Offset-free Receding Horizon Control of Constrained Linear Systems subject to Time-varying Setpoints and Persistent Unmeasured Disturbances

Gabriele Pannocchia and Eric C. Kerrigan

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Gabriele Pannocchia
Department of Chemical Engineering
University of Pisa
Pisa 56126, Italy
Email: pannocchia@ing.unipi.it

Eric C. Kerrigan
Department of Engineering
University of Cambridge
Cambridge CB2 1PZ, UK
Email: erickerrigan@ieee.org

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Abstract

This paper addresses the design of a nonlinear time-invariant, dynamic state feedback receding horizon controller, which guarantees constraint satisfaction, robust stability and offset-free control of constrained, linear time-invariant systems in the presence of time-varying setpoints and unmeasured, persistent, additive disturbances. First, this objective is obtained by designing a dynamic, linear time-invariant, offset-free controller and an appropriate domain of attraction for this linear controller is defined. The linear (unconstrained) controller is then modified by adding a perturbation term, which is computed by a robust receding horizon controller. It is shown that the domain of attraction of the receding horizon controller contains that of the linear controller and an efficient implementation of the receding horizon controller is proposed. Proofs of robust constraint satisfaction, robust stability and offset-free control are given. The effectiveness of the proposed controller is illustrated on an example of a continuous stirred tank reactor.

Keywords: Offset-free control, receding horizon control, set invariance, dynamic state feedback control, nonlinear control, constrained systems.

1 Introduction

The control of systems in the presence of constraints is an important task in many application fields because constraints "always" arise from physical limitations and quality or safety reasons. Moreover, in practical applications, disturbances are usually present and often they are not measurable or predictable. For example, in the chemical industries disturbances arise from interactions between different plant units, from changes in the raw materials and in the operating conditions (such as ambient temperature, humidity, etc.).

It is well-known that if an unmeasured, persistent disturbance is stationary (e.g. if it is white), then offset-free control is not possible, whereas if a disturbance is non-stationary (e.g. if it is integrating or periodic), offset-free control can be an achievable goal. In many practical applications, especially in the process industries, disturbances are often non-stationary. In particular, they are often integrating and reach, after some transient, a constant value. Hence, one basic objective of an effective control algorithm is that it

*Technical Report CUED/F-INFENG/TR.468, Department of Engineering, University of Cambridge, UK.

guarantees offset-free control whenever this is possible. Moreover, an effective control algorithm is also applicable to cases in which the setpoints of the controlled variables are allowed to be changed.

In the field of classical, linear (unconstrained) feedback control, the problem of offset-free control can be considered mature [1, 2, 3, 4]. However, it is clear that linear controllers have a limited range of application because of the presence of constraints.

The design of control algorithms able to stabilize linear plants subject to unknown, but bounded disturbances in the presence of input and state constraints has been the subject of several works over the last half century; a number of excellent surveys are available [5, 6, 7] which discuss how the important goal of guaranteeing closed-loop stability and constraint satisfaction can be obtained. Existing control algorithms, which address the problem of robust control of constrained systems, are usually based on ideas from set invariance [8, 9], reference governors [10, 11, 12, 13] or receding horizon control [14, 15, 16, 17, 18, 19, 20, 21, 22]. It is interesting to note that, despite the practical importance of guaranteeing offset-free control in the presence of integrating disturbances, none of the existing receding horizon control algorithms with robust stability *and* robust constraint satisfaction guarantees are able to guarantee offset-free control.

Compared to linear (unconstrained) control, the rigorous study of designing controllers that guarantee offset-free control has received very little attention in the constrained control community, until relatively recently [23, 24, 25, 26, 27]. Though the receding horizon control algorithms presented in [23, 24, 25, 26, 27] guarantee offset-free control and robust constraint satisfaction around a neighborhood of the steady-state, they do not guarantee robust constraint satisfaction for all initial states over which the receding horizon controller is defined. Furthermore, with the exception of [11, 16], none of the existing receding horizon control algorithms that guarantee robust stability and robust constraint satisfaction, address the problem of tracking arbitrary setpoints (rather than those generated by a finite-dimensional exogenous system).

In this paper, a novel receding horizon control algorithm for controlling constrained linear systems subject to unmeasured, bounded disturbances is presented. The proposed algorithm is guaranteed to remove steady-state offset in the controlled variables whenever the disturbances reach an (unknown) constant value, and the algorithm is guaranteed to satisfy input and state constraints. None of the existing receding horizon control algorithms are able to provide similar guarantees. Moreover, in the algorithm proposed here, the setpoints of the controlled variables are allowed to vary arbitrarily with time, provided they also converge to some limit point.

This paper is organized as follows. In Section 2 the problem definition is given and in Section 3 the design of a linear offset-free controller is presented along with detailed discussions about its closed-loop properties and its domain of attraction. As is well-known, the design of an effective offset-free control algorithm requires one to use an auxiliary system for estimating the non-stationary disturbances. This is the approach adopted in this and the subsequent section.

In Section 4 a nonlinear controller is designed, using ideas from model predictive control, in order to enlarge the domain of attraction. The effect of the inclusion of the auxiliary system in the definition of the receding horizon controller is carefully analyzed. The added complexity calls for the derivation of results that are analogous to existing results in the literature on robust receding horizon control. Because of the many new assumptions made in this paper, we believe that the details of the proofs of the main results are important. In the interest of rigor, nearly all of the details of the proofs have therefore been included in the Appendix.

The main characteristics of the receding horizon control algorithm proposed in this paper are illustrated in Section 5 through an example of a continuous stirred tank reactor. Finally, the main contributions of this paper are summarized in Section 6.

NOTATION: $\text{abs}(M)$ is the matrix of the absolute values of the corresponding components of the matrix

M and $|M|$ is the determinant of M. $L \otimes M$ is the Kronecker product of L and M. Given column vectors a and b, the column vector $(a,b) := [a^T \ b^T]^T$ and $a \leq b$ denotes component-wise inequality between a and b. Given a set Ω, $\mathscr{M}_\Omega$ is the set of infinite sequences $\omega(\cdot) := \{\omega(0), \omega(1), \ldots\}$ that take on values in Ω, i.e. $\mathscr{M}_\Omega := \{\omega(\cdot) \mid \omega(k) \in \Omega, \forall k \in \mathbb{N}\}$. Where it is clear from the context, $\omega(k)$ will denote the *actual* value of the infinite sequence $\omega(\cdot)$ at time k, while ω_k will be used to denote the *prediction* of $\omega(\tau+k)$ at a time instant k steps into the future if $\omega(\tau) = \omega_0 = \omega$ is the value of the variable at current time τ. Given a positive integer N, I_N is the identity matrix with N rows and N columns, $\mathbf{1}_N := [1 \ 1 \cdots \ 1]^T$ and $\tilde{\mathbf{1}}_N = [1 \ 0 \ 0 \ \cdots \ 0]^T$ are column vectors of length N. Given a positive scalar r, $\mathbb{B}_r$ denotes the norm-ball of radius r, i.e. $\mathbb{B}_r := \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$. If the set $\Omega \subset X \times Y$, then the projection of Ω onto X is defined as $\text{Proj}_X(\Omega) := \{x \in X \mid \exists y \in Y \text{ such that } (x,y) \in \Omega\}$. Given a set Ω, the Cartesian product $\Omega^N := \underbrace{\Omega \times \Omega \times \cdots \times \Omega}_{N \text{ times}}$.

2 Problem Description and Preliminary Results

In this paper we consider a discrete-time linear time-invariant plant:

$$x^+ = Ax + Bu + Ed\,, \tag{1a}$$

$$z = C_z x\,, \tag{1b}$$

in which $x \in \mathbb{R}^n$ is the plant state, x^+ is the plant successor state, $u \in \mathbb{R}^m$ is the control input (manipulated variable), $d \in \mathbb{R}^r$ is a persistent, unmeasured disturbance and $z \in \mathbb{R}^p$ is the controlled variable, i.e. the variable to be controlled to a given (time-varying) setpoint s. Affine inequality constraints are given on the state and input, i.e.

$$x \in \mathscr{X} \subset X\,, \quad u \in \mathscr{U} \subset U\,, \tag{2}$$

where $X := \mathbb{R}^n$ is the state space, $U := \mathbb{R}^m$ is the input space, $\mathscr{X}$ is a polyhedron (i.e. a closed and convex set that can be described by a finite number of affine inequality constraints) and $\mathscr{U}$ is a polytope (i.e. a bounded polyhedron); the interior of $\mathscr{X} \times \mathscr{U}$ contains the origin[1].

Assumption 1 (General). A measurement of the plant state is available at each sample instant, (A,B) is stabilizable, (A,C_z) is detectable and

$$\text{rank} \begin{bmatrix} I-A & -B \\ C_z & 0 \end{bmatrix} = n + p\,. \tag{3}$$

Remark 1. Notice that the last condition implies that the dimension of the controlled variable cannot exceed the dimension of either the state or the input, i.e. $p \leq \min\{n,m\}$. This condition will be used to guarantee the existence of an offset-free steady-state.

A *dynamic* nonlinear time-invariant state feedback controller is to be designed and is to assume the following structure:

$$\sigma^+ = \alpha(x, \sigma, s)\,, \tag{4a}$$

$$u = \gamma(x, \sigma, s)\,, \tag{4b}$$

where $\sigma \in \mathbb{R}^l$ is the controller state, σ^+ is the controller successor state, $\alpha : \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^p \to \mathbb{R}^l$ is the controller state dynamics map and $\gamma : \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^p \to \mathbb{R}^m$ is the controller output map.

Remark 2. In this paper, both $\alpha(\cdot)$ and $\gamma(\cdot)$ will be nonlinear.

[1] Note that the results in this paper can easily be extended to the case with mixed constraints on the state and input.

The plant dynamics (1a), together with the controller (4), forms a closed-loop system

$$\xi^+ = f(\xi, s, d), \tag{5}$$

where the closed-loop system state is

$$\xi := \begin{bmatrix} x \\ \sigma \end{bmatrix} \tag{6}$$

and the closed-loop dynamics are given by

$$f(\xi, s, d) := \begin{bmatrix} Ax + B\gamma(x, \sigma, s) \\ \alpha(x, \sigma, s) \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} d. \tag{7}$$

Let $s(\cdot)$ and $d(\cdot)$ denote an infinite setpoint sequence and an infinite disturbance sequence, respectively. Also, let $\phi(k, \xi, s(\cdot), d(\cdot))$ be the solution to (5) at time k when the augmented state is ξ at time $k = 0$, the controller is defined by (4), the setpoint sequence is $s(\cdot)$ and the disturbance sequence is $d(\cdot)$, i.e.

$$\phi(k, \xi, s(\cdot), d(\cdot)) := \underbrace{f(f(\ldots(f}_{k \text{ times}}(\xi, s(0), d(0)), s(1), d(1))\ldots), s(k-1), d(k-1)). \tag{8}$$

By definition, $\phi(0, \xi, s(\cdot), d(\cdot)) = \xi$.

We also define the following:

$$\xi(k) := \phi(k, \xi, s(\cdot), d(\cdot)), \tag{9a}$$

$$x(k) := \begin{bmatrix} I_n & 0 \end{bmatrix} \phi(k, \xi, s(\cdot), d(\cdot)), \tag{9b}$$

$$\sigma(k) := \begin{bmatrix} 0 & I_l \end{bmatrix} \phi(k, \xi, s(\cdot), d(\cdot)), \tag{9c}$$

$$u(k) := \gamma(\phi(k, \xi, s(\cdot), d(\cdot)), s(k)), \tag{9d}$$

$$z(k) := \begin{bmatrix} C_z & 0 \end{bmatrix} \phi(k, \xi, s(\cdot), d(\cdot)). \tag{9e}$$

Given a controller defined by (4), an infinite setpoint sequence $s(\cdot)$ and an infinite disturbance sequence $d(\cdot)$, the resulting closed-loop trajectories of the individual variables are then denoted by $\xi(\cdot)$, $x(\cdot)$, $\sigma(\cdot)$, $u(\cdot)$ and $z(\cdot)$.

Assumption 2 (Setpoint). At each time instant, the current setpoint is known but future setpoint values are unknown. The setpoint sequence $s(\cdot)$ takes on values in a polytope $\mathscr{S} \subset \mathbb{R}^p$ containing the origin and asymptotically reaches a steady-state value, i.e. $s(k) \in \mathscr{S}$ for all $k \in \mathbb{N}$ and there exists an $\bar{s} \in \mathscr{S}$ such that $\lim_{k\to\infty} s(k) = \bar{s}$.

In general, since the disturbance is persistent and unknown it is impossible to drive the controlled variable to the asymptotic setpoint $\bar{s}$. However, we consider the following restriction on the disturbance:

Assumption 3 (Disturbance). At each time instant, current and future disturbances are unknown. The disturbance sequence $d(\cdot)$ takes on values in a polytope $\mathscr{D} \subset \mathbb{R}^r$ containing the origin and asymptotically reaches an unknown steady-state value, i.e. $d(k) \in \mathscr{D}$ for all $k \in \mathbb{N}$ and there exists a $\bar{d} \in \mathscr{D}$ such that $\lim_{k\to\infty} d(k) = \bar{d}$.

Under the above assumptions we present a novel method for designing a dynamic, nonlinear, time-invariant state feedback controller (4) that, for any allowable disturbance and setpoint sequence (i.e. any infinite disturbance and setpoint sequence that satisfy Assumptions 2 and 3), accomplishes the goal of driving the controlled variable to any given allowable asymptotic setpoint, while respecting the state and input constraints, i.e.

$$\lim_{k\to\infty} z(k) = \bar{s} \tag{10a}$$

and

$$x(k) \in \mathscr{X}, \quad u(k) \in \mathscr{U} \tag{10b}$$

for all $k \in \mathbb{N}$.

2.1 Eliminating the Universal Quantifier from a Set of Affine Inequality Constraints

We present here the following well-known result [15, 8, 19, 28, 20], which will be useful later on:

Proposition 1. *Let the polyhedron $\mathscr{P}$ be given by*

$$\mathscr{P} := \left\{ v \in \mathbb{R}^t \mid Fv \leq g + Hw \text{ for all } w \in \mathscr{W} \right\}, \tag{11}$$

where $F \in \mathbb{R}^{q\times t}$ and $H \in \mathbb{R}^{q\times s}$ are matrices, $g \in \mathbb{R}^q$ is a vector and $\mathscr{W}$ is a compact (i.e. closed and bounded) subset of $\mathbb{R}^s$, then

$$\mathscr{P} = \left\{ v \in \mathbb{R}^t \;\middle|\; Fv \leq g + \min_{w\in\mathscr{W}} Hw \right\}, \tag{12}$$

where the minimization is performed row-wise, i.e. if H_i denotes the i'th row of H, then $\min_{w\in\mathscr{W}} Hw := [\min_{w\in\mathscr{W}} H_1 w \ \cdots \ \min_{w\in\mathscr{W}} H_q w]^T$. *Furthermore, if*

$$\mathscr{W} := \{ w \in \mathbb{R}^s \mid \|w\|_\infty \leq \eta \}, \tag{13}$$

then

$$\mathscr{P} = \left\{ v \in \mathbb{R}^t \mid Fv \leq g - \eta \operatorname{abs}(H)\mathbf{1}_s \right\}. \tag{14}$$

2.2 Robust Stability of Discrete-time Systems with Perturbations

Since we are interested in robust stability results, we review the following definitions and results for a generic nonlinear, perturbed discrete-time system [29]:

$$\zeta^+ = F(\zeta) + w, \tag{15}$$

in which $F : \mathbb{R}^\ell \to \mathbb{R}^\ell$ and $F(0) = 0$. Let $\Phi(k, \zeta, w(\cdot))$ denote the solution to (15) at time k, given the initial state ζ and an infinite perturbation sequence $w(\cdot)$.

Definition 1. The origin is a robustly asymptotically stable fixed point of (15) if the following two conditions are satisfied:

1. (*Robust stability*) For all $\varepsilon > 0$, there exist a $\delta > 0$ and a $\mu > 0$ such that if the initial condition $\zeta \in \mathbb{B}_\delta$ and the perturbation sequence $w(\cdot)$ satisfies $w(k) \in \mathbb{B}_\mu$ for all $k \in \mathbb{N}$, then $\Phi(k, \zeta, w(\cdot)) \in \mathbb{B}_\varepsilon$ for all $k \in \mathbb{N}$;

2. (*Robust convergence*) For all initial conditions $\zeta \in \mathbb{B}_\delta$ and perturbation sequences $w(\cdot)$ satisfying $w(k) \in \mathbb{B}_\mu$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} w(k) = 0$, the solution of (15) satisfies $\lim_{k\to\infty} \Phi(k, \zeta, w(\cdot)) = 0$.

Definition 2. If $\bar{w} := \lim_{k\to\infty} w(k)$ is the limit point of the perturbation sequence $w(\cdot)$, then a vector $\bar{\zeta}$ satisfying $\bar{\zeta} = F(\bar{\zeta}) + \bar{w}$ is a robustly asymptotically stable fixed point of (15) if the origin is a robustly asymptotically stable fixed point of the system $\chi^+ = G(\chi) + \omega$, in which $\chi := \zeta - \bar{\zeta}$, $\omega := w - \bar{w}$ and $G(\chi) := F(\bar{\zeta} + \chi) - F(\bar{\zeta})$.

Note that Definition 1 is used in [29] when proving the following theorem[2]:

Theorem 1. *[29, Th. 3] Let $F : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be a Lipschitz continuous function in a neighborhood of the origin with $F(0) = 0$. If the origin is an exponentially stable fixed point of the unperturbed system $\zeta^+ = F(\zeta)$, then it is a robustly asymptotically stable fixed point of the perturbed system $\zeta^+ = F(\zeta) + w$.*

Corollary 1. *If all the eigenvalues of the matrix $\mathscr{A}$ are strictly inside the unit disk, then the origin is a robustly asymptotically stable fixed point of the perturbed LTI system $\zeta^+ = \mathscr{A}\zeta + w$.*

3 Linear Controller Design

3.1 The Augmented System

In order to address the problem we make use of the following auxiliary system to define the controller state dynamics:

$$\hat{x}^+ = Ax + Bu + (\hat{d} + x - \hat{x}), \tag{16a}$$

$$\hat{d}^+ = \hat{d} + x - \hat{x}. \tag{16b}$$

Remark 3. The system (16) corresponds to using a dead-beat observer for the following system:

$$\begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix}^+ = \begin{bmatrix} A & I \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u,$$

$$x = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix},$$

in which it is clear that $\hat{d} \in \mathbb{R}^n$ is an integrated (step) disturbance acting on the state $\hat{x} \in \mathbb{R}^n$. The role of $\hat{d}$ is essential in removing steady-state offset in the presence of an unknown persistent disturbance [25, 26] and will be clarified later. As will be seen later, the dimensions of $\hat{d}$ and d need not be the same in order to guarantee offset-free control. It is also important to point out that the disturbance $\hat{d}$ does not integrate the tracking error, i.e. the difference between the setpoint s and the controlled variable z.

By combining the plant dynamics (1) and the auxiliary system (16), we obtain the following augmented system:

$$\xi^+ = \mathscr{A}\xi + \mathscr{B}u + \mathscr{E}d, \tag{17a}$$

$$z = \mathscr{C}\xi, \tag{17b}$$

in which

$$\xi := \begin{bmatrix} x \\ \hat{x} \\ \hat{d} \end{bmatrix}, \quad \mathscr{A} := \begin{bmatrix} A & 0 & 0 \\ I+A & -I & I \\ I & -I & I \end{bmatrix}, \quad \mathscr{B} := \begin{bmatrix} B \\ B \\ 0 \end{bmatrix}, \quad \mathscr{E} := \begin{bmatrix} E \\ 0 \\ 0 \end{bmatrix}, \quad \mathscr{C} := \begin{bmatrix} C_z & 0 & 0 \end{bmatrix}. \tag{17c}$$

We also define the controller state $\sigma \in \mathbb{R}^l$, with $l := 2n$, to be the states of the auxiliary system (16), i.e.

$$\sigma := \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix}. \tag{18}$$

[2] [29, Def. 2] contains a typographical error, hence the reason why the proof of [29, Th. 3] is inconsistent with [29, Def. 2]. However, the proof of [29, Th. 3] is correct and consistent with the definition of stability given in this paper. The authors would like to thank Prof. James Rawlings for confirming this.

3.2 Target Calculation and Unconstrained Offset-free Controller Design

When a non-zero persistent disturbance affects a system (and/or the current setpoint s is different from the origin), the origin of the state and input needs to be shifted in order to cancel the effect of such a disturbance on the controlled variable [2, 30]. To this aim, at each sample instant we use the estimate of the future disturbance and compute the steady-state target $(\bar{x}, \bar{u})$ such that one can drive the controlled variable to the current setpoint. When the dimension of the input is equal to the dimension of the controlled variable ($m = p$) these targets are uniquely defined by:

$$\begin{bmatrix} I - A & -B \\ C_z & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} \hat{d}^+ \\ s \end{bmatrix} = \begin{bmatrix} I & -I & I \\ 0 & 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ I \end{bmatrix} s \, . \tag{19}$$

Notice that this corresponds to finding the pair $(\bar{x}, \bar{u})$ such that $C_z \bar{x} = s$ and $\bar{x} = A\bar{x} + B\bar{u} + \hat{d}^+$, i.e. the state and input that cancel the effect of the disturbance. If, instead, there are extra degrees of freedom ($m > p$) these targets are non-unique. However, one can address both cases [30] by solving the following equality-constrained quadratic program (i.e. least-squares problem), in which $\bar{R} \in \mathbb{R}^{m \times m}$ is a positive definite matrix:

$$\left(\bar{x}^* \left(\xi, s\right), \bar{u}^* \left(\xi, s\right)\right) := \underset{(\bar{x}, \bar{u})}{\operatorname{argmin}} \frac{1}{2} \bar{u}^T \bar{R} \bar{u}, \tag{20a}$$

subject to

$$\begin{bmatrix} I - A & -B \\ C_z & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} I & -I & I \\ 0 & 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ I \end{bmatrix} s \, . \tag{20b}$$

For a given augmented state ξ and a given setpoint s, one can think of $\left(\bar{x}^* \left(\xi, s\right), \bar{u}^* \left(\xi, s\right)\right)$ as the new 'origin' around which the system should be regulated. Solving for $\left(\bar{x}^* \left(\xi, s\right), \bar{u}^* \left(\xi, s\right)\right)$ is trivial:

Lemma 1 (Target calculation). *If Assumption 1 holds, the minimizer of the equality-constrained quadratic program* (20) *is linear with respect to the augmented state ξ and the setpoint s, and is given by*

$$\begin{bmatrix} \bar{x}^* \left(\xi, s\right) \\ \bar{u}^* \left(\xi, s\right) \end{bmatrix} = \begin{bmatrix} \Pi_{13} & -\Pi_{13} & \Pi_{13} \\ \Pi_{23} & -\Pi_{23} & \Pi_{23} \end{bmatrix} \xi + \begin{bmatrix} \Pi_{14} \\ \Pi_{24} \end{bmatrix} s \, , \tag{21a}$$

where $\Pi_{13} \in \mathbb{R}^{n \times n}$, $\Pi_{23} \in \mathbb{R}^{m \times n}$, $\Pi_{14} \in \mathbb{R}^{n \times p}$ and $\Pi_{24} \in \mathbb{R}^{m \times p}$ are the relevant block matrix components of

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} \\ \Pi_{21} & \Pi_{22} & \Pi_{23} & \Pi_{24} \\ \Pi_{31} & \Pi_{32} & \Pi_{33} & \Pi_{34} \\ \Pi_{41} & \Pi_{42} & \Pi_{43} & \Pi_{44} \end{bmatrix} := \begin{bmatrix} 0 & 0 & -I + A^T & -C_z^T \\ 0 & \bar{R} & B^T & 0 \\ I - A & -B & 0 & 0 \\ C_z & 0 & 0 & 0 \end{bmatrix}^{-1} . \tag{21b}$$

Note that the square matrix in (21b) *has $m + 2n + p$ columns.*

Proof. See Appendix A. □

We now consider what would happen if one were to choose a gain matrix K such that $A + BK$ is strictly stable and let the control input in the augmented system (17a) be given by

$$u = \bar{u}^*(\xi, s) + K(x - \bar{x}^*(\xi, s)) \, . \tag{22}$$

Before proceeding, we need the following result:

Lemma 2 (Stability). *Suppose that Assumption 1 holds and $K \in \mathbb{R}^{m\times n}$ is such that $A+BK$ is strictly stable. If $\mathscr{A}$ and $\mathscr{B}$ are given by* (17c), *$\Gamma \in \mathbb{R}^{m\times n}$ is any constant matrix and*

$$\mathscr{K} := \begin{bmatrix} K+\Gamma & -\Gamma & \Gamma \end{bmatrix}, \tag{23}$$

then

$$\mathscr{A}_{\mathscr{K}} := \mathscr{A} + \mathscr{B}\mathscr{K} \tag{24}$$

is strictly stable.

Proof. See Appendix B. □

By defining

$$\Gamma := \Pi_{23} - K\Pi_{13}, \quad \mathscr{L} := \Pi_{24} - K\Pi_{14}, \tag{25}$$

and substituting (21a) into (22) it follows that

$$u = \Pi_{23}x - \Pi_{23}\hat{x} + \Pi_{23}\hat{d} + \Pi_{24}s + K(x - \Pi_{13}x + \Pi_{13}\hat{x} - \Pi_{13}\hat{d} - \Pi_{14}s) \tag{26a}$$

$$= (K+\Gamma)x - \Gamma\hat{x} + \Gamma\hat{d} + \mathscr{L}s \tag{26b}$$

$$= \mathscr{K}\xi + \mathscr{L}s. \tag{26c}$$

After substituting (26) into (17a), one can write an expression for the augmented system (17a) under the linear control $u = \mathscr{K}\xi + \mathscr{L}s$ as

$$\xi^+ = \mathscr{A}_{\mathscr{K}}\xi + \mathscr{E}d + \mathscr{F}s, \tag{27}$$

where

$$\mathscr{F} := \begin{bmatrix} B\mathscr{L} \\ B\mathscr{L} \\ 0 \end{bmatrix}. \tag{28}$$

Let $\psi(k,\xi,s(\cdot),d(\cdot))$ be the solution of the closed-loop system (27) at time k, given the state ξ at time $k=0$, the setpoint sequence $s(\cdot)$ and the disturbance sequence $d(\cdot)$.

As a consequence of the above results, we introduce the following standing assumption:

Assumption 4 (Stabilizing gain). The matrix $K \in \mathbb{R}^{m\times n}$ is chosen such that $A+BK$ is strictly stable, $\mathscr{K}$ is given by (23) with Γ given by (25), $\mathscr{A}_{\mathscr{K}} := \mathscr{A} + \mathscr{B}\mathscr{K}$ and $\mathscr{L}$ is given by (25).

The following result states that if the control is given by $u = \mathscr{K}\xi + \mathscr{L}s$, then the value of the controlled variable for (27) is guaranteed to converge to the asymptotic setpoint $\bar{s}$, given any allowable infinite setpoint and disturbance sequence:

Lemma 3 (Offset-free control). *If Assumptions 1–4 hold, then the solution of the closed-loop system* (27) *satisfies*

$$\lim_{k\to\infty} \mathscr{C}\psi(k,\xi,s(\cdot),d(\cdot)) = \bar{s} \tag{29}$$

for all $\xi \in \mathbb{R}^{3n}$.

Proof. See Appendix C. □

3.3 The Maximal Constraint-admissible Robustly Positively Invariant Set

We now consider the problem of computing the maximal constraint-admissible robustly positively invariant set in the space of the augmented state $\xi := (x, \hat{x}, \hat{d})$.

Let the *constraint-admissible set* Ξ be defined as all augmented states for which the constraints on the plant state and plant input are satisfied, for any choice of setpoint $s \in \mathscr{S}$, if the control is given by $u = \mathscr{K}\xi + \mathscr{L}s$:

$$\Xi := \left\{ \xi \in \mathbb{R}^{3n} \mid x \in \mathscr{X} \text{ and } \mathscr{K}\xi + \mathscr{L}s \in \mathscr{U} \text{ for all } s \in \mathscr{S} \right\}. \tag{30}$$

Remark 4. Note that, since $\mathscr{X}$ and $\mathscr{U}$ are polyhedra given by affine inequalities, Ξ is easily computed by applying Proposition 1 to the above definition.

The *maximal constraint-admissible robustly positively invariant set* $\mathscr{O}_\infty$ for the closed-loop system (27) is defined as all initial states in Ξ for which the evolution of the system remains in Ξ for all allowable infinite setpoint and disturbance sequences, i.e.

$$\mathscr{O}_\infty := \{ \xi \in \Xi \mid \psi(k, \xi, s(\cdot), d(\cdot)) \in \Xi \text{ for all } s(\cdot) \in \mathscr{M}_\mathscr{S}, \text{ all } d(\cdot) \in \mathscr{M}_\mathscr{D} \text{ and all } k \in \mathbb{N} \}. \tag{31}$$

Since (27) is linear and time-invariant and Ξ is given by a finite number of affine inequality constraints, $\mathscr{O}_\infty$ is easily computed by solving a finite number of LPs [28].

Assumption 5 (Maximal invariant set). The set $\mathscr{O}_\infty$ as defined in (31) is non-empty, contains the origin in its interior and is finitely determined (i.e. $\mathscr{O}_\infty$ can be described by a finite number of affine inequality constraints).

Remark 5. Except for a few pathological cases, Assumption 5 is met if $\mathscr{A}_\mathscr{K}$ is strictly stable, $\mathscr{X}$ is bounded, $(\mathscr{A}_\mathscr{K}, [I_n\ 0])$ is observable and $\mathscr{D}$ and $\mathscr{S}$ are sufficiently small [28]; however, observability of $(\mathscr{A}_\mathscr{K}, [I_n\ 0])$ and boundedness of $\mathscr{X}$ are not guaranteed under the assumptions in this paper. Despite this, in all test cases we have found that Assumption 5 holds. If Assumption 5 is violated, then it is easy to compute an approximation to $\mathscr{O}_\infty$ in finite time, e.g. by intersecting Ξ or $\mathscr{X}$ with a sufficiently large bounded polyhedron. The reader is referred to [8, 28] for alternative methods of computing an approximation to $\mathscr{O}_\infty$ in finite time if $\mathscr{O}_\infty$ is not finitely determined.

The following result states that, provided the augmented state is in $\mathscr{O}_\infty$ at time $k = 0$, then the evolution of the augmented system under the linear control $u = \mathscr{K}\xi + \mathscr{L}s$ is such that offset-free control is guaranteed and the state and input constraints are satisfied for all allowable setpoint and disturbance sequences:

Theorem 2 (Linear controller). *Suppose that Assumptions 1–5 hold. The solution of the closed-loop system* (27) *satisfies* (29) *and*

$$\begin{bmatrix} I_n & 0 \end{bmatrix} \psi(k, \xi, s(\cdot), d(\cdot)) \in \mathscr{X} \text{ and } \mathscr{K}\psi(k, \xi, s(\cdot), d(\cdot)) + \mathscr{L}s(k) \in \mathscr{U} \tag{32}$$

for all $\xi \in \mathscr{O}_\infty$ and all $k \in \mathbb{N}$. Furthermore, if $\bar{\xi} := (I - \mathscr{A}_\mathscr{K})^{-1}(\mathscr{E}\bar{d} + \mathscr{F}\bar{s})$ is in the interior of $\mathscr{O}_\infty$, then $\bar{\xi}$ is the robustly asymptotically stable fixed point of (27).

Proof. See Appendix D. □

Because of the assumptions in Theorem 2, it is important to correctly initialize the controller state $\sigma(0) := (\hat{x}(0), \hat{d}(0))$ such that $\xi(0) := (x(0), \sigma(0)) \in \mathscr{O}_\infty$. A sensible way of choosing the initial controller state is to compute the minimizer of the following quadratic program, given the initial plant state $x(0)$:

$$\left(\hat{x}(0), \hat{d}(0)\right) := \operatorname*{argmin}_{(\hat{x}, \hat{d})} \left\{ (x - \hat{x})^T (x - \hat{x}) + \hat{d}^T \hat{d} \mid \xi \in \mathscr{O}_\infty \text{ and } x = x(0) \right\}. \tag{33}$$

We can now also define X_0, the set of plant states for which there exists a controller state such that the augmented state is in $\mathscr{O}_\infty$, as

$$X_0 := \left\{ x \in \mathbb{R}^n \;\middle|\; \exists \sigma \in \mathbb{R}^{2n} \text{ such that } \xi \in \mathscr{O}_\infty \right\} . \tag{34}$$

Clearly, (33) is feasible if and only if $x(0) \in X_0$.

Remark 6. For analysis purposes, one might want to compute X_0 explicitly. Note that since $\mathscr{O}_\infty$ is a polyhedron, the set X_0 is easily computed as the projection [31, 32] of $\mathscr{O}_\infty$ onto the plant state space X, i.e.

$$X_0 = \operatorname{Proj}_X \left(\mathscr{O}_\infty \right) . \tag{35}$$

4 Receding Horizon Controller Design

The set X_0 is the set of initial plant states for which the controlled variable will be ultimately driven to the asymptotic setpoint $\bar{s}$ by the linear control $u = \mathscr{K}\xi + \mathscr{L}s$. This section presents an efficient approach for computing a nonlinear controller, which enlarges the set of initial plant states for which the controlled variable can ultimately be driven to the asymptotic setpoint. This will be achieved by using ideas from model predictive control of constrained systems [33, 6, 7].

4.1 Definition and Properties of the Receding Horizon Controller

We follow the same approach as in [14, 17, 16, 19, 20] of "pre-stabilizing" the plant by letting the linear control in (26) be modified with a perturbation term as follows:

$$u = \mathscr{K}\xi + \mathscr{L}s + v , \tag{36}$$

where $v \in \mathbb{R}^m$ is the perturbation term. The solution to the finite horizon optimal control problem (FHOCP), defined below, is a finite sequence of input perturbations that guarantees robust constraint satisfaction over the horizon and optimizes some cost function. Under the control in (36), the augmented state dynamics in (17a) become

$$\xi^+ = \mathscr{A}_\mathscr{K}\xi + \mathscr{B}v + \mathscr{E}d + \mathscr{F}s . \tag{37}$$

Before proceeding, let the horizon length N be a positive integer and the block vectors $\mathbf{v} \in \mathbb{R}^{mN}$, $\mathbf{s} \in \mathbb{R}^{p(N-1)}$, $\mathbf{d} \in \mathbb{R}^{rN}$ be defined as

$$\mathbf{v} := \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}, \quad \mathbf{s} := \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_{N-1} \end{bmatrix}, \quad \mathbf{d} := \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{N-1} \end{bmatrix} . \tag{38}$$

Remark 7. In the sequel, note that $\mathbf{s}$ and all related terms are present only if $N > 1$.

Let

$$\xi_k := \chi(k, \xi, \mathbf{v}, s, \mathbf{s}, \mathbf{d}) := \mathscr{A}_\mathscr{K}^k \xi + \mathscr{A}_\mathscr{K}^{k-1} \mathscr{F}s + \sum_{i=0}^{k-1} \mathscr{A}_\mathscr{K}^i (\mathscr{B}v_{k-1-i} + \mathscr{E}d_{k-1-i}) + \sum_{i=0}^{k-2} \mathscr{A}_\mathscr{K}^i \mathscr{F}s_{k-1-i} \tag{39}$$

denote the solution to (37) for all $k \in \{1, \ldots, N\}$, given the current augmented state ξ, a finite sequence of control perturbations $\mathbf{v}$, the current setpoint $s_0 := s$, a finite sequence of future setpoints $\mathbf{s}$ and a finite sequence of disturbances $\mathbf{d}$. The corresponding predicted plant state and input are similarly defined as

$$x_k := \begin{bmatrix} I_n & 0 \end{bmatrix} \chi(k, \xi, \mathbf{v}, s, \mathbf{s}, \mathbf{d}) , \qquad \forall k \in \{1, \ldots, N\} , \tag{40a}$$

$$u_k := \mathscr{K}\chi(k, \xi, \mathbf{v}, s, \mathbf{s}, \mathbf{d}) + \mathscr{L}s_k + v_k , \qquad \forall k \in \{0, \ldots, N-1\} . \tag{40b}$$

Given the above definitions, we now define the set of admissible input perturbations $\mathscr{V}_N(\xi,s)$ as the set of input perturbations of length N such that for all allowable future setpoint sequences of length $N-1$ and disturbance sequences of length N, the input constraints $\mathscr{U}$ are satisfied over the horizon $k=0,\ldots,N-1$, the state constraints $\mathscr{X}$ are satisfied over the horizon $k=1,\ldots,N-1$ and the augmented state at the end of the horizon is in $\mathscr{O}_\infty$ (hence the predicted plant state at the end of the horizon is also in $\mathscr{X}$), i.e.

$$\mathscr{V}_N(\xi,s) := \left\{ \mathbf{v}\in\mathbb{R}^{mN} \;\middle|\; \begin{array}{c} \xi_0=\xi,\ s_0=s,\ x_k\in\mathscr{X},\ k=1,\ldots,N-1,\ \xi_N\in\mathscr{O}_\infty, \\ u_k\in\mathscr{U},\ k=0,\ldots,N-1 \text{ for all } \mathbf{s}\in\mathscr{S}^{N-1} \text{ and all } \mathbf{d}\in\mathscr{D}^N \end{array} \right\}. \tag{41}$$

Remark 8. Note that $\mathscr{V}_N(\xi,s)$ is defined by an *infinite* number of constraints. Obtaining an equivalent expression for $\mathscr{V}_N(\xi,s)$ in terms of a *finite* number of affine inequality constraints is straightforward and a method that allows one to do this efficiently is described in Section 4.2.

In order to compute the receding horizon controller, we need to define an associated finite horizon optimal control problem (FHOCP). We choose to define $\mathbb{P}_N(\xi,s)$, the FHOCP to be solved for a given ξ and s, as

$$\mathbb{P}_N(\xi,s): \quad V_N^*(\xi,s) := \min_{\mathbf{v}\in\mathscr{V}_N(\xi,s)} V_N(\xi,s,\mathbf{v}), \tag{42}$$

where the cost to be minimized is defined as

$$\begin{aligned} V_N(\xi,s,\mathbf{v}) := {} & (\tilde{x}_N-\overline{x}^*(\xi,s))^T P(\tilde{x}_N-\overline{x}^*(\xi,s)) \\ & + \sum_{k=0}^{N-1} (\tilde{x}_k-\overline{x}^*(\xi,s))^T Q(\tilde{x}_k-\overline{x}^*(\xi,s)) + (\tilde{u}_k-\overline{u}^*(\xi,s))^T R(\tilde{u}_k-\overline{u}^*(\xi,s)), \end{aligned} \tag{43}$$

with the matrices $Q\in\mathbb{R}^{n\times n}$, $R\in\mathbb{R}^{m\times m}$ and $P\in\mathbb{R}^{n\times n}$ positive definite, and the vectors $\tilde{x}_k\in\mathbb{R}^n$ and $\tilde{u}_k\in\mathbb{R}^m$ defined as

$$\tilde{x}_0 = x \tag{44a}$$

$$\tilde{x}_{k+1} = A\tilde{x}_k + B\tilde{u}_k + (x-\hat{x}+d) \qquad \forall k\in\{0,\ldots,N-1\}, \tag{44b}$$

$$\tilde{u}_k = \overline{u}^*(\xi,s) + K(\tilde{x}_k-\overline{x}^*(\xi,s)) + v_k \qquad \forall k\in\{0,\ldots,N-1\}. \tag{44c}$$

The minimizer of $\mathbb{P}_N(\xi,s)$ is similarly defined:

$$\mathbf{v}^*(\xi,s) := \left(v_0^*(\xi,s),\ldots,v_{N-1}^*(\xi,s)\right) := \operatorname*{argmin}_{\mathbf{v}\in\mathscr{V}_N(\xi,s)} V_N(\xi,s,\mathbf{v}). \tag{45}$$

We assume here that the minimizer of (42) exists; this assumption is justified in Section 4.2.

Remark 9. The cost function $V_N(\cdot)$ can be regarded as the "nominal cost" in which the setpoint and the plant disturbance remain constant over the horizon N. Also note from (44b) that the disturbance affecting the plant, i.e. Ed, is assumed to be equal to its last deadbeat estimate $(x-\hat{x}+\hat{d})$.

As is standard in receding horizon control [33, 6, 7], for a given state ξ and a given setpoint s, we only keep the first element $v_0^*(\xi,s)$ of the solution to the FHOCP. Using this receding horizon principle, we define our controller in (4) by substituting

$$u = \mathscr{K}\xi + \mathscr{L}s + v_0^*(\xi,s) \tag{46}$$

into the equation for the augmented system (17a) and comparing it with the expression for the closed-loop dynamics (7). In other words, the controller state dynamics map in (4a) is given by

$$\alpha(\xi,s) := \begin{bmatrix} I+A & -I & I \\ I & -I & I \end{bmatrix}\xi + \begin{bmatrix} B\mathscr{K} \\ 0 \end{bmatrix}\xi + \begin{bmatrix} B\mathscr{L} \\ 0 \end{bmatrix}s + \begin{bmatrix} B \\ 0 \end{bmatrix} v_0^*(\xi,s) \tag{47a}$$

and the controller output map in (4b) is

$$\gamma(\xi,s) := \mathscr{K}\xi + \mathscr{L}s + v_0^*(\xi,s)\,. \tag{47b}$$

It is important to be able to determine all the plant states for which one can guarantee that problem $\mathbb{P}_N(\xi,s)$ has a solution. The set of plant states $X_N^{\mathbf{v}}$ for which one can initialize the controller state such that the set of admissible input perturbations $\mathscr{V}_N(\xi,s)$ is non-empty for all $s \in \mathscr{S}$ (and hence $\mathbb{P}_N(\xi,s)$ has a solution), is given by

$$X_N^{\mathbf{v}} := \left\{x \in \mathscr{X} \;\middle|\; \exists \sigma \in \mathbb{R}^{2n} \text{ such that } \mathscr{V}_N(\xi,s) \neq \emptyset \text{ for all } s \in \mathscr{S}\right\}. \tag{48}$$

As will be shown below, $X_N^{\mathbf{v}}$ is the set of plant states in $\mathscr{X}$ for which the controlled variable will ultimately be driven to the setpoint $\bar{s}$ by the controller (4), if α and γ are given by (47).

We can now give our first main result:

Theorem 3 (Domain of RHC law). *Suppose that Assumptions 1–5 hold. If X_0 is defined in* (34) *and each $X_i^{\mathbf{v}}$, $i \in \{1,\dots,N\}$, is defined as in* (48) *with $N = i$, then all the sets in $\{X_0, X_1^{\mathbf{v}},\dots,X_N^{\mathbf{v}}\}$ contain the origin in their interior and satisfy*

$$X_0 \subseteq X_1^{\mathbf{v}} \subseteq \cdots \subseteq X_{N-1}^{\mathbf{v}} \subseteq X_N^{\mathbf{v}}\,. \tag{49}$$

Proof. See Appendix E. □

Theorem 3 is very important because it shows that, under the above assumptions, an increase in the horizon length does not decrease the size of the set of initial plant states for which the controlled variable can be driven to the setpoint.

Assumption 6. The matrices Q and R are chosen to be positive definite, the matrix P is the positive definite solution of the following discrete algebraic Riccati equation:

$$P = Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA\,, \tag{50a}$$

and the matrix K is the corresponding gain:

$$K = -(R + B^T PB)^{-1} B^T PA\,. \tag{50b}$$

$\mathscr{K}$ is given by (23) with Γ given by (25), $\mathscr{A}_{\mathscr{K}} := \mathscr{A} + \mathscr{B}\mathscr{K}$ and $\mathscr{L}$ is given by (25).

Remark 10. It is clear that the matrix K defined in (50b) is such that $A + BK$ is strictly stable.

Before giving our second main result, we need the following:

Lemma 4 (FHOCP equivalence). *Suppose that Assumptions 1 and 6 hold and let $J_N(\mathbf{v})$ be defined as:*

$$J_N(\mathbf{v}) := \sum_{k=0}^{N-1} v_k^T W v_k\,. \tag{51}$$

If the (positive definite) matrix $W \in \mathbb{R}^{m\times m}$ is given by

$$W := R + B^T PB\,, \tag{52}$$

then the cost function $V_N(\cdot)$ satisfies

$$V_N(\xi,s,\mathbf{v}) = J_N(\mathbf{v}) + (x - \bar{x}^*(\xi,s))^T P(x - \bar{x}^*(\xi,s)) \tag{53}$$

and

$$\mathbf{v}^*(\xi,s) := \operatorname*{argmin}_{\mathbf{v}\in\mathscr{V}_N(\xi,s)} V_N(\xi,s,\mathbf{v}) = \operatorname*{argmin}_{\mathbf{v}\in\mathscr{V}_N(\xi,s)} J_N(\mathbf{v})\,. \tag{54}$$

Proof. See Appendix F. □

Lemma 5 (Robust feasibility and perturbation sequence). *Suppose that Assumptions 1–3 and 5–6 hold. If the controller* (4) *is defined by* (47) *and* $\mathscr{V}_N(\xi(0),s(0))$ *is non-empty, then* $\mathscr{V}_N(\xi(k),s(k))$ *is non-empty for all* $k\in\mathbb{N}$ *and*

$$\lim_{k\to\infty} v_0^*(\xi(k),s(k)) = 0\,. \tag{55}$$

Proof. See Appendix G. □

We can now state our second main result:

Theorem 4 (Offset-free control, robust constraint satisfaction and stability of RHC law). *Suppose that Assumptions 1–3 and 5–6 hold and that the controller* (4) *is defined by* (47)*. One can choose the initial controller state* $\sigma(0)$ *such that* $\mathbb{P}_N(\xi(0),s(0))$ *has a solution and the evolution of the closed-loop system* (5) *satisfies* (10) *for all* $k\in\mathbb{N}$ *if and only if the initial plant state* $x(0)\in X_N^{\mathbf{v}}$*. Furthermore, if* $\bar{\xi} := (I-\mathscr{A}_{\mathscr{K}})^{-1}(\mathscr{E}\bar{d}+\mathscr{F}\bar{s})$ *is in the interior of* $\mathscr{O}_\infty$*, then* $\bar{\xi}$ *is the robustly asymptotically stable fixed point of* (5)*.*

Proof. See Appendix H. □

As in Section 3.3, we need to initialize the controller state correctly such that $\mathbb{P}_N(\xi(0),s(0))$ has a solution. A sensible method for simultaneously obtaining an optimal initial controller state and input perturbation sequence is to solve the following optimization problem, given the initial plant state $x(0)$ and the initial setpoint $s(0)$:

$$\begin{aligned}&\left(\hat{x}(0),\hat{d}(0),\mathbf{v}^*(\xi(0),s(0))\right) :=\\ &\qquad \operatorname*{argmin}_{(\hat{x},\hat{d},\mathbf{v})}\left\{J_N(\mathbf{v})+\lambda\left((\hat{x}-x)^T(\hat{x}-x)+\hat{d}^T\hat{d}\right)\;\middle|\;\mathbf{v}\in\mathscr{V}_N(\xi,s),\;x=x(0),\;s=s(0)\right\}\,,\end{aligned} \tag{56}$$

where λ is a strictly positive scalar.

4.2 Efficient Implementation of the Receding Horizon Controller

Since $\mathscr{X}$, $\mathscr{U}$ and $\mathscr{O}_\infty$ are polyhedral sets with non-empty interiors, they are given by a finite number of affine inequality constraints. As a consequence, it is easy to obtain an equivalent expression for the set of admissible input perturbations $\mathscr{V}_N(\xi,s)$ as

$$\mathscr{V}_N(\xi,s)=\left\{\mathbf{v}\in\mathbb{R}^{mN}\;\middle|\;F\mathbf{v}\le b+G^d\mathbf{d}+G^s\mathbf{s}+H^\xi\xi+H^s s\text{ for all }\mathbf{s}\in\mathscr{S}^{N-1}\text{ and all }\mathbf{d}\in\mathscr{D}^N\right\}\,, \tag{57}$$

where the matrices $F\in\mathbb{R}^{q\times mN}$, $G^d\in\mathbb{R}^{q\times rN}$, $G^s\in\mathbb{R}^{q\times p(N-1)}$, $H^\xi\in\mathbb{R}^{q\times 3n}$, $H^s\in\mathbb{R}^{q\times p}$ and the vector $b\in\mathbb{R}^q$ depend on the augmented system dynamics (37) and are given in Appendix I.

By using the results of Proposition 1 one can compute an equivalent expression for $\mathscr{V}_N(\xi,s)$ in terms of a finite number of affine inequality constraints:

$$\mathscr{V}_N(\xi,s)=\left\{\mathbf{v}\in\mathbb{R}^{mN}\;\middle|\;F\mathbf{v}\le c+H^\xi\xi+H^s s\right\}\,, \tag{58}$$

where

$$c := b+\min_{\mathbf{d}\in\mathscr{D}^N}G^d\mathbf{d}+\min_{\mathbf{s}\in\mathscr{S}^{N-1}}G^s\mathbf{s}\,. \tag{59}$$

Remark 11. Since $\mathscr{D}$ and $\mathscr{S}$ (and hence $\mathscr{D}^N$ and $\mathscr{S}^{N-1}$) are polyhedra and can therefore be described by a finite number of affine inequality constraints, c can be computed efficiently by solving q LPs.

Remark 12. If $\mathscr{D}$ and $\mathscr{S}$ are given only by upper and lower bounds on the components of d and s, respectively, then it is not necessary to solve LPs in order to compute c; checking the signs of the components of G^d and G^s is sufficient. For example, if the disturbance and the setpoint are assumed to take on values in the hypercubes

$$\mathscr{D} := \{d \in \mathbb{R}^r \mid \|d\|_\infty \leq \beta\}, \quad \mathscr{S} := \{s \in \mathbb{R}^p \mid \|s\|_\infty \leq \eta\},$$

then it follows from Proposition 1 that

$$c = b - \beta\,\mathrm{abs}(G^d)\mathbf{1}_{rN} - \eta\,\mathrm{abs}(G^s)\mathbf{1}_{p(N-1)}.$$

Remark 13. From Appendix I it is clear that the number of constraints q in (58) is not dependent on the description for $\mathscr{S}$ and $\mathscr{D}$, but only dependent on N and the number of constraints that describe $\mathscr{X}$, $\mathscr{U}$ and $\mathscr{O}_\infty$. Note also that q increases only linearly with the horizon length N.

Given all of the above, it is now clear that the minimizer to $\mathbb{P}_N(\xi,s)$ exists if and only if $\mathscr{V}_N(\xi,s) \neq \emptyset$. The minimizer of $\mathbb{P}_N(\xi,s)$ is the solution to the following finite-dimensional, strictly convex quadratic program (QP):

$$\mathbf{v}^*(\xi,s) = \operatorname*{argmin}_{\mathbf{v}} \left\{ J_N(\mathbf{v}) \;\middle|\; F\mathbf{v} \leq c + H^\xi \xi + H^s s \right\}. \tag{60}$$

Clearly, (56) is also a finite-dimensional, strictly convex QP.

There are essentially two ways in which one can compute $v_0^*(\xi,s)$ (and hence the control input) for a given ξ and s:

- As is standard in conventional model predictive control [33, 6, 7], given the current value for ξ and s, one can compute $v_0^*(\xi,s)$ on-line by solving the QP defined in (60) using standard QP solvers [34].
- The QP in (60) is a so-called *parametric* QP, since the constraints (and hence the solution) of the QP in (60) are dependent on the *parameters* ξ and s. This observation allows one to compute the explicit expression for $v_0^*(\cdot)$ off-line using recent results presented in [35]. The results in [35] can be used to show that $v_0^*(\cdot)$ is a piecewise affine function of (ξ,s) and is defined over a polyhedral partition, i.e. the domain of $v_0^*(\cdot)$ is the union of a finite number of polyhedra and $v_0^*(\cdot)$ is affine in each polyhedron. Computing the value of $v_0^*(\xi,s)$ on-line amounts to looking up the polyhedron in which (ξ,s) is contained and substituting (ξ,s) into the associated affine function.

Remark 14. For analysis purposes, one might want to compute an explicit expression for $X_N^{\mathbf{v}}$. Since one can obtain a polyhedral expression for $\mathscr{V}_N(\xi,s)$, it is possible to compute a polyhedral expression for $X_N^{\mathbf{v}}$ by using standard projection algorithms [32, 36], i.e.

$$X_N^{\mathbf{v}} = \mathscr{X} \cap \mathrm{Proj}_X \left\{ (\xi,s) \in \mathbb{R}^{3n} \times \mathbb{R}^p \;\middle|\; F\mathbf{v} \leq c + H^\xi \xi + H^s s \text{ for all } s \in \mathscr{S} \right\} \tag{61a}$$

$$= \mathscr{X} \cap \mathrm{Proj}_X \left\{ (\xi,s) \in \mathbb{R}^{3n} \times \mathbb{R}^p \;\middle|\; F\mathbf{v} \leq c + H^\xi \xi + \min_{s\in\mathscr{S}} H^s s \right\}, \tag{61b}$$

where the last step clearly follows from Proposition 1.

5 Illustrative example

As an example, we consider a jacketed continuous stirred tank reactor (CSTR) studied by Henson and Seborg [37] in which an irreversible liquid-phase reaction occurs. A detailed nonlinear model has two states (reactant concentration and reactor temperature), one input (cooling liquid temperature) and two disturbances (feed temperature and feed reactant concentration). This CSTR shows three steady states,

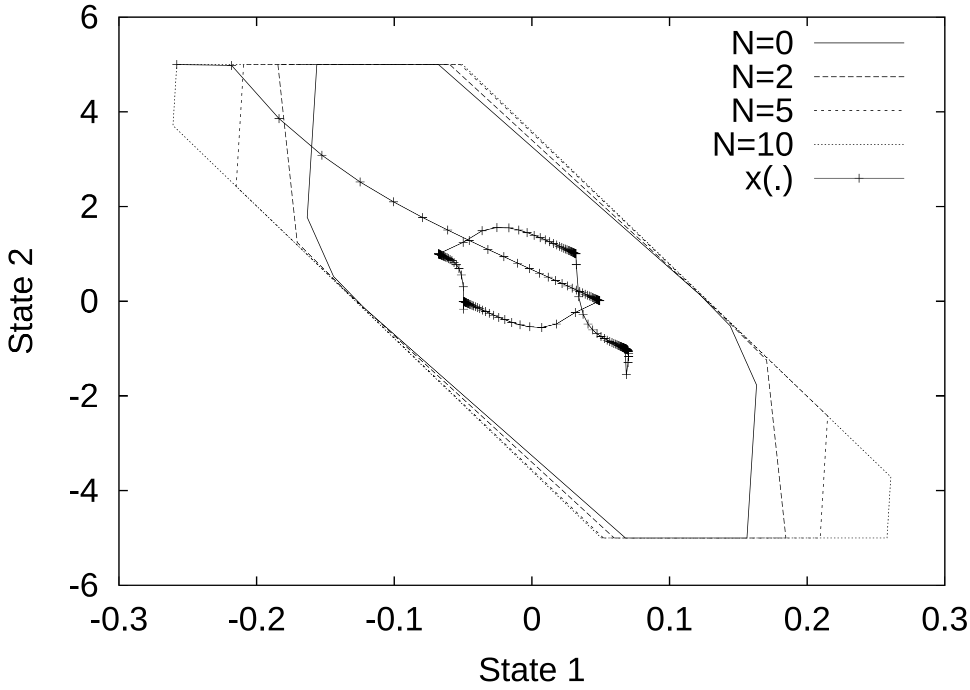

Figure 1: Domain of attraction ($X_N^{\mathbf{v}}$) for different fixed horizons

two of which are open-loop unstable, and for quality and safety reasons the middle conversion open-loop unstable steady-state is chosen as a nominal operating setpoint. Using a sampling time of $t_s = 0.1$ min and introducing deviation variables (from the corresponding steady state) a linearized model is as follows:

$$\begin{bmatrix} x^1 \\ x^2 \end{bmatrix}^+ = \begin{bmatrix} 0.7776 & -0.0045 \\ 26.6185 & 1.8555 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} + \begin{bmatrix} -0.0004 \\ 0.2907 \end{bmatrix} u + \begin{bmatrix} -0.0002 & 0.0893 \\ 0.1390 & 1.2267 \end{bmatrix} \begin{bmatrix} d^1 \\ d^2 \end{bmatrix}$$

$$z = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix},$$

in which x^1 and x^2 represent the reactant concentration and the reactor temperature, respectively; u represents the coolant temperature; d^1 and d^2 represent the feed temperature and the feed reactant concentration, respectively. Notice from the structure of C_z that the controlled variable is the reactor temperature, for which offset-free control to the setpoint s is required. Also notice that the system matrix A has one stable and one unstable eigenvalue. The following constraints on the plant states and input and on the admissible disturbances and setpoint are considered:

$$\begin{bmatrix} -0.5 \\ -5 \end{bmatrix} \leq \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \leq \begin{bmatrix} 0.5 \\ 5 \end{bmatrix}, \qquad -15 \leq u \leq 15, \qquad \begin{bmatrix} -2 \\ -0.1 \end{bmatrix} \leq \begin{bmatrix} d^1 \\ d^2 \end{bmatrix} \leq \begin{bmatrix} 2 \\ 0.1 \end{bmatrix}, \qquad -1 \leq s \leq 1 .$$

We present in Figure 1 the domain of attraction (i.e. $X_N^{\mathbf{v}}$) of four receding horizon controllers using different fixed horizons (specified in the figure) and the same penalty matrices: $Q = I_2$ and $R = 0.2$. Notice that X_0 is the domain of attraction of the linear controller. As expected from Theorem 3 we have that an increase in the fixed horizon length results in a larger feasible region and also that the domain of attraction of the linear controller is included in that of the receding horizon controllers.

We present in Figure 2 the domain of attraction of four receding horizon controllers using the same fixed horizon, $N = 10$, and different stabilizing gain matrices. These gains were computed as the optimal LQR

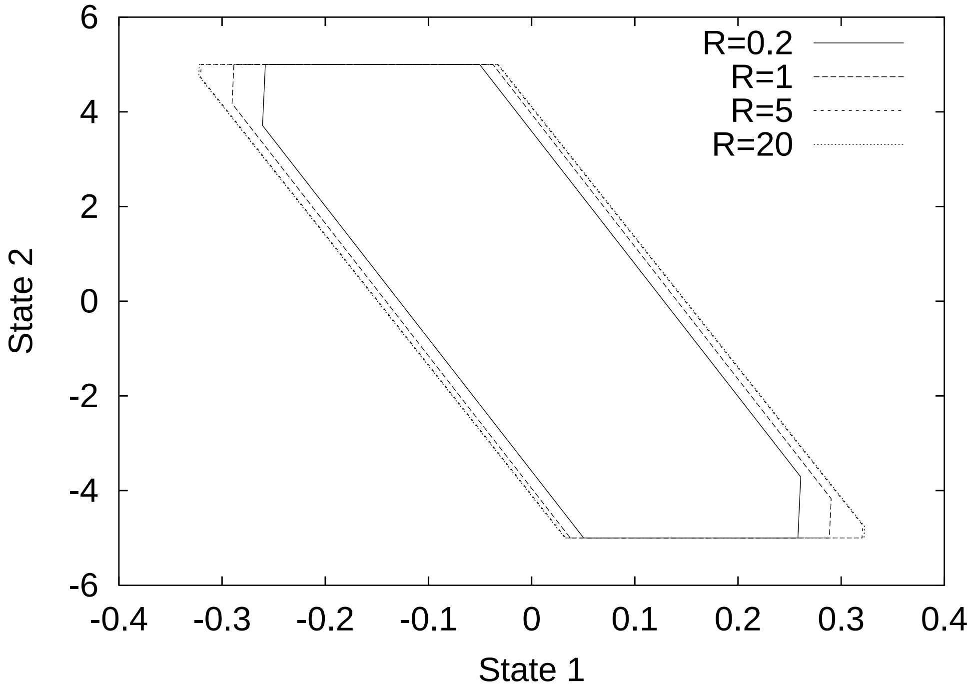

Figure 2: Domain of attraction ($X_{10}^{\mathbf{v}}$) for different stabilizing gain

Table 1: Disturbances and setpoint

t (min)	$[0,4)$	$[4,8)$	$[8,12)$	$[12,16)$	$[16,24)$	$[20,24]$
d	$[2\ 0.1]^T$	$[2\ -0.1]^T$	$[-2\ -0.1]^T$	$[-2\ 0.1]^T$	$[2\ 0.1]^T$	$[-2\ 0.1]^T$
s	0	0	1	1	-1	-1

gain with $Q = I_2$ and different R (specified in the figure) as penalty matrices. It is interesting to notice that when the input penalty matrix R used to compute the stabilizing gain is reduced, i.e. a more aggressive controller is chosen, the corresponding domain of attraction is larger. However, it is important to remark that this result is not general and depends on the system parameters and on the fixed horizon. To clarify this point, we present in Figure 3 the domain of attraction of the corresponding receding horizon controllers using a fixed horizon of $N = 2$.

We present in Figure 4 the closed-loop simulation results (controlled variable and input) obtained with four receding horizon controllers using the same fixed horizon, $N = 10$, different penalty matrices ($Q = I_2$ for all controllers and R specified in the figure), and the scalar used in (56) was $\lambda = 1000$.

The initial plant state is $x(0) = \begin{bmatrix} -0.258 & 5 \end{bmatrix}^T$, the disturbances and the setpoint vary during the simulation time as reported in Table 1. For the receding horizon controller based on $Q = I_2$ and $R = 0.2$ the plant state sequence, $x(\cdot)$, is also reported in Figure 1. Notice that the state sequence $x(\cdot)$ initially starts at the boundary of the domain of attraction $X_{10}^{\mathbf{v}}$ and enters the domain of attraction of the linear controller X_0 in finite time. As expected from Theorem 4 the proposed controllers asymptotically drive the controlled variable to the asymptotic setpoint despite the presence of persistent unmeasured disturbances. Also, when the setpoint is changed the controllers drive the controlled variable to the new setpoint. Moreover, it is interesting to notice that the choice of penalty matrices has a direct impact on the closed-loop performance.

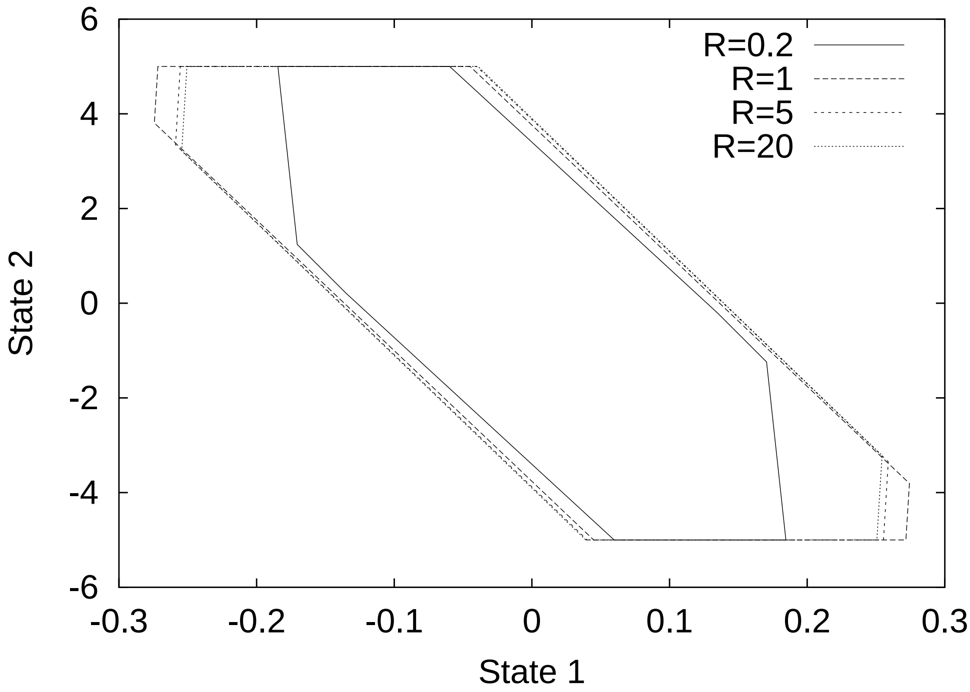

Figure 3: Domain of attraction ($X_2^{\mathbf{v}}$) for different stabilizing gain

That is, when a lower input penalty R is chosen, the disturbance is rejected (and the setpoint is reached) more quickly and a larger control input is used.

We finally present in Figure 5 a comparison of the proposed receding horizon controller with a "standard" (i.e. non offset-free) robust receding horizon controller. As an example we chose the approach in [17], which is similar to the one proposed in this paper, in the sense that a pre-stabilizing gain matrix is used and the plant state prediction at the end of the horizon in restricted to be in the maximal disturbance invariant set $\mathscr{O}_\infty$. Both controllers are based on the same stabilizing gain matrix K, which is the optimal LQR gain with $Q = I_2$ and $R = 0.2$. The fixed horizon used for both controllers is $N = 10$ and the perturbation penalty for the "standard" controller is chosen as $W = R + B^T PB$ with P the solution to the corresponding steady-state Riccati equation. The initial plant state is $x(0) = \begin{bmatrix} -0.258 & 5 \end{bmatrix}^T$ and the disturbance varies as specified in Table 1. In this comparison the setpoint is the origin since the method in [17] does not apply to setpoints different from the origin (an extension of [17] to the setpoint tracking problem has been proposed in [16]; however, the controller proposed in [16] still does not guarantee offset-free control). As expected, the goal of offset-free control is achieved by the proposed method whereas the controller of [17] leaves a significant and undesired steady-state offset.

6 Conclusions

This paper has shown how one can design a nonlinear time-invariant, dynamic state feedback controller that guarantees robust constraint satisfaction, robust stability and offset-free control in the presence of time-varying setpoints and persistent, non-stationary, additive disturbances on the state. The design of the controller was split into two parts:

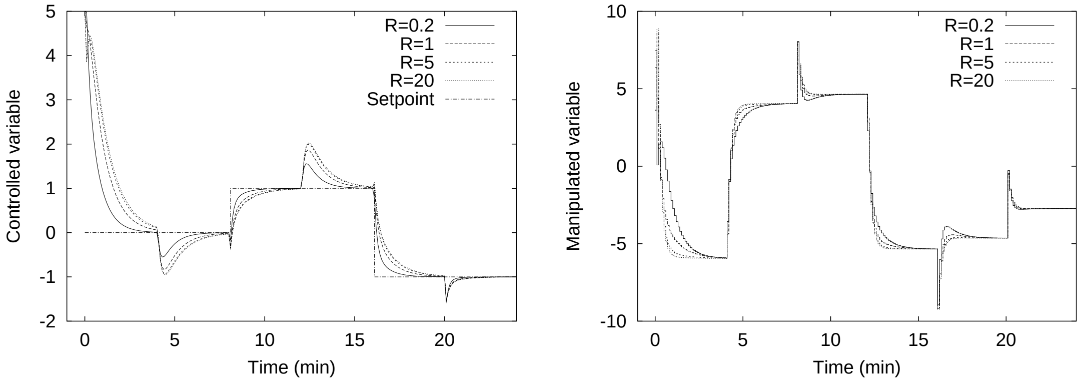

Figure 4: Closed-loop comparison of different receding horizon controllers: controlled variable (left) and input (right)

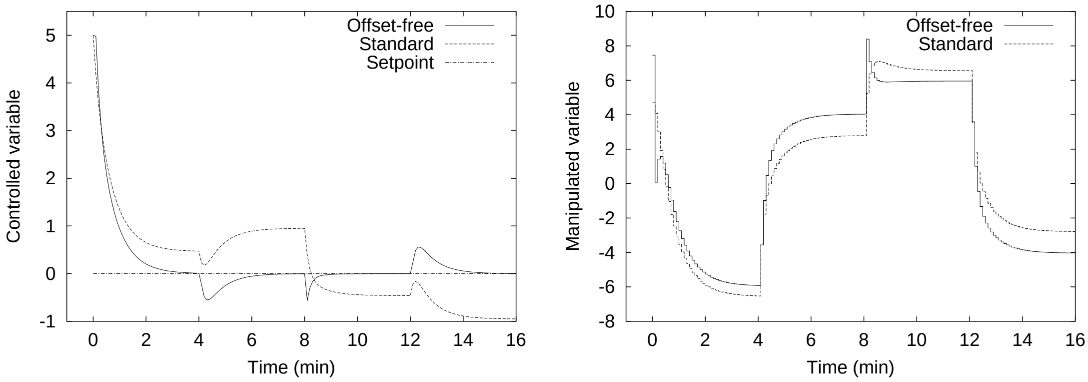

Figure 5: Closed-loop comparison of offset-free and standard robust receding horizon controllers: controlled variable (left) and input (right).

- *The design of a dynamic, linear, time-invariant controller*. A deadbeat observer is used to estimate the disturbance, the new steady state is given as a linear function of the current plant and observer states and of the current setpoint, and the controller aims to regulate the plant state and input to the new target steady state. In order to estimate the region of attraction of the linear controller, it was proposed that the maximal constraint-admissible robustly positively invariant set $\mathscr{O}_\infty$ associated with the linear controller be computed.

- *The design of a dynamic, nonlinear, time-invariant receding horizon controller*. In order to increase the region of attraction of the linear controller, a robust receding horizon controller, which computes perturbations to the linear control law, was proposed. The receding horizon controller includes the state and input constraints explicitly in its computations as well as the effect of the unknown persistent disturbance, thereby guaranteeing robust constraint satisfaction. It was proposed that the set $\mathscr{O}_\infty$ be included as a terminal constraint in the prediction horizon and it was shown that the specific formulation of the proposed receding horizon controller improves on the linear controller in terms of the domain of attraction.

The robust receding horizon controller presented in this paper can be implemented in an efficient manner and is computationally tractable. The incorporation of the effect of the disturbance and of future unknown setpoints has very little effect on the computational complexity since the number of decision variables and constraints increases only linearly with an increase in the horizon length.

The paper also demonstrated the effectiveness of using the results in this paper in designing a controller for guaranteeing offset-free control of a continuous stirred tank reactor with respect to existing non offset-free algorithms. The simulation results were shown to be in agreement with the theory.

We conclude this paper with some recommendations on how the results in this paper may be extended:

- The choice of auxiliary system has an impact on the region of attraction and closed-loop performance of the system. A more detailed investigation into this topic could be undertaken.
- The constraints on the state and input were not included in the target calculation in Section 3.2. If the constraints are included in the target calculation, then the optimal steady-state target is no longer a linear function of the augmented state and setpoint. Clearly, this complicates the receding horizon controller design. However, the inclusion of constraints in the target calculation will enlarge the domain of attraction and increase the size of the disturbance and setpoints that can be handled by the controller. An extension of this paper, which includes constraints in the target calculation, could combine the results in [26] with those in [16].
- Clearly, the rank condition in (3) is not always satisfied. If this assumption is violated, then one might have to relax the requirement that offset-free control be achieved on all controlled variables. One possible approach to resolving this problem is to prioritize the controlled variables when performing the target calculation. The framework proposed in [38] may be useful in this context.
- Rather than optimizing over perturbations to a pre-stabilizing control law, one could consider optimizing over arbitrary, nonlinear feedback policies [15, 18, 6, 7, 22]. This will enlarge the region of attraction of the receding horizon controller at the expense of an increase in computational complexity.
- The important problem of guaranteeing robust stability, performance, constraint satisfaction and offset-free control when output feedback (rather than state feedback) is used, remains to be addressed.

Acknowledgements

The authors would like to thank Colin Jones for sharing some of his software, which allowed for the efficient computation of $X_N^{\mathbf{v}}$ in the example.

Appendix

A Proof of Lemma 1

The statement follows immediately from the Lagrangian/KKT conditions for (20) [34, Sect. 16.1]. It is important to verify that the matrix to be inverted in (21b) is non-singular.

In order to see why this is the case, let Z be a matrix of dimension $(n+m)\times(m-p)$ (if the system is square, i.e. $m=p$, the proof of non-singularity is trivial) whose columns are an orthonormal basis for the null space of $\begin{bmatrix} I-A & -B \\ C_z & 0 \end{bmatrix}$. Consider any vector $v \in \mathbb{R}^{m-p}$ with $v \neq 0$, and let

$$z = \begin{bmatrix} x^* \\ u^* \end{bmatrix} = Zv\,.$$

Notice that since the columns of Z are independent, $z \neq 0$.

We now show by contradiction that $u^* \neq 0$. Suppose that $u^* = 0$. We can write

$$\begin{bmatrix} I-A \\ C_z \end{bmatrix} x^* = \begin{bmatrix} Bu^* \\ 0 \end{bmatrix} = 0\,.$$

From Assumption 1 we have that (A, C_z) is detectable. Hence, from the Hautus Lemma [39, Sect. 7.1] it follows that the matrix $\begin{bmatrix} I-A \\ C_z \end{bmatrix}$ has full column rank. But this implies that $x^* = 0$ which is in contradiction with the fact that $z \neq 0$. Hence, it must be $u^* \neq 0$.

Therefore, since $z = Zv$, we can write:

$$v^T Z^T \begin{bmatrix} 0 & 0 \\ 0 & \bar{R} \end{bmatrix} Zv = \begin{bmatrix} x^* \\ u^* \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & \bar{R} \end{bmatrix} \begin{bmatrix} x^* \\ u^* \end{bmatrix} = (u^*)^T \bar{R} u^* > 0\,,$$

where the last inequality comes from the fact that $\bar{R}$ is positive definite and that $u^* \neq 0$. This implies that the reduced Hessian defined as

$$Z^T \begin{bmatrix} 0 & 0 \\ 0 & \bar{R} \end{bmatrix} Z$$

is positive definite, and we can apply the results in [34, Lemma 16.1] to deduce that

$$\begin{bmatrix} 0 & 0 & -I+A^T & -C_z^T \\ 0 & \bar{R} & B^T & 0 \\ I-A & -B & 0 & 0 \\ C_z & 0 & 0 & 0 \end{bmatrix}$$

is non-singular and the target calculation (20) has a unique minimizer.

B Proof of Lemma 2

From the definitions, it follows that

$$\mathscr{A}_{\mathscr{K}} := \mathscr{A} + \mathscr{B}\mathscr{K} = \begin{bmatrix} A+BK+B\Gamma & -B\Gamma & B\Gamma \\ I_n+A+BK+B\Gamma & -I_n-B\Gamma & I_n+B\Gamma \\ I_n & -I_n & I_n \end{bmatrix}. \tag{62}$$

The eigenvalues of $\mathscr{A}+\mathscr{B}\mathscr{K}$ are the roots of $|\mathscr{A}+\mathscr{B}\mathscr{K}-\lambda I_{3n}|=0$. Note that

$$\begin{aligned}
|\mathscr{A}+\mathscr{B}\mathscr{K}-\lambda I_{3n}| &= \begin{vmatrix} A+BK+B\Gamma-\lambda I_n & -B\Gamma & B\Gamma \\ I_n+A+BK+B\Gamma & -I_n-B\Gamma-\lambda I_n & I_n+B\Gamma \\ I_n & -I_n & I_n-\lambda I_n \end{vmatrix} \\
&= \begin{vmatrix} A+BK+B\Gamma-\lambda I_n & -B\Gamma & B\Gamma \\ \lambda I_n & -\lambda I_n & \lambda I_n \\ I_n & -I_n & I_n-\lambda I_n \end{vmatrix} \text{ (subtract rows 1 and 3 from 2)} \\
&= \begin{vmatrix} A+BK-\lambda I_n & -B\Gamma & B\Gamma \\ 0 & -\lambda I_n & \lambda I_n \\ 0 & -I_n & I_n-\lambda I_n \end{vmatrix} \text{ (add column 2 to column 1)} \\
&= \begin{vmatrix} A+BK-\lambda I_n & 0 & B\Gamma \\ 0 & 0 & \lambda I_n \\ 0 & -\lambda I_n & I_n-\lambda I_n \end{vmatrix} \text{ (add column 3 to column 2)} \\
&= (-1)^n \cdot \begin{vmatrix} A+BK-\lambda I_n & 0 & B\Gamma \\ 0 & -\lambda I_n & I_n-\lambda I_n \\ 0 & 0 & \lambda I_n \end{vmatrix} \text{ (exchange rows 2 and 3)} \\
&= (-1)^n \cdot |A+BK-\lambda I_n| \cdot |\lambda I_n| \cdot |\lambda I_n| \text{ (determinant of block triangular matrix)} \\
&= (-1)^n \cdot |A+BK-\lambda I_n| \cdot \lambda^n \cdot \lambda^n \\
&= (-1)^n \cdot \lambda^{2n} \cdot |A+BK-\lambda I_n|\,.
\end{aligned}$$

This implies that $2n$ of the eigenvalues of $\mathscr{A}+\mathscr{B}\mathscr{K}$ are at the origin and the rest are equal to the eigenvalues of $A+BK$. Hence, if $A+BK$ has all its eigenvalues strictly inside the unit disk, then the eigenvalues of $\mathscr{A}+\mathscr{B}\mathscr{K}$ are strictly inside the unit disk.

C Proof of Lemma 3

Since $\lim_{k\to\infty} s(k)=\bar{s}$ and $\lim_{k\to\infty} d(k)=\bar{d}$ we have from (26)–(27) and from the results of Lemma 2 that

$$\xi_\infty := \lim_{k\to\infty} \psi(k,\xi,s(\cdot),d(\cdot)) = \mathscr{A}_{\mathscr{K}}\xi_\infty + \mathscr{E}\bar{d} + \mathscr{F}\bar{s} = \mathscr{A}\xi_\infty + \mathscr{B}u_\infty + \mathscr{E}\bar{d}, \tag{63}$$

in which $u_\infty = \mathscr{K}\xi_\infty + \mathscr{L}\bar{s}$. Let ξ_∞ be partitioned as follows:

$$\xi_\infty = \begin{bmatrix} x_\infty \\ \hat{x}_\infty \\ \hat{d}_\infty \end{bmatrix},$$

in which each block is a column vector of length n. We can rewrite (63) explicitly as follows:

$$x_\infty = Ax_\infty + Bu_\infty + E\bar{d} \tag{64a}$$
$$\hat{x}_\infty = Ax_\infty + Bu_\infty + (x_\infty - \hat{x}_\infty + \hat{d}_\infty) \tag{64b}$$
$$\hat{d}_\infty = x_\infty - \hat{x}_\infty + \hat{d}_\infty\,. \tag{64c}$$

From (64c) we immediately obtain that $x_\infty = \hat{x}_\infty$ which, combined with (64b), leads to

$$x_\infty = Ax_\infty + Bu_\infty + (x_\infty - \hat{x}_\infty + \hat{d}_\infty)\,. \tag{65}$$

Let $(\bar{x}_\infty, \bar{u}_\infty)$ denote the solution to the target calculation problem (20) for the augmented state ξ_∞ and the setpoint $\bar{s}$. From (20b) we can write:

$$\bar{x}_\infty = A\bar{x}_\infty + B\bar{u}_\infty + (x_\infty - \hat{x}_\infty + \hat{d}_\infty)\,, \tag{66}$$

which, subtracted from (65), leads to:

$$x_\infty - \bar{x}_\infty = A(x_\infty - \bar{x}_\infty) + B(u_\infty - \bar{u}_\infty) = (A+BK)(x_\infty - \bar{x}_\infty)\,, \tag{67}$$

where the last step comes from (22). It is important to notice that (67) and Assumption 4 implies that

$$x_\infty = \bar{x}_\infty\,. \tag{68}$$

In order to see this, note that (67) can be rewritten as $(I_n - A - BK)(x_\infty - \bar{x}_\infty) = 0$, which is certainly satisfied if either (68) holds or if $x_\infty - \bar{x}_\infty \in \text{null}(I_n - A - BK)$. It is also clear that (68) is the unique solution if $(I_n - A - BK)$ is full rank. Suppose that $(I_n - A - BK)$ is not full rank and let $x^* \in \mathbb{R}^n$ be such that $x^* \neq 0$ and $(I_n - A - BK)x^* = 0$. We would have $x^* = (A+BK)x^*$, that is x^* is an eigenvector of $(A+BK)$ associated with the eigenvalue $\lambda^* = 1$, which is in contradiction with Assumption 4 because all eigenvalues of $(A+BK)$ are strictly inside the unit circle. Hence, $(I_n - A - BK)$ is full rank and (68) holds. Finally, from (68) and from (20b) we obtain:

$$\begin{aligned}\bar{s} &= C_z\bar{x}_\infty = C_z x_\infty = \mathscr{C}\xi_\infty \\ &= \lim_{k\to\infty} \mathscr{C}\psi(k,\xi,s(\cdot),d(\cdot))\,.\end{aligned}$$

D Proof of Theorem 2

Robust constraint satisfaction follows immediately from the fact that $\mathscr{O}_\infty$ is robustly positively invariant for the closed-loop system (27) and the fact that $\mathscr{O}_\infty$ is constraint-admissible.

Note now that, since $\mathscr{A}_\mathscr{K}$ is strictly stable, $(I - \mathscr{A}_\mathscr{K})^{-1}$ exists and hence $\bar{\xi}$ is well-defined and unique. Note also from the proof of Lemma 3 that $\bar{\xi} = \xi_\infty := \lim_{k\to\infty} \psi(k,\xi,s(\cdot),d(\cdot))$. Furthermore, if $\bar{\xi} \in \text{int}(\mathscr{O}_\infty)$, then there exists a non-empty ball, centered around $\bar{\xi}$, which is contained in $\mathscr{O}_\infty$.

Robust asymptotic stability follows from Corollary 1 by defining

$$\zeta := \xi - \bar{\xi}, \qquad w := \mathscr{E}(d - \bar{d}) + \mathscr{F}(s - \bar{s})\,.$$

Hence, we can write the closed-loop system dynamics in terms of the "shifted" variables as $\zeta^+ = \mathscr{A}_\mathscr{K}\zeta + w$. The proof is completed by noting that $\lim_{k\to\infty} w(k) = 0$.

E Proof of Theorem 3

Though a result, similar to the one stated here, appears to be well-known [17, Sect. 4.2], we have been unable to find a detailed proof in the literature. Classical robust "open-loop" receding horizon control [7, Sect. 4.5] is well-known to exhibit "infeasibility" problems if the plant is open-loop unstable and no pre-stabilizing policy is used in the predictions [22]. However, it is a remarkable fact that one can remove this problem by optimizing over a sequence of perturbations to a pre-stabilizing control law. To show that this is indeed still true for the control algorithm proposed in this paper, we present a detailed proof.

It follows trivially from Assumption 5 that X_0 contains the origin in its interior. The rest of the proof is by induction.

Let the plant state $x \in X_i^{\mathbf{v}}$, where $i \in \{1,\ldots,N-1\}$, the controller state σ be such that $\mathscr{V}_i(\xi,s)$ is non-empty and $\mathbf{v}_i := (v_0,\ldots,v_{i-1}) \in \mathscr{V}_i(\xi,s)$ be an admissible perturbation sequence of length i. Also let $\mathbf{s}_{i-1} := (s_1,\ldots,s_{i-1}) \in \mathscr{S}^{i-1}$ and $\mathbf{d}_i := (d_0,\ldots,d_{i-1}) \in \mathscr{D}^i$ be a setpoint and a disturbance admissible sequences of length $i-1$ and i, respectively.

From the definition of $\mathscr{V}_i(\xi,s)$, it follows that $\chi(i,\xi,\mathbf{v}_i,s,\mathbf{s}_{i-1},\mathbf{d}_i)\in\mathscr{O}_\infty$ for all $\mathbf{s}_{i-1}\in\mathscr{S}^{i-1}$ and all $\mathbf{d}_i\in\mathscr{D}^i$. Recall that $\mathscr{O}_\infty$ is disturbance invariant and constraint-admissible for the closed-loop system (27), hence $\mathscr{O}_\infty$ is disturbance invariant and constraint-admissible for system (37) under the infinite perturbation sequence $\{v(k)\}_{k=0}^{\infty}:=\{0,0,\ldots\}$.

It follows that if $\chi(i,\xi,\mathbf{v}_i,s,\mathbf{s}_{i-1},\mathbf{d}_i)\in\mathscr{O}_\infty$ for all $\mathbf{s}_{i-1}\in\mathscr{S}^{i-1}$ and all $\mathbf{d}_i\in\mathscr{D}^i$, then the solution $\chi(i+1,\xi,(\mathbf{v}_i,0),s,\mathbf{s}_i,\mathbf{d}_{i+1})\in\mathscr{O}_\infty$ for all $\mathbf{s}_i\in\mathscr{S}^i$ and all $\mathbf{d}_{i+1}\in\mathscr{D}^{i+1}$.

This implies that if $\mathbf{v}_i\in\mathscr{V}_i(\xi,s)$, then $(\mathbf{v}_i,0)\in\mathscr{V}_{i+1}(\xi,s)$. Hence, if $\mathscr{V}_i(\xi,s)$ is non-empty, then $\mathscr{V}_{i+1}(\xi,s)$ is non-empty. It follows from the definition of $X_i^{\mathbf{v}}$ that if $x\in X_i^{\mathbf{v}}$, then $x\in X_{i+1}^{\mathbf{v}}$, hence $X_i^{\mathbf{v}}\subseteq X_{i+1}^{\mathbf{v}}$.

Using similar arguments as above, the result is completed by noticing that $X_0\subseteq X_1^{\mathbf{v}}$.

F Proof of Lemma 4

A similar result, for robust receding horizon controllers that do not provide offset-free control, is well-known [17, Rem. 3]. However, since different assumptions are made in this paper, a detailed proof is included. As will be seen, the proof is slightly involved.

From (20b) we can write

$$\overline{x}^*(\xi,s)=A\overline{x}^*(\xi,s)+B\overline{u}^*(\xi,s)+(x-\hat{x}+\hat{d}),$$

which, subtracted from (44b), leads to:

$$\tilde{x}_{k+1}-\overline{x}^*(\xi,s)=A(\tilde{x}_k-\overline{x}^*(\xi,s))+B(\tilde{u}_k-\overline{u}^*(\xi,s)),\quad \forall k\in\{0,1,\ldots,N-1\}. \tag{69}$$

Let

$$\begin{aligned} w_k &:= \tilde{x}_k-\overline{x}^*(\xi,s), && \forall k\in\{0,\ldots,N\},\\ \rho_k &:= \tilde{u}_k-\overline{u}^*(\xi,s), && \forall k\in\{0,\ldots,N-1\}. \end{aligned}$$

Notice that it immediately follows from (44c) that $\rho_k=Kw_k+v_k$. Hence, (69) can be rewritten as

$$w_{k+1}=Aw_k+B\rho_k=(A+BK)w_k+Bv_k=A_Kw_k+Bv_k,$$

where $A_K:=A+BK$.

We will now proceed to show that (a similar relation for the case of $N=\infty$ is given in [17, Rem. 3]):

$$V_N(\xi,s,\mathbf{v})=w_N^TPw_N+\sum_{k=0}^{N-1}w_k^TQw_k+\rho_k^TR\rho_k=w_0^TPw_0+\sum_{k=0}^{N-1}v_k^T(R+B^TPB)v_k=w_0^TPw_0+J_N(\mathbf{v}).$$

As a first step, note that

$$\begin{aligned} V_N(\xi,s,\mathbf{v}) &= w_N^TPw_N+\sum_{k=0}^{N-1}w_k^TQw_k+(Kw_k+v_k)^TR(Kw_k+v_k)\\ &= (A_Kw_{N-1}+Bv_{N-1})^TP(A_Kw_{N-1}+Bv_{N-1})\\ &\quad +w_{N-1}^TQw_{N-1}+(Kw_{N-1}+v_{N-1})^TR(Kw_{N-1}+v_{N-1})\\ &\quad +\sum_{k=0}^{N-2}w_k^TQw_k+(Kw_k+v_k)^TR(Kw_k+v_k)\\ &= w_{N-1}^T(Q+K^TRK+A_K^TPA_K)w_{N-1}+v_{N-1}^T(R+B^TPB^T)v_{N-1}\\ &\quad +2w_{N-1}^T(K^TR+A_K^TPB)v_{N-1}+\sum_{k=0}^{N-2}w_k^TQw_k+(Kw_k+v_k)^TR(Kw_k+v_k). \end{aligned}$$

From Assumption 6, P is the solution of the Riccati equation, hence we can write (after simple algebraic manipulations) that

$$P = Q + K^T RK + A_K^T PA_K .$$

Moreover, notice that

$$\begin{aligned} K^T R + A_K^T PB &= -A^T PB(R+B^T PB)^{-1}R + (A - B(R+B^T PB)^{-1}B^T PA)^T PB \\ &= -A^T PB(R+B^T PB)^{-1}R + A^T PB - A^T PB(R+B^T PB)^{-1}B^T PB \\ &= A^T PB\left[I_m - (R+B^T PB)^{-1}(R+B^T PB)\right] \\ &= 0 . \end{aligned}$$

Thus, we can write:

$$V_N(\xi,s,\mathbf{v}) = v_{N-1}^T(R+B^T PB)v_{N-1} + w_{N-1}^T P w_{N-1} + \sum_{k=0}^{N-2} w_k^T Q w_k + (Kw_k+v_k)^T R(Kw_k+v_k) .$$

In a similar way we can show that

$$\begin{aligned} w_{N-1}^T P w_{N-1} + \sum_{k=0}^{N-2} w_k^T Q w_k + (Kw_k+v_k)^T R(Kw_k+v_k) = v_{N-2}^T(R+B^T PB)v_{N-2} + w_{N-2}^T P w_{N-2} + \\ \sum_{k=0}^{N-3} w_k^T Q w_k + (Kw_k+v_k)^T R(Kw_k+v_k) , \end{aligned}$$

obtaining that

$$\begin{aligned} V_N(\xi,s,\mathbf{v}) = v_{N-1}^T(R+B^T PB)v_{N-1} + v_{N-2}^T(R+B^T PB)v_{N-2} + w_{N-2}^T P w_{N-2} \\ + \sum_{k=0}^{N-3} w_k^T Q w_k + (Kw_k+v_k)^T R(Kw_k+v_k) . \end{aligned}$$

By repeating these calculations to replace all terms in the sum we finally obtain

$$V_N(\xi,s,\mathbf{v}) = \sum_{k=0}^{N-1} v_k^T(R+B^T PB)v_k + w_0^T P w_0 = J_N(\mathbf{v}) + w_0^T P w_0 .$$

The fact that (54) holds, trivially comes from the observation that $V_N(\xi,s,\mathbf{v})$ and $J_N(\mathbf{v})$ differ from each other by a term, independent of the decision variable $\mathbf{v}$, i.e. $(x-\bar{x}^*(\xi,s))^T P(x-\bar{x}^*(\xi,s))$.

G Proof of Lemma 5

The proof for the first part is similar to the proof of [17, Lem. 7]. However, since different assumptions are made in this paper, a detailed proof is included.

Assume $\mathscr{V}_N(\xi,s)$ is non-empty and let $\mathbf{v}^*(\xi,s) := \left(v_0^*(\xi,s),\ldots,v_{N-1}^*(\xi,s)\right)$ be the associated minimizer of problem $\mathbb{P}_N(\xi,s)$. Consider also the candidate perturbation sequence for the augmented state ξ^+ and the setpoint s^+ at the next time instant, i.e.

$$\tilde{\mathbf{v}}(\xi,s) := \left(v_1^*(\xi,s),\ldots,v_{N-1}^*(\xi,s),0\right) .$$

Using similar arguments as in the proof of Theorem 3, given the set of possible augmented states $f(\xi,s,\mathscr{D})$ at the next time instant, it follows that if $\xi^+ \in f(\xi,s,\mathscr{D})$, then $\tilde{\mathbf{v}}(\xi,s)$ is an admissible input perturbation sequence for all $s^+ \in \mathscr{S}$, i.e. $\tilde{\mathbf{v}}(\xi,s) \in \mathscr{V}_N(f(\xi,s,d),s^+)$ for all $s^+ \in \mathscr{S}$ and all $d \in \mathscr{D}$. This proves that if $\mathscr{V}_N(\xi(0),s(0))$ is non-empty, then $\mathscr{V}_N(\xi(k),s(k))$ is non-empty for all $k \in \{1,2,\ldots\}$ and all allowable setpoint and disturbance sequences.

Note that the second part of the proof, except for the last few lines, is similar to the proof of [17, Thm. 8]. Once again, since different assumptions are made in this paper, a detailed proof is included.

If we let $J_N^*(\xi,s) := J_N(\mathbf{v}^*(\xi,s))$, then it is clear that

$$J_N^*(\xi,s) = J_N(\mathbf{v}^*(\xi,s)) \geq J_N(\tilde{\mathbf{v}}(\xi,s)) \geq J_N(\mathbf{v}^*(\xi^+,s^+)) = J_N^*(\xi^+,s^+)$$

for all $\xi^+ \in f(\xi,s,\mathscr{D})$ and all $s^+ \in \mathscr{S}$. This implies that, for all allowable setpoint and disturbance sequences, the sequence $\{J_N^*(\xi(k),s(k))\}_{k=0}^{\infty}$ is a non-negative, non-increasing sequence. Hence, the sequence converges to some non-negative value, which implies that

$$\lim_{k\to\infty} J_N^*(\xi(k),s(k)) - J_N^*(\xi(k+1),s(k+1)) = 0\,.$$

However, we can write (recalling that W is positive definite)

$$\begin{aligned} 0 \leq v_0^*(\xi(k),s(k))^T W v_0^*(\xi(k),s(k)) &= J_N^*(\xi(k),s(k)) - J_N(\tilde{\mathbf{v}}(\xi(k),s(k))) \\ &\leq J_N^*(\xi(k),s(k)) - J_N^*(\xi(k+1),s(k+1))\,, \end{aligned}$$

which implies that

$$\lim_{k\to\infty} v_0^*(\xi(k),s(k))^T W v_0^*(\xi(k),s(k)) = 0\,.$$

Since W is positive definite, it follows that

$$\lim_{k\to\infty} v_0^*(\xi(k),s(k)) = 0\,.$$

H Proof of Theorem 4

Sufficiency. Suppose that $x(0) \in X_N^{\mathbf{v}}$, then it immediately follows from (48) that for any initial setpoint $s(0) \in \mathscr{S}$ one can choose a controller state $\sigma(0)$ such that $\mathscr{V}_N(\xi(0),s(0)) \neq \emptyset$ and hence $\mathbb{P}_N(\xi(0),s(0))$ has a solution. This implies from Lemma 5 we have that $\mathscr{V}_N(\xi(k)) \neq \emptyset$ for all $k \in \mathbb{N}$ and also that

$$v_\infty := \lim_{k\to\infty} v(k) := \lim_{k\to\infty} v_0^*(\xi(k),s(k)) = 0\,. \tag{70}$$

The fact that (10a) holds can now be shown exactly as in the proof of Lemma 3, since from (37) and (70) it follows that

$$\begin{aligned} \lim_{k\to\infty} \phi(k,\xi,s(\cdot),d(\cdot)) &= \mathscr{A}_{\mathscr{K}}\xi_\infty + \mathscr{B}v_\infty + \mathscr{E}\bar{d} + \mathscr{F}\bar{s} \\ &= \mathscr{A}_{\mathscr{K}}\xi_\infty + \mathscr{E}\bar{d} + \mathscr{F}\bar{s} \\ &= \mathscr{A}\xi_\infty + \mathscr{B}u_\infty + \mathscr{E}\bar{d} \\ &= \xi_\infty, \end{aligned}$$

in which $u_\infty = \mathscr{K}\xi_\infty + \mathscr{L}\bar{s} + v_\infty = \mathscr{K}\xi_\infty + \mathscr{L}\bar{s}$.

The fact that (10b) holds follows trivially from Lemma 5 and the definition of $\mathscr{V}_N(\cdot)$.

Necessity. This is obvious because if $x(0) \notin X_N^{\mathbf{v}}$, then we either have that $x(0) \notin \mathscr{X}$ or that there exists an $s(0) \in \mathscr{S}$ such that for all $\sigma(0) \in \mathbb{R}^{2n}$, $\mathscr{V}_N(\xi(0),s(0)) = \emptyset$ and hence the control input is undefined at time 0.

Finally, robust asymptotic stability can be shown in a similar fashion as in the proof of Theorem 2. This is because it is easy to show that, for any $\xi \in \mathscr{O}_\infty$, the optimal perturbation $v_0^*(\xi, s) = 0$ for all $s \in \mathscr{S}$. Hence, we can write the closed-loop system dynamics in a neighborhood of $\bar{\xi}$, in terms of the "shifted" variables, as $\zeta^+ = \mathscr{A}_{\mathscr{K}}\zeta + w$.

Note that the condition that $\bar{\xi} \in \text{int}(\mathscr{O}_\infty)$ is fundamental to the proof; we do not yet have a method for relaxing this assumption. If $\bar{\xi}$ is on the boundary of $\mathscr{O}_\infty$, then it is possible that the optimal perturbation $v_0^*(\xi, s)$ is non-zero in a neighborhood of $\bar{\xi}$; a non-zero $v_0^*(\xi, s)$ may "destabilize" the system for a subset of initial states in a neighborhood of $\bar{\xi}$ (though robust convergence to $\bar{\xi}$ is, of course, still guaranteed).

I Computation of Matrices in Section 4.2

Let the polyhedra $\mathscr{X}$, $\mathscr{U}$ and $\mathscr{O}_\infty$ be defined by

$$\mathscr{X} := \{x \in \mathbb{R}^n \mid S_x x \leq b_x\}, \tag{71}$$

$$\mathscr{U} := \{u \in \mathbb{R}^m \mid S_u u \leq b_u\}, \tag{72}$$

$$\mathscr{O}_\infty := \left\{\xi \in \mathbb{R}^{3n} \;\middle|\; S_\xi \xi \leq b_\xi \right\}, \tag{73}$$

where $S_x \in \mathbb{R}^{q_x \times n}$, $S_u \in \mathbb{R}^{q_u \times m}$, $S_\xi \in \mathbb{R}^{q_\xi \times 3n}$, $b_x \in \mathbb{R}^{q_x}$, $b_u \in \mathbb{R}^{q_u}$, $b_\xi \in \mathbb{R}^{q_\xi}$ and let the matrices $T_x \in \mathbb{R}^{q_x \times 3n}$, $T_u \in \mathbb{R}^{q_u \times 3n}$ and $T_s \in \mathbb{R}^{q_u \times p}$ be defined as

$$T_x := \begin{bmatrix} S_x & 0 \end{bmatrix}, \quad T_u := S_u \mathscr{K}, \quad T_s := S_u \mathscr{L}. \tag{74}$$

Given the above, it follows from (41) that

$$\mathscr{V}_N(\xi, s) = \left\{ \mathbf{v} \in \mathbb{R}^{mN} \;\middle|\; \begin{array}{c} \xi_0 = \xi,\ s_0 = s,\ T_x \xi_k \leq b_x,\ k = 1, \ldots, N-1,\ S_\xi \xi_N \leq b_\xi \text{ and} \\ T_u \xi_k + T_s s_k + S_u v_k \leq b_u,\ k = 0, \ldots, N-1 \text{ for all } \mathbf{s} \in \mathscr{S}^{N-1},\ \mathbf{d} \in \mathscr{D}^N \end{array} \right\}. \tag{75}$$

Let

$$q := (N-1)q_x + Nq_u + q_\xi \tag{76}$$

and the matrices $L \in \mathbb{R}^{q \times mN}$, $M \in \mathbb{R}^{q \times (N+1)3n}$, $M_{\mathbf{s}} \in \mathbb{R}^{q \times (N-1)p}$, $M_s \in \mathbb{R}^{q \times p}$ be given by

$$L := \begin{bmatrix} 0 \\ I_N \otimes S_u \end{bmatrix}, \quad M_{\mathbf{s}} := \begin{bmatrix} 0 \\ I_{N-1} \otimes T_s \end{bmatrix}, \quad M_s := \begin{bmatrix} 0 \\ \tilde{\mathbf{1}}_N \otimes T_s \end{bmatrix}, \tag{77a}$$

$$M := \begin{bmatrix} 0 & I_{N-1} \otimes T_x & 0 \\ 0 & 0 & S_\xi \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I_N \otimes T_u & 0 \end{bmatrix}. \tag{77b}$$

If we let the block vectors $b \in \mathbb{R}^q$ and $\mathbf{x} \in \mathbb{R}^{3n(N+1)}$ be defined as

$$b := \begin{bmatrix} \mathbf{1}_{N-1} \otimes b_x \\ b_\xi \\ \mathbf{1}_N \otimes b_u \end{bmatrix}, \quad \mathbf{x} := \begin{bmatrix} \xi_0 \\ \vdots \\ \xi_N \end{bmatrix}, \tag{78}$$

then it is easy to verify from (75) that

$$\mathscr{V}_N(\xi, s) = \left\{ \mathbf{v} \in \mathbb{R}^{mN} \;\middle|\; \xi_0 = \xi,\ L\mathbf{v} + M\mathbf{x} + M_s s + M_{\mathbf{s}}\mathbf{s} \leq b \text{ for all } \mathbf{s} \in \mathscr{S}^{N-1},\ \mathbf{d} \in \mathscr{D}^N \right\}. \tag{79}$$

If we now let the block matrices $\mathbf{A} \in \mathbb{R}^{3n(N+1)\times 3n}$, $\mathbf{B} \in \mathbb{R}^{3n(N+1)\times mN}$, $\mathbf{E} \in \mathbb{R}^{3n(N+1)\times rN}$, $\mathbf{F}_s \in \mathbb{R}^{3n(N+1)\times p}$ and $\mathbf{F_s} \in \mathbb{R}^{3n(N+1)\times p(N-1)}$ be defined as

$$\mathbf{A} = \begin{bmatrix} I \\ \mathscr{A}_{\mathscr{K}} \\ \mathscr{A}_{\mathscr{K}}^2 \\ \vdots \\ \mathscr{A}_{\mathscr{K}}^N \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \mathscr{B} & 0 & \dots & 0 \\ \mathscr{A}_{\mathscr{K}}\mathscr{B} & \mathscr{B} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathscr{A}_{\mathscr{K}}^{N-1}\mathscr{B} & \mathscr{A}_{\mathscr{K}}^{N-2}\mathscr{B} & \dots & \mathscr{B} \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \mathscr{E} & 0 & \dots & 0 \\ \mathscr{A}_{\mathscr{K}}\mathscr{E} & \mathscr{E} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathscr{A}_{\mathscr{K}}^{N-1}\mathscr{E} & \mathscr{A}_{\mathscr{K}}^{N-2}\mathscr{E} & \dots & \mathscr{E} \end{bmatrix}, \tag{80a}$$

$$\mathbf{F}_s = \begin{bmatrix} 0 \\ \mathscr{F} \\ \mathscr{A}_{\mathscr{K}}\mathscr{F} \\ \vdots \\ \mathscr{A}_{\mathscr{K}}^{N-1}\mathscr{F} \end{bmatrix}, \quad \mathbf{F_s} = \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \mathscr{F} & \dots & 0 \\ \vdots & \ddots & \vdots \\ \mathscr{A}_{\mathscr{K}}^{N-2}\mathscr{F} & \dots & \mathscr{F} \end{bmatrix}, \tag{80b}$$

then it follows that

$$\mathbf{x} = \mathbf{A}\xi_0 + \mathbf{B}\mathbf{v} + \mathbf{E}\mathbf{d} + \mathbf{F}_s s + \mathbf{F_s}\mathbf{s}. \tag{81}$$

Finally, by substituting (81) into (79) it follows that

$$\mathscr{V}_N(\xi, s) = \left\{ \mathbf{v} \in \mathbb{R}^{mN} \;\middle|\; F\mathbf{v} \leq b + G^d\mathbf{d} + G^s\mathbf{s} + H^\xi\xi + H^s s \text{ for all } \mathbf{s} \in \mathscr{S}^{N-1},\ \mathbf{d} \in \mathscr{D}^N \right\}, \tag{82}$$

where

$$F := L + M\mathbf{B}, \quad G^d := -M\mathbf{E}, \quad G^s := -M\mathbf{F_s} - M_\mathbf{s}, \quad H^\xi := -M\mathbf{A}, \quad H^s := -M\mathbf{F}_s - M_s. \tag{83}$$

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