AIR TRAFFIC CONTROL WITH AN EXPECTED VALUE CRITERION [1]

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Abstract: In this contribution we discuss a stochastic framework for air traffic conflict resolution. The conflict resolution task is posed as the problem of optimising an expected value criterion. Optimisation is carried out by Monte Carlo Markov Chain (MCMC) simulation. A numerical example illustrates the proposed strategy.

Keywords: Air Traffic Control, Monte Carlo Markov Chain, Model Predictive Control

1. INTRODUCTION

In the current organisation of Air Traffic Management the centralised Air Traffic Control (ATC) is in complete control of the air traffic and ultimately responsible for safety. Aircraft, before take off, receive flight plans which cover the entire flight. During the flight, ATC sends additional instructions to them, depending on the actual traffic, in order to avoid dangerous encounters. The main objective of ATC is to maintain safe separation. The level of accepted minimum safe separation can vary with the density of the traffic and the region of airspace. For example, a largely accepted value for horizontal minimum safe separation is 5 nmi in general en-route airspace which is reduced to 3 nmi during in approach sectors with aircraft landing and departing. A conflict is defined as the situation of loss of minimum safe separation between two aircraft. If it is possible, ATC tries also to fulfil, the, possibly conflicting, requests of aircraft and airlines (desired path to avoid turbulence, desired time of arrivals to meet schedule, etc..).

In order to improve performance of ATC, mainly in view of increasing levels of traffic, research effort has been spent in the last decade to create tools for Conflict Detection and Conflict Resolution. A review of research work on ATC is presented in (Kuchar and Yang, 2000).

In Conflict Detection one has to evaluate the possibility of future conflict starting from the current state of the airspace and taking into account uncertainty in the future position of aircraft while they follow given nominal paths. In doing Conflict Detection one needs a model to predict the future. In a probabilistic setting, the model could be either an empirical distribution of future position or a stochastic differential equation that describes the aircraft motion and defines implicitly a distribution for future aircraft positions. The stochastic part enters the system as the action of the wind field and several uncertainties in the physics of the aircraft. On the basis of the prediction model one can evaluate metrics related to safety. One example of a possible metric is conflict probability over a certain time horizon. Several methods have been developed to

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estimate different metrics related to safety for a number of prediction models, e.g (Blom and Bakker, 2002; Hu *et al.*, 2003; Irvine, 2001; Paielli and Erzberger, 1997; Prandini *et al.*, 2000).
Among other methods, Monte Carlo (MC) methods have the main advantage of allowing flexibility in the complexity of the prediction model since the model is used only as a simulator and, in principle, it is not involved in explicit calculations. In all methods a trade off exists between computational effort (simulation time in the case of MC methods) and complexity of the model. Techniques to accelerate MC methods by saving computational time are under development, see e.g. (Krystul and Blom, 2004).
In Conflict Resolution one wants to calculate suitable maneuvers to avoid a predicted conflict. A number of Conflict Resolution algorithms has been proposed for a deterministic setting, see (Kuchar and Yang, 2000). In a stochastic setting, the research effort has been concentrated mainly on Conflict Detection while few resolution strategies dedicated to a stochastic setting have been proposed, the main reason being the complexity of stochastic prediction models. Simple conflict resolution maneuvers have been considered in (Paielli and Erzberger, 1997; Prandini *et al.*, 2000).
In this paper we present a Monte Carlo Markov Chain (MCMC) framework (Robert and Casella, 1999) for Conflict Resolution in a stochastic setting. The approach is borrowed from Bayesian statistics (Mueller, 1999; Mueller *et al.*, 2002). We will consider a resolution criterion that takes into account separation and other factors (e.g. aircraft requests). Then, the procedure of (Mueller, 1999) is employed to estimate the resolution maneuver that optimises the expected value criterion through MCMC simulation. The interesting point in this approach is that it extends the advantages of Monte Carlo techniques, in terms of flexibility and complexity of the problems that can be tackled, to Conflict Resolution.
In this contribution, we restrict our attention to *level flight*. The case of level flight is meaningful from an application point of view since aircraft typically tend to fly at the same altitude most of the time. However, even if it is common ATC practice to solve conflicts between aircraft flying at the same altitude through lateral maneuvers (EUROCONTROL Experimental Centre, 2002*a*), also vertical resolution maneuvers (ie maneuvers involving one aircraft climbing or descending) are frequently used. Therefore the extension to three dimensional context will be crucial to make the contribution valuable for applicability in ATC. The extension to a three dimensional context goes beyond the scope of this paper, which is devoted to present the framework that we propose for Conflict Resolution. The approach to Conflict Resolution presented in this paper extends to three

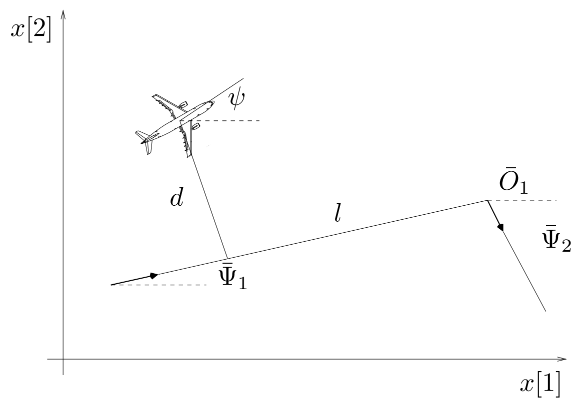

Fig. 1. The flight plan

dimensional flight without additional theoretical issues, though more work would be needed to identify the action of ATC in three dimensional context.
The paper is organised as follows. In the next section we recall modelling of the motion of commercial aircraft in level flight. In Section 3 we discuss the choice of resolution criteria. The MCMC procedure for optimisation is described in Section 4. In Section 5 we show effectiveness of the algorithm in the resolution of a simple conflict. Section 6 contains conclusions and future objectives.

2. MODELLING OF AIRCRAFT MOTION

We recall modelling of the motion of commercial aircraft in level flight from the point of view of ATC. The model is based on description of commercial aircraft contained in the Base of Aircraft Data (BADA) database (EUROCONTROL Experimental Centre, 2002*b*). The reader is referred to (Glover and Lygeros, 2003) for a detailed presentation of the model.
Commercial aircraft receive a *flight plan* (Fig. 1), which covers the entire flight, before take off. Aircraft are equipped with a Flight Management System (FMS) that assists the pilot in following the flight plan. A flight plan contains a sequence of waypoints $\{\bar{O}_i\}_{i=1}^{M}$ which in the case of level flight are expressed as coordinates in a 2D reference frame. The *reference path* is the sequence of straight lines joining each waypoint to the next. Correspondingly, for each segment of the reference path, the *reference heading* is defined as $\bar{\Psi}_i = \angle[\bar{O}_i - \bar{O}_{i-1}]$. Each time a waypoint is reached, the waypoint is eliminated from the flight plan and the aircraft heads to the next one according to the corresponding new reference heading. The first segment of the flight plan is therefore defined by the current reference heading $\bar{\Psi}_1$ and the first waypoint $\bar{O}_1$. In the current system the aircraft travel between waypoints with constant airspeed (i.e. speed relative to the air surrounding the aircraft) dictated by altitude dependent speed

profiles which can be found in BADA.

The motion of the aircraft from the point of view of ATC is determined by the aircraft dynamics plus the action of the FMS that keeps the aircraft in track with the flight plan.

In BADA the dynamics of aircraft is described by Point Mass Model differential equations in the form

$$\dot{\alpha} = f(\alpha, u, w)$$

where: α (position (x), heading (ψ), airspeed and mass) are the states, u (bank angle and engine thrust) are the control inputs and w (wind velocity) is a disturbance. In general the wind velocity can be modelled as a random field $W(x,t)$ with space time autocorrelation (i.e. $W(x_1,t_1)$, $W(x_2,t_1)$ $W(x_1,t_2)$, $W(x_2,t_2)$ are correlated random variables). This model of the aircraft dynamics has been implemented and used for the simulations in Section 5 - see (Glover and Lygeros, 2003) for more details.

The FMS controls the motion of the aircraft, i.e. it corrects errors with respect to the reference path and executes turns. In order to describe the action of FMS, assume that the aircraft is directed to waypoint $\bar{O}_1$ with reference heading $\bar{\Psi}_1$ and let us introduce l and d defined as

$$\begin{bmatrix} d \\ l \end{bmatrix} = \begin{bmatrix} -\sin(\bar{\Psi}_1) & \cos(\bar{\Psi}_1) \\ \cos(\bar{\Psi}_1) & \sin(\bar{\Psi}_1) \end{bmatrix} \begin{bmatrix} x[1] - \bar{O}_1[1] \\ x[2] - \bar{O}_1[2] \end{bmatrix} .$$

The moduli of l and d represents respectively the distance between the projection of the aircraft position on the nominal trajectory and the waypoint $\bar{O}_1$ and the distance of the aircraft position from the nominal trajectory. We can assume that the FMS receives as an input the error signals l, d and $\psi - \bar{\Psi}_1$. Several control strategies can be implemented in the FMS. A 3D FMS regulates only the cross track error d by controlling the heading through the bank angle. The airspeed is fixed for level flight and is defined from look up tables depending on the altitude. In the simulation example of Section 5 a 3D FMS is implemented. In the case of 3.5D FMS the waypoints are stamped with a time of arrival. The FMS regulates the error with the expected time of arrival and adjusts the engine thrust to eliminate this error. In the case of a 4D FMS the error with respect to a continuous 4D reference path (position + time) is considered.

The aircraft trajectory is then defined by the stochastic differential equation describing the control system aircraft + FMS. Let us remark that in general the space time correlation of the wind field makes it impossible to calculate exact quantities such as, for example, the probability of conflict in a multi aircraft system. The Conflict Detection methods in the literature that are not simulation based MC methods generally make the approximation that the effect of the wind field can be described as Brownian motion, see e.g. (Blom and Bakker, 2002; Hu *et al.*, 2003). In MC methods the space time correlation of the wind field in principle does not harm applicability.

3. AIR TRAFFIC CONTROL WITH OPTIMISATION OF AN EXPECTED VALUE CRITERION

The role of ATC is to monitor the traffic and detect possible dangerous encounters in the future. Indeed, flight plans are calculated before take off and cannot take into account the actual traffic configuration during the flight. The role of ATC is to intervene by sending suitable maneuver instructions in order to resolve predicted conflicts.

Let us consider a multiaircraft system. Without loss of generality, we assume that ATC monitors a future time horizon $[0, T]$ where $t = 0$ denotes the present. We model ATC instructions to each aircraft as a set of waypoints valid over the time horizon $[0, T]$. We denote this set of waypoints for all the aircraft as Ω. We assume that Ω determines the nominal paths in $[0, T]$. If no ATC intervention is required then $\Omega = \bar{\Omega}$ where $\bar{\Omega}$ denotes the set of the waypoints of the original flight plan in $[0, T]$. Let us introduce also a sample time ΔT (e.g. $\Delta T = 6$ sec which is tipically the time interval between two successive radar measurements) and denote X the vector of the time sequence of positions of all aircraft in $[0, T]$ at the sampled instants. Vector X is a random variable with probability density function $X \sim p_\Omega(x)$. The probability density $p_\Omega(x)$ is determined by the SDE describing the aircraft + FMS closed loop system and by the initial conditions (i.e. the positions and headings at time 0). The subscript Ω denotes that the distribution of X depends on the instructions received from ATC.

The objective of ATC is to select Ω in such a way that the aircraft trajectories will be conflict free and efficient. A conflict is defined as the loss of a minimum safe separation, say $\bar{c}$ (a typical value is $\bar{c} = 5$ nmi), between two aircraft. If we denote $x^i(t)$ and $x^j(t)$ the positions of two different aircraft then a conflict is defined as the event

$$\exists t \in [0,T] \ : \ \|x^i(t) - x^j(t)\| < \bar{c} .$$

In general, for any realization of the random variable X one can define a criterion $u(\Omega, X)$ that penalizes conflicting trajectories and measures the efficiency of conflict-free trajectories. Efficiency can be measured, for example, in terms of distances of the trajectories from desired paths. Once a criterion has been chosen, a sensible choice of Ω is then determined by the optimization of the expected value criterion

$$U(\Omega) = \int u(\Omega, x) p_\Omega(x) dx . \qquad (1)$$

A Monte Carlo Markov Chain (MCMC) procedure to find an approximate solution to this problem is described in the next section.

4. SIMULATION BASED OPTIMISATION

In this section we recall a simulation based procedure to optimise expected value criteria. This procedure has been proposed in Bayesian statistics literature. The original idea has been presented in (Mueller, 1999). In (Mueller *et al.*, 2002) results on asymptotic convergence are derived.
Consider the problem of optimising the expected value criterion (1) where Ω is the optimisation parameter and $p_\Omega(x)$ is a probability density function which depends on the optimisation parameter. The procedure presented below addresses the approximate optimisation of $U(\Omega)$ through extensive use of simulations. Apart from the possibility of evaluating $u(\Omega, X)$ no other particular assumptions are imposed on the optimisation criterion. Here we consider maximisation of $U(\Omega)$, i.e. $U(\Omega)$ is an expected utility. Obviously no modifications of the procedure are required in the case of minimisation of an expected cost.
The optimisation procedure relies on the definition of an augmented stochastic model in which also Ω is a random variable. The stochastic model is formed by Ω and J independent replicas of X. We denote $h(\omega, x_1, x_2, \ldots, x_J)$ the joint distribution of $(\Omega, X_1, X_2, X_3, \ldots, X_J)$. It is straightforward to see that if

$$h(\omega, x_1, x_2, \ldots, x_J) \propto \prod_j u(\omega, x_j) p_\omega(x_j) \qquad (2)$$

then

$$\Omega \sim h(\omega) \propto \left[\int u(\omega, x) p_\omega(x) dx \right]^J . \qquad (3)$$

This means that if we can extract from the augmented model $(\Omega, X_1, X_2, X_3, \ldots, X_J)$ then the extracted Ω's will cluster around the optimal points of $U(\Omega)$ for a sufficient high J. These extractions can be used to find an approximate solution to the original optimisation problem.
Extractions from the augmented stochastic model, with the desired joint probability density given by (2), can be obtained through a MCMC scheme. The algorithm is presented below. In the following algorithm $g(\omega|\bar{\omega})$ is an instrumental (or *proposal*) distribution which is freely chosen by the user. The only requirement is that $g(\omega|\bar{\omega})$ covers the support of $h(\omega)$.

MCMC algorithm (Metropolis-Hastings)

Initial state $(\bar{\omega}, \bar{x}_j\, j = 1, \ldots, J)$ and $\bar{u}_J = \prod_j u(\bar{\omega}, \bar{x}_j)$

1 Extract

$$\tilde{\Omega} \sim g(\omega|\bar{\omega})$$

2 Extract

$$\tilde{X}_j \sim p_{\tilde{\Omega}}(x) \quad j = 1 \ldots J$$

and calculate

$$\tilde{U}_J = \prod_j u(\tilde{\Omega}, \tilde{X}_j)$$

3 Extract the new state of the chain as

$$(\bar{\Omega}, \bar{U}_J) = \begin{cases} (\tilde{\Omega}, \tilde{U}_J) \text{with probability } \rho(\bar{\omega}, \bar{u}_J, \tilde{\Omega}, \tilde{U}_J) \\ (\bar{\omega}, \bar{u}_J) \text{with probability } 1 - \rho(\bar{\omega}, \bar{u}_J, \tilde{\Omega}, \tilde{U}_J) \end{cases}$$

where

$$\rho(\bar{\omega}, \bar{u}_J, \tilde{\omega}, \tilde{u}_J) = \min \left\{ 1, \frac{\tilde{u}_J}{\bar{u}_J} \frac{g(\bar{\omega}|\tilde{\omega})}{g(\tilde{\omega}|\bar{\omega})} \right\}$$

4 Repeat steps 1 through 3

The algorithm is a formulation of the Metropolis-Hasting algorithm for a desired distribution given by $h(\omega, x_1, x_2, \ldots, x_J)$ with proposal distribution given by

$$g(\omega|\bar{\omega}) \prod_j p_\omega(x_j) \, .$$

In fact, in this case, the acceptance probability for the Metropolis-Hastings algorithm is (Robert and Casella, 1999)

$$\frac{h(\tilde{\omega}, \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_J)}{h(\bar{\omega}, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_J)} \frac{g(\bar{\omega}|\tilde{\omega}) \prod_j p_\omega(\bar{x}_j)}{g(\tilde{\omega}|\bar{\omega}) \prod_j p_\omega(\tilde{x}_j)} \, .$$

and by inserting (2) in the above expression one obtains exactly $\rho(\bar{\omega}, \bar{u}_J, \tilde{\omega}, \tilde{u}_J)$. The distribution of $\bar{\Omega}$ then converges to a stationary distribution given by (3) (Robert and Casella, 1999).
Interestingly enough, in the case in which one wants to consider a discrete version of the above MCMC then only discretisation of U_J and Ω is needed and not of X.
In the following section, we illustrates effectiveness of this algorithm through a numerical simulation example. Open research issues are pointed out in the conclusions.

5. SIMULATION EXAMPLE

In this section we illustrate the conflict resolution algorithm in a two aircraft encounter. The model used in simulation is the one presented in (Glover and Lygeros, 2003) as anticipated in Section 2. The reader is referred to (Glover and Lygeros, 2003) also for a discussion on several implementation issues.
The FMS executes turns following a smooth circular path from one reference path to the next. In order to do so the aircraft will begin tracking the next flight segment a certain distance before it reaches the next waypoint.
The wind is modelled as the sum of two components, nominal and stochastic. The nominal wind represents forecast data available to air traffic controllers. Here the nominal wind is assumed to

be zero and all wind is considered to be stochastic. The stochastic wind component is modelled as a random field $W(x,t) : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$. In this example we assume that the wind field is stationary and jointly Gaussian with correlation function $E[W(x_1,t_1)W(x_2,t_2)^T] = R(\Delta x, \Delta t)$, with $\Delta x = \|x_1 - x_2\|$, $\Delta t = |t_1 - t_2|$. The behaviour of the FMS and the statistic of the wind have been tuned according of two sources of information: experimental statistics of aircraft deviations from their flight plans and a comparison of forecast wind and real wind measured from aircraft. The reader is referred to (Glover and Lygeros, 2003) for relevant references.

Let us now describe the two aircraft encounter. The initial configuration at time $t = 0$ is as follows, in the notation coordinates are expressed in meters. Aircraft 1 has position $x_0^1 = [-110000\ 0]$, heading $\psi_0^1 = 0$ and next waypoint $\bar{O}_1^1 = [110000\ 0]$ with reference heading $\bar{\Psi}_1^1 = 0°$. This aircraft will not change its flight plan. Aircraft 2 has position $x_0^2 = [0\ -110000]$, heading $\psi_0^2 = 90°$ and next waypoint $\bar{O}_1^2 = [0\ -100000]$ with reference heading $\bar{\Phi}_1^2 = 90°$. The second waypoint $0_2^2 = \Omega$ must be chosen in $[-100000\ 100000] \times [-100000\ 100000]$ to prevent conflict with Aircraft 1. The third and fourth waypoints of Aircraft 2 are then $\bar{O}_3^2 = [100000\ 0]$ and $\bar{O}_4^2 = [110000\ 0]$. Notice that the last waypoints of the two aircraft are the same. Both aircraft fly at constant airspeed $v = 150$ m/sec.

We assume that the requirement for conflict resolution is that Aircraft 2 arrives after Aircraft 1 with a time separation of 300 sec.

Let us denote T_1 and T_2 the times of arrival of the two aircraft at the last waypoint [110000 0]. The following resolution criterion has then been formulated

$$u(\Omega, X, \Delta T) = \begin{cases} \varepsilon & \text{if (conflict)} \vee (T_1 > T_2) \\ \varepsilon + e^{-a|\Delta T - 300|} & \text{otherwise} \end{cases}$$

where X contains the time sequence of positions of the two aircraft, $\Delta T = T_2 - T_1$, $a = 0.01$ and $\varepsilon = 0.00001$. The event conflict is defined as the loss of 5 nmi = 9260 m separation.

The MCMC algorithm of Section 4 has been employed in order to choose Ω that maximises the expected value criterion. The proposal distribution $g(\omega|\bar{\omega})$ has been chosen as a uniform distribution $g(\omega|\bar{\omega}) = const$ with $\omega \in [-100000\ 100000] \times [-100000\ 100000]$.

Three values of J have been considered: $J = 1, 5, 10$. Each time 4000 iterations of the MCMC algorithm have been performed. The scatter plots of accepted states are depicted in Fig. 2. For each session the first ten samples have been discarded in order to allow convergence of the Markov Chain to the stationary distribution ("*burn in*" period). The case $J = 20$ with 12000 iterations is also

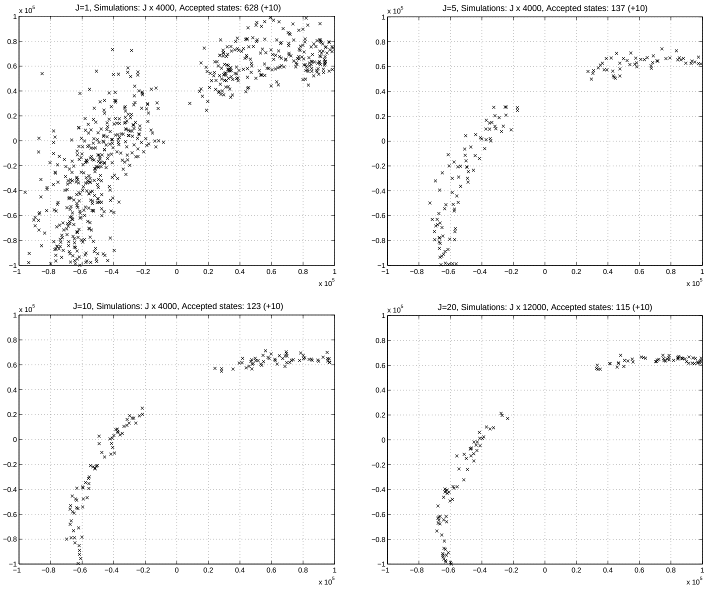

Fig. 2. Accepted states during MCMC simulation

displayed to illustrate the behaviour of the algorithm for a great number of simulations. From the figures it can be clearly seen that, for the resolution criterion that has been chosen, there exist two regions of nearly optimal solutions.

In the remainder of this section we illustrate two resolution maneuvers which belong respectively to the two different regions. For the two maneuvers, conflict probability (P_c) and expected delay between arrivals ($E[\Delta T]$) have been estimated with Monte Carlo simulation by using 10000 trajectory realizations. The first maneuver is determined by

$$\Omega = [-60000\ \ -40000]$$

for which we estimated $\hat{P}_c = 0$ and $\hat{E}[\Delta T] = 298$ sec. The second maneuver is instead determined by

$$\Omega = [38000\ 60000]$$

and we obtained the estimates $\hat{P}_c = 0.008$ and $\hat{E}[\Delta T] = 304$ sec. In Fig.s 3 and 4 trajectory realizations for the two maneuvers are displayed.

6. CONCLUSIONS

In this contribution we have presented a stochastic framework for air traffic conflict resolution from the point of view of ATC. Ongoing research is focused on possible improvements of the resolution algorithm. The degrees of freedom in the resolution procedure are the search distribution and the resolution criterion itself. These elements can be properly selected in order to increase the efficiency of the procedure in terms of computational time. In the simulation example of Section 5 we have used a uniform search distribution. This resulted in time spent to search over regions with a low criterion value and therefore in a great

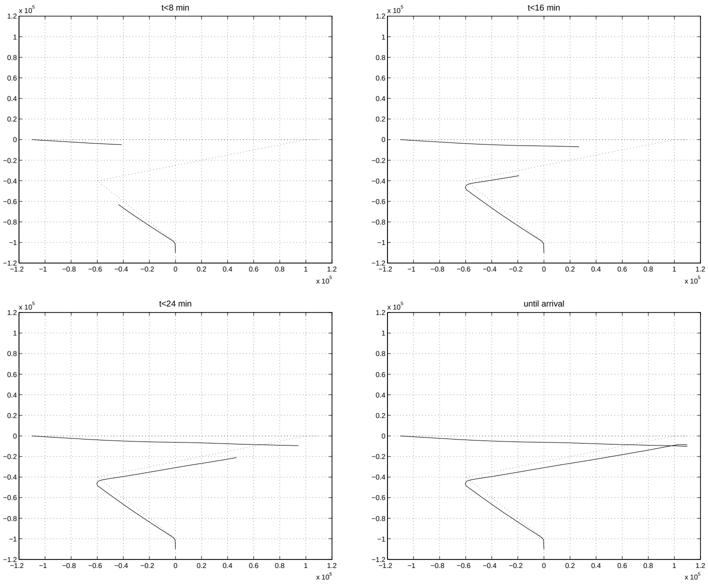

Fig. 3. First resolution maneuver: trajectory realizations (continuous) and reference path (dotted)

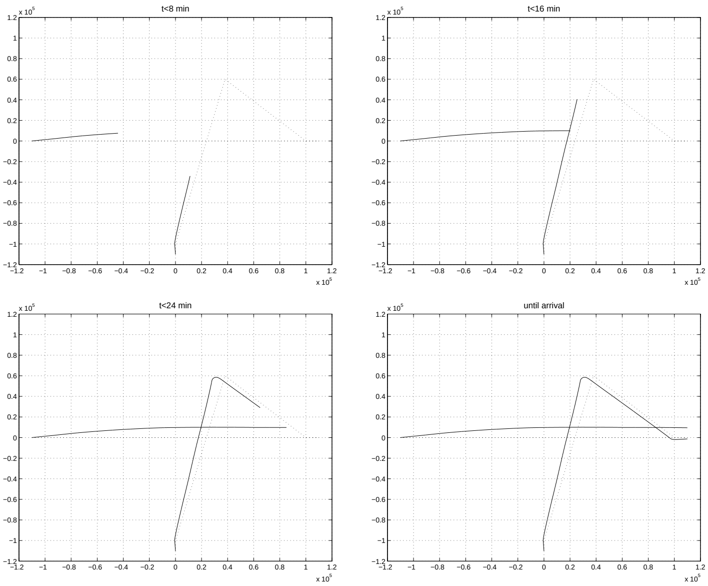

Fig. 4. Second resolution maneuver: trajectory realizations (continuous) and reference path (dotted)

number of rejected samples. In general, the search distribution could include clues on the expected / desired resolution maneuver in order to increase the efficiency of the search. Formulation of the the conflict resolution procedure in the Sequential Monte Carlo (Doucet *et al.*, 2001) framework is also under investigation.

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