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Designing robustly stabilising controllers for LTI spatially distributed systems using coprime factor synthesis☆

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Abstract

This paper considers the design of feedback controllers for linear, time-invariant, spatially distributed systems in an approach which generalises the H^∞-framework and in particular the H^∞ loop-shaping method. To this end, we introduce a class of spatially distributed system models called finite dimensional, distributed, linear, time-invariant systems. Sensors and actuators are considered to be part of the controller, rather than part of the plant, and thus the controller we wish to design is itself a spatially distributed system. Optimising over placements and shapes of the sensor and actuator spatial distribution functions is an integrated part of the controller design procedure. As an illustrative design example, we present the feedback stabilisation of an electrostatically destabilised, electrically conducting membrane.

Keywords: H^∞ control; Robust stabilization; Coprime factorization; Distributed parameter systems

1. Motivation

In this paper we propose a method for designing a feedback controller that robustly stabilises an open-loop unstable, linear, time-invariant (LTI), spatially distributed plant, based around an extension of H^∞ loop-shaping (Glover & McFarlane, 1989; McFarlane & Glover, 1990, 1992) and gap-metric ideas to the spatially distributed context. By a spatially distributed system we mean a system whose input and output signals may depend on spatial variables as well as time. As a consequence, the controller we wish to design is itself a spatially distributed system. The design of such a controller proceeds as follows:

(i) calculate an approximate plant model in the class of (finite-dimensional, distributed, linear, time-invariant) FD-DLTI systems (see Definition 10) and associate a gap-metric uncertainty ball with that finite-dimensional plant model,

(ii) calculate an optimally stabilising controller for the approximate plant model, i.e., a controller that achieves the largest possible stability margin, and

(iii) compute a controller whose spatial distribution functions can be realised by the sensors and actuators available and which has a sufficiently small gap distance to the optimal controller of (ii). It is in this third step that the issue of optimal sensor and actuator placement (and possibly shaping) is addressed.

This paper is structured as follows. In Section 2 we collect all the preliminary results needed for the design method. Section 3 is devoted to the design method itself, and Section 4 gives a design example.

2. Preliminaries

2.1. Mathematical notation

$\mathbb{Z}_+$ denotes the set of positive integers, and $\mathbb{C}_+ := \{s \in \mathbb{C}_+ \mid \mathrm{Re}(s) > 0\}$. The Laplace transform operator is represented by $L\{.\}$. The symbol "$\sharp$" stands for "countably infinitely many". If $G(s)$ is a real-rational transfer matrix, then $\|G(s)\|_H$ denotes the Hankel norm of $G(s)$ (see, e.g.,

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Glover, 1984). Let $\mathscr{U}, \mathscr{Y}$ be Hilbert spaces. For $\mathscr{U}$, and analogously for any other Hilbert space, let $\|.\|_{\mathscr{U}}$, $\langle .,. \rangle_{\mathscr{U}}$ and $\boldsymbol{I}_{\mathscr{U}}$ denote norm, scalar product and identity operator in $\mathscr{U}$. The space of all bounded, linear operators from $\mathscr{U}$ to $\mathscr{Y}$, equipped with the induced norm $\|.\|_{\text{ind}}$, is denoted by $\mathscr{B}(\mathscr{U}, \mathscr{Y})$. The *Hardy* ∞*-space* $H^{\infty}_{\mathscr{B}(\mathscr{U},\mathscr{Y})}(\mathbb{C}_+)$ is the Banach space of $\mathscr{B}(\mathscr{U}, \mathscr{Y})$-valued functions $\boldsymbol{T}$ of a complex variable that are analytic on $\mathbb{C}_+$ and bounded in the norm $\|\boldsymbol{T}\|_{\infty} := \sup_{s \in \mathbb{C}_+} \{\|\boldsymbol{T}(s)\|_{\text{ind}}\}$. The *para-hermitian conjugate* is denoted as $\boldsymbol{T}^{\sim}(s) = (\boldsymbol{T}(-\bar{s}))^*$, where $(.)^*$ denotes the adjoint and $\bar{s}$ represents the complex-conjugate of s. Given the Hilbert space $\mathscr{V}$ and the (row) vector $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_n) \in \mathscr{V}^n$. Applying the Gram-Schmidt orthonormalisation procedure to $\hat{v}$ yields the uniquely defined (row) vector $v = (v_1, \ldots, v_n) \in \mathscr{V}^n$. The map $\hat{v} \mapsto v$ shall be denoted by $v = \text{GSO}(\hat{v})$. To distinguish signals and systems in the time and frequency domain, subscripts 't' are used to indicate the time domain, and no subscripts are used in the frequency domain.

2.2. View of spatially distributed systems

In this paper we consider feedback control systems whose inputs, outputs, disturbances, etc. may be spatially distributed signals, i.e., signals that depend on spatial variables as well as time. We seek to retain the spatial element of signals in the definition of input-output performance measures by making use of the 2-norm for both the space- and time-dependence of signals and thereby define the induced norm of a system. As we are concerned with LTI systems, Fourier/Laplace transforms of signals may be taken with respect to time, and systems can typically be represented as integral operators with a frequency-dependent kernel.

As an example of a LTI spatially distributed system, consider an elastic string stretched between $x = 0$ and $x = 1$ and clamped at both ends. Denote the string's deflection from the equilibrium position by $\mathbf{y_t}(x, t)$ and assume the string is set in motion under the action of a distributed load $\mathbf{u_t}(x, t)$ ($\mathbf{u_t}(x, t) \equiv 0$ for $t < 0$). The dynamics of the string are governed by the PDE

$$\frac{\partial^2 \mathbf{y_t}(x,t)}{\partial t^2} + \delta \frac{\partial \mathbf{y_t}(x,t)}{\partial t} - \tau \frac{\partial^2 \mathbf{y_t}(x,t)}{\partial x^2} = \mathbf{u_t}(x,t), \tag{1}$$

$x \in (0, 1)$, $t \geqslant 0$, together with the boundary conditions $\mathbf{y_t}(0, t) = \mathbf{y_t}(1, t) = 0$. In (1), $\delta > 0$ is a frictional coefficient, and $\tau > 0$ represents the tension per unit mass of the string. At any given time instant we assume that the system's spatially distributed input signal, $\mathbf{u_t}(x, t)$, is square-integrable in x over the spatial domain $\mathscr{D}^i = (0, 1)$, i.e., $\mathbf{u_t}(., t) \in L^2(\mathscr{D}^i) =: \mathscr{U}$ for fixed t. Bringing in the time dependence, we consider $\mathbf{u_t}(., t) =: \boldsymbol{u_t}(t)$ to belong to the Lebesgue space of $\mathscr{U}$-valued, square-integrable functions, $L^2_{\mathscr{U}}[0, \infty)$, which is defined as $L^2_{\mathscr{U}}[0, \infty) := \{\boldsymbol{u_t}(t) \in \mathscr{U}$ for all $t \in [0, \infty) \mid \int_0^{\infty} \langle \boldsymbol{u_t}(t), \boldsymbol{u_t}(t) \rangle_{\mathscr{U}} \, dt < \infty\}$. Similarly, a suitable space for the output signal, $\mathbf{y_t}(x, t)$, is $L^2_{\mathscr{Y}}[0, \infty)$, where $\mathscr{Y} := L^2(\mathscr{D}^o)$ with $\mathscr{D}^o = (0, 1)$.

Thus we consider the system (1) as defining an operator from $L^2_{\mathscr{U}}[0, \infty)$ to $L^2_{\mathscr{Y}}[0, \infty)$. Taking Laplace transforms of (1) with zero initial conditions gives $((s^2 + \delta s) - \tau \mathrm{d}^2/\mathrm{d}x^2)\mathbf{y}(x, s) = \mathbf{u}(x, s)$, $x \in (0, 1)$, $s \in \mathbb{C}$, plus the boundary conditions $\mathbf{y}(0, s) = \mathbf{y}(1, s) = 0$. After taking Laplace transforms, the input and output signal spaces become the Hardy 2-spaces $H^2_{\mathscr{U}}(\mathbb{C}_+)$ and $H^2_{\mathscr{Y}}(\mathbb{C}_+)$, respectively. The system's frequency-domain input-output relationship is given by $\boldsymbol{y}(s) = \boldsymbol{P}^{\infty}(s)\boldsymbol{u}(s)$, where $\boldsymbol{P}^{\infty}(s)$ is an infinite-dimensional integral operator depending on s. The kernel of $\boldsymbol{P}^{\infty}(s)$ is given by

$$\boldsymbol{P}^{\infty}(x, \xi, s) = \sum_{k=1}^{\infty} \frac{2 \sin(k\pi x) \sin(k\pi\xi)}{s^2 + \delta s + \omega_k^2}, \quad \omega_k := k\pi\sqrt{\tau}, \tag{2}$$

and represents the string's Laplace-transformed Green's function.

Remark 1. The theory developed in this paper applies to any kind of LTI spatially distributed system and can still be used if no explicit expression of the system's Green's function is known. In the present example the system's Laplace-transformed Green's function was introduced primarily to motivate the definition of FDDLTI systems in Definition 10 below. In the case study of Section 4 the system's Green's function will be used to compute the gap distance between the infinite-dimensional system and a finite-dimensional system approximation exactly. When the Green's function is not known, one can use approximative methods instead (see Reinschke, 1999).

2.3. Stability of feedback systems

We present in this Section 2.3 only those definitions and the background theory that are needed to develop the design method. For a full development of an input-output approach to LTI spatially distributed systems see Reinschke (1999) and Reinschke, Cantoni and Smith (2001).

To be able to state mathematically rigorous results on the robust stability of feedback loops, we need to assume that all spatially distributed systems under consideration belong to the following class of systems.

Definition 2. Let $\mathscr{U}, \mathscr{Y}$ be Hilbert spaces. $\mathscr{S}_t(\mathscr{U}, \mathscr{Y}; [0, \infty))$ is defined to be the set of closed, linear, shift-invariant, causally extendible (see Reinschke et al. (2001)) operators $\boldsymbol{P_t}: \text{dom}(\boldsymbol{P_t}) \subseteq L^2_{\mathscr{U}}[0, \infty) \to L^2_{\mathscr{Y}}[0, \infty)$. The frequency-domain equivalent of $\mathscr{S}_t(\mathscr{U}, \mathscr{Y}; [0, \infty))$ shall be denoted by $\mathscr{S}(\mathscr{U}, \mathscr{Y}; \mathbb{C}_+)$ and is given by $\mathscr{S}(\mathscr{U}, \mathscr{Y}; \mathbb{C}_+) := \{\boldsymbol{P}(s) \mid L\{\boldsymbol{P_t u_t}\}(s) = \boldsymbol{P}(s) L\{\boldsymbol{u_t}\}(s), \quad \boldsymbol{P_t} \in \mathscr{S}_t(\mathscr{U}, \mathscr{Y}; [0, \infty)), \boldsymbol{u_t} \in \text{dom}(\boldsymbol{P_t})\}$.

Remark 3. Throughout the remainder of this paper, except for Section 4.1, we will work in the frequency-domain, interchanging the time-domain operator $\boldsymbol{P_t}: \text{dom}(\boldsymbol{P_t}) \subseteq$

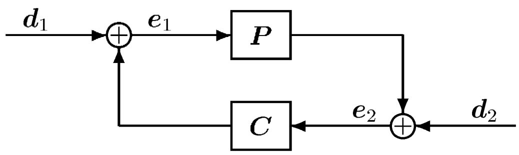

Fig. 1. Standard distributed feedback configuration $[\boldsymbol{P},\boldsymbol{C}]$ with external disturbance signals $\boldsymbol{d}_1,\boldsymbol{d}_2$.

$L^2_{\mathscr{U}}[0,\infty)\to L^2_{\mathscr{Y}}[0,\infty)$ with its frequency-domain equivalent $\boldsymbol{P}\colon \operatorname{dom}(\boldsymbol{P})\subseteq H^2_{\mathscr{U}}(\mathbb{C}_+)\to H^2_{\mathscr{Y}}(\mathbb{C}_+)$.

Consider the spatially distributed feedback system of Fig. 1, which we denote by $[\boldsymbol{P},\boldsymbol{C}]$, where $\boldsymbol{P}\colon \operatorname{dom}\boldsymbol{P}\subseteq H^2_{\mathscr{U}}(\mathbb{C}_+)\to H^2_{\mathscr{Y}}(\mathbb{C}_+)$ and $\boldsymbol{C}\colon \operatorname{dom}\boldsymbol{C}\subseteq H^2_{\mathscr{Y}}(\mathbb{C}_+)\to H^2_{\mathscr{U}}(\mathbb{C}_+)$ are linear, shift-invariant operators.

Definition 4. Let $\mathscr{U},\mathscr{Y}$ be Hilbert spaces. Suppose that $\boldsymbol{P}\in\mathscr{S}(\mathscr{U},\mathscr{Y};\mathbb{C}_+)$ and $\boldsymbol{C}\in\mathscr{S}(\mathscr{Y},\mathscr{U};\mathbb{C}_+)$. The feedback configuration $[\boldsymbol{P},\boldsymbol{C}]$ in Fig. 1 is said to be stable if

$$\boldsymbol{F}_{\boldsymbol{P},\boldsymbol{C}} := \begin{pmatrix} \boldsymbol{I}_{\mathscr{U}} & \boldsymbol{C} \\ \boldsymbol{P} & \boldsymbol{I}_{\mathscr{Y}} \end{pmatrix} : \operatorname{dom}(\boldsymbol{P})\times\operatorname{dom}(\boldsymbol{C})$$

$$\to H^2_{\left[\substack{\mathscr{U}\\ \mathscr{Y}}\right]}(\mathbb{C}_+) : \begin{pmatrix} \boldsymbol{e}_1 \\ -\boldsymbol{e}_2 \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{d}_1 \\ -\boldsymbol{d}_2 \end{pmatrix}$$

has a bounded inverse on $H^2_{\left[\substack{\mathscr{U}\\ \mathscr{Y}}\right]}(\mathbb{C}_+)$.

Right coprime factorisations, the gap-metric $\delta_g(.,.)$ and the stability margin $b_{\boldsymbol{P},\boldsymbol{C}}$, which are to be introduced next, are defined in complete analogy to the standard lumped-parameter case (Georgiou & Smith, 1990).

Definition 5. Let $\mathscr{U}$, $\mathscr{Y}$ be Hilbert spaces, and assume $\boldsymbol{P}\in\mathscr{S}(\mathscr{U},\mathscr{Y};\mathbb{C}_+)$. Two (operator-valued) functions $\boldsymbol{M}\in H^\infty_{\mathscr{B}(\mathscr{U},\mathscr{U})}(\mathbb{C}_+)$ and $\boldsymbol{N}\in H^\infty_{\mathscr{B}(\mathscr{U},\mathscr{Y})}(\mathbb{C}_+)$ are called (strongly) right coprime if there exist (operator-valued) functions $\tilde{\boldsymbol{X}}\in H^\infty_{\mathscr{B}(\mathscr{U},\mathscr{U})}(\mathbb{C}_+)$ and $\tilde{\boldsymbol{Y}}\in H^\infty_{\mathscr{B}(\mathscr{Y},\mathscr{U})}(\mathbb{C}_+)$ such that $\tilde{\boldsymbol{X}}\boldsymbol{M}-\tilde{\boldsymbol{Y}}\boldsymbol{N}=\boldsymbol{I}_{\mathscr{U}}$. A factorisation of the form $\boldsymbol{P}=\boldsymbol{N}\boldsymbol{M}^{-1}$ is called a right coprime factorisation (RCF) of $\boldsymbol{P}$ if $\boldsymbol{M}$ and $\boldsymbol{N}$ are (strongly) right coprime; the factorisation is called a normalised RCF if, in addition, $\left[\begin{smallmatrix}\boldsymbol{M}\\ \boldsymbol{N}\end{smallmatrix}\right]$ is inner, i.e., $\boldsymbol{M}^\sim\boldsymbol{M}+\boldsymbol{N}^\sim\boldsymbol{N}=\boldsymbol{I}_{\mathscr{U}}$.

Remark 6. Given two systems $\boldsymbol{P}_1,\boldsymbol{P}_2\in\mathscr{S}(\mathscr{U},\mathscr{Y};\mathbb{C}_+)$ with $\boldsymbol{P}_1=\boldsymbol{N}_1\boldsymbol{M}_1^{-1}$ being a normalised RCF and $\boldsymbol{P}_2=\boldsymbol{N}_2\boldsymbol{M}_2^{-1}$ being a (not necessarily normalised) RCF, the gap distance between the two systems can be calculated using the Ober–Sefton formula (Sefton & Ober, 1993; see also Reinschke et al., 2001).

$$\delta_g(\boldsymbol{P}_1,\boldsymbol{P}_2)=\inf_{\boldsymbol{Q},\boldsymbol{Q}^{-1}\in H^\infty_{\mathscr{B}(\mathscr{U},\mathscr{U})}(\mathbb{C}_+)}\left\|\begin{bmatrix}\boldsymbol{M}_1\\ \boldsymbol{N}_1\end{bmatrix}-\begin{bmatrix}\boldsymbol{M}_2\\ \boldsymbol{N}_2\end{bmatrix}\boldsymbol{Q}\right\|_\infty.$$

Definition 7. Let $\mathscr{U}$, $\mathscr{Y}$ be Hilbert spaces. Suppose that $\boldsymbol{P}\in\mathscr{S}(\mathscr{U},\mathscr{Y};\mathbb{C}_+)$ and $\boldsymbol{C}\in\mathscr{S}(\mathscr{Y},\mathscr{U};\mathbb{C}_+)$, and assume the feedback configuration $[\boldsymbol{P},\boldsymbol{C}]$ in Fig. 1 to be stable.

$$b_{\boldsymbol{P},\boldsymbol{C}} := \left\|\begin{bmatrix}\boldsymbol{I}_{\mathscr{U}}\\ \boldsymbol{P}\end{bmatrix}(\boldsymbol{I}_{\mathscr{U}}-\boldsymbol{C}\boldsymbol{P})^{-1}[\boldsymbol{I}_{\mathscr{U}}\quad \boldsymbol{C}]\right\|_\infty^{-1}$$

is called the *stability margin* of the feedback configuration $[\boldsymbol{P},\boldsymbol{C}]$. The *optimal stability margin* for a given plant $\boldsymbol{P}\in\mathscr{S}(\mathscr{U},\mathscr{Y};\mathbb{C}_+)$ is defined as $b_{\mathrm{opt}}(\boldsymbol{P}) := \sup_{\boldsymbol{C}\in\mathscr{S}(\mathscr{Y},\mathscr{U};\mathbb{C}_+)} b_{\boldsymbol{P},\boldsymbol{C}}$.

The following theorem states a sufficient condition for the stability of feedback loops with simultaneous plant and controller uncertainties, both uncertainties being measured in the gap-metric.

Theorem 8 (Foias, Georgiou, & Smith, 1993). *Let $\boldsymbol{P}\in\mathscr{S}(\mathscr{U},\mathscr{Y};\mathbb{C}_+)$ and $\boldsymbol{C}\in\mathscr{S}(\mathscr{Y},\mathscr{U};\mathbb{C}_+)$, where $\mathscr{U},\mathscr{Y}$ are Hilbert spaces. Assume b_1 and b_2 to be fixed, non-negative numbers such that $b_1^2+b_2^2<1$. Then $[\boldsymbol{P},\boldsymbol{C}]$ being stable with $b_1\sqrt{1-b_2^2}+b_2\sqrt{1-b_1^2}<b_{\boldsymbol{P},\boldsymbol{C}}$ implies that $[\boldsymbol{P}_1,\boldsymbol{C}_1]$ is stable for all $\boldsymbol{P}_1\in\mathscr{S}(\mathscr{U},\mathscr{Y};\mathbb{C}_+)$ and all $\boldsymbol{C}_1\in\mathscr{S}(\mathscr{Y},\mathscr{U};\mathbb{C}_+)$ which satisfy $\delta_g(\boldsymbol{P},\boldsymbol{P}_1)\leqslant b_1$ and $\delta_g(\boldsymbol{C},\boldsymbol{C}_1)\leqslant b_2$.*

The equivalence between gap uncertainty balls and coprime factor uncertainty balls holds for LTI spatially distributed systems (cf. Reinschke, 1999) as it does for LTI lumped-parameter systems (Georgiou & Smith, 1990; Sefton & Ober, 1993).

Theorem 9. *Let $\mathscr{U},\mathscr{Y}$ be Hilbert spaces, $\mathscr{B}_{\mathrm{ncfu}}(\boldsymbol{P},b) := \{\hat{\boldsymbol{P}}\in\mathscr{S}(\mathscr{U},\mathscr{Y};\mathbb{C}_+)\,|\,\hat{\boldsymbol{P}}=(\boldsymbol{N}+\boldsymbol{\Delta}_N)(\boldsymbol{M}+\boldsymbol{\Delta}_M)^{-1}$ is a RCF with $\|\left[\begin{smallmatrix}\boldsymbol{\Delta}_M\\ \boldsymbol{\Delta}_N\end{smallmatrix}\right]\|_\infty<b\}$ and $\mathscr{B}_g(\boldsymbol{P},b) := \{\hat{\boldsymbol{P}}\in\mathscr{S}(\mathscr{U},\mathscr{Y};\mathbb{C}_+)\,|\,\delta_g(\boldsymbol{P},\hat{\boldsymbol{P}})<b\}$. For all b: $0<b\leqslant 1$ we have $\mathscr{B}_g(\boldsymbol{P},b)=\mathscr{B}_{\mathrm{ncfu}}(\boldsymbol{P},b)$.*

2.4. FDDLTI systems

Returning to the vibrating string example in Section 2.2, note that the integral operator $\boldsymbol{P}^{(N)}(s)$ with the kernel $\boldsymbol{P}^{(N)}(x,\xi,s) := \sum_{k=1}^N \alpha_k(x)(s^2+\delta s+\omega_k^2)^{-1}\beta_k(\xi)$, where $\alpha_k(.)=\beta_k(.)=\sqrt{2}\sin(k\pi.)$, represents a finite-rank approximation of $\boldsymbol{P}^\infty(s)$ whose kernel is given by (2). We now introduce a class of finite-rank systems which represents a natural generalisation of the system described by $\boldsymbol{P}^{(N)}(s)$.

Definition 10. Let $\mathscr{U} := L^2(\mathscr{D}^i)$ and $\mathscr{Y} := L^2(\mathscr{D}^o)$. A FDDLTI system is a LTI operator $\boldsymbol{P_t}: L^2_{\mathscr{U}}[0,\infty)\to L^2_{\mathscr{Y}}[0,\infty)$ (possibly unbounded) whose frequency domain input-output relationship is given by $\mathbf{y}(x,s)=\int_{\mathscr{D}^i}\mathrm{d}\xi\,\mathbf{P}(x,\xi,s)\mathbf{u}(\xi,s)$, $x\in\mathscr{D}^o$, where the kernel $\mathbf{P}(x,\xi,s)$ has the form

$$\mathbf{P}(x,\xi,s)=E^\alpha(x)P(s)(E^\beta(\xi))^{\mathrm{T}}. \tag{3}$$

In (3), $E^\alpha(x) := (\alpha_1(x),\ldots,\alpha_p(x))$ and $E^\beta(\xi) := (\beta_1(\xi),\ldots,\beta_m(\xi))$ are finite-dimensional row vectors of real-valued spatial functions, with $\{\alpha_1,\ldots,\alpha_p\} \subset \mathscr{Y}$, $\langle\alpha_k,\alpha_l\rangle_{\mathscr{Y}} = \delta_{kl}$, $\{\beta_1,\ldots,\beta_m\} \subset \mathscr{U}$, $\langle\beta_k,\beta_l\rangle_{\mathscr{U}} = \delta_{kl}$, and $P(s)$ is a proper, real-rational, $(p\times m)$-transfer matrix with (minimal) state-space representation $[A,B,C,D]$, i.e., $P(s) = C(sI-A)^{-1}B + D$.

Remark 11. Note that using sets of orthonormal spatial projection functions in the spatial vectors E^α and E^β does not restrict the class of kernels $\mathbf{P}(x,\xi,s)$ in Definition 10. If one starts from an approximate, finite-dimensional plant model that was obtained using the Finite Element Method or some other Galerkin-type approximation method with non-orthonormal spatial projection functions, one can subsequently orthonormalise the spatial projection functions using the Gram–Schmidt orthonormalisation procedure and absorb the constant Gram–Schmidt matrices into $P(s)$.

As one might expect from the definition of FDDLTI systems, the computation of a RCF, the optimal stability margin and a stabilising (distributed) controller achieving the optimal stability margin carries over from lumped-parameter systems to FDDLTI systems, with minor modifications only.

Theorem 12. *Let $\mathscr{U},\mathscr{Y}$ be Hilbert spaces and assume that $\boldsymbol{P}$ is an FDDLTI system as defined in Definition 10. Let $P(s) = N(s)(M(s))^{-1}$ be a (normalised) RCF of the lumped-parameter transfer matrix $P(s)$. Then $\boldsymbol{P} = \boldsymbol{N}\boldsymbol{M}^{-1}$ is a (normalised) RCF of the FDDLTI system $\boldsymbol{P}$ if $\boldsymbol{N}\in H^\infty_{\mathscr{B}(\mathscr{U},\mathscr{Y})}(\mathbb{C}_+)$ has the kernel $\boldsymbol{N}(x,\xi,s) := E^\alpha(x)N(s)(E^\beta(\xi))^{\mathrm{T}}$, $\boldsymbol{D}\in H^\infty_{\mathscr{B}(\mathscr{U},\mathscr{U})}(\mathbb{C}_+)$ has the kernel $\mathbf{D}(x,\xi,s) := E^\beta(x)(M(s)-I_m)(E^\beta(\xi))^{\mathrm{T}}$ and $\boldsymbol{M} := \boldsymbol{I}_{\mathscr{U}} + \boldsymbol{D}\in H^\infty_{\mathscr{B}(\mathscr{U},\mathscr{U})}(\mathbb{C}_+)$.*

Proof. Extend the orthonormal sets $\{\beta_1,\ldots,\beta_m\} \subset \mathscr{U}$ and $\{\alpha_1,\ldots,\alpha_p\} \subset \mathscr{Y}$ to complete orthonormal sequences $\{\beta_k\}_{k\in\mathbb{Z}_+} \subset \mathscr{U}$ and $\{\alpha_j\}_{j\in\mathbb{Z}_+} \subset \mathscr{Y}$. Define the (infinite-dimensional) vectors $E^{\alpha^\perp} := (\alpha_{p+1},\alpha_{p+2},\ldots)$ and $E^{\beta^\perp} := (\beta_{m+1},\beta_{m+2},\ldots)$. Now observe that, for any $\varphi\in\mathscr{U}$, $\boldsymbol{N}(s)\varphi = E^\alpha N(s)\langle E^\beta,\varphi\rangle_{\mathscr{U}}$, $\boldsymbol{M}(s)\varphi = E^\beta M(s)\langle E^\beta,\varphi\rangle_{\mathscr{U}} + E^{\beta^\perp}\langle E^{\beta^\perp},\varphi\rangle_{\mathscr{U}}$, and $\boldsymbol{M}^{-1}(s)\varphi = E^\beta M^{-1}(s)\langle E^\beta,\varphi\rangle_{\mathscr{U}} + E^{\beta^\perp}\langle E^{\beta^\perp},\varphi\rangle_{\mathscr{U}}$. It follows that $\boldsymbol{N}(s)\boldsymbol{M}^{-1}(s)\varphi = E^\alpha N(s)M^{-1}(s)\langle E^\beta,\varphi\rangle_{\mathscr{U}} = \boldsymbol{P}(s)\varphi$, i.e., $\boldsymbol{P} = \boldsymbol{N}\boldsymbol{M}^{-1}$.

To show that $[{}^{\boldsymbol{M}}_{\boldsymbol{N}}] = [{}^{\boldsymbol{I}_{\mathscr{U}}+\boldsymbol{D}}_{\boldsymbol{N}}]\in H^\infty_{\mathscr{B}(\mathscr{U},[{}^{\mathscr{U}}_{\mathscr{Y}}])}(\mathbb{C}_+)$ satisfies $\boldsymbol{M}^\sim(\jmath\omega)\boldsymbol{M}(\jmath\omega)+\boldsymbol{N}^\sim(\jmath\omega)\boldsymbol{N}(\jmath\omega)=\boldsymbol{I}_{\mathscr{U}}$ for all $\omega\in\mathbb{R}$, define $\boldsymbol{W}(\jmath\omega) := \boldsymbol{M}^\sim(\jmath\omega)\boldsymbol{M}(\jmath\omega)+\boldsymbol{N}^\sim(\jmath\omega)\boldsymbol{N}(\jmath\omega)$ and note that $\boldsymbol{W}(\jmath\omega)$ has the kernel

$$\begin{aligned}\mathbf{W}(x,\xi,\jmath\omega) &= [E^\beta(x)\ E^{\beta^\perp}(x)] \\ &\quad\times\begin{bmatrix} M^\sim(\jmath\omega) & 0 \\ 0 & I_\sharp\end{bmatrix}\begin{bmatrix} M(\jmath\omega) & 0 \\ 0 & I_\sharp\end{bmatrix} \\ &\quad\times\begin{bmatrix}(E^\beta(\xi))^{\mathrm{T}} \\ (E^{\beta^\perp}(\xi))^{\mathrm{T}}\end{bmatrix} \\ &\quad + E^\beta(x)N^\sim(\jmath\omega)N(\jmath\omega)(E^\beta(\xi))^{\mathrm{T}} \\ &= [E^\beta(x)\quad E^{\beta^\perp}(x)] \\ &\quad\times\begin{bmatrix} M^\sim(\jmath\omega)M(\jmath\omega)+N^\sim(\jmath\omega)N(\jmath\omega) & 0 \\ 0 & I_\sharp\end{bmatrix} \\ &\quad\times\begin{bmatrix}(E^\beta(\xi))^{\mathrm{T}} \\ (E^{\beta^\perp}(\xi))^{\mathrm{T}}\end{bmatrix}.\end{aligned}$$

Since $M^\sim(\jmath\omega)M(\jmath\omega)+N^\sim(\jmath\omega)N(\jmath\omega) = I_m$ for all $\omega\in\mathbb{R}$, it follows that $\boldsymbol{W}(\jmath\omega) = \boldsymbol{I}_{\mathscr{U}}$ for all $\omega\in\mathbb{R}$.

Theorem 13. *For the FDDLTI plant $\boldsymbol{P}\in\mathscr{S}(\mathscr{U},\mathscr{Y};\mathbb{C}_+)$ with kernel $\mathbf{P}(x,\xi,s) = E^\alpha(x)P(s)(E^\beta(\xi))^{\mathrm{T}}$, $P(s)\in\mathbb{C}^{p\times m}$, let $P(s) = N(s)(M(s))^{-1}$ be a normalised RCF. Then the optimal stability margin can be evaluated as $b_{\mathrm{opt}}(\boldsymbol{P}) = \sqrt{1-\left\|\begin{smallmatrix}M(s)\\N(s)\end{smallmatrix}\right\|_H^2}$, and a (stabilising, distributed) controller $\hat{\boldsymbol{C}}\in\mathscr{S}(\mathscr{Y},\mathscr{U};\mathbb{C}_+)$ achieving the optimal stability margin is given by the kernel $\hat{\mathbf{C}}(\xi,x,s) = E^\beta(\xi)\hat{C}(s)(E^\alpha(x))^{\mathrm{T}}$, where $\hat{C}(s)$ is a solution to the lumped-parameter H^∞-optimisation*

$$\hat{C}(s) = \arg\min_{C(s)\,\mathrm{stblsng}}\left\|\begin{bmatrix} I_m \\ P(s)\end{bmatrix}(I_m - C(s)P(s))^{-1} \times [I_m\quad C(s)]\right\|_\infty.$$

Proof. Let $E^\alpha(x)$, $E^\beta(\xi)$, $E^{\alpha^\perp}(x)$ and $E^{\beta^\perp}(\xi)$ be defined as in the proof of Theorem 12. Now take a $\boldsymbol{C}\in\mathscr{S}(\mathscr{Y},\mathscr{U};\mathbb{C}_+)$ and write its kernel as

$$\boldsymbol{C}(\xi,x,s) = [E^\beta(\xi)\quad E^{\beta^\perp}(\xi)]\begin{bmatrix} C_{11}(s) & C_{12}(s) \\ C_{21}(s) & C_{22}(s)\end{bmatrix} \times\begin{bmatrix}(E^\alpha(x))^{\mathrm{T}} \\ (E^{\alpha^\perp}(x))^{\mathrm{T}}\end{bmatrix},$$

where $C(s) := \begin{bmatrix} C_{11}(s) & C_{12}(s) \\ C_{21}(s) & C_{22}(s)\end{bmatrix}$ is partitioned compatibly with the spatial vectors E^α, $E^{\alpha^\perp}$, E^β and $E^{\beta^\perp}$. One can show that $b_{\boldsymbol{P},\boldsymbol{C}}$ as defined in Definition 7 satisfies

$$b^{-1}_{\boldsymbol{P},\boldsymbol{C}} = \left\|\begin{bmatrix} I_m & 0_{m\times\sharp} \\ 0_{\sharp\times m} & I_\sharp \\ P & 0_{p\times\sharp}\end{bmatrix} \times\begin{bmatrix}(I_m - C_{11}P)^{-1} & 0_{m\times\sharp} \\ C_{21}P(I_m-C_{11}P)^{-1} & I_\sharp\end{bmatrix}^{-1} \times\begin{bmatrix} I_m & 0_{m\times\sharp} & C_{11} & C_{12} \\ 0_{\sharp\times m} & I_\sharp & C_{21} & C_{22}\end{bmatrix}\right\|_\infty. \quad (4)$$

By swapping the second and the third row as well as the second and the third column on the RHS of (4) and using the substitution $S(s) := (I_m - C_{11}(s)P(s))^{-1}$, we obtain

$$(b_{\text{opt}}(\boldsymbol{P}))^{-1} = \inf_{\boldsymbol{C}\in\mathscr{S}(\mathscr{Y},\mathscr{U};\mathbb{C}_+)} b^{-1}_{\boldsymbol{P},\boldsymbol{C}}$$

$$= \inf_{\substack{C_{11}\\C_{12}\\C_{21}\\C_{22}}} \left\| \begin{array}{cc|cc} S & SC_{11} & 0_{m\times\sharp} & SC_{12} \\ PS & PSC_{11} & 0_{p\times\sharp} & PSC_{12} \\ \hline C_{21}PS & C_{21}PSC_{11}+C_{21} & I_\sharp & C_{21}PSC_{12}+C_{22} \end{array} \right\|_\infty .$$

Since matrix dilations are not norm-decreasing, we can put $C_{21}(s) := 0_{\sharp\times m}$, $C_{12}(s) := 0_{m\times\sharp}$ and $C_{22}(s) := 0_{\sharp\times\sharp}$. In doing so, we get

$$(b_{\text{opt}}(\boldsymbol{P}))^{-1} = \inf_{C_{11}} \left\| \begin{array}{c|c} \begin{bmatrix} I_m \\ P \end{bmatrix} (I_m - C_{11}P)^{-1}[I_m \quad C_{11}] & 0 \\ \hline 0 & I_\sharp \end{array} \right\|_\infty$$

$$= \max\{(b_{\text{opt}}(P))^{-1}, 1\} = (b_{\text{opt}}(P))^{-1}. \qquad (5)$$

We thus have $b_{\text{opt}}(\boldsymbol{P}) = b_{\text{opt}}(P) = \sqrt{1 - \left\| \begin{smallmatrix} M(s) \\ N(s) \end{smallmatrix} \right\|_H^2}$ (the latter equality having been proved in McFarlane and Glover, 1990), and an optimal distributed controller $\hat{\boldsymbol{C}}$ achieving a stability margin of $b_{\text{opt}}(\boldsymbol{P})$ is given by the kernel $\hat{\boldsymbol{C}}(\xi,x,s) = E^\beta(\xi)\hat{C}(s)(E^\alpha(x))^{\mathrm{T}}$ with $\hat{C}(s)$ as stated in the theorem.

Theorem 14 in conjunction with Assumption 1, to be stated next, says how, under certain conditions, the gap between an infinite-dimensional, distributed LTI system, $\boldsymbol{P}^\infty$, and a FDDLTI system, $\boldsymbol{P}$, can be computed.

Assumption 1. Given two LTI spatially distributed systems $\boldsymbol{P}^\infty, \boldsymbol{P} \in \mathscr{S}(\mathscr{U},\mathscr{Y};\mathbb{C}_+)$, where $\mathscr{U},\mathscr{Y}$ are Hilbert spaces. Assume $\{\alpha_j\}_{j\in\mathbb{Z}_+} \subset \mathscr{Y}$ and $\{\beta_k\}_{k\in\mathbb{Z}_+} \subset \mathscr{U}$ to be complete orthonormal sequences. For given $p,m\in\mathbb{Z}_+$, define the spatial function vectors $E^\alpha(x) := (\alpha_1(x),\ldots,\alpha_p(x))$, $E^{\alpha^\perp}(x) := (\alpha_{p+1}(x),\alpha_{p+2}(x),\ldots)$, $E^\beta(\xi) := (\beta_1(\xi),\ldots,\beta_m(\xi))$, $E^{\beta^\perp}(\xi) := (\beta_{m+1}(\xi),\ldots)$.

(i) There is a *stable* $\boldsymbol{C}^\infty \in \mathscr{S}(\mathscr{Y},\mathscr{U};\mathbb{C}_+)$ stabilising $\boldsymbol{P}^\infty$.

(ii) $\boldsymbol{P}$ is a FDDLTI system with kernel $\boldsymbol{P}(x,\xi,s) = E^\alpha(x)P(s)(E^\beta(\xi))^{\mathrm{T}}$.

(iii) $\langle E^\beta, (\boldsymbol{I}_\mathscr{U} - \boldsymbol{C}^\infty\boldsymbol{P}^\infty)^{-1}E^{\beta^\perp}\rangle_\mathscr{U} = 0$, $\langle E^{\beta^\perp}, (\boldsymbol{I}_\mathscr{U} - \boldsymbol{C}^\infty\boldsymbol{P}^\infty)^{-1}E^\beta\rangle_\mathscr{U} = 0$, $\langle E^\alpha, \boldsymbol{P}^\infty(\boldsymbol{I}_\mathscr{U} - \boldsymbol{C}^\infty\boldsymbol{P}^\infty)^{-1}E^{\beta^\perp}\rangle_\mathscr{Y} = 0$ as well as $\langle E^{\alpha^\perp}, \boldsymbol{P}^\infty(\boldsymbol{I}_\mathscr{U} - \boldsymbol{C}^\infty\boldsymbol{P}^\infty)^{-1}E^\beta\rangle_\mathscr{Y} = 0$.

(iv) There exist integers p_0, m_0 such that (iii) holds for all p,m with $p > p_0$ and $m > m_0$. Furthermore, $\|\langle E^{\alpha^\perp}, \boldsymbol{P}^\infty E^{\beta^\perp}\rangle_\mathscr{Y}\|_\infty \to 0$ as $p,m\to\infty$.

Theorem 14. (a) *Let Assumption* 1 (i) *hold. Then a RCF of* $\boldsymbol{P}^\infty$ *is given by* $\boldsymbol{P}^\infty = \boldsymbol{N}^\infty(\boldsymbol{M}^\infty)^{-1}$, *where* $\boldsymbol{M}^\infty := (\boldsymbol{I}_\mathscr{U} - \boldsymbol{C}^\infty\boldsymbol{P}^\infty)^{-1}$ *and* $\boldsymbol{N}^\infty := \boldsymbol{P}^\infty(\boldsymbol{I}_\mathscr{U} - \boldsymbol{C}^\infty\boldsymbol{P}^\infty)^{-1}$.

(b) *Let Assumption* 1 (i) *to* (iii) *hold and let* $P(s) = N(s)(M(s))^{-1}$ *be a normalised RCF. With* $\boldsymbol{M}^\infty, \boldsymbol{N}^\infty$ *as defined in* (a), *call* $M_{11}^\infty := \langle E^\beta, \boldsymbol{M}^\infty E^\beta\rangle_\mathscr{U}$, $M_{22}^\infty := \langle E^{\beta^\perp}, \boldsymbol{M}^\infty E^{\beta^\perp}\rangle_\mathscr{U}$, $N_{11}^\infty := \langle E^\alpha, \boldsymbol{N}^\infty E^\beta\rangle_\mathscr{Y}$ *and* $N_{22}^\infty := \langle E^{\alpha^\perp}, \boldsymbol{N}^\infty E^{\beta^\perp}\rangle_\mathscr{Y}$. *Then we have*

$$\delta_g(\boldsymbol{P}^\infty, \boldsymbol{P}) = \max\left\{ \inf_{Q_{11},Q_{11}^{-1}\in H^\infty_{m\times m}(\mathbb{C}_+)} \left\| \begin{bmatrix} M \\ N \end{bmatrix} - \begin{bmatrix} M_{11}^\infty \\ N_{11}^\infty \end{bmatrix} Q_{11} \right\|_\infty , \inf_{Q_{22},Q_{22}^{-1}\in H^\infty_{\sharp\times\sharp}(\mathbb{C}_+)} \left\| \begin{bmatrix} I_{\sharp\times\sharp} \\ 0_{\sharp\times\sharp} \end{bmatrix} - \begin{bmatrix} M_{22}^\infty \\ N_{22}^\infty \end{bmatrix} Q_{22} \right\|_\infty \right\}, \quad (6)$$

provided the RHS of (6) *is less than one. If furthermore Assumption* 1(iv) *holds, then as* $p,m\to\infty$ *we have*

$$\inf_{Q_{22},Q_{22}^{-1}\in H^\infty_{\sharp\times\sharp}(\mathbb{C}_+)} \left\| \begin{bmatrix} I_{\sharp\times\sharp} \\ 0_{\sharp\times\sharp} \end{bmatrix} - \begin{bmatrix} M_{22}^\infty \\ N_{22}^\infty \end{bmatrix} Q_{22} \right\|_\infty \to 0. \qquad (7)$$

Proof. (a) $\boldsymbol{M}^\infty$ and $\boldsymbol{N}^\infty$ satisfy $\boldsymbol{N}^\infty(\boldsymbol{M}^\infty)^{-1} = \boldsymbol{P}^\infty(\boldsymbol{I}_\mathscr{U} - \boldsymbol{C}^\infty\boldsymbol{P}^\infty)^{-1}(\boldsymbol{I}_\mathscr{U} - \boldsymbol{C}^\infty\boldsymbol{P}^\infty) = \boldsymbol{P}^\infty$. This factorisation is coprime since $[\boldsymbol{I}_\mathscr{U}\ -\boldsymbol{C}^\infty]$ is a stable left inverse of $\left[\begin{smallmatrix}\boldsymbol{M}^\infty\\ \boldsymbol{N}^\infty\end{smallmatrix}\right]$: $[\boldsymbol{I}_\mathscr{U}\ -\boldsymbol{C}^\infty]\left[\begin{smallmatrix}\boldsymbol{M}^\infty\\ \boldsymbol{N}^\infty\end{smallmatrix}\right] = [\boldsymbol{I}_\mathscr{U}\ -\boldsymbol{C}^\infty]\left[\begin{smallmatrix}\boldsymbol{I}_\mathscr{U}\\ \boldsymbol{P}^\infty\end{smallmatrix}\right](\boldsymbol{I}_\mathscr{U} - \boldsymbol{C}^\infty\boldsymbol{P}^\infty)^{-1} = (\boldsymbol{I}_\mathscr{U} - \boldsymbol{C}^\infty\boldsymbol{P}^\infty)(\boldsymbol{I}_\mathscr{U} - \boldsymbol{C}^\infty\boldsymbol{P}^\infty)^{-1} = \boldsymbol{I}_\mathscr{U}$.

(b) Let $\boldsymbol{P} = \boldsymbol{N}\boldsymbol{M}^{-1}$ be a normalised RCF as described in Theorem 12. Then the integral operators $\boldsymbol{M}$ and $\boldsymbol{N}$ have kernels of the form

$$\mathbf{M}(\xi,\zeta,s) = [E^\beta(\xi) \quad E^{\beta^\perp}(\xi)] \begin{bmatrix} M(s) & 0 \\ 0 & I_\sharp \end{bmatrix} \begin{bmatrix} (E^\beta(\zeta))^{\mathrm{T}} \\ (E^{\beta^\perp}(\zeta))^{\mathrm{T}} \end{bmatrix},$$

$$\mathbf{N}(x,\zeta,s) = [E^\alpha(x) \quad E^{\alpha^\perp}(x)] \begin{bmatrix} N(s) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (E^\beta(\zeta))^{\mathrm{T}} \\ (E^{\beta^\perp}(\zeta))^{\mathrm{T}} \end{bmatrix}.$$

Taking into account Assumption 1(iii), the kernels of $\boldsymbol{M}^\infty$ and $\boldsymbol{N}^\infty$ can be written as

$$\mathbf{M}^\infty(\xi,\zeta,s) = [E^\beta(\xi) \quad E^{\beta^\perp}(\xi)] \begin{bmatrix} M_{11}^\infty(s) & 0 \\ 0 & M_{22}^\infty(s) \end{bmatrix} \times \begin{bmatrix} (E^\beta(\zeta))^{\mathrm{T}} \\ (E^{\beta^\perp}(\zeta))^{\mathrm{T}} \end{bmatrix},$$

$$\boldsymbol{N}^\infty(x,\zeta,s) = [E^\alpha(x) \quad E^{\alpha^\perp}(x)] \begin{bmatrix} N_{11}^\infty(s) & 0 \\ 0 & N_{22}^\infty(s) \end{bmatrix} \times \begin{bmatrix} (E^\beta(\zeta))^{\mathrm{T}} \\ (E^{\beta^\perp}(\zeta))^{\mathrm{T}} \end{bmatrix}.$$

In the Ober–Sefton gap formula (cf. Remark 6) write $\boldsymbol{Q}$ as $\boldsymbol{Q}(s) = [E^\beta\ E^{\beta^\perp}]Q(s)\langle[E^\beta\ E^{\beta^\perp}],.\rangle_\mathscr{U}$, where $Q(s)=$

$\langle [E^{\beta}\ E^{\beta^{\perp}}], \boldsymbol{Q}(s)[E^{\beta}\ E^{\beta^{\perp}}]\rangle_{\mathscr{U}}$, to obtain

$$\delta_g(\boldsymbol{P}^{\infty},\boldsymbol{P}) = \inf_{Q,Q^{-1}\in H^{\infty}_{\sharp\times\sharp}(\mathbb{C}_+)} \left\| \begin{bmatrix} M & 0 \\ 0 & I_{\sharp} \\ N & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} M^{\infty}_{11} & 0 \\ 0 & M^{\infty}_{22} \\ N^{\infty}_{11} & 0 \\ 0 & N^{\infty}_{22} \end{bmatrix} Q \right\|_{\infty},$$

from which (6) follows by a similar argument to that in the proof of Theorem 13 (see Reinschke, 1999, for details). Finally, (7) can be verified by noting that if Assumption 1(iv) holds, then

$$\inf_{Q_{22},Q_{22}^{-1}\in H^{\infty}_{\sharp\times\sharp}(\mathbb{C}_+)} \left\| \begin{bmatrix} I_{\sharp} \\ 0 \end{bmatrix} - \begin{bmatrix} M^{\infty}_{22} \\ N^{\infty}_{22} \end{bmatrix} Q_{22} \right\|_{\infty} \leqslant \left\| \begin{bmatrix} I_{\sharp} \\ 0 \end{bmatrix} - \begin{bmatrix} M^{\infty}_{22} \\ N^{\infty}_{22} \end{bmatrix} \right\|_{\infty} \to 0 \quad \text{as } p,m\to\infty.$$

Remark 15. Suppose Assumption 1(iii) is relaxed in the following way: $\langle E^{\beta}, (\boldsymbol{I}_{\mathscr{U}} - \boldsymbol{C}^{\infty}\boldsymbol{P}^{\infty})^{-1}E^{\beta^{\perp}}\rangle_{\mathscr{U}} \leqslant \epsilon$, $\langle E^{\beta^{\perp}}, (\boldsymbol{I}_{\mathscr{U}} - \boldsymbol{C}^{\infty}\boldsymbol{P}^{\infty})^{-1}E^{\beta}\rangle_{\mathscr{U}} \leqslant \epsilon$, $\langle E^{\alpha}, \boldsymbol{P}^{\infty}(\boldsymbol{I}_{\mathscr{U}} - \boldsymbol{C}^{\infty}\boldsymbol{P}^{\infty})^{-1} E^{\beta^{\perp}}\rangle_{\mathscr{Y}} \leqslant \epsilon$ as well as $\langle E^{\alpha^{\perp}}, \boldsymbol{P}^{\infty}(\boldsymbol{I}_{\mathscr{U}} - \boldsymbol{C}^{\infty}\boldsymbol{P}^{\infty})^{-1}E^{\beta}\rangle_{\mathscr{Y}} \leqslant \epsilon$. If Assumptions 1(i) and (ii) hold with $\kappa := (\|[\boldsymbol{I}_{\mathscr{U}} - \boldsymbol{C}^{\infty}]\|_{\infty}^{-1} - 2\epsilon)^{-1} > 0$, then it can be shown that for $\delta > 4$ the following inequality holds instead of equality (6):

$$\delta_g(\boldsymbol{P}^{\infty},\boldsymbol{P}) \leqslant \delta\epsilon\kappa + \max\left\{ \inf_{Q_{11},Q_{11}^{-1}\in H^{\infty}_{m\times m}(\mathbb{C}_+)} \left\| \begin{bmatrix} M \\ N \end{bmatrix} - \begin{bmatrix} M^{\infty}_{11} \\ N^{\infty}_{11} \end{bmatrix} Q_{11} \right\|_{\infty}, \inf_{Q_{22},Q_{22}^{-1}\in H^{\infty}_{\sharp\times\sharp}(\mathbb{C}_+)} \left\| \begin{bmatrix} I_{\sharp\times\sharp} \\ 0_{\sharp\times\sharp} \end{bmatrix} - \begin{bmatrix} M^{\infty}_{22} \\ N^{\infty}_{22} \end{bmatrix} Q_{22} \right\|_{\infty} \right\}.$$

2.5. Controller structure

Given a LTI spatially distributed plant $\boldsymbol{P}\colon \mathrm{dom}(\boldsymbol{P}) \subseteq H^2_{\mathscr{U}}(\mathbb{C}_+) \to H^2_{\mathscr{Y}}(\mathbb{C}_+)$, we wish to design a spatially distributed feedback controller $\boldsymbol{C}\colon \mathrm{dom}(\boldsymbol{C}) \subseteq H^2_{\mathscr{Y}}(\mathbb{C}_+) \to H^2_{\mathscr{U}}(\mathbb{C}|_+)$ that is FDDLTI, i.e., its frequency-domain kernel takes the form $\boldsymbol{C}(\xi,x,s) = E^{\beta^c}(\xi)C(s)(E^{\alpha^c}(x))^{\mathrm{T}}$, where $E^{\beta^c}(.) = (\beta^c_1(.),\ldots,\beta^c_{m_c}(.)) \in \mathscr{U}^{m_c}$ and $E^{\alpha^c}(.) = (\alpha^c_1(.),\ldots,\alpha^c_{p_c}(.)) \in \mathscr{Y}^{p_c}$, and $C(s)$ is a proper, real-rational, $(m_c \times p_c)$ transfer matrix. The spatial functions $\alpha^c_1(.),\ldots,\alpha^c_{p_c}(.)$ represent the p_c spatial distribution functions of the sensors whilst $\beta^c_1(.),\ldots,\beta^c_{m_c}(.)$ are the m_c spatial distribution functions of the actuators. The dynamic parts of the sensors and actuators are assumed to be absorbed into $C(s)$. We will furthermore assume that the locations and spatial shapes of the sensors and the actuators are determined by a parameter vector $\Pi = (\pi_1,\ldots,\pi_{n_\pi})$ which can be varied (continuously) within the set of all "feasible" parameter vectors, $\boldsymbol{\Pi}$. The parameter dependence of the spatial functions in the controller's kernel may be indicated by writing

$$\boldsymbol{C}(\xi,x,s) = E^{\beta^c}(\xi;\Pi)C(s)(E^{\alpha^c}(x;\Pi))^{\mathrm{T}}, \quad \Pi\in\boldsymbol{\Pi}.$$

The controller's frequency-domain input-output relationship is given by $\mathbf{u}(\xi,s) = \int_{\mathscr{D}^o} \mathrm{d}x\, \mathbf{C}(\xi,x,s)\mathbf{y}(x,s)$, $\xi\in\mathscr{D}^i$. In our design framework we will seek to optimise the locations and shapes of the sensor and actuator spatial distribution functions (i.e., optimise over the parameter vector $\Pi\in\boldsymbol{\Pi}$) as well as the (lumped) controller transfer matrix, $C(s)$, connecting the sensors and actuators.

3. Coprime factor synthesis

3.1. Controller design procedure

Step 1: Let $\boldsymbol{P}^{\infty} \in \mathscr{S}(\mathscr{U},\mathscr{Y};\mathbb{C}_+)$ be an open-loop unstable, linear, time-invariant, (typically infinite-dimensional) spatially distributed plant model for which we wish to design a stabilising feedback controller. Find an FDDLTI plant model $\boldsymbol{P}$ approximating $\boldsymbol{P}^{\infty}$.

General methods for computing an FDDLTI plant model given an infinite-dimensional plant model in terms of either PDEs or Laplace-transformed Green's functions are described in Reinschke (1999).

Step 2: Let $W(s)$ be a scalar, real-rational, stable and stably invertible transfer function. Set $\boldsymbol{P}_W(s) := \boldsymbol{P}(s)W(s)$, $\boldsymbol{P}^{\infty}_W(s) := \boldsymbol{P}^{\infty}(s)W(s)$ and $b_{\boldsymbol{P}_W} := \delta_g(\boldsymbol{P}^{\infty}_W,\boldsymbol{P}_W)$. Select $W(s)$ such that (i) the "desired loop shape" $\boldsymbol{P}_W(s)$ is an appropriate one as is customary in H^{∞} loop-shaping, and (ii) the controller error margin

$$b^{\mathrm{marg}}_{\boldsymbol{C}} := b_{\mathrm{opt}}(\boldsymbol{P}_W)\sqrt{1-b^2_{\boldsymbol{P}_W}} - b_{\boldsymbol{P}_W}\sqrt{1-b^2_{\mathrm{opt}}(\boldsymbol{P}_W)} \qquad (8)$$

is sufficiently large. Let $\hat{\boldsymbol{C}}$ be an optimally stabilising controller for $\boldsymbol{P}_W$.

Making $b^{\mathrm{marg}}_{\boldsymbol{C}}$ large means finding a weighting function $W(s)$ that makes $b_{\mathrm{opt}}(\boldsymbol{P}_W)$ large whilst keeping the weighted finite-dimensional approximation error $b_{\boldsymbol{P}_W} = \delta_g(\boldsymbol{P}^{\infty}_W,\boldsymbol{P}_W)$ reasonably small. The controller $\hat{\boldsymbol{C}}$ (cf. Theorem 13) has a kernel of the form $\hat{\mathbf{C}}(\xi,x,s) = E^{\beta}(\xi)\hat{C}(s)(E^{\alpha}(x))^{\mathrm{T}}$. The spatial distribution functions in $\hat{\mathbf{C}}(\xi,x,s)$ are the same as in the kernel of the FDDLTI plant $\boldsymbol{P}$ and these spatial distribution functions will usually not be realizable with real sensors and actuators.

Step 3: Find an implementable controller, $\boldsymbol{C}^{\mathrm{imp}}$, such that the gap distance $\delta_g(\hat{\boldsymbol{C}},\boldsymbol{C}^{\mathrm{imp}})$ is less than the value of $b^{\mathrm{marg}}_{\boldsymbol{C}}$ obtained in Step 2.

A numerical procedure for this step is outlined below in Section 3.2. Note that, by Theorem 8, the feedback loop $[\boldsymbol{P}^{\infty}_W,\boldsymbol{C}^{\mathrm{imp}}]$ is stable if $b_{\boldsymbol{P}_W}\sqrt{1-b^2_{\boldsymbol{C}}} + b_{\boldsymbol{C}}\sqrt{1-b^2_{\boldsymbol{P}_W}} < b_{\mathrm{opt}}(\boldsymbol{P}_W)$, or equivalently, $b_{\boldsymbol{C}} < b^{\mathrm{marg}}_{\boldsymbol{C}}$, where $b^{\mathrm{marg}}_{\boldsymbol{C}}$ is as defined in (8). The implementable controller, $\boldsymbol{C}^{\mathrm{imp}}$, has a kernel of the form $\mathbf{C}^{\mathrm{imp}}(\xi,x,s) = E^{\beta^c}(\xi;\Pi)C(s)(E^{\alpha^c}(x;\Pi))^{\mathrm{T}}$ (cf. Section 2.5). That is, the number of sensors and actuators are chosen a priori, and the shape and location of their spatial distribution functions can only be varied as permitted by the parameter set $\boldsymbol{\Pi}$.

Usually, we will want the number of sensors and actuators, p_c and m_c, to be (substantially) smaller than p and m, respectively. If, for a particular number of sensors and actuators and a particular choice of $\mathbf{\Pi}$, no implementable controller $\boldsymbol{C}^{\text{imp}}$ can be found that satisfies the condition $\delta_g(\hat{\boldsymbol{C}}, \boldsymbol{C}^{\text{imp}}) < b_{\boldsymbol{C}}^{\text{marg}}$, then the set $\mathbf{\Pi}$ needs to be altered, if necessary by increasing the number of sensors and actuators.

Step 4: If $\boldsymbol{C}^{\text{imp}}$ has the kernel $\mathbf{C}^{\text{imp}}(\xi,x,s) = E^{\beta^c}(\xi)C^{\text{imp}}(s)(E^{\alpha^c}(x))^{\mathrm{T}}$, take $\boldsymbol{C}$ with kernel $\mathrm{C}(\xi,x,s) = E^{\beta^c}(\xi)C(s)(E^{\alpha^c}(x))^{\mathrm{T}}$, where $C(s) := C^{\text{imp}}(s)W(s)$, as the robustly stabilising controller for the infinite-dimensional plant model $\boldsymbol{P}^{\infty}$.

3.2. Computing an implementable controller

Let $\hat{\boldsymbol{C}} = \hat{\boldsymbol{N}}\hat{\boldsymbol{M}}^{-1}$ be a normalised RCF. By Theorem 9 we have $\delta_g(\hat{\boldsymbol{C}}, \boldsymbol{C}^{\text{imp}}) < b_{\boldsymbol{C}}^{\text{marg}}$ iff there exists a RCF $\boldsymbol{C}^{\text{imp}} = \boldsymbol{N}^{\text{imp}}(\boldsymbol{M}^{\text{imp}})^{-1}$ such that $\| {\hat{\boldsymbol{M}}-\boldsymbol{M}^{\text{imp}} \atop \hat{\boldsymbol{N}}-\boldsymbol{N}^{\text{imp}}} \|_\infty < b_{\boldsymbol{C}}^{\text{marg}}$. Hence, we wish to find

$$b_{\boldsymbol{C}}^{\text{opt}} := \min_{\substack{\boldsymbol{M}^{\text{imp}},\ \boldsymbol{N}^{\text{imp}} \text{ stable \& coprime} \\ \boldsymbol{C}^{\text{imp}}=\boldsymbol{N}^{\text{imp}}(\boldsymbol{M}^{\text{imp}})^{-1} \text{ impl.}}} \left\| \begin{bmatrix} \hat{\boldsymbol{M}} \\ \hat{\boldsymbol{N}} \end{bmatrix} - \begin{bmatrix} \boldsymbol{M}^{\text{imp}} \\ \boldsymbol{N}^{\text{imp}} \end{bmatrix} \right\|_\infty \tag{9}$$

and a pair of operator-valued functions $\{\boldsymbol{M}_{\text{opt}}^{\text{imp}}, \boldsymbol{N}_{\text{opt}}^{\text{imp}}\}$ achieving this minimum. Since the coprimeness requirement in (9) is automatically satisfied if $\|[{\hat{\boldsymbol{M}} \atop \hat{\boldsymbol{N}}}] - [{\boldsymbol{M}^{\text{imp}} \atop \boldsymbol{N}^{\text{imp}}}]\|_\infty$ is small enough (cf. Vidyasagar, 1985, p. 235), we consider the simpler optimisation problem

$$b_{\boldsymbol{C}}^{\text{opt}} := \min_{\substack{\boldsymbol{M}^{\text{imp}},\boldsymbol{N}^{\text{imp}} \text{ stable} \\ \boldsymbol{C}^{\text{imp}}=\boldsymbol{N}^{\text{imp}}(\boldsymbol{M}^{\text{imp}})^{-1} \text{ impl.}}} \left\| \begin{bmatrix} \hat{\boldsymbol{M}} \\ \hat{\boldsymbol{N}} \end{bmatrix} - \begin{bmatrix} \boldsymbol{M}^{\text{imp}} \\ \boldsymbol{N}^{\text{imp}} \end{bmatrix} \right\|_\infty \tag{10}$$

and check the coprimeness of the optimal solution, $\{\boldsymbol{M}_{\text{opt}}^{\text{imp}}, \boldsymbol{N}_{\text{opt}}^{\text{imp}}\}$, afterwards. By Theorem 12 the normalised right coprime factors $\{\hat{\boldsymbol{M}}, \hat{\boldsymbol{N}}\}$ of $\hat{\boldsymbol{C}}$ and the right coprime factors $\{\boldsymbol{M}^{\text{imp}}, \boldsymbol{N}^{\text{imp}}\}$ of $\boldsymbol{C}^{\text{imp}}$ are of the form $\hat{\boldsymbol{M}}(s) = E^{\alpha}(\hat{M}(s)-I_p)\langle E^{\alpha}, .\rangle_{\mathscr{Y}} + \boldsymbol{I}_{\mathscr{Y}}$, $\hat{\boldsymbol{N}}(s) = E^{\beta}\hat{N}(s)\langle E^{\alpha}, .\rangle_{\mathscr{Y}}$ and $\boldsymbol{M}^{\text{imp}}(s) = E^{\alpha^c}(\Pi)(M^{\text{imp}}(s) - I_{p_c})\langle E^{\alpha^c}(\Pi), .\rangle_{\mathscr{Y}} + \boldsymbol{I}_{\mathscr{Y}}$, $\boldsymbol{N}^{\text{imp}}(s) = E^{\beta^c}(\Pi)N^{\text{imp}}(s)\langle E^{\alpha^c}(\Pi), .\rangle_{\mathscr{Y}}$, where $\hat{C}(s) = \hat{N}(s)(\hat{M}(s))^{-1}$ is a normalised RCF and $C^{\text{imp}}(s) = N^{\text{imp}}(s)(M^{\text{imp}}(s))^{-1}$ is a RCF. Using the Gram-Schmidt orthonormalisation process, define

$$(E^{\alpha}(x), E^{\alpha^{\perp}}(x;\Pi)) := \mathrm{GSO}(E^{\alpha}(x), E^{\alpha^c}(x;\Pi)),$$

$$(E^{\alpha^c}(x;\Pi), E^{\alpha^{c\perp}}(x;\Pi)) := \mathrm{GSO}(E^{\alpha^c}(x;\Pi), E^{\alpha}(x)),$$

$$(E^{\beta}(\xi), E^{\beta^{\perp}}(\xi;\Pi)) := \mathrm{GSO}(E^{\beta}(\xi), E^{\beta^c}(\xi;\Pi)),$$

$$(E^{\beta^c}(\xi;\Pi), E^{\beta^{c\perp}}(\xi;\Pi)) := \mathrm{GSO}(E^{\beta^c}(\xi;\Pi), E^{\beta}(\xi))$$

and let p_a and m_a denote the dimensions of the vectors defined in the first and the second two lines, respectively. Form the following scalar products of the α-functions,

$$\langle E^{\alpha}(.), E^{\alpha^c}(.;\Pi)\rangle_{\mathscr{Y}} =: T_{11}^{\alpha}(\Pi),$$

$$\langle E^{\alpha}(.), E^{\alpha^{c\perp}}(.;\Pi)\rangle_{\mathscr{Y}} =: T_{12}^{\alpha}(\Pi),$$

$$\langle E^{\alpha^{\perp}}(.), E^{\alpha^c}(.;\Pi)\rangle_{\mathscr{Y}} =: T_{21}^{\alpha}(\Pi),$$

$$\langle E^{\alpha^{\perp}}(.), E^{\alpha^{c\perp}}(.;\Pi)\rangle_{\mathscr{Y}} =: T_{22}^{\alpha}(\Pi)$$

and similarly of the β-functions, i.e.,

$$\langle E^{\beta}(.), E^{\beta^c}(.;\Pi)\rangle_{\mathscr{U}} =: T_{11}^{\beta}(\Pi),$$

etc. Eq. (10) can now be re-written as

$$b_{\boldsymbol{C}}^{\text{opt}} = \min_{Q(s) \text{ stable},\ \Pi\in\mathbf{\Pi}} \|R(s;\Pi) - T_1Q(s)T_2\|_\infty, \tag{11}$$

where $R(s;\Pi) := \begin{bmatrix} R_{11}(s;\Pi) & R_{12}(s;\Pi) \\ R_{21}(s;\Pi) & R_{22}(s;\Pi) \end{bmatrix}$,

$$\hat{D}(s) := \hat{M}(s) - I_p,$$

$$R_{11}(s;\Pi) := \begin{bmatrix} (T_{11}^{\alpha}(\Pi))^{\mathrm{T}}\hat{D}(s) \\ (T_{11}^{\beta}(\Pi))^{\mathrm{T}}\hat{N}(s) \end{bmatrix} T_{11}^{\alpha}(\Pi) + \begin{bmatrix} I_{p_c} \\ 0_{m_c\times p_c} \end{bmatrix},$$

$$R_{12}(s;\Pi) := \begin{bmatrix} (T_{11}^{\alpha}(\Pi))^{\mathrm{T}}\hat{D}(s) \\ (T_{11}^{\beta}(\Pi))^{\mathrm{T}}\hat{N}(s) \end{bmatrix} T_{12}^{\alpha}(\Pi),$$

$$R_{21}(s;\Pi) := \begin{bmatrix} (T_{12}^{\alpha}(\Pi))^{\mathrm{T}}\hat{D}(s) \\ (T_{12}^{\beta}(\Pi))^{\mathrm{T}}\hat{N}(s) \end{bmatrix} T_{11}^{\alpha}(\Pi),$$

$$R_{22}(s;\Pi) := \begin{bmatrix} (T_{12}^{\alpha}(\Pi))^{\mathrm{T}}\hat{D}(s) \\ (T_{12}^{\beta}(\Pi))^{\mathrm{T}}\hat{N}(s) \end{bmatrix} T_{12}^{\alpha}(\Pi);$$

$$T_1 := \begin{bmatrix} I_{p_c+m_c} \\ 0_{((p_a-p_c)+(m_a-m_c))\times(p_c+m_c)} \end{bmatrix}, \quad Q(s) := \begin{bmatrix} M^{\text{imp}}(s) \\ N^{\text{imp}}(s) \end{bmatrix},$$

and $T_2 := [I_{p_c}\ \ 0_{p_c\times(p_a-p_c)}]$. For fixed $\Pi\in\mathbf{\Pi}$, the RHS of (11) is a standard H^{∞}-optimisation problem. So, once the sensors and actuators have been placed, finding an optimal controller transfer matrix connecting the sensors and actuators is a problem with a known solution.

3.3. Sensor and actuator placement

In view of (11), we would ideally like to solve for Π^{opt} (representing the optimal locations and shapes of the sensor and actuator spatial distribution functions), where $\Pi^{\text{opt}} := \arg\min_{\Pi\in\mathbf{\Pi}} \gamma(\Pi)$ and $\gamma(\Pi) := \min_{Q(s)\text{ stable}} \|R(s;\Pi) - T_1Q(s)T_2\|_\infty$. In the sequel we will propose two methods which both find an approximate value for Π^{opt}. The first one (Algorithm 1 below) is iterative and "decouples" the two minimisations.

Algorithm 1 (Q-Π-interation).

Step 0: (Initialisation) Given a tolerance $\epsilon > 0$, an initial parameter vector $\Pi_0 \in \mathbf{\Pi}$, and a real number $\gamma_{\text{old}} \geqslant 1$.

Step 1: $\gamma_{\text{new}} := \min_{Q(s)\,\text{stable}} \|R(s;\Pi_0) - T_1Q(s)T_2\|_\infty$, and store $\hat{Q}(s)$ achieving the minimum.

Step 2: IF $(\gamma_{\text{old}} - \gamma_{\text{new}}) < \epsilon$ THEN $\hat{\Pi} := \Pi_0$ is taken as optimal; STOP.

Step 3: Find $\hat{\Pi} := \arg\min_{\Pi\in\mathbf{\Pi}} \|R(s;\Pi) - T_1\hat{Q}(s)T_2\|_\infty$.

Step 4: Set $\Pi_0 := \hat{\Pi}$, $\gamma_{\text{old}} := \gamma_{\text{new}}$, and loop to Step 1.

Remark 16. Step 3 of Algorithm 1 can be implemented using Algorithm 3 stated below, with $g(\Pi)$ re-defined as $g(\Pi) := \|R(s;\Pi) - T_1\hat{Q}(s)T_2\|_\infty$.

The second method of finding an approximate value for Π^{opt} is not an iterative scheme. The basic idea is that, instead of minimising $\gamma(\Pi)$, we minimise the lower bound $g(\Pi)$ of $\gamma(\Pi)$, which is obtained by applying Parrott's Theorem (see Zhou, Doyle, and Glover, 1996, p. 40):

$$g(\Pi) := \max\{g_1(\Pi), g_2(\Pi)\} \leqslant \gamma(\Pi), \tag{12}$$

where

$$g_1(\Pi) := \left\| \begin{bmatrix} (T_{12}^{\alpha}(\Pi))^{\mathrm{T}}\hat{D}(s) \\ (T_{12}^{\beta}(\Pi))^{\mathrm{T}}\hat{N}(s) \end{bmatrix} [T_{11}^{\alpha}(\Pi) \;\; T_{12}^{\alpha}(\Pi)] \right\|_\infty$$

and

$$g_2(\Pi) := \left\| \begin{matrix} \begin{bmatrix} (T_{11}^{\alpha}(\Pi))^{\mathrm{T}} \\ (T_{12}^{\alpha}(\Pi))^{\mathrm{T}} \end{bmatrix} \hat{D}(s)T_{12}^{\alpha}(\Pi) \\ \begin{bmatrix} (T_{11}^{\beta}(\Pi))^{\mathrm{T}} \\ (T_{12}^{\beta}(\Pi))^{\mathrm{T}} \end{bmatrix} \hat{N}(s)T_{12}^{\alpha}(\Pi) \end{matrix} \right\|_\infty .$$

Exploiting the fact that $T^{\alpha}(\Pi)$ and $T^{\beta}(\Pi)$ are orthogonal matrices for all $\Pi \in \mathbf{\Pi}$, the expressions for $g_1(\Pi)$ and $g_2(\Pi)$ can be simplified:

$$g_1(\Pi) = \left\| \begin{matrix} (T_{12}^{\alpha}(\Pi))^{\mathrm{T}}\hat{D}(s) \\ (T_{12}^{\beta}(\Pi))^{\mathrm{T}}\hat{N}(s) \end{matrix} \right\|_\infty ,$$

$$g_2(\Pi) = \left\| \begin{bmatrix} \hat{D}(s) \\ \hat{N}(s) \end{bmatrix} T_{12}^{\alpha}(\Pi) \right\|_\infty . \tag{12a}$$

Thus we arrive at the following algorithm.

Algorithm 2 (Parrott lower bound optimisation).

Step 1: $\gamma_0 := \min_{Q(s)\,\text{stable}} \|R(s;\Pi_0) - T_1Q(s)T_2\|_\infty$, where $\Pi_0 \in \mathbf{\Pi}$ is the initial parameter vector.

Step 2: By means of a local search about $\Pi = \Pi_0$ find

$$\hat{\Pi} := \arg\min_{\Pi\in\mathbf{\Pi}} g(\Pi), \tag{13}$$

where $g(\Pi)$ was defined in (12) and (12a).

Step 3: $\gamma_1 := \min_{Q(s)\,\text{stable}} \|R(s;\hat{\Pi}) - T_1Q(s)T_2\|_\infty$. If $\gamma_1 < \gamma_0$, then take Π_1, otherwise Π_0, as being (approximately) optimal.

Remark 17. Algorithm 2 does not guarantee to yield an improved parameter vector $\Pi \in \mathbf{\Pi}$ since, at least in principle, it is possible that $\gamma(\Pi)$ increases as Π is varied whilst its Parrott lower bound, $g(\Pi)$, decreases. However, in the example of Section 4 we always found the first four digits of $\gamma(\Pi)$ and $g(\Pi)$ to coincide, so there are examples for which Algorithm 2 is very suitable.

Recall that Π was assumed to be n_π-dimensional. The RHS of (13) therefore represents an n_π-dimensional, nonsmooth, not necessarily convex, optimisation problem. To solve it, we will now outline a descent algorithm that is based on linear matrix inequalities (the idea of using LMIs is due to Vinnicombe, 1998) see also Vinnicombe & Miyamoto (1997) and that will converge to a local minimum.

Algorithm 3.

Step 0: (Initialisation). Given a $\Pi_0 \in \mathbf{\Pi}$.

Step 1: Find a descent direction ζ of $g(\Pi)$ at $\Pi = \Pi_0$. STOP if there is no descent direction; $\hat{\Pi} := \Pi_0$ is taken as optimal.

Step 2: Perform a line search of $g(\Pi_0 + \lambda\zeta)$ for $\lambda \in (0,\infty)$ and find a suitable $\hat{\lambda} \in (0,\infty)$ such that $g(\hat{\Pi}) < g(\Pi_0)$, where $\hat{\Pi} := \Pi_0 + \hat{\lambda}\zeta$.

Step 3: Set $\Pi_0 := \hat{\Pi}$ and loop to Step 1.

Finding a descent direction or deciding that there is none is generally the difficult part in nonsmooth optimisation (Hiriart-Urruty & Lemarechal, 1993). In the particular nonsmooth optimisation problem (13), however, the structure of the nonsmooth function $g(.)$ can be exploited. We will assume that the matrices $T^{\alpha}(\Pi), T^{\beta}(\Pi)$ are continuously differentiable with respect to Π for all $\Pi \in \mathbf{\Pi}$. Correspondingly, if we define

$$\begin{bmatrix} (T_{12}^{\alpha}(\Pi))^{\mathrm{T}}\hat{D}(s) \\ (T_{12}^{\beta}(\Pi))^{\mathrm{T}}\hat{N}(s) \end{bmatrix} =: T^{(1)}(\Pi) \begin{bmatrix} \hat{D}(s) \\ \hat{N}(s) \end{bmatrix} =: Z^{(1)}(s;\Pi)$$

and

$$\begin{bmatrix} \hat{D}(s) \\ \hat{N}(s) \end{bmatrix} T_{12}^{\alpha}(\Pi) =: \begin{bmatrix} \hat{D}(s) \\ \hat{N}(s) \end{bmatrix} T^{(2)}(\Pi) =: Z^{(2)}(s;\Pi),$$

then $Z^{(1)}(s;\Pi)$ and $Z^{(2)}(s;\Pi)$ can be approximated as

$$Z^{(1)}(s;\Pi_0 + \eta) \approx Z_a^{(1)}(s;\Pi_0,\eta) := \left(T^{(1)}(\Pi_0) + \sum_{i=1}^{n_\pi} \eta_i \cdot \left.\frac{\partial T^{(1)}(\pi_1,\ldots,\pi_{n_\pi})}{\partial \pi_i}\right|_{\Pi_0} \right) \begin{bmatrix} \hat{D}(s) \\ \hat{N}(s) \end{bmatrix},$$

$$Z^{(2)}(s;\Pi_0 + \eta) \approx Z_a^{(2)}(s;\Pi_0,\eta) := \begin{bmatrix} \hat{D}(s) \\ \hat{N}(s) \end{bmatrix} \left(T^{(2)}(\Pi_0) + \sum_{i=1}^{n_\pi} \eta_i \cdot \left.\frac{\partial T^{(2)}(\pi_1,\ldots,\pi_{n_\pi})}{\partial \pi_i}\right|_{\Pi_0} \right),$$

where $\eta = (\eta_1 \;\; \eta_2 \;\; \ldots \;\; \eta_{n_\pi})^{\mathrm{T}}$. For $\|\eta\|$ small, the ∞-norms of $Z^{(1)}(s;\Pi_0+\eta)$ and $Z^{(2)}(s;\Pi_0+\eta)$ are (approximately) equal to the ∞-norms of $Z_a^{(1)}(s;\Pi_0,\eta)$ and $Z_a^{(2)}(s;\Pi_0,\eta)$, respectively. If $[A_0,B_0,C_0,D_0]$ is a minimal state-space realizations of $\left[\begin{smallmatrix}\hat{D}(s)\\ \hat{N}(s)\end{smallmatrix}\right]$, then state-space realizations of $Z_a^{(1)}(s;\Pi_0,\eta)$ and $Z_a^{(2)}(s;\Pi_0,\eta)$ are given by $[A^{(1)},B^{(1)},C^{(1)},D^{(1)}]$ and $[A^{(2)},B^{(2)},C^{(2)},D^{(2)}]$, respectively, where

$$A^{(1)} := A_0, \quad B^{(1)} := B_0,$$

$$C^{(1)} := \left(T^{(1)}(\Pi_0) + \sum_{i=1}^{n_\pi} \eta_i \cdot \left.\frac{\partial T^{(1)}}{\partial \pi_i}\right|_{\Pi_0}\right) C_0,$$

$$D^{(1)} := \left(T^{(1)}(\Pi_0) + \sum_{i=1}^{n_\pi} \eta_i \cdot \left.\frac{\partial T^{(1)}}{\partial \pi_i}\right|_{\Pi_0}\right) D_0;$$

$$A^{(2)} := \begin{pmatrix} A_0 & & \mathbf{0} \\ & A_0 & \\ \mathbf{0} & & \ddots & \\ & & & A_0 \end{pmatrix},$$

$$B^{(2)} := \begin{pmatrix} B_0\, T^{(2)}(\Pi_0) \\ B_0 \left.\dfrac{\partial T^{(2)}}{\partial \pi_1}\right|_{\Pi_0} \\ \vdots \\ B_0 \left.\dfrac{\partial T^{(2)}}{\partial \pi_{n_\pi}}\right|_{\Pi_0} \end{pmatrix},$$

$$C^{(2)} := (\, C_0 \quad \eta_1 \cdot C_0 \quad \ldots \quad \eta_{n_\pi} \cdot C_0 \,),$$

$$D^{(2)} := D_0 \left(T^{(2)}(\Pi_0) + \sum_{i=1}^{n_\pi} \left.\frac{\partial T^{(2)}}{\partial \pi_i}\right|_{\Pi_0}\right).$$

By the LMI expression for the ∞-norm (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994; Gahinet, Nemirovski, Laub, & Chilali, 1995), for given η and ϵ, there holds $\|Z_a^{(j)}(s;\Pi_0,\eta)\|_\infty < \epsilon$ iff there are positive definite matrices $X^{(j)},\ j=1,2$, such that

$$\begin{pmatrix} (A^{(j)})^{\mathrm{T}}X^{(j)} + X^{(j)}A^{(j)} & X^{(j)}B^{(j)} & (C^{(j)})^{\mathrm{T}} \\ (B^{(j)})^{\mathrm{T}}X^{(j)} & -\epsilon I & (D^{(j)})^{\mathrm{T}} \\ C^{(j)} & D^{(j)} & -\epsilon I \end{pmatrix} < 0. \tag{14}$$

Since the two matrices (for $j=1,2$) on the LHS of (14) are linearly dependent on the parameters $\eta_1,\ldots,\eta_{n_\pi}$ and ϵ, we can find $\hat{\epsilon} := \min \epsilon$ such that there exist $\hat{X}^{(j)} > 0,\ j=1,2$, and $\hat{\eta} \in \mathbb{R}^{n_\pi}$ satisfying the two LMIs (14) with $\epsilon := \hat{\epsilon}$, $X^{(1)} := \hat{X}^{(1)}$, $X^{(2)} := \hat{X}^{(2)}$ and $\eta := \hat{\eta}$. If $\hat{\epsilon} < g(\Pi_0)$, then $\hat{\eta} = (\eta_1\ \eta_2\ \ldots\ \eta_{n_\pi})^{\mathrm{T}}$ is a (local) (steepest) descent direction for $g(\Pi)$ at $\Pi = \Pi_0$. Otherwise there is no descent direction.

We have thus outlined a procedure that can be used for Step 1 of Algorithm 3. Efficient line search algorithms for Step 2 can be found in Hiriart-Urruty and Lemarechal, 1993, and references therein. This completes the description of how to solve the optimisation problem (13).

4. Design example

4.1. Formulation of the control problem

The design example—stabilisation of an electrostatically destabilised, electrically conducting membrane—is taken from Lang and Staelin (1982). The experimental setup is schematically depicted in Fig. 2 and can be described as follows. A rectangular ($L_1 \times L_2 = 1.04$ m $\times$ 1.12 m), flexible, electrically conducting membrane is suspended vertically, clamped at its boundaries and biased by a high-voltage source. With no voltage bias the membrane would be level-flat and motionless; this is the equilibrium state of zero membrane deflection. As the voltage bias is increased from zero beyond a threshold voltage $V = V_c$, the equilibrium state becomes unstable. To one side of the membrane a parallel surface of equal dimension is placed at a short distance normal to the membrane. This surface consists of a three-by-three array of fixed conducting plates, each independently addressable through a low-voltage source, and acts as a distributed control input. On the other side of the membrane a similar array of nine plates allows a capacitive measurement of the membrane deflection. (The capacitance between the membrane and each sensor plate in conjunction with a fixed inductor form an oscillator whose resonant frequency gives the membrane deflection averaged across the area of each sensor plate.) The spatial distribution function of each sensor and actuator is assumed to be equal to a constant inside the area of the corresponding sensor or actuator plate, and zero everywhere outside.

We make the following simplifying assumptions:

(i) long wave limit: the membrane deflection wavelengths of significance are much larger than the distance between the membrane and the actuator (respectively, sensor) plates;
(ii) small amplitudes: the membrane deflection is small compared to the distance between membrane and actuator as well as membrane and sensor plates; the control voltages $v_{t1}(t),\ldots,v_{t9}(t)$ in Fig. 2 are much smaller than the high-voltage bias V.

With these assumptions, the membrane deflection $\mathbf{y_t}(x_1, x_2, t)$ — defined as positive toward the sensor plates — obeys the PDE (see Lang and Staelin, 1982)

$$\mu\, \frac{\partial^2 \mathbf{y_t}(x_1,x_2,t)}{\partial t^2} + \delta\, \frac{\partial \mathbf{y_t}(x_1,x_2,t)}{\partial t}$$

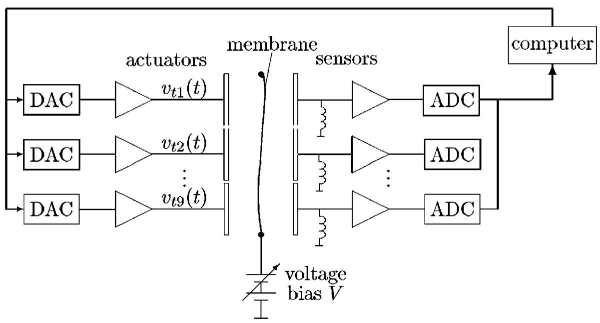

Fig. 2. Functional diagram of Lang and Staelin's experimental setup (adapted version of Fig. 1 in Lang and Staelin, 1982).

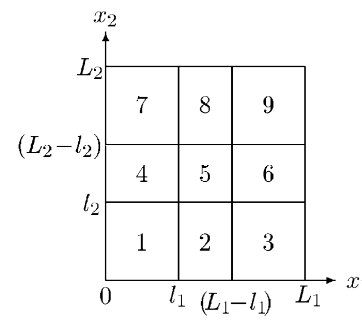

Fig. 3. Parametrised partitioning of the sensor and the actuator surfaces (l_1 and l_2 being the parameters).

$$- \tau_1 \frac{\partial^2 \mathbf{y_t}(x_1,x_2,t)}{\partial x_1^2} - \tau_2 \frac{\partial^2 \mathbf{y_t}(x_1,x_2,t)}{\partial x_2^2}$$

$$- \frac{2\varepsilon_0 V^2}{H^3}\,\mathbf{y_t}(x_1,x_2,t) = -\frac{\varepsilon_0 V}{H^2}\,\mathbf{u_t}(x_1,x_2,t), \qquad (15)$$

where $\{x_1,x_2\} \in \mathscr{D} := (0,L_1)\times(0,L_2)$, in conjunction with the boundary condition $\mathbf{y_t}(\xi_1,\xi_2,t)=0$ for all $\{\xi_1,\xi_2\}\in\partial\mathscr{D}$. The distributed variable $\mathbf{u_t}(x_1,x_2,t)$ on the RHS of (15) stands for a distributed control voltage to be realized by the actuators. (The fact that the form of $\mathbf{u_t}(x_1,x_2,t)$ is restricted due to the discreteness of the actuators is disregarded for the moment.) The meaning and the numerical values (again taken from Lang and Staelin (1982)) of the physical parameters in (15) are as follows: membrane mass density $\mu = 0.033$ kg/m^2, viscous damping coefficient $\delta = 0.48$ kg/m^2/s, membrane tension in x_1-direction $\tau_1 = 8.4$ N/m, membrane tension in x_2-direction $\tau_2 = 7.8$ N/m, distance between membrane and actuator plates $H = 9.2$ mm, free space permittivity $\varepsilon_0 = 8.85$ pAs/V/m.

It is not difficult to show that the PDE (15), after Laplace transformation and with zero initial conditions, is equivalent to the integral relationship $\mathbf{y}(x_1,x_2;s) = \int_0^{L_1} \mathrm{d}\xi_1 \int_0^{L_2} \mathrm{d}\xi_2\, \mathbf{P}^\infty(x_1,x_2,\xi_1,\xi_2,s)\mathbf{u}(\xi_1,\xi_2,s)$, where

$$\mathbf{P}^\infty(x_1,x_2,\xi_1,\xi_2,s) = \sum_{j=1}^{\infty} \alpha_j(x_1,x_2)\,\frac{-\varepsilon_0 V/H^2}{s^2\mu + s\delta + \omega_j^2 - \Omega}\,\beta_j(\xi_1,\xi_2) \qquad (16)$$

is the plant's Laplace-transformed Green's function. In (16), $\alpha_j(x_1,x_2)=\beta_j(x_1,x_2) := (2/\sqrt{L_1L_2})\sin(i\pi x_1/L_1)\sin(l\pi x_2/L_2)$, $\omega_j := \sqrt{\tau_1(i\pi/L_1)^2 + \tau_2(l\pi/L_2)^2}$, and $\Omega := 2\varepsilon_0 V^2/H^3$. The mapping from the double index (i,l) to the single index j is defined such that $\{\omega_j\}_{j\in\mathbb{Z}_+}$ is an increasing sequence. From (16) it is clear that the first k modes are unstable if the high-voltage bias V is such that $2\varepsilon_0 V^2/H^3 \geqslant \omega_k^2$. The first mode thus becomes unstable at $V = V_c$, where $V_c := \pi\sqrt{H^3/2\varepsilon_0(\tau_1/L_1^2 + \tau_2/L_2^2)\cdot H^{3/(2\varepsilon_0)}} = 2460$ V, the second mode goes unstable at $V = 3760$ V, etc. From now on we will assume that V has a fixed value of 2500 V, whence precisely the first mode of the open-loop plant is unstable. A finite-dimensional approximation $\mathbf{P}^{(m)}(x_1,x_2,\xi_1,\xi_2,s)$ of $\mathbf{P}^\infty(x_1,x_2,\xi_1,\xi_2,s)$ is obtained from (16) by truncating the infinite sum after the first $m = 10$ terms:

$$\mathbf{P}^{(10)}(x_1,x_2,\xi_1,\xi_2,s) = E^\alpha(x_1,x_2)P^{(10)}(s)\,(E^\beta(\xi_1,\xi_2))^{\mathrm{T}}, \qquad (17)$$

$E^\alpha(x_1,x_2) := (\alpha_1(x_1,x_2),\ldots,\alpha_{10}(x_1,x_2))$, $E^\beta(\xi_1,\xi_2) := (\beta_1(\xi_1,\xi_2),\ldots,\beta_{10}(\xi_1,\xi_2))$, $P^{(10)}(s) := (-\varepsilon_0 V/H^2)\cdot \mathrm{diag}((s^2\mu+s\delta+\omega_1^2-\Omega)^{-1},\ldots,(s^2\mu+s\delta+\omega_{10}^2-\Omega)^{-1})$. Let $\boldsymbol{P}^\infty$ and $\boldsymbol{P}^{(10)}$ denote the integral operators whose kernels are given by (16) and (17), respectively, and notice that we have accomplished Step 1 of the design procedure outlined in Section 3.1.

4.2. Maximum controller error margin

The scalar, stable, stably invertible, weighting function $W(s)$, as introduced in Step 2 of the design procedure of Section 3.1, is assumed to be of the form $W(s)=(s+w_1)/(s+w_2)$ with $w_1 > 0$ and $w_2 > 0$. Define $\boldsymbol{P}_W^\infty := \boldsymbol{P}^\infty \cdot W$ as well as $\boldsymbol{P}_W^{(10)} := \boldsymbol{P}^{(10)} \cdot W$. To determine the maximum controller error margin, $b_{\boldsymbol{C}}^{\mathrm{marg}}$, (cf. (8)) we need to evaluate $\delta_g(\boldsymbol{P}_W^\infty, \boldsymbol{P}_W^{(10)})$, which can be done using Theorem 14. (It is easy to see that the Assumptions 1(i) to (v) are satisfied by the pair $\{\boldsymbol{P}_W^\infty, \boldsymbol{P}_W^{(10)}\}$.) $b_{\boldsymbol{C}}^{\mathrm{marg}}$ depends on the choice of the weighting function $W(s)$, i.e., on w_1 and w_2. It was found that $w_1 = 3.1158\cdot 10^6$ and $w_2 = 70$ are nearly optimal with respect to maximizing $b_{\boldsymbol{C}}^{\mathrm{marg}}$. For these values of w_1 and w_2 the finite-dimensional approximation error, $\delta_g(\boldsymbol{P}_W^\infty, \boldsymbol{P}_W^{(10)})$, is 0.0458, and the optimally stabilising controller $\hat{\boldsymbol{C}}$, which is of McMillan degree 30, achieves $b_{\mathrm{opt}}(\boldsymbol{P}_W^{(10)}) = 0.3732$ and hence $b_{\boldsymbol{C}}^{\mathrm{marg}} = 0.3303$.

4.3. Computing an implementable controller

Characterising the implementable, spatially distributed controller $\boldsymbol{C}^{\mathrm{imp}}$, it is assumed that the sensor and the actuator surfaces are partitioned in the same way (i.e., $\alpha_i^c(x_1,x_2) = \beta_i^c(x_1,x_2)$ for all $\{x_1,x_2\}\in\mathscr{D}$ and $i = 1,\ldots,9$), and that the partitioning is parametrised using two parameters, l_1 and l_2, as indicated in Fig. 3. By means of a direct, global search, in which l_1 was varied between 0 and 0.36, and l_2 between 0 and 0.40, it was found that the values $l_1 = 0.20$ and $l_2 = 0.22$ are nearly optimal, achieving $\delta_g(\hat{\boldsymbol{C}}, \boldsymbol{C}^{\text{imp}}) = 0.3195 < b_{\boldsymbol{C}}^{\text{marg}}$. The McMillan degree of the implementable controller $\boldsymbol{C}^{\text{imp}}$, initially the same as the McMillan degree of $\hat{\boldsymbol{C}}$, was found to be reducible to just three without altering the first four digits of $\delta_g(\hat{\boldsymbol{C}}, \boldsymbol{C}^{\text{imp}})$. To validate that the (reduced McMillan degree) implementable controller $\boldsymbol{C}^{\text{imp}}$ is indeed stabilising, it was connected with the weighted, finite-dimensional plant approximations $\boldsymbol{P}_W^{(10)}$ and $\boldsymbol{P}_W^{(20)}$. In both cases the closed loop was found to be stable.

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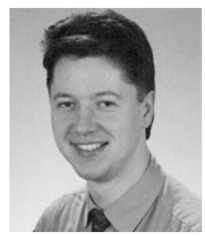

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