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CUED/F-INFENG/TR.440

3 October 2002

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Abstract

In order to ensure robust feasibility and stability of model predictive control (MPC) schemes, it is often necessary to optimise over feedback policies rather than open-loop trajectories. All specific proposals to date have required the solution of nonlinear programs and/or the solution of a large number of optimisation problems. In this paper we introduce a new stage cost and show that the use of this cost allows one to formulate a robustly stable MPC problem that can be solved using a single linear program. Furthermore, this is a multi-parametric linear program, which implies that the receding horizon control (RHC) law is piecewise affine, and can be explicitly pre-computed, so that the linear program does not have to be solved on-line. Two numerical examples are presented; one of these is taken from the literature, so that a direct comparison of solutions and computational complexity with earlier proposals is possible.

Keywords: min-max problems, robust control, optimal control, predictive control, receding horizon control, parametric programming, piecewise linear control

1 Introduction

This paper is concerned with the practical real-time implementability of robustly stable model predictive control (MPC) when constraints are present on the inputs and the states. We assume that the plant model is known, except for unknown but bounded state disturbances, and that the states of the system are measurable.

We consider a discrete-time, linear, time-invariant plant

$$x_{k+1} = Ax_k + Bu_k + w_k \,, \tag{1}$$

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where $x_k \in \mathbb{R}^n$ is the system state, $u_k \in \mathbb{R}^m$ is the control input and $w_k \in \mathbf{W}$ is a persistent disturbance that only takes on values in the polytope $\mathbf{W} \subset \mathbb{R}^n$. It is assumed that the disturbance w_k can jump between arbitrary values within $\mathbf{W}$ and that no stochastic description for it is postulated. Therefore, a worst-case approach is taken in this paper. It is assumed that (A, B) is stabilisable and that polytopic[1] constraints on the state and input, that are either due to physical, safety or performance considerations, are also given:

$$x_k \in \mathbf{X}, \quad u_k \in \mathbf{U}, \quad \forall k \in \mathbb{N}.$$

We assume that $\mathbf{W}$ contains the origin and that $\mathbf{X} \subset \mathbb{R}^n$ and $\mathbf{U} \subset \mathbb{R}^m$ contain the origin in their interiors.

Since a persistent, unknown disturbance is present, it is impossible to drive the state to the origin. Instead, it is only possible to drive the system to a bounded target set $\mathbf{T}$ contained inside $\mathbf{X}$. The goal is to obtain a (time-invariant) nonlinear feedback control law $u = \kappa(x)$ such that the system is robustly steered to the target set, while also satisfying the state and input constraints, and minimising some worst case cost.

It is by now well-established that with polytopic disturbance bounds, a linear model and a convex cost, in order to solve such min-max problems it is sufficient to consider only the disturbance realisations that take on values at the vertices of $\mathbf{W}$ [31]. However, the number of extreme disturbance realisations typically grows exponentially with the length of the prediction horizon used in MPC. Since the optimisation in MPC is required to be performed on-line in real time, the practical feasibility of implementing robust MPC formulated along these lines is questionable.

In this paper we introduce a new type of *stage cost*

$$L(x, u) := \min_{y \in \mathbf{T}} \|Q(x - y)\|_p + \|R(u - Kx)\|_p, \tag{2}$$

where $Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{m \times m}$, $K \in \mathbb{R}^{m \times n}$ and $\mathbf{T} \subset \mathbb{R}^n$. We will show that, if $p = 1$ or $p = \infty$, the use of this stage cost allows the robustly stable feedback min-max MPC problem to be solved using a *single* linear program (LP). Furthermore, we will show that this LP is in fact a *multi-parametric* LP (mp-LP), that allows the receding horizon control (RHC) law to be pre-computed off-line along the lines developed by [6], and from which it follows that this law is in fact piecewise affine[2]. These facts make robust MPC/RHC, using the stage cost (2), a practical proposition.

Remark 1 *A similar stage cost to* (2) *was independently proposed in [23] and briefly discussed within the context of guaranteeing robust stability of a new type of MPC scheme. The stage cost proposed in [23] is* $L(x, u) := (1/2)\|x - \mathrm{Proj}_{\mathbf{T}}(x)\|_2^2 + (1/2)\|u - Kx\|_2^2$, *where* $\mathrm{Proj}_{\mathbf{T}}(x)$ *denotes the orthogonal projection of* x *onto* $\mathbf{T}$. *The difference between this stage cost and* (2) *is minor, but the formulation in* (2) *is perhaps more natural. More importantly, the MPC scheme proposed in [23] is fundamentally different from the feedback min-max MPC scheme considered here and [23] only briefly discusses the properties of their proposed stage cost. As such, this paper makes a contribution by analysing and discussing the properties of* (2) *in detail with regards to its use in feedback min-max MPC.*

The paper is organised as follows. In Section 2 we review recent developments in robust MPC/RHC, motivate the problem setup that was outlined above, and define it precisely. In Section 3 we review

[1] A polytope is defined to be a bounded polyhedron given by the intersection of a finite number of closed half-spaces. In other words, the sets $\mathbf{W}$, $\mathbf{X}$ and $\mathbf{U}$ are compact, convex sets that can be described by a finite number of linear inequalities.

[2] In this paper, MPC will be used to refer to the on-line computation of the solution to the feedback min-max optimal control problem P_N defined in the next section. RHC will be used to denote that the explicit expression for the solution to the feedback min-max problem P_N is pre-computed off-line.

known requirements for MPC/RHC to be robustly stable, and show how the stage cost (2) satisfies those requirements. We also point out some advantages of this cost, over a previously proposed cost. In Section 4 we show in detail how the problem can be solved as a single LP, and exploit its multi-parametric nature. Section 5 is devoted to numerical examples and Section 6 presents the conclusions.

2 Background and problem formulation

The problem of steering a constrained system subject to persistent disturbances to a target set, while also minimising some worst case cost, was considered as early as the 1960s and [9, 10, 13, 15, 33] contain some of the first, and perhaps also some of the most insightful, results. More recent attempts at the control of constrained systems are based on set invariance [8, 16].

In [9, 15] set-based solutions to the robust time-optimal problem were presented, but the unsolved problem was how to keep the state evolution inside the target set once it had been reached. The latter problem was solved in [30] by requiring that the target set be robustly controlled invariant. Once inside the target set the control input is determined by a pre-computed control law that ensures that the state trajectory never leaves the target set. Furthermore, [30] continues by decomposing the state space into simplices and computing an explicit affine expression for the control law in each simplex. All that is required on-line is to determine in which simplex the current state lies and the control input is then given by the pre-computed affine control law.

In general, solving a min-max problem subject to constraints and disturbances is computationally too demanding for practical implementation. However, various attempts have been made at presenting approximate solutions to this problem. Most of these solutions appear to have come from the field of robust MPC [26, 29]. Usually, MPC schemes obtain on-line the solution to a finite-horizon approximation of the infinite-horizon problem. For a given state only the initial segment of the optimal sequence is implemented; at the next time instant a new measurement is taken and a new finite-horizon min-max problem is solved.

Due to the various assumptions and approximations made, it is difficult to compare different min-max MPC schemes with one another. However, most robust MPC schemes can be classified into two categories [29]: (i) *open-loop* min-max MPC [1, 2, 11, 34], where a single control input sequence (or sequence of perturbations to a given stabilising control law [21, 25]) is used to minimise the worst case cost, and (ii) *feedback* min-max MPC [4, 20, 24, 31], where the worst case cost is minimised over a sequence of feedback control laws. In general, the open-loop formulation is too conservative and often severely under-estimates the set of feasible trajectories. As such, the feedback MPC formulation was proposed in [27] as an improvement over open-loop MPC.

In order to determine a suitable control law an optimal control problem P_N (defined below) with horizon N is solved. Let $\mathbf{w} := \{w_0, w_1, \ldots, w_{N-1}\}$ denote a disturbance sequence over the interval 0 to $N-1$. Effective control in the presence of the disturbance requires state feedback [29, §4.6], so that the decision variable in the optimal control problem (for a given initial state) is a control policy π defined by

$$\pi := \{u(0), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}, \tag{3}$$

where $u(0) \in \mathbf{U}$ and $\mu_k : \mathbf{X} \to \mathbf{U}$, $k \in \{1, \ldots, N-1\}$; $u(0)$ is a control *action* (since the current state is known) and each $\mu_k(\cdot)$ is a state feedback control *law*. Let $\phi(k; x, \pi, \mathbf{w})$ denote the solution to (1) at time k when the state is x at time 0, the control is determined by policy π ($u = \mu_k(x)$ at *event* (x, k), i.e. state x, time k) and the disturbance sequence is $\mathbf{w}$.

Given a target set (often also called terminal constraint) $\mathbf{T} \subset \mathbf{X}$, for each initial state $x \in \mathbf{X}$, let $\Pi_N(x)$ denote the set of *admissible* policies, i.e.

$$\Pi_N(x) := \{\pi \mid u(0) \in \mathbf{U},\ \mu_k(\phi(k; x, \pi, \mathbf{w})) \in \mathbf{U},\ \phi(k; x, \pi, \mathbf{w}) \in \mathbf{X},\ \phi(N; x, \pi, \mathbf{w}) \in \mathbf{T}, \\ \forall k \in \{1, \dots, N-1\}, \forall \mathbf{w} \in \mathbf{W}^N\Big\} \quad (4)$$

and let

$$X_N := \{x \in \mathbf{X} \,|\, \Pi_N(x) \neq \emptyset\} \quad (5)$$

denote the set of states in $\mathbf{X}$ that can be robustly steered (steered for all $\mathbf{w} \in \mathbf{W}^N$) to the target set $\mathbf{T}$ in N steps.

In order to define an optimal control problem, a cost $V_N(\cdot)$ that is dependent on the policy π and current state x, but not dependent on $\mathbf{w}$, is defined; the conventional choice is

$$V_N(x, \pi) := \max_{\mathbf{w} \in W^N} \left[\sum_{k=0}^{N-1} L(x_k, u_k) + F(x_N) \right], \quad (6)$$

where $x_k := \phi(k; x, \pi, \mathbf{w})$ if $k \in \{0, \dots, N\}$, $u_k := \mu_k(\phi(i; x, \pi, \mathbf{w}))$ if $k \in \{1, \dots, N-1\}$ and $u_0 := u(0)$.

The target set $\mathbf{T}$, stage cost $L(\cdot)$ and terminal cost $F(\cdot)$ have to satisfy certain conditions in order to ensure that the solution of the feedback min-max optimal control problem, when implemented in a receding horizon fashion, is robustly stabilising. These conditions will be set out in the following section.

The feedback min-max optimal control problem P_N can now be defined as

$$P_N(x): \qquad V_N^0(x) := \inf_{\pi} \{V_N(x, \pi) \,|\, \pi \in \Pi_N(x)\}. \quad (7)$$

Let $\pi_N^0(x)$ denote the solution to $P_N(x)$, i.e.

$$\pi_N^0(x) = \left\{u_0^0(x), \mu_1^0(\cdot; x), \dots, \mu_{N-1}^0(\cdot; x)\right\} := \arg\inf_{\pi} \{V_N(x, \pi) \,|\, \pi \in \Pi_N(x)\}, \quad (8)$$

where the notation $\mu_i^0(\cdot; x)$ shows the dependence of the optimal policy on the current state x.

It should be noted that the solution to problem P_N is frequently not unique — that is, there can be a whole set of minimisers, from which one must be selected. Thus the time-invariant, *set-valued* MPC/RHC law $\kappa_N : X_N \to 2^U$ (2^U is the set of all subsets of U) is defined by the first element of $\pi_N^0(x)$:

$$\kappa_N(x) := u_0^0(x),\ \forall x \in X_N. \quad (9)$$

Typically, but not always, $u_0^0(x)$ is a singleton.

The feedback min-max problem P_N defined in (7) is an infinite dimensional optimisation problem and impossible to solve directly. Methods for solving P_N using finite dimensional optimisation techniques have been proposed in [4, 17, 31] and this paper can be seen as an immediate extension of [31].

In [4, 17] it is proposed that a combined dynamic- and parametric programming approach be used to obtain an explicit expression for the RHC law. Provided the stage cost is piecewise affine (e.g. if a 1-norm or ∞-norm is used), a piecewise affine expression for $\kappa_N(\cdot)$ can be computed off-line. All that is required on-line is, given the current state x, to look up the control input from the explicit expression.

Stability is not proven for the stage and terminal costs proposed in [4] nor do the costs satisfy the stability conditions given in [28, §3.3] and [29, §4.4]. However, robust stability can be guaranteed if the stage cost

$$L(x,u) := \begin{cases} \|Qx\| + \|Ru\| & \text{if } (x,u) \in (\mathbf{X} \setminus \mathbf{T}) \times \mathbf{U} \\ 0 & \text{if } (x,u) \in \mathbf{T} \times \mathbf{U} \end{cases}, \tag{10}$$

proposed in [17, 28], is used. Though this choice of cost solves the stability problem, it should be noted that (10) is not continuous (on the boundary of $\mathbf{T}$).

The use of such a discontinuous stage cost is a major obstacle to implementation using standard solvers for linear, quadratic, semi-definite or other smooth, convex nonlinear programming problems. As such, (2) is proposed as an alternative that solves the problem of obtaining a continuous stage cost that can be implemented using smooth, convex programming solvers, while still guaranteeing robust stability of the closed-loop system.

Remark 2 *This paper investigates the use of* (2) *in solving* P_N *using the method proposed in [31]. Though not discussed here, it is possible to use* (2) *in solving* P_N *using the methods described in [4, 17].*

3 Requirements for robust stability

It is well-known that, for an MPC/RHC law that assumes a finite horizon, an arbitrary choice of terminal constraint, stage cost and terminal cost does not guarantee stability of the closed-loop system. In the absence of state disturbances, conventional MPC/RHC schemes employ a terminal cost $F(x) := \|Px\|$, that is a control Lyapunov function inside $\mathbf{T}$, in order to guarantee robust stability of the origin for the closed-loop system [28, 29]. However, if the interior of $\mathbf{W}$ is non-empty and the disturbance is persistent, then one can easily show that there does not exist a so-called *robust* control Lyapunov function [28, 29] in a neighbourhood of the origin. Since it is no longer possible to drive the system to the origin, but only to some set containing the origin, the conventional choice of stage and terminal cost cannot guarantee stability or convergence [28, §3.3.2] and a new type of stage and terminal cost is needed.

Before proceeding to set up conditions for robust stability some definitions, taken from [17], are in order. If $d(z, Z) := \inf_{y \in Z} \|z - y\|$ for any set $Z \subset \mathbb{R}^n$ and $\|\cdot\|$ denotes any norm, then the set $\mathbf{T}$ is *robustly stable* iff, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(x_0, \mathbf{T}) \leq \delta$ implies $d(x_i, \mathbf{T}) \leq \varepsilon$, for all $i \geq 0$ and all admissible disturbance sequences. The set $\mathbf{T}$ is *robustly asymptotically (finite-time) attractive* with domain of attraction X iff for all $x_0 \in X$, $d(x_i, \mathbf{T}) \to 0$ as $i \to \infty$ (there exists a time M such that $x_i \in \mathbf{T}$ for all $i \geq M$) for all admissible disturbance sequences. The set $\mathbf{T}$ is *robustly asymptotically (finite-time) stable* with domain of attraction X iff it is robustly stable and robustly asymptotically (finite-time) attractive with domain of attraction X.

Consider now the following assumptions, adapted from [17, 31, 32]:

A1: The terminal constraint set $\mathbf{T} \subset \mathbf{X}$ contains the origin in its interior. A linear, time-invariant control law $K : \mathbb{R}^n \to \mathbb{R}^m$ is given such that the terminal constraint set $\mathbf{T}$ is disturbance invariant [19] for the closed-loop system, i.e. $(A + BK)x + w \in \mathbf{T}$ for all $x \in \mathbf{T}$ and all $w \in \mathbf{W}$. In addition, $Kx \in \mathbf{U}$ for all $x \in \mathbf{T}$.
A2: The terminal cost $F(x) := 0$ for all $x \in \mathbb{R}^n$.
A3: The stage cost $L(x, u) := 0$ if $x \in \mathbf{T}$ and $u = Kx$.
A4a: $L(\cdot)$ is continuous over $\mathbf{X} \times \mathbf{U}$ and there exists a $c > 0$ such that $L(x, u) \geq c\,(d\,(x, \mathbf{T}))$ for all

$(x, u) \in (\mathbf{X} \setminus \mathbf{T}) \times \mathbf{U}$.
A4b: $L(\cdot)$ is continuous over $(\mathbf{X} \setminus \mathbf{T}) \times \mathbf{U}$ and there exists a $c > 0$ such that $L(x, u) \geq c\|x\|$ for all $(x, u) \in (\mathbf{X} \setminus \mathbf{T}) \times \mathbf{U}$.

A1, A2, A3, A4a and A4b satisfy the assumptions on the stage cost, terminal cost and terminal constraint given in [28, §3.3] and [29, §4.4]. Hence, one can follow a standard procedure of using the optimal value function as a candidate Lyapunov function [28, 29] and show that:

Theorem 1 *If A1, A2, A3 and A4a (and A4b) hold, then* $\mathbf{T}$ *is robustly asymptotically (finite-time) stable for the closed-loop system* $x_{k+1} = Ax_k + B\kappa_N(x_k) + w_k$ *with a region of attraction* X_N.

In [29, §4.6.3] and [31] it is argued that one need only consider the set of extreme disturbance realisations if the following assumption holds in addition to those given above:

A5: $L(\cdot)$ is convex over $\mathbf{X} \times \mathbf{U}$.

It is shown in [31] how, provided A1, A2, A3, A4a (and A4b) and A5 hold, one can associate a different control input sequence with each extreme disturbance realisation and, using a *causality constraint* that prevents the optimiser from assuming knowledge of future disturbances, one can compute a control input $u \in \kappa_N(x)$ on-line using standard finite-dimensional convex programming solvers. However, in [29, §4.6.3] and [31], an exact expression for the stage cost that allows one to implement the proposed method is not given; only general conditions on $L(\cdot)$ as in A3, A4a and A4b are given.

Our main concern here is to point out that the stage cost (2) satisfies assumptions A3 and A4a (but not A4b) if Q is non-singular. Using this stage cost in computing $\kappa_N(\cdot)$ thus assures that $\mathbf{T}$ is robustly asymptotically stable (but not necessarily finite-time stable) for the closed-loop system.

Furthermore, the stage cost (2) satisfies assumption A5 (for proof, see the Appendix). Its use thus allows the robustly stable MPC problem to be solved as a finite-dimensional problem, as will be shown in more detail in the next section.

Remark 3 *We once again point out that the stage cost* (10)*, that was proposed in [17, 28], is not continuous and hence not convex. As such, it does not satisfy assumption A5 and therefore cannot be used with the approach proposed in [31].*

Remark 4 *The second term in the stage cost* (2) *follows the idea of pre-stabilising predictions in MPC, that was introduced in [22] and developed further by those authors for use in robust MPC [21]. If* $Q := 0$ *and* $R := I$*, then the stage cost* (2) *is similar to the one used in [21]. However, it is important to note A4a and A4b are not satisfied if* Q *is singular (as is the case if* $Q := 0$*). As such, it is not yet clear how the assumptions in this paper need to be modified in order to use the method proposed in [21] for proving convergence.*

In order to justify this statement, an example for which the state of the closed-loop system does not converge to $\mathbf{T}$ *if* $Q := 0$ *and* $R := I$ *in* (2)*, follows. Let the system be given by* $x_{k+1} = x_k + u_k + w_k$ *and let* $\mathbf{X} := \{x \in \mathbb{R} \,|\, |x| \leq 2\}$, $\mathbf{U} := \{u \in \mathbb{R} \,|\, |u| \leq 1.5\}$, $\mathbf{T} := \{x \in \mathbb{R} \,|\, |x| \leq 0.5\}$, $\mathbf{W} := \{w \in \mathbb{R} \,|\, |w| \leq 0.1\}$, $K := -0.1$, $Q := 0$, $R := 1$, $p = \infty$ *and* $N := 2$*. If the initial state* $x_0 = 1$ *and the disturbance sequence is given by* $w_k := 0.1$ *for all* $k \in \mathbb{N}$*, then the state sequence satisfies* $x_{k+1} = x_k + \kappa_N(x_k) + w_k = 1$ *for all* $k \in \mathbb{N}$.

Remark 5 *It is interesting to observe that A3, A4a and A4b are satisfied if* R *is singular or* $R := 0$

in (2). *As such, the use of the second term is not necessary in guaranteeing robust stability*[3]*; the second term only affects the performance of the closed-loop system.*

Consider now the "dual-mode" control law

$$\Gamma(x) := \begin{cases} \kappa_N(x) & \text{if } x \in X_N \backslash \mathbf{T} \\ Kx & \text{if } x \in \mathbf{T} \end{cases} \tag{11}$$

where $\kappa_N(\cdot)$ is defined in (9). By definition, problem P_N satisfies assumption A2. If $\mathbf{T}$, K and $L(\cdot)$ are chosen such that assumptions A1, A3 and A4 are satisfied, then $\Gamma(\cdot)$ is a robustly stabilising control law, by Theorem 1.

For methods of computing a terminal constraint $\mathbf{T}$ that satisfies A1, see [16, 18, 19, 30, 31]. However, some further observations regarding K and $\mathbf{T}$ are in order.

The choice of K in (2) is arbitrary, but typically it is chosen such that $A + BK$ has all its eigenvalues inside the unit disk and the control law is optimal via some norm. Another factor that needs to be taken into consideration is how the choice of K affects the size of $\mathbf{T}$ that one can use. This problem is not yet fully understood, but some proposals have been put forward for computing a sequence of linear control laws and an associated sequence of disturbance invariant sets of increasing size [12].

The exact choice of disturbance invariant $\mathbf{T}$ is perhaps also arbitrary, but as discussed in detail in [23, 30, 31], a sensible choice for $\mathbf{T}$ is the *minimal* disturbance invariant set [18, 19]

$$\mathcal{O}^K_{\min} := \sum_{i=0}^{\infty} (A + BK)^i \mathbf{W} \tag{12}$$

for the system $x_{k+1} = (A + BK)x_k + w_k$ that is contained inside

$$X_K := \{x \in \mathbf{X} \,|\, Kx \in \mathbf{U}\}\,. \tag{13}$$

The problem, however, with setting $\mathbf{T} = \mathcal{O}^K_{\min}$, is that the region of attraction X_N can be quite small.

One way of enlarging X_N is to set $\mathbf{T}$ equal to the *maximal* disturbance invariant set $\mathcal{O}^K_\infty$ [18, 19] for the system $x_{k+1} = (A + BK)x_k + w_k$ that is contained inside X_K, i.e.

$$\mathcal{O}^K_\infty := \{x_0 \in X_K \,|\forall k \in \mathbb{N}, \forall w_k \in \mathbf{W} : x_{k+1} = (A + BK)x_k + w_k \in X_K\}\,. \tag{14}$$

This has the benefit that *if* the state enters $\mathbf{T}$ in finite time, then one can guarantee that the state of the system $x_{k+1} = Ax_k + B\Gamma(x_k) + w_k$ will robustly converge to the minimal disturbance invariant set $\mathcal{O}^K_{\min}$ (this is a consequence of the properties of state trajectories of $x_{k+1} = (A + BK)x_k + w_k$ that start inside $\mathcal{O}^K_\infty$ [18, §3]). Recall, however, that with the stage cost (2) one cannot guarantee that the state of the system will enter $\mathbf{T}$ in finite time.

A compromise that results in a smaller X_N, but still guarantees convergence to the minimal disturbance invariant set $\mathcal{O}^K_{\min}$, is to set $\mathbf{T}$ equal to any subset of the interior of $\mathcal{O}^K_\infty$ that is a disturbance invariant set for the system $x_{k+1} = (A + BK)x_k + w_k$. Since $\mathbf{T}$ is robustly asymptotically stable, this guarantees that the state of the system $x_{k+1} = Ax_k + B\kappa_N(x_k) + w_k$ will enter $\mathcal{O}^K_\infty$ in finite time. As soon as the

[3] In conventional MPC with a quadratic cost and no disturbance [26, 29], R is often chosen to be positive definite in order to guarantee *uniqueness* of the solution of the optimal control problem. In contrast, uniqueness of the solution is not guaranteed if R is positive definite and $p = 1$ or $p = \infty$ in (2).

state enters $\mathcal{O}_\infty^K$, one can switch to the control law $u = Kx$, thereby guaranteeing robust convergence of the state of the system $x_{k+1} = (A + BK)x_k + w_k$ to $\mathcal{O}_{\min}^K$. More precisely, if the "dual-mode" control law

$$\psi(x) := \begin{cases} \kappa_N(x) & \text{if } x \in X_N \backslash \mathcal{O}_\infty^K \\ Kx & \text{if } x \in \mathcal{O}_\infty^K \end{cases} \tag{15}$$

then the following result follows:

Theorem 2 *If A1, A2, A3 and A4a hold, the eigenvalues of $A + BK$ have magnitude less than* 1 *and $\mathbf{T} \subseteq \text{int}\mathcal{O}_\infty^K$, then the minimal disturbance invariant set $\mathcal{O}_{\min}^K$ is robustly asymptotically stable for the closed-loop system $x_{k+1} = Ax_k + B\psi(x_k) + w_k$ with a region of attraction X_N. If, in addition, $\mathcal{O}_{\min}^K \subseteq \text{int}\mathbf{T}$, then $\mathbf{T}$ is robustly finite-time stable for the closed-loop system $x_{k+1} = Ax_k + B\psi(x_k) + w_k$ with a region of attraction X_N.*

Proof: This is a consequence of the above discussion and the fact that $(A + BK)^k x \to 0$ as $k \to \infty$. Hence, for large k, the state trajectories of the system are determined almost entirely by the disturbance sequence and $\mathcal{O}_{\min}^K$ is a limit set for the trajectories of $x_{k+1} = (A + BK)x_k + w_k$ [18, §3]. See also [19] for details regarding the properties of the maximal and minimal disturbance invariant sets. □

Remark 6 *Note that Theorem 2 does not require that A4b hold.*

Remark 7 *The new stage cost* (2) *can be interpreted in a similar fashion to the stage cost $L(x, u) := \|Qx\|_p + \|Ru\|_p$ that is typically used in conventional MPC schemes without disturbances. In the new stage cost* (2)*, deviations of the state trajectory from $\mathbf{T}$ as well as deviations from some "ideal" control law $u = Kx$ are penalised instead of penalising deviations from the origin. The minimal disturbance invariant set $\mathcal{O}_{\min}^K$ can be thought of as the "origin" of the system. If $\mathbf{T} = \mathcal{O}_{\min}^K$, then one can interpret* (2) *as penalising deviations from the "origin". Similarly, if $\mathbf{T} \supset \mathcal{O}_{\min}^K$, then one can think of the terminal constraint as containing the "origin" (though the stage cost does not penalise deviations from the "origin" anymore).*

4 Solution via linear programming

Following the same approach as in [31], let $\mathbf{w}^\ell := \{w_0^\ell, \ldots, w_{N-1}^\ell\}$ denote an admissible disturbance sequence over the finite horizon $k = 0, \ldots, N-1$ and let $\ell \in \mathcal{L}$ index these realisations[4]. Also let $\mathbf{u}^\ell := \{u_0^\ell, \ldots, u_{N-1}^\ell\}$ denote a control sequence associated with the ℓ'th disturbance realisation and let $\mathbf{x}^\ell := \{x_0^\ell, \ldots, x_N^\ell\}$ represent the sequence of solutions of the model equation

$$x_{k+1}^\ell = Ax_k^\ell + Bu_k^\ell + w_k^\ell, \quad \ell \in \mathcal{L} \tag{16}$$

with $x_0^\ell = x$, where x denotes the current state.

4.1 Causality constraint

As a first step towards an implementable solution we follow [31] in replacing problem P_N by the following equivalent problem, in which the optimisation over feedback policies is achieved by optimising over control sequences, but with the *causality constraint* (17e) enforced:

[4]This is a slight abuse of notation, because the set of possible realisations is uncountable.

Problem 1 (Infinite Dimensional Feedback Min-Max) *Given the current state x, if*

$$\mathbf{u}_\infty := \left\{u_k^\ell \mid k = 0, \dots, N-1,\ \ell \in \mathcal{L}\right\}$$

find a solution to the problem

$$\mathbf{u}_\infty^0(x) := (\arg\min_{\mathbf{u}_\infty}) \max_{\ell \in \mathcal{L}} \left[F\left(x_N^\ell\right) + \sum_{k=0}^{N-1} L\left(x_k^\ell, u_k^\ell\right) \right], \tag{17a}$$

where the optimisation is subject to (16), $x_0^\ell = x$ *for all* $\ell \in \mathcal{L}$ *and*

$$x_k^\ell \in \mathbf{X}, \qquad k = 1, \dots, N-1, \qquad \forall \ell \in \mathcal{L} \tag{17b}$$

$$u_k^\ell \in \mathbf{U}, \qquad k = 0, \dots, N-1, \qquad \forall \ell \in \mathcal{L} \tag{17c}$$

$$x_N^\ell \in \mathbf{T}, \qquad \forall \ell \in \mathcal{L} \tag{17d}$$

$$x_k^{\ell_1} = x_k^{\ell_2} \Rightarrow u_k^{\ell_1} = u_k^{\ell_2} \qquad k = 0, \dots, N-1 \qquad \forall \ell_1, \ell_2 \in \mathcal{L}\,. \tag{17e}$$

As explained in more detail in [26, 29, 31], this problem is equivalent to the feedback min-max problem P_N due to two facts: (i) a *different* control input sequence is associated with each disturbance sequence, thereby overcoming the problem of open-loop MPC that associates a *single* control input sequence with all disturbance sequences; (ii) the *causality constraint* (17e) associates each predicted state at time j with a single control input, thereby reducing the degrees of freedom and making the control law independent of the control and disturbance sequence taken to reach that state.

Let the *finite* subset $\mathcal{L}_v \subset \mathcal{L}$ index those disturbance sequences $\mathbf{w}^\ell$ that take on values at the vertices of the polytope $\mathbf{W}^N$ and consider the following *finite* dimensional optimisation problem:

Problem 2 (Finite Dimensional Feedback Min-Max) *Given the current state x, if*

$$\mathbf{u} := \left\{\mathbf{u}^1, \mathbf{u}^2, \dots, \mathbf{u}^V\right\}$$

where V is the cardinality of $\mathcal{L}_v$, then find a solution to the problem

$$\mathbf{u}^0(x) := (\arg\min_{\mathbf{u}}) \max_{\ell \in \mathcal{L}_v} \left[F\left(x_N^\ell\right) + \sum_{k=0}^{N-1} L\left(x_k^\ell, u_k^\ell\right) \right], \tag{18a}$$

where the optimisation is subject to (16), $x_0^\ell = x$ *for all* $\ell \in \mathcal{L}_v$ *and*

$$x_k^\ell \in \mathbf{X}, \qquad k = 1, \dots, N-1, \qquad \forall \ell \in \mathcal{L}_v \tag{18b}$$

$$u_k^\ell \in \mathbf{U}, \qquad k = 0, \dots, N-1, \qquad \forall \ell \in \mathcal{L}_v \tag{18c}$$

$$x_N^\ell \in \mathbf{T}, \qquad \forall \ell \in \mathcal{L}_v \tag{18d}$$

$$x_k^{\ell_1} = x_k^{\ell_2} \Rightarrow u_k^{\ell_1} = u_k^{\ell_2} \qquad k = 0, \dots, N-1, \qquad \forall \ell_1, \ell_2 \in \mathcal{L}_v\,. \tag{18e}$$

At first sight, it might not be clear how the the causality constraint (18e) translates into linear constraints. However, note that for all $k \in \{0, \dots, N-2\}$, if $x_0^{\ell_1} = x_0^{\ell_2}$, $w_j^{\ell_1} = w_j^{\ell_2}$ and $u_j^{\ell_1} = u_j^{\ell_2}$ for all $j \in \{0, \dots, k\}$, then $x_j^{\ell_1} = x_j^{\ell_2}$ for all $j \in \{1, \dots, k+1\}$ and hence one needs to set $u_{k+1}^{\ell_1} = u_{k+1}^{\ell_2}$ in order to satisfy the causality constraint. Therefore, as discussed in [26, 31], the causality constraint (18e) can be replaced by associating the same control input with each node of the resulting extreme disturbance/state trajectory tree[5]. This observation reduces the original number of control inputs that need to be computed from Nv^N to $1 + v + \ldots + v^{N-1}$, where v is the number of vertices of $\mathbf{W}$. A similar observation holds for the number of constraints and slack variables that need to be considered.

For example, if $v = 2$ and $N = 2$, then there are $V = v^N = 4$ extreme disturbance sequences and if $\mathcal{L}_v$ has been defined such that $w_0^1 = w_0^2$ and $w_0^3 = w_0^4$, then (21e) can be substituted with $u_0^1 = u_0^2 = u_0^3 = u_0^4$, $u_1^1 = u_1^2$ and $u_1^3 = u_1^4$.

The question one can now ask is under what conditions the first element of $\mathbf{u}^0(x)$ is equal to the first element of $\mathbf{u}^0_\infty(x)$. As noted in [29, §4.6.3], if the system is linear, $\mathbf{X}$, $\mathbf{U}$, $\mathbf{W}$ and $\mathbf{T}$ are polytopes and $F(\cdot)$ and $L(\cdot)$ are convex functions, then using similar convexity arguments as in [31, Thm. 2], it can be shown that the first element of $\mathbf{u}^0(x)$ is equal to the first element of $\mathbf{u}^0_\infty(x)$ and hence also equal to $\kappa_N(x)$.

The next result follows:

Theorem 3 (Robustly Stable Feedback Min-Max MPC) *If the stage cost is given by* (2)*, Q is non-singular, $F(x) := 0$ and $\mathbf{T}$ satisfies A1, then $\kappa_N(x)$ is equal to the first element of $\mathbf{u}^0(x)$ and $\mathbf{T}$ is robustly asymptotically stable for the closed-loop system $x_{k+1} = Ax_k + B\kappa_N(x_k) + w_k$ with a region of attraction X_N.*

4.2 Setting up as an LP problem

In [31] it was suggested that the solution to Problem 2 should be computed on-line using standard convex, nonlinear programming solvers. We will now describe how this problem can be solved using linear programming if stage cost (2) is used. This will involve setting up a linear program that is equivalent to Problem 2.

Let the total cost $J(x, \mathbf{u}^\ell, \mathbf{w}^\ell)$ for the current state x and a sequence of control inputs $\mathbf{u}^\ell$ associated with a given disturbance realisation $\mathbf{w}^\ell$ be defined as[6]:

$$J(x, \mathbf{u}^\ell, \mathbf{w}^\ell) := \sum_{k=0}^{N-1} L\left(x_k^\ell, u_k^\ell\right) .$$

As in [31], the optimisation (18) can be written as

$$\min_{\mathbf{u}\in\mathbf{C}(x)} \max_{\ell\in\mathcal{L}_v} J(x, \mathbf{u}^\ell, \mathbf{w}^\ell), \tag{19}$$

which is equivalent to the convex program

$$\min_{\mathbf{u},\gamma} \left\{\gamma \,\middle|\, \mathbf{u} \in \mathbf{C}(x)\,,\, J(x, \mathbf{u}^\ell, \mathbf{w}^\ell) \le \gamma, \forall \ell \in \mathcal{L}_v \right\}, \tag{20}$$

where $\mathbf{C}(x)$ is a polytope implicitly defined by the constraints in (18).

If one uses the stage cost (2) with $p = 1$ then the value of $\min_{u\in\mathbf{U}} L(x, u)$ can be computed by solving the linear program

$$\min_{u\in\mathbf{U}} L(x, u) = \min_{u,y,\alpha,\beta,\gamma} \gamma$$

[5] Using standard causality arguments, it should be clear that this substitution results in an equivalent problem in the sense that the optimal cost and the first element of the optimal input sequence remains unchanged.

[6] Recall that $F(x) := 0$.

subject to

$$-\alpha \le Q(x-y) \le \alpha, \quad -\beta \le R(u-Kx) \le \beta, \quad u \in \mathbf{U}, \quad y \in \mathbf{T}, \quad \mathbf{1}'\alpha + \mathbf{1}'\beta \le \gamma\,,$$

where $\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}^m$ and the unit vector $\mathbf{1} := [1, 1, \ldots, 1]'$ has appropriate length.

The above procedure is fairly standard and has been used in converting standard and open-loop min-max MPC problems with 1-norm and ∞-norm costs to linear programs [1, 2, 3, 4, 11, 26, 34]. We now use it to set up a linear program equivalent to (20). Let

$$J(x, \mathbf{u}^\ell, \mathbf{w}^\ell) := \min_{\mathbf{y}^\ell} \sum_{k=0}^{N-1} \|Q(x_k^\ell - y_k^\ell)\|_1 + \|R(u_k^\ell - Kx_k^\ell)\|_1\,,$$

and $\mathbf{y}^\ell$, μ^ℓ, η^ℓ and $\mathbf{y}$, μ, η be defined similarly to $\mathbf{u}^\ell$ and $\mathbf{u}$. It now follows that (20) is equivalent to

$$\min_{\mathbf{u},\mathbf{y},\mu,\eta,\gamma} \gamma \tag{21a}$$

subject to

$$\begin{aligned}
&x_{k+1}^\ell = Ax_k^\ell + Bu_k^\ell + w_k^\ell, && x_0^\ell = x, \quad k = 0,\ldots,N-1, && \forall \ell \in \mathcal{L}_v && \text{(21b)}\\
& && x_k^\ell \in \mathbf{X}, \quad k = 1,\ldots,N-1, && \forall \ell \in \mathcal{L}_v && \text{(21c)}\\
& && x_N^\ell \in \mathbf{T}, && \forall \ell \in \mathcal{L}_v && \text{(21d)}\\
&x_k^{\ell_1} = x_k^{\ell_2} \Rightarrow u_k^{\ell_1} = u_k^{\ell_2} && k = 0,\ldots,N-1, && \forall \ell_1,\ell_2 \in \mathcal{L}_v && \text{(21e)}\\
&-\mu_k^\ell \le Q(x_k^\ell - y_k^\ell) \le \mu_k^\ell, && y_k^\ell \in \mathbf{T}, \quad k = 0,\ldots,N-1, && \forall \ell \in \mathcal{L}_v && \text{(21f)}\\
&-\eta_k^\ell \le R(u_k^\ell - Kx_k^\ell) \le \eta_k^\ell, && u_k^\ell \in \mathbf{U}, \quad k = 0,\ldots,N-1, && \forall \ell \in \mathcal{L}_v && \text{(21g)}\\
&\sum_{k=0}^{N-1} \mathbf{1}'\mu_k^\ell + \mathbf{1}'\eta_k^\ell \le \gamma, && && \forall \ell \in \mathcal{L}_v. && \text{(21h)}
\end{aligned}$$

Remark 8 *Note that it is also possible to convert the feedback min-max MPC problem to a linear program if* $p = \infty$ *is chosen in the stage cost* (2)*. This is achieved in a similar fashion as above by noting that if* $\min_{u\in\mathbf{U}} L(x,u) := \min_{y\in\mathbf{T}} \|Q(x-y)\|_\infty + \|R(u-Kx)\|_\infty$*, then*

$$\min_{u\in\mathbf{U}} L(x,u) = \min_{u,y,\alpha,\beta,\gamma} \gamma$$

subject to

$$-\mathbf{1}\alpha \le Q(x-y) \le \mathbf{1}\alpha, \quad -\mathbf{1}\beta \le R(u-Kx) \le \mathbf{1}\beta, \quad u \in \mathbf{U}, \quad y \in \mathbf{T}, \quad \alpha + \beta \le \gamma\,,$$

where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ *and the unit vector* $\mathbf{1}$ *has appropriate length.*

It is interesting to observe that the use of the ∞*-norm results in less variables and constraints than in the case of the* 1*-norm. The former choice of norm is therefore probably preferred if computational speed is an issue. However, the latter norm might be preferred if a control action is sought that is closer to having used the quadratic norm, as in conventional MPC.*

4.3 Explicit solution of the RHC law via parametric programming

The development in the previous section allows the on-line solution of the robust MPC problem, providing that the available computing resources and the required update interval are such that the LP can be solved quickly enough. If this is not possible, an alternative is to pre-compute the solution, to store this solution in a database, and to read out the appropriate part of the solution (which can be done relatively quickly) as required.

By substituting (21b) into the rest of the constraints it is possible to show, as in [3, 4, 6], that (21) can be written in the form

$$\min_{\theta} \left\{ c'\theta \,|F\theta \leq g + Gx \right\}, \tag{22}$$

where θ is the decision variable and consists of the non-redundant components of $(\mathbf{u}, \mathbf{y}, \mu, \eta, \gamma)$; the vectors c, g and matrices F, G are of appropriate dimensions and do not depend on x. The key observation here is that the constraints are dependent on the current state x in the affine manner shown. This means that the feedback min-max problem falls into the class of *multi-parametric* linear programs (mp-LPs) [14], where each component of x represents a parameter that will affect the solution. This class of problems can be solved *off-line* for all allowable values of x and results in a *piecewise affine* expression for the solution in terms of x [7, 14].

The polyhedron $X_F := \{x \in \mathbb{R}^n \,|\exists\theta : F\theta \leq g + Gx\}$ is the set of states for which a solution to (22) exists. Given a polytope of states $\mathcal{X} \subseteq X_F$ and using the algorithm described in [7], one can compute the explicit solution of the feedback min-max control law for all $x \in \mathcal{X}$. The resulting feedback min-max RHC law is then of the following piecewise affine form:

$$\kappa_N(x) = K_i x + h_i, \ \forall x \in \mathcal{CR}_i \,,$$

where each $K_i \in \mathbb{R}^{m\times n}$ and $h_i \in \mathbb{R}^m$ are associated with a so-called *critical region* $\mathcal{CR}_i$. The critical regions $\mathcal{CR}_i$ are polytopes with mutually disjoint interiors such that $\mathcal{X} = \bigcup_i \mathcal{CR}_i$. All that is required on-line is to determine in which critical region the current state lies and then compute the control action using only matrix multiplication and addition, as in [3, 4, 6, 30].

Remark 9 *The solution to the control law presented here is of the same piecewise affine structure as the one given in [4]. However, the derivation in [4] is based on dynamic programming and requires the solution of* $2N$ *multi-parametric* mixed-integer *linear programs (mp-MILPs) (by exploiting the convex, piecewise affine nature of the optimal cost, this has since been improved to solving N mp-LPs [5]). The scheme presented in this paper requires the solution of a single mp-LP instead, though this is perhaps of more significance for the on-line computation of the MPC solution than for off-line pre-computation of the RHC law.*

Finally, as mentioned earlier, robust stability is not guaranteed for the stage cost used in [4]. However, robust stability is guaranteed using the new stage cost (2) *proposed in this paper.*

5 Examples

The following two examples were implemented in Matlab 6.0 using the LP solver provided with Matlab Optimization Toolbox 2.1. The mp-LP solver was implemented using the algorithm described in [7].

5.1 Case $n = 1$

The first example is taken from [4, 31]. The system is given by

$$x_{k+1} = x_k + u_k + w_k,$$

with

$$\mathbf{X} := \{x \in \mathbb{R} \,|\, -1.2 \leq x \leq 2\}, \; \mathbf{T} := \{x \in \mathbb{R} \,|\, -1 \leq x \leq 1\}, \; \mathbf{W} := \{w \in \mathbb{R} \,|\, -1 \leq w \leq 1\}, \mathbf{U} := \mathbb{R}.$$

For an initial comparison, the same stage and terminal cost as in [4] were used, i.e.

$$L(x, u) := |Qx| + |Ru|, \;\; F(x) := 0, \;\; \forall x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

with $Q = 1$ and $R = 10$. With $N = 2$ and $\mathcal{X} := \{x \in \mathbb{R} \,|\, -1.2 \leq x \leq 2\}$, by solving a single mp-LP as described in this paper, the robust RHC law $\kappa_N(\cdot)$ was found to be

$$\kappa_N(x) = -x \text{ if } -1.2 \leq x \leq 2, \tag{23}$$

which is the same as [4, Eqn. 24].

The computation of $\kappa_N(\cdot)$ took 1.1 s on a Pentium III. This is a considerable improvement to the 55 s it took in [4] to solve 4 mp-MILPs on a similarly-specified computer (though it is reported in [5] that the same problem took 1.27s to solve using 2 mp-LPs).

When the new stage cost (2) was used, i.e.

$$L(x, u) := \min_{y \in \mathbf{T}} |Q(x - y)| + |R(u - Kx)|$$

with $K := -1$ (as proposed in [31, §F]), the robust control law $\kappa_N(\cdot)$ was computed in 1.2 s on a Pentium III and found to be the same as in (23).

5.2 Case $n = 2$

For the second example, the system is given by

$$x_{k+1} = \begin{bmatrix} 1 & 0.8 \\ 0 & 0.7 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k + w_k,$$

with

$$\mathbf{X} := \left\{x \in \mathbb{R}^2 \,|\, \|x\|_\infty \leq 10\right\}, \; \mathbf{W} := \left\{w \in \mathbb{R}^2 \,|\, \|w\|_\infty \leq 0.1\right\}, \mathbf{U} := \{u \in \mathbb{R} \,|\, -3 \leq u \leq 3\}.$$

Given $K := -[1\;1]$, the target set was chosen to be the maximal disturbance invariant set $\mathcal{O}_\infty^K$ contained inside $X_K := \{x \in \mathbf{X} \,|\, Kx \in \mathbf{U}\}$ for the closed-loop system $x_{k+1} = (A + BK)x_k + w_k$, i.e.

$$\mathbf{T} := \mathcal{O}_\infty^K = \left\{ x \in \mathbb{R}^2 \;\middle|\; -\begin{bmatrix} 3 \\ 2.8 \\ 2.75 \end{bmatrix} \leq \begin{bmatrix} 1 & 1 \\ 0 & 0.5 \\ 0.5 & 0.15 \end{bmatrix} x \leq \begin{bmatrix} 3 \\ 2.8 \\ 2.75 \end{bmatrix} \right\}.$$

The stage cost was chosen to be

$$L(x, u) := \min_{y \in \mathbf{T}} \|Q(x - y)\|_\infty + \|R(u - Kx)\|_\infty,$$

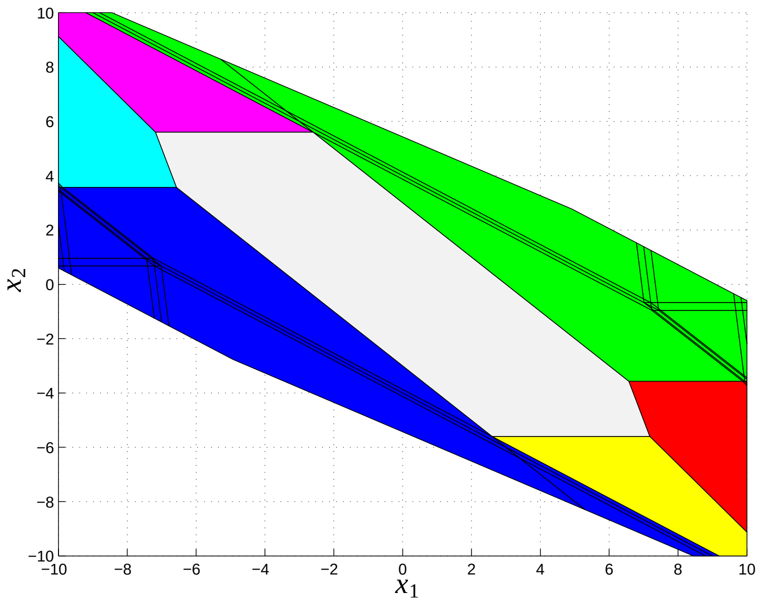

Figure 1: The critical regions that define the explicit expression for $\kappa_N(\cdot)$ for the second example

with $Q = I$ and $R = 0.1$. The control horizon was set to $N = 2$ and $\mathcal{X} := X_N$ was computed using the software developed in [16].

The LP that solves the feedback min-max MPC problem has 190 inequalities and 39 decision variables. The computation of the explicit expression for the RHC law $\kappa_N(\cdot)$ was completed in under 4 minutes[7] on an AMD Athlon processor. The critical regions that define the explicit solution of the associated mp-LP are shown in Figure 1 (in order to save space, the expressions for the associated critical regions are not listed). Though 71 separate critical regions were computed, it was found that only 7 distinct affine control laws were defined over different parts of X_N (critical regions with the same affine control law are plotted with the same shade in Figure 1). Post-processing might therefore reduce the number of regions that need to be stored on-line. The 7 affine control laws that, together with the critical regions shown in Figure 1, define $\kappa_N(\cdot)$ are:

$$\begin{aligned} \kappa_N^{1,2}(x) &= \begin{bmatrix} 0 & 0 \end{bmatrix} x \pm 3, \\ \kappa_N^{3,4}(x) &= \begin{bmatrix} 0 & -0.7 \end{bmatrix} x \pm 5.5, \\ \kappa_N^{5,6}(x) &= \begin{bmatrix} -1 & -1.5 \end{bmatrix} x \pm 2.8, \\ \kappa_N^{7}(x) &= \begin{bmatrix} -1 & -1 \end{bmatrix} x. \end{aligned}$$

Finally, Figure 2 shows part of the response of the closed-loop system $x_{k+1} = Ax_k + B\Gamma(x_k) + w_k$ to a random, persistent disturbance satisfying $w_k \in \mathbf{W}$ for all $k \in \mathbb{N}$, starting from initial state $x_0 = \begin{bmatrix} 10 & -10 \end{bmatrix}'$. As can be seen, the presence of the persistent disturbance prevents the state of the system from converging to the origin. Note that in this example the state enters $\mathbf{T}$ in finite time, despite the fact that only robust asymptotic convergence to the target set $\mathbf{T}$ was guaranteed. Recall also that $\Gamma(\cdot)$ and $\mathbf{T}$ have been defined such that if the state enters $\mathbf{T}$ in finite time, then the state of the system is

[7] Since by far most of the computational effort actually goes into removing redundant inequalities from the newly computed critical regions and the partitioning of the state space, it is expected that this time can be reduced by a few orders of magnitude using state-of-the-art LP solvers, rather than using Matlab's Optimization Toolbox.

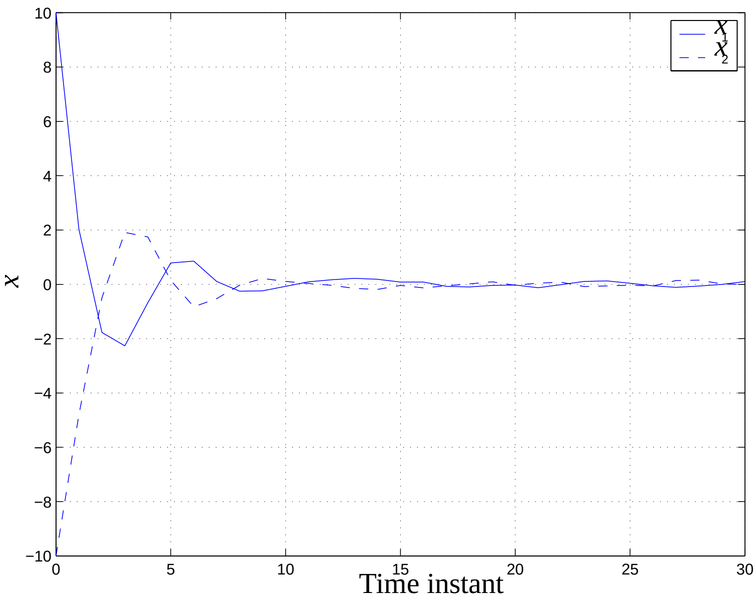

Figure 2: Closed-loop response of the second example to a random, persistent disturbance

guaranteed to remain inside **T** for all future admissible disturbance sequences. Furthermore, using the arguments presented in Section 3, it follows that if the state enters **T** in finite time, then the control law $\Gamma(\cdot)$ is such that the state of the closed-loop system will robustly converge to the minimal disturbance invariant set $\mathcal{O}^K_{\min}$. Finally, if the state enters $\mathcal{O}^K_{\min}$ in finite time, then the trajectory of the closed-loop system $x_{k+1} = Ax_k + B\Gamma(x_k) + w_k$ is guaranteed to remain inside $\mathcal{O}^K_{\min}$.

6 Conclusions

Robust MPC requires optimisation over feedback policies, rather than the more traditional optimisation over open-loop sequences, if excessive conservativeness, and hence infeasibility and/or instability, is to be avoided. But this is difficult to implement with reasonable computational effort, and hence its practicality has been questionable, particularly if on-line optimisation in real time is envisaged.

In this paper we have introduced a new stage cost, that allows one to compute the solution of the full robust MPC problem — that is, optimisation over feedback policies with guaranteed robust convergence to the target set in the face of persistent disturbances — using only one linear program. This is in contrast with previous proposals that have required the solution of nonlinear programs and/or the solution of a large number of optimisation problems.

A detailed comparison of the competing proposals is not straightforward, however, because the dimensions of the optimisations involved vary in complicated ways. It is therefore not yet possible to say conclusively which scheme will be more efficient for on-line implementation, or which one would be preferred for off-line pre-computation. The answers may well depend on problem-specific details.

Acknowledgements

The authors would like to thank David Mayne for valuable feedback and discussions during the preparation of this paper, especially with regards Section 3.

This research was supported by the Royal Academy of Engineering, UK.

Appendix: Proof that (2) is convex

We need to prove that

$$L(\lambda x_1 + [1-\lambda]x_2, \lambda u_1 + [1-\lambda]u_2) \leq \lambda L(x_1, u_1) + [1-\lambda]L(x_2, u_2) \tag{24}$$

for all $\lambda \in [0, 1]$. Note that the proof relies on the convexity of $\mathbf{T}$ and that it is easy to demonstrate that $L(.,.)$ is not convex if $\mathbf{T}$ is not convex.

Proof:

$$\begin{aligned} &L(\lambda x_1 + [1-\lambda]x_2, \lambda u_1 + [1-\lambda]u_2) = \\ &\qquad \min_{y \in \mathbf{T}} \|Q(\lambda x_1 + [1-\lambda]x_2 - y)\|_p + \|R(\lambda u_1 + [1-\lambda]u_2 - K\{\lambda x_1 + [1-\lambda]x_2\})\|_p \end{aligned} \tag{25}$$

Let

$$y_i = \arg\min_{y \in \mathbf{T}} \|Q(x_i - y)\|_p$$

and consider the first term on the right hand-side of (25), noting that $\lambda y_1 + [1-\lambda]y_2 \in \mathbf{T}$ since $\mathbf{T}$ is convex:

$$\begin{aligned} \min_{y \in \mathbf{T}} \|Q(\lambda x_1 + [1-\lambda]x_2 - y)\|_p &\leq \|Q(\lambda x_1 + [1-\lambda]x_2 - \lambda y_1 - [1-\lambda]y_2)\|_p \\ &\leq \lambda\|Q(x_1 - y_1)\|_p + [1-\lambda]\|Q(x_2 - y_2)\|_p \end{aligned} \tag{26}$$

(Minkowski's inequality).
Now consider the second term on the right hand-side of (25):

$$\begin{aligned} \|R(\lambda u_1 + [1-\lambda]u_2 - K\{\lambda x_1 + [1-\lambda]x_2\})\|_p &= \|\lambda R(u_1 - Kx_1) + [1-\lambda]R(u_2 - Kx_2)\|_p \\ &\leq \lambda\|R(u_1 - Kx_1)\|_p + [1-\lambda]\|R(u_2 - Kx_2)\|_p \end{aligned} \tag{27}$$

Adding together (26) and (27) proves (24).

References

[1] J.C. Allwright, On min-max model-based predictive control, in: D. Clarke (Ed.), *Advances in Model-Based Predictive Control*, Oxford University Press, 1994.

[2] J.C. Allwright and G.C. Papavasiliou, On linear programming and robust model-predictive control using impulse-responses, *Systems & Control Letters* **18** (1992) 159–164.

[3] A. Bemporad, F. Borrelli and M. Morari, The explicit solution of constrained LP-based receding horizon control, *Proceedings of the 39th IEEE Conference on Decision and Control*, Sydney, Australia (2000).

[4] A. Bemporad, F. Borrelli and M. Morari, Robust model predictive control: Piecewise linear explicit solution, *Proceedings of the European Control Conference*, Porto, Portugal (2001).

[5] A. Bemporad, F. Borrelli and M. Morari, Robust constrained optimal control and its state-feedback piecewise-affine solution, Preprint, 2001.

[6] A. Bemporad, M. Morari, V. Dua, E.N. Pistikopoulos, The explicit linear quadratic regulator for constrained systems, *Automatica* **38:1** (2002) 3–20.

[7] F. Borrelli, A. Bemporad and M. Morari, A geometric algorithm for multi-parametric linear programming, Submitted to *Journal of Optimization Theory and Applications* (2001). Technical report available from `http://www.dii.unisi.it/~bemporad/`.

[8] F. Blanchini, Set invariance in control, *Automatica* **35** (1999) 1747–1767.

[9] D.P. Bertsekas and I.B. Rhodes, On the minimax reachability of target sets and target tubes, *Automatica* **7** (1971) 233–247.

[10] D.P. Bertsekas and I.B. Rhodes, Sufficiently informative functions and the minimax feedback control of uncertain dynamic systems, *IEEE Transactions on Automatic Control* **AC-18** (1973) 117–124.

[11] P.J. Campo and M. Morari, Robust model predictive control, *Proceedings of the American Control Conference*, Green Valley AZ, USA (1987) 1021–1026.

[12] M. Cannon, V. Deshmukh and B. Kouvaritakis, Nonlinear model predictive control with polytopic invariant sets, *Proceedings of the 15th IFAC World Congress on Automatic Control*, Barcelona, Spain (2002).

[13] M.C. Delfour and S.K. Mitter, Reachability of perturbed systems and min sup problems, *SIAM Journal of Control* **47:4** (1969) 521–533.

[14] T. Gal, *Postoptimal Analyses, Parametric Programming, and Related Topics*, 2nd Edition, de Gruyter, 1995.

[15] J.D. Glover and F.C. Schweppe, Control of linear dynamic systems with set constrained disturbances, *IEEE Transactions on Automatic Control* **AC-16:5** (1971), 411–423.

[16] E.C. Kerrigan, *Robust Constraint Satisfaction: Invariant Sets and Predictive Control*, PhD Thesis, University of Cambridge, UK, 2000. Thesis and associated Matlab Invariant Set Toolbox available for download at `http://www-control.eng.cam.ac.uk/eck21/`.

[17] E.C. Kerrigan and D.Q. Mayne, Optimal control of constrained, piecewise affine systems with bounded disturbances, *Proceedings of the 41st IEEE Conference on Decision and Control*, Las Vegas NV, USA (2002).

[18] I. Kolmanovsky and E.G. Gilbert, Maximal output admissible sets for discrete-time systems with disturbance inputs, *Proceedings of the American Control Conference* (1995) 1995–2000.

[19] I. Kolmanovsky and E.G. Gilbert, Theory and computation of disturbance invariant sets for discrete-time linear systems, *Mathematical Problems in Engineering: Theory, Methods and Applications* **4** (1998) 317–367.

[20] M.V. Kothare, V. Balakrishnan and M. Morari, Robust constrained model predictive control using linear matrix inequalities, *Automatica* **32:10** (1996), 1361–79.

[21] B. Kouvaritakis, J.A. Rossiter and J. Schuurmans, Efficient robust predictive control, *IEEE Transactions on Automatic Control* **45:8** (2000) 1545–1549.

[22] B. Kouvaritakis and J.A. Rossiter, Stable generalized predictive control, *IEE Proceedings, Part D* **139** (1992) 349–362.

[23] W. Langson, I. Chryssochoos and D. Mayne, Robust model predictive control using tubes, Technical Report EEE/C&P/DQM/11/2001, Imperial College, London, UK (2001). Submitted to Automatica.

[24] J.H. Lee and Z. Yu, Worst-case formulations of model predictive control for systems with bounded parameters, *Automatica* **33:5** (1997) 763–781.

[25] Y.I. Lee and B. Kouvaritakis, A linear programming approach to constrained robust predictive control, *IEEE Transactions on Automatic Control* **45:9** (2000) 1765–1770.

[26] J.M. Maciejowski, *Predictive Control with Constraints*, Prentice Hall, UK, 2001.

[27] D.Q. Mayne, Nonlinear model predictive control: An assessment. In: J.C. Kantor, C.E. Garcia and B. Carnahan (eds.) *Fifth International Conference on Chemical Process Control (CPC V)*, CACHE, AIChE, Tahoe City CA, USA, 1996.

[28] D.Q. Mayne, Control of constrained dynamic systems, *European Journal of Control* **7** (2001) 87–99.

[29] D.Q. Mayne, J.B. Rawlings, C.V. Rao and P.O.M. Scokaert, Constrained model predictive control: Stability and optimality, *Automatica* **36** (2000) 789–814.

[30] D.Q. Mayne and W.R. Schroeder, Robust time-optimal control of constrained linear systems, *Automatica*, **33:12** (1997) 2103–2118.

[31] P.O.M. Scokaert and D.Q. Mayne, Min-max feedback model predictive control, *IEEE Transactions on Automatic Control* **43:8** (1998) 1136–1142.

[32] P.O.M. Scokaert, D.Q. Mayne and J.B. Rawlings, Suboptimal model predictive control (Feasibility implies stability), *IEEE Transactions on Automatic Control* **44:3** (1999) 648–654.

[33] H.S. Witsenhausen, A minimax control problem for sampled linear systems, *IEEE Transactions on Automatic Control*, **13:1** (1968) 5–21.

[34] Z.Q. Zheng, *Robust Control of Systems Subject to Constraints*, PhD Thesis, California Institute of Technology, Pasadena CA, USA, 1995.