$\begin{aligned} \|P_{nT-1}y\|_2 &= \sqrt{\|Mf\|_2^2 + \|x_1\|_2^2 + \|Mf\|_2^2 + \|x_2\|_2^2 + \cdots + \|Mf\|_2^2 + \|x_n\|_2^2} \\ &\geq \sqrt{n}\|Mf\|_2 \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\|P_{nT-1}y\|_2}{\left(\frac{1}{k}\|P_{nT-1}\xi\|_2\right)^{\frac{1}{x}}} &\geq \frac{\sqrt{n}\|Mf\|_2}{\left(\frac{1}{k}\sqrt{n}\|f\|_2\right)^{\frac{1}{x}}} \\ &= \frac{\sqrt{n}\|Mf\|_2}{(\sqrt{n})^{\frac{1}{x}}\ \|Mf\|_2} \\ &= n^{\frac{x-1}{2x}} > 1 \end{aligned}$$

for $x > 1$. This means that in fact $\eta_M(s) > (\frac{s}{k})^{\frac{1}{x}}$ which is a contradiction.

Theorem 5.1 *Let $x \leq 1$ and $\beta > 0$. Assume that M is finite memory, monotone stable, causal, and NLTI. The system in figure 4 achieves robust stability for all $\Delta \in C^{\beta}_{\mathbf{\Delta}_{NC},2,x}$ only if there exists an $s^* \geq 0$ such that $\eta_M(s) \leq (\frac{s}{k})^{\frac{1}{x}}$ for all $s \geq s^*$.*

Proof: The method of proof will again be similar to the one in Dahleh and Diaz-Bobillo (1995) or Shamma (1991) and Shamma and Zhao (1993). We will show that one can construct a destabilizing perturbation $\Delta \in C^{\beta}_{\mathbf{\Delta}_{NC},2,x}$ whenever the conditions of the theorem are not satisfied. So, assume that $\forall_{s^* \geq 0}, \exists_{s \geq s^*}: \eta_M(s) > (\frac{s}{k})^{\frac{1}{x}}$.

A particular signal $\xi \in \ell \setminus \ell_2$ is constructed for which there is an admissible Δ such that one has $(I - \Delta M)\xi \in \ell_2$. The lack of invertibility of $(I - \Delta M)$ then follows immediately.

This will be done in two steps. The first step is to construct that signal ξ. The next step is to use this signal to construct a destabilizing perturbation.

As is the proof of theorem 4.1, we have the assumption that M is finite memory. This means that there exists an increasing integer function $FM(\cdot; M): Z_+ \to Z_+$ according to definition 2.3.

Construction of the unbounded signals

The signal ξ to be constructed has to satisfy (a) be unbounded and (b) if y is the output of M to ξ then $\|P_t y\|_2 > \left(\frac{1}{(1+\beta)k}\|P_t \xi\|_2\right)^{\frac{1}{x}}$ for all t.

Assume that $s_0 = 1$ and $t_0 = 0$. The construction of ξ proceeds as follows. For all $n = 1, 2, 3, \cdots$, choose $\alpha_n > 1$ big enough (we will see what we mean by big enough soon) and let $s_n^* = \alpha_n s_{n-1}$. Then $\exists_{s_n \geq s_n^*}: \eta_M(s_n) > (\frac{s_n}{k})^{\frac{1}{x}}$. Let $\epsilon_n = \eta_M(s_n) - (\frac{s_n}{k})^{\frac{1}{x}}$. Also, let $a_n = \frac{s_n}{s_n^*} \geq 1$ and $c_n = a_n \alpha_n > 1$. Then,

$$\begin{aligned} s_n &= a_n s_n^* = a_n \alpha_n s_{n-1} \\ &= c_n s_{n-1} = c_n c_{n-1} s_{n-2} \\ &= c_n c_{n-1} \cdots c_2 c_1 s_0 \\ &= c_n c_{n-1} \cdots c_2 c_1 \end{aligned}$$

Now, choose $N_n > 0$ and $f_n \in \ell_2$ with $\|f_n\|_2 = s_n$ and $supp(f_n) = [0, N_n]$ such that $\eta_M(s_n) - \|Mf_n\|_{2|_{[0,N_n]}} < \epsilon_n$. This means that

$$\|Mf_n\|_2 > \left(\frac{s_n}{k}\right)^{\frac{1}{x}} = \left(\frac{c_n c_{n-1} \cdots c_2 c_1}{k}\right)^{\frac{1}{x}}$$

Note that $\|f_n\|_{2|_{[0,N_n]}} = \|f_n\|_2 = s_n$. For simplicity, from now on, let $\|Mf_i\|_2 = \|Mf_i\|_{2|_{[0,N_i]}}$. Also, let x_i represent some signals in ℓ_2 of appropriate length for $i = 1, 2, 3, \cdots$.

Let $t_n = FM(N_n; M) + t_{n-1} + 2$ and

$$P_{t_n - 1}\xi = (f_1, 0, f_2, 0, \cdots, f_n, 0)$$

From proposition 2.1 it follows that the response $M\xi = M\sum_n f_n = \sum_n Mf_n$ despite the nonlinearity of M. Given this decomposition, $M\xi$ may be block partitioned as follows

$$P_{t_n - 1}y = (Mf_1, x_1, Mf_2, x_2, \cdots, Mf_n, x_n) \tag{15}$$

Therefore, we have $\|P_{t_n-1}\xi\|_2$ given by

$$\begin{aligned} \|P_{t_n-1}\xi\|_2 &= \sqrt{\|f_1\|_2^2 + \|f_2\|_2^2 + \cdots + \|f_n\|_2^2} \\ &= \sqrt{c_1^2 + c_2^2 c_1^2 + \cdots + (c_n c_{n-1} \cdots c_2 c_1)^2} \\ &= s_1 \sqrt{1 + c_2^2 + \cdots + (c_n c_{n-1} \cdots c_2)^2} \end{aligned}$$

and $\|P_{t_n-1}y\|_2$ given by

$$\begin{aligned}
\|P_{t_n-1}y\|_2 &= \sqrt{\|Mf_1\|_2^2 + \|x_1\|_2^2 + \|Mf_2\|_2^2 + \|x_2\|_2^2 + \cdots + \|Mf_n\|_2^2 + \|x_n\|_2^2} \\
&> \sqrt{\left(\frac{c_1}{k}\right)^{\frac{2}{x}} + \left(\frac{c_2c_1}{k}\right)^{\frac{2}{x}} + \cdots + \left(\frac{c_nc_{n-1}\cdots c_2c_1}{k}\right)^{\frac{2}{x}}} \\
&= \left(\frac{s_1}{k}\right)^{\frac{1}{x}} \sqrt{1 + c_2^{2/x} + \cdots + (c_nc_{n-1}\cdots c_2)^{2/x}} \\
&= \left(\frac{1}{k}\|P_{t_n-1}\xi\|_2\right)^{\frac{1}{x}} \sqrt{\frac{1 + c_2^{2/x} + \cdots + (c_nc_{n-1}\cdots c_2)^{2/x}}{\left(1 + c_2^2 + \cdots + (c_nc_{n-1}\cdots c_2)^2\right)^{1/x}}} \\
&= \left(\frac{1}{(1+\beta_n)k}\|P_{t_n-1}\xi\|_2\right)^{\frac{1}{x}}
\end{aligned}$$

where

$$\beta_n = \left(\frac{1 + c_2^{2/x} + \cdots + (c_nc_{n-1}\cdots c_2)^{2/x}}{\left(1 + c_2^2 + \cdots + (c_nc_{n-1}\cdots c_2)^2\right)^{1/x}}\right)^{-\frac{x}{2}} - 1 > 0$$

It is easy to see that when $\alpha_n \to \infty$, $c_n \to \infty$, and therefore $\beta_n \to 0$. So, we need to choose α_n big enough so that $0 < \beta_n < \beta$.

Also, $s_1 > 1$ and for all $i = 1, 2, \cdots, n$, $c_i > 1$. This means that

$$\begin{aligned}
\|P_{t_n-1}\xi\|_2 &= s_1\sqrt{1 + c_2^2 + \cdots + (c_nc_{n-1}\cdots c_2)^2} \\
&> \sqrt{1 + 1 + \cdots + 1} \\
&= \sqrt{n}
\end{aligned}$$

Therefore, when $n \to \infty$, $\|P_{t_n-1}\xi\|_2 \to \infty$ and therefore ξ is unbounded. So, both requirements for ξ are met.

Construction of the destabilizing perturbation

Given the signals y and ξ, we show that a nonlinear, non-causal perturbation can be constructed to destabilize the closed-loop system.

Let the signals y and ξ be given as before. Δ must be constructed such that $\eta_\Delta(s) < ks^x$ and $\xi = \Delta(y)$. Consider the perturbation defined as follows

$$(\Delta f)(t) = \begin{cases} 0, & \text{if } t < t_1, \\ \xi(t-j), & \text{if for some integer } \; j \geq 0,\; P_tf = P_tS_jy, \\ 0, & \text{otherwise.} \end{cases}$$

It can be verified that Δ is a nonlinear and non-causal perturbation. We notice that the maximum amplification occurs when the input signal of Δ is y. We also know that

$$\|P_{t_n-1}y\|_2 > \left(\frac{1}{(1+\beta_n)k}\|P_{t_n-1}\xi\|_2\right)^{\frac{1}{x}}$$

or equivalent

$$\|P_{t_n-1}\xi\|_2 < (1+\beta_n)k\|P_{t_n-1}y\|_2^x$$

Since $\beta_n < \beta$ we have

$$\|P_{t_n-1}\xi\|_2 < (1+\beta)k\|P_{t_n-1}y\|_2^x$$

which means that $\eta_\Delta(s) < (1+\beta)ks^x$ and therefore $\Delta \in C^\beta_{\mathbf{\Delta}_{NC},2,x}$ and maps y to ξ.

So, Δ is constructed to have $\Delta(y) = \xi - (f_1, 0, 0, 0, \cdots) = (0, 0, f_2, 0, f_3, 0, f_4, \cdots)$.

Now, we just need to show that this is indeed a destabilizing perturbation. If we let ξ be the input to $(I - \Delta M)$ then we have

$$\begin{aligned}(I-\Delta M)(\xi) &= \xi - \Delta(M\xi) \\ &= (f_1, 0, f_2 - f_2, 0, \cdots, 0, f_n - f_n, 0, \cdots) \\ &= (f_1, 0, 0, 0, 0, \cdots) \in \ell_2\end{aligned}$$

This implies that the system in figure 4 is not $\ell_2 - stable$ because it maps a signal in ℓ_2 to a signal in $\ell \setminus \ell_2$. Therefore, as in the case of the ℓ_∞ proof, we conclude that the system is not monotone stable. This completes the proof. ■

Comment: Since, by construction, Δ is infinite memory, we can have $\eta_\Delta(s) < (1+\beta)ks^x$ with $x \leq 1$.

Comment: In the ℓ_2 case, the construction of a causal perturbation Δ instead of a non-causal one, like in the ℓ_∞ case where the conditions for stability hold for both NLTV and NLTI causal perturbations, is under investigation.

6 Example – Linear system with a nonlinear feedforward term

In this section we will give an example where the theorems presented in the previous sections can be applied to conclude stability or instability of closed loop systems in the form of the one in figure 4.

6.1 System description

Consider the following SISO system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + bu(t) \\ y(t) &= cx(t) + f(u_t)\end{aligned}$$

where $t \geq 0$, $x(t) \in \Re^{n\times 1}$, $u(t), y(t) \in \Re$, $A \in \Re^{n\times n}$, $b \in \Re^{n\times 1}$, $c \in \Re^{1\times n}$, and the only information we have of f is that $\|f(u_T)\|_\infty < \|u_T\|^2_\infty$, for every $u \in \ell$ and $T \geq 0$.

Suppose that only the output is available for feedback and that some control law $u = K(y)$ was designed. In this example, we will show that the controller K needs to be nonlinear in order to have the feedback system robustly stable.

We need to rearrange the system to make it look like the one in figure 4. For a given control law $u = K(y)$, the result is in figure 8.

Let P represent the initial system but with $f(u) = 0$, that is, P is a linear system with matrices $(A, b, c, 0)$. Then, we redraw the closed loop system to a simpler form (figure 9).

So, in figure 4, we define M as the system that maps the signal w to u and Δ as the system that maps u to w. The dynamics of M, at any time $t \geq 0$, are

$$\begin{aligned}\dot{x}(t) &= Ax(t) + bu(t) \\ u(t) &= K(w(t) + cx(t))\end{aligned}$$

and for Δ are

$$w(t) = f(u_t)$$

Figure 8: Closed loop system

Figure 9: Simplified closed loop system

M can be written as the operator $M = (1-PK)^{-1}K$. For Δ it is easy to see that $\eta_\Delta(s) < s^2$ and therefore $\Delta \in C_{\mathbf{\Delta},\infty} = \{\Delta \in \mathbf{\Delta} : \ \eta_\Delta(s) < s^2\}$. In definition 2.5, $\Omega(s) = s^2$ which is a monotonic increasing homeomorphism.

Let's now study the stability of the closed loop system for different control laws.

6.2 Linear controller

Assume that K is a linear controller (static or dynamic) such that $(1 - PK)^{-1}K$ is stable. This means that $\eta_M(s) = \alpha s$ where $\alpha = \|M\|_{\ell_\infty - ind}$. In this case, for any $N > 1/\alpha$ there is no $s^* \geq 0$ such that $\eta_M(s) \leq \sqrt{s}$ for $s^* < s \leq (\sqrt{s^*} + N)^2 = s^* + \bar{N}$. This can be seen by noticing that $\bar{N} = (\sqrt{s^*} + N)^2 - s^* = 2\sqrt{s^*}N + N^2 > 2\sqrt{s^*}\frac{1}{\alpha} + \frac{1}{\alpha^2} \geq \frac{1}{\alpha^2}$, or $\bar{N} > \frac{1}{\alpha^2}$. Since there is no interval on the positive real bigger than $\frac{1}{\alpha^2}$ such that $\eta_M(s) \leq \sqrt{s}$, the result follows. Therefore, using theorem 4.1, we conclude that the closed loop system is unstable. Moreover, there is no linear controller K that can stabilize the system.

6.3 Saturation controller

Assume now that in the previous section we add a saturation after the controller K (in the feedback loop). This way, $u(t) = sat(Ky(t))$, for every $t \geq 0$, where $sat(\cdot)$ denotes the usual saturation function.

In order to prove the stability of the closed loop system, we need to find η_M or an upper bound

m of η_M such that the conditions of theorem 4.1 are satisfied. Here, it is assumed that η_M exists, that is, it is assumed that M is stable.

If the input w is such that it produces an output u with $\|u\|_\infty \leq 1$ then M can be viewed as a linear system because the saturation is in the linear region. In this case, as seen before, $\|M\|_{\ell_\infty - ind} = \alpha$, and therefore $\eta_M(s) \leq m(s) = \alpha s$ for $0 \leq s \leq \frac{1}{\alpha}$. For $\|w\|_\infty > \frac{1}{\alpha}$ we can bound $\|u\|_\infty$ by 1.

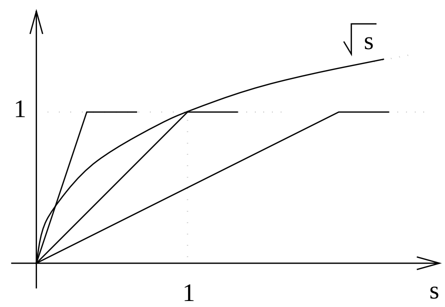

Figure 10: $m(s)$ for different values of α

From figure 10 it is easy to see that for every $\alpha > 0$ there always exists $s^* \geq 0$ such that $\eta_M(s) \leq m(s) \leq \sqrt{s}$ for all $s \geq s^*$ (note that $s^* = 1$ will work for every $\alpha > 0$). Therefore, using theorem 4.1, we conclude that the closed loop system is stable[3].

7 Concluding Remarks and Future Work

This paper presented *necessary* conditions for robust stability of a system M perturbed by a family of disturbances Δ. Sufficient conditions were also included for completeness. It was shown that, for the vector space $(\ell_\infty, \|\cdot\|_\infty)$, those sufficient conditions are *arbitrarily close* to the necessary conditions under appropriate assumptions on the system M.

While in the vector space $(\ell_\infty, \|\cdot\|_\infty)$ the results were completely analyzed, in the vector space $(\ell_2, \|\cdot\|_2)$ several problems arose. The fact that only finite memory systems were being considered imposed immediately, in this vector space, restrictions on the gain function of the system M. Also, while necessary conditions with either NLTV or NLTI perturbations were presented for the vector space $(\ell_\infty, \|\cdot\|_\infty)$, only necessary conditions with non-causal perturbations were found for the vector space $(\ell_2, \|\cdot\|_2)$.

For future work, these results should be extended to the general case where M is not finite memory but it is infinite memory. In particular, we believe that it should not be very different from the proofs we have here, to show that these results still hold when M is fading memory (see Shamma and Zhao (1993)) instead of finite memory. This is not done here to make the proofs more readable.

In Vidyasagar (1993) results for the case where the nonlinearities are sector bounded by linear functions are given. For future work, this results can be extended to the case where the nonlinearities are sector bounded by a certain class of monotonic increasing functions. Also, as future work, in the ℓ_2 case, a causal perturbation Δ should be constructed instead of non-causal one like in the ℓ_∞ case where the conditions for stability hold for both NLTV and NLTI causal perturbations.

[3] Note that this notion of stability is the one given in definition 2.2.

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