Approximation of the minimal robustly positively invariant set for discrete-time LTI systems with persistent state disturbances

S.V. Raković*†, E.C. Kerrigan‡§, K.I. Kouramas*¶ and D.Q. Mayne*

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Abstract

This paper provides a solution to the problem of computing a robustly positively invariant outer approximation of the minimal robustly positively invariant set for a discrete-time linear time-invariant system. It is assumed that the disturbance is additive and persistent, but bounded.

Keywords: Set invariance, invariant approximations, constrained control, robust control, linear systems.

1 Introduction

The theory of set invariance plays a fundamental role in the control of constrained systems and has been a subject of research by many authors — see for instance [Bla99, Ker00] and the references therein. Two important issues are the calculation of the minimal robustly positively invariant (mRPI) set and the maximal robustly positively invariant (MRPI) set.

The mRPI set is used as a target set in robust time-optimal control [MS97], in the design of robust predictive controllers [ML01, LCRM04, KM03, KMar] and in understanding the properties of the *maximal* robustly positively invariant set [KG98, Kou02]. The MRPI set has been used extensively for the design of reference governors [GK95], for the regulation problem and for the calculation of the region of feasibility in model predictive control problems [Ker00, CRZ01].

Despite this wide use of the mRPI and MRPI sets, there are still unresolved issues. For the case of the mRPI set, there exists no method for the exact computation of the mRPI set, except

*Department of Electrical and Electronic Engineering, Imperial College London, London SW7 2BT, United Kingdom. Tel: +44-(0)20-7594-6295/87/81. Fax: +44-(0)20-7594-6282, E-mail: sasa.rakovic@imperial.ac.uk, d.mayne@imperial.ac.uk, kkouramas@imperial.ac.uk

†Corresponding author.

‡Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, United Kingdom. Tel: +44-(0)1223-332600. Fax: +44-(0)1223-332662. Email: erickerrigan@ieee.org

§Royal Academy of Engineering Post-doctoral Research Fellow.

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those given in [Las93, Sect. 3.3], [MS97, Thm. 3], [SM98, Sect. II.B], where it is assumed that the closed-loop system dynamics are nilpotent.

It is the purpose of this paper to provide methods for computation of invariant approximation of the minimal robustly positively invariant set.

Moreover, we will discuss a set of useful *a priori* efficient tests and computation of upper bounds relevant to the proposed method.

This paper is organized as follows. Section 2 is concerned with the definitions of the mRPI and MRPI sets and the problem formulation. Section 3 deals with the problem of calculating a robustly positively invariant (RPI) approximation of the mRPI set for systems with disturbance inputs which are bounded. Computational issues are discussed in Section 4. A set of illustrative examples is provided in Section 5. Finally, Section 6 presents conclusions.

Notation: Let $\mathbb{N} \triangleq \{0, 1, 2, \ldots\}$ be the set of non-negative integers, $\mathbb{N}_+ \triangleq \{1, 2, \ldots\}$ the set of integers greater than 0, the set of integers $\mathbb{N}_{[a,b]} \triangleq \{a, a+1, \ldots, b-1, b\}$ if $0 \leq a \leq b$. Given a vector $v \in \mathbb{R}^n$ and matrix $M \in \mathbb{R}^{m \times n}$, $\|v\|_p$ is the vector p-norm and $\|M\|_p$ is the induced matrix p-norm. If M is a square matrix, then $\rho(M)$ is the spectral radius of M. Let $\mathbb{B}_p^n(r) \triangleq \{x \in \mathbb{R}^n \mid \|x\|_p \leq r\}$ be a p-norm ball in $\mathbb{R}^n$, where $r \geq 0$. Given two sets $\mathcal{U}$ and $\mathcal{V}$, such that $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^n$, the Minkowski (vector) sum is defined by $\mathcal{U} \oplus \mathcal{V} \triangleq \{u+v \mid u \in \mathcal{U},\ v \in \mathcal{V}\}$, and the Pontryagin difference as $\mathcal{U} \ominus \mathcal{V} \triangleq \{u \in \mathbb{R}^n \mid u + v \in \mathcal{U},\ \forall v \in \mathcal{V}\} = \{u \in \mathbb{R}^n \mid u \oplus \mathcal{V} \subseteq \mathcal{U}\}$. Given the collection of sets $\{\mathcal{U}_i \subset \mathbb{R}^n \mid i \in \mathbb{N}_{[a,b]}\}$, where $a \leq b$, we denote $\bigoplus_{i=a}^{b} \mathcal{U}_i \triangleq \mathcal{U}_a \oplus \mathcal{U}_{a+1} \oplus \ldots \oplus \mathcal{U}_b$. Let $v(\cdot) \triangleq \{v(0), v(1), \ldots\}$ denote an infinite sequence of variables, where $v(k)$ is the k'th element in the sequence. The set $\mathcal{M}_{\mathcal{V}} \triangleq \{v(\cdot) \mid v(k) \in \mathcal{V},\ \forall k \in \mathbb{N}\}$ is the set of all infinite sequences that take on values in $\mathcal{V} \subseteq \mathbb{R}^n$ (equivalently $\mathcal{M}_{\mathcal{V}}$ is the set of all maps $v : \mathbb{N} \to \mathcal{V}$). We use $\lceil x \rceil$ to denote the least integer greater or equal to x.

2 Preliminary Definitions and Problem Formulation

We consider the following autonomous discrete-time linear time-invariant (DLTI) system:

$$x^+ = Ax + w, \tag{1}$$

where $x \in \mathbb{R}^n$ is the current state, x^+ is the successor state and $w \in \mathbb{R}^n$ is an unknown disturbance. A standing assumption is that $A \in \mathbb{R}^{n \times n}$ is a strictly stable matrix (all the eigenvalues of A are strictly inside the unit disk, i.e. $\rho(A) < 1$). The disturbance w is persistent, but contained in a convex and compact (i.e. closed and bounded) set $W \subset \mathbb{R}^n$, which contains the origin.

If the initial state is x at time 0 (note that since the system is time-invariant, the current time can always be taken to be zero), then we denote by $\phi(k, x, w(\cdot))$ the solution to (1) at time instant k, given the infinite disturbance sequence $w(\cdot) \triangleq \{w(0), w(1), \ldots\}$.

The motivation for this paper comes from the fact that often one would like to determine whether the state trajectory of the system will be contained in a set $X \subset \mathbb{R}^n$, given any allowable disturbance sequence. For this purpose, recall the following definition:

Definition 1 (RPI set). [Bla99] The set $\Omega \subset \mathbb{R}^n$ is a *robustly positively invariant* (RPI) set of (1) if $Ax + w \subseteq \Omega$ for all $x \in \Omega$ and all $w \in W$.

Remark 1. It is useful to note that, by definition, Ω is RPI if and only if $A\Omega \oplus W \subseteq \Omega$. Note also that Ω is RPI if and only if $A\Omega \subseteq \Omega \ominus W$.

Definition 2 (Constraint-admissible set). The set $\Omega \subset \mathbb{R}^n$ is a *constraint-admissible* set if it is contained in $X \subset \mathbb{R}^n$.

Remark 2. Clearly, the set Ω is a *constraint-admissible*, RPI set if it is contained in X and Ω is RPI.

The existence of RPI sets is very important for the satisfaction of constraints. For example, it is well-known [Bla99] that the solution of the system will satisfy $\phi(k, x, w(\cdot)) \in X$ for all time $k \in \mathbb{N}$ and all allowable disturbance sequences $w(\cdot) \in \mathcal{M}_W$ *if and only if* there exists a constraint-admissible, RPI set Ω and the initial state $x \in \Omega$.

An important set in the analysis and synthesis of controllers for constrained systems is the minimal RPI set:

Definition 3 (mRPI set). The *minimal robustly positively invariant* (mRPI) set F_∞ of (1) is the RPI set of (1) that is contained in every closed, RPI set of (1).

The properties of the mRPI set F_∞ are well-known. It is possible to show [KG98, Sect. IV] that the mRPI set F_∞ exists, is unique, compact and contains the origin. It is also easy to show that the zero initial condition response of (1) is bounded in F_∞, i.e. $\phi(k, 0, w(\cdot)) \in F_\infty$ for all $w(\cdot) \in \mathcal{M}_W$ and all $k \in \mathbb{N}$. It therefore follows, from the linearity and asymptotic stability of system (1), that F_∞ is the limit set of all trajectories of (1). In particular, F_∞ is the smallest closed set in $\mathbb{R}^n$ that has the following property: given any $r > 0$ and $\varepsilon > 0$, there exists a $\bar{k} \in \mathbb{N}$ such that if $x \in \mathbb{B}_p^n(r)$, then the solution of (1) satisfies $\phi(k, x, w(\cdot)) \in F_\infty \oplus \mathbb{B}_p^n(\varepsilon)$ for all $w(\cdot) \in \mathcal{M}_W$ and all $k \geq \bar{k}$.

We now turn our attention to methods for computing F_∞. If we were to define the (convex and compact) set F_s as

$$F_s \triangleq \bigoplus_{i=0}^{s-1} A^i W, \tag{2}$$

then it is possible to show [KG98, Sect. IV] that $F_s \subseteq F_\infty$ and that $F_s \to F_\infty$ as $s \to \infty$, i.e. for every $\varepsilon > 0$, there exists an $s \in \mathbb{N}$ such that $F_\infty \subseteq F_s \oplus \mathbb{B}_p^n(\varepsilon)$. Clearly, F_∞ is then given by

$$F_\infty = \bigoplus_{i=0}^{\infty} A^i W. \tag{3}$$

Since F_∞ is a Minkowski sum of infinitely many terms, it is generally impossible to obtain an explicit characterization of it. However, as noted in [Las93, Sect. 3.3] and [KG98, Rem. 4.2], it is possible to show that if there exist an integer $s \in \mathbb{N}_+$ and a scalar $\alpha \in [0, 1)$ such that $A^s = \alpha I$, then $F_\infty = (1-\alpha)^{-1} \bigoplus_{i=0}^{s-1} A^i W$. It therefore follows trivially [MS97, Thm. 3] that if A is nilpotent with index s ($A^s = 0$), then $F_\infty = \bigoplus_{i=0}^{s-1} A^i W$.

In this paper, we relax the assumption that there exists an $s \in \mathbb{N}_+$ and a scalar $\alpha \in [0, 1)$ such that $A^s = \alpha I$. Since we can no longer compute F_∞ exactly, we address the problem of computing an RPI, outer approximation of the mRPI set F_∞.

We will continue to address the problem of computing an RPI set that contains the mRPI set F_∞. This will be achieved by scaling F_s by a suitable amount.

3 Approximating the minimal robustly positively invariant set

Our main result is as follows:

Theorem 1. *[Kou02] If $0 \in \text{int}(W)$, then there exist a finite integer $s \in \mathbb{N}_+$ and scalar $\alpha \in [0,1)$ that satisfies*

$$A^s W \subseteq \alpha W. \tag{4}$$

Furthermore, if (4) is satisfied, then

$$F(\alpha, s) \triangleq (1-\alpha)^{-1} F_s \tag{5}$$

is a convex, compact, RPI set of (1). Furthermore, $0 \in \text{int}(F(\alpha, s))$ and $F_\infty \subseteq F(\alpha, s)$.

Proof. Existence of an $s \in \mathbb{N}_+$ and $\alpha \in [0,1)$ that satisfies (4) follows from the fact that the origin is in the *interior* of W and that A is strictly stable.

Convexity and compactness of $F(\alpha, s)$ follows directly from the fact that F_s (and hence $F(\alpha, s)$) is the Minkowski sum of a finite set of convex and compact sets.

Let $G(\alpha, j, k) \triangleq (1-\alpha)^{-1} \bigoplus_{i=j}^{k} A^i W$. It follows that

$$\begin{aligned}
AG(\alpha, 0, s-1) \oplus W &= G(\alpha, 1, s) \oplus W && \text{(6a)}\\
&= (1-\alpha)^{-1} A^s W \oplus G(\alpha, 1, s-1) \oplus W && \text{(6b)}\\
&\subseteq (1-\alpha)^{-1} \alpha W \oplus W \oplus G(\alpha, 1, s-1) && \text{(6c)}\\
&= [(1-\alpha)^{-1}\alpha + 1] W \oplus G(\alpha, 1, s-1) && \text{(6d)}\\
&= (1-\alpha)^{-1} W \oplus G(\alpha, 1, s-1) && \text{(6e)}\\
&= G(\alpha, 0, s-1). && \text{(6f)}
\end{aligned}$$

In going from (6b) to (6c) we have used the fact that $P \subseteq Q \Rightarrow P \oplus R \subseteq Q \oplus R$ for arbitrary sets $P \subset \mathbb{R}^n$, $Q \subset \mathbb{R}^n$ and $R \subset \mathbb{R}^n$.

Since $F(\alpha, s) = G(\alpha, 0, s-1)$, it follows that $AF(\alpha, s) \oplus W \subseteq F(\alpha, s)$ holds, hence $F(\alpha, s)$ is RPI. It follows trivially from the definition of the mRPI set that $F(\alpha, s)$ contains F_∞. Note also that $0 \in \text{int}(F_\infty)$ if $0 \in \text{int}(W)$.

□

Note that

$$F(\alpha_0, s) \subset F(\alpha_1, s) \Leftrightarrow \alpha_0 < \alpha_1. \tag{7}$$

Furthermore, if A is not nilpotent, then

$$F(\alpha, s_0) \subset F(\alpha, s_1) \Leftrightarrow s_0 < s_1. \tag{8}$$

Clearly, based on these observations, one can obtain a better approximation of the mRPI set F_∞, given an initial pair (α, s). Let

$$s^o(\alpha) \triangleq \inf \{ s \in \mathbb{N}_+ \mid A^s W \subseteq \alpha W \}, \tag{9a}$$

$$\alpha^o(s) \triangleq \inf \{ \alpha \in [0, 1) \mid A^s W \subseteq \alpha W \} \tag{9b}$$

be the smallest values of s and α such that (4) holds for a given α and s, respectively.

Remark 3. The infimum in (9a) exists for any choice of $\alpha \in (0, 1)$; $s^o(0)$ is finite if and only if A is nilpotent. Note that $s^o(\alpha) \to \infty$ as $\alpha \searrow 0$ if and only if A is not nilpotent. The infimum in (9b) is also guaranteed to exist if s is sufficiently large. Note that there exists a finite s such that $\alpha^o(s) = 0$ if and only if A is nilpotent. However, if A is not nilpotent, then $\alpha^o(s) \searrow 0$ as $s \to \infty$.

By a process of iteration one can use the above definitions and results to compute a pair (α, s) such that $F(\alpha, s)$ is a sufficiently good RPI, outer approximation of F_∞.

For example, by starting with $s = 1$, one can increment s until there exists an $\alpha \in [0, 1)$ such that (4) holds. One can then compute $\alpha^o(s)$ and use $F(\alpha^o(s), s)$ as an RPI aproximation to F_∞. If necessary, one can increase s until $F(\alpha^o(s), s)$ is a sufficiently good approximation of F_∞.

Alternatively, one can take an initial value for α, compute $s^* \triangleq s^o(\alpha)$, proceed to compute $\alpha^* \triangleq \alpha^o(s^*)$ and test whether $F(\alpha^*, s^*)$ is small enough. It is clear that this iteration results in

$$F_\infty \subseteq F(\alpha^*, s^*) \subseteq F(\alpha, s^*) \subseteq F(\alpha, s). \tag{10}$$

If $F(\alpha^*, s^*)$ is not a good enough approximation of F_∞, then this procedure could be restarted by choosing a smaller value for α.

Of course, any other iteration can be implemented until a fixed point is reached or $F(\alpha, s)$ is deemed to be a sufficiently good approximation of F_∞.

Clearly, the case when the origin is in the interior of W does not pose any problems with regards the existence of an $\alpha \in [0, 1)$ and a finite $s \in \mathbb{N}_+$ that satisfy (4), provided one bear in mind whether or not A is nilpotent. This result can be extended to a more general case, when the interior of W is empty, but the origin is in the relative interior of W. This is a subject of ongoing research.

4 Efficient computations and *a priori* upper bounds

This section will present results that allow for the development of efficient tests and computations of an *a priori* upper bound for the condition in (4). Results will also be given that allow for the efficient computation of $s^o(\alpha)$ and $\alpha^o(s)$ in (9).

Note that if all the sets mentioned in this paper are polyhedra or polytopes (bounded polyhedra), then efficient computations are possible. Computations are also much simpler if the sets contain the origin in their interiors. As such, we will assume throughout this section that W and X are polyhedra, which contain the origin in their interiors.

If X and W are polyhedra/polytopes, then the computation of the Minkowski sum, Pontryagin difference, linear maps and inverses of linear maps can be done by using standard software for manipulating polytopes. These packages therefore allow one to compute F_s and $F(\alpha, s)$.

However, often we are not interested in the explicit computation of these sets, but only whether the condition presented in (4) is satisfied or whether $F(\alpha, s) \subseteq X$, where X is any polyhedron. This is the case we will mainly be addressing in this section.

For this purpose, recall the following definition:

Definition 4 (Support function). The *support function* of a set $\Pi \subset \mathbb{R}^n$, evaluated at $z \in \mathbb{R}^n$, is defined as

$$h(\Pi, z) \triangleq \sup_{\pi \in \Pi} z^T \pi. \tag{11}$$

Our main interest in the support function is the well-known fact that the support function of a set allows one to write equivalent conditions for the set to be a subset of another. In particular:

Proposition 1. *Let Π be a non-empty set in $\mathbb{R}^n$ and the polyhedron*

$$\Psi = \left\{ \psi \in \mathbb{R}^n \;\middle|\; f_i^T \psi \leq g_i, \; i \in \mathcal{I} \right\}, \tag{12}$$

where $f_i \in \mathbb{R}^n$, $g_i \in \mathbb{R}$ and $\mathcal{I}$ is a finite index set:

(i) $\Pi \subseteq \Psi$ *if and only if* $h(\Pi, f_i) \leq g_i$ *for all* $i \in \mathcal{I}$.

(ii) $\Pi \subseteq \operatorname{int}(\Psi)$ *if and only if* $h(\Pi, f_i) < g_i$ *for all* $i \in \mathcal{I}$.

The following result allows one to compute the support function of a set that is the Minkowski sum of a finite sequence of linear maps of non-empty, compact sets:

Proposition 2. *Let each matrix $L_k \in \mathbb{R}^{n \times m}$ and each Φ_k be a non-empty, compact set in $\mathbb{R}^m$ for all $k \in \{1, \dots, K\}$. If*

$$\Pi = \bigoplus_{k=1}^{K} L_k \Phi_k, \tag{13}$$

then

$$h(\Pi, z) = \sum_{k=1}^{K} \max_{\phi \in \Phi_k} (z^T L_k) \phi. \tag{14}$$

Furthermore, if $\Phi_k = \mathbb{B}_\infty^m(1)$, then

$$\max_{\phi \in \Phi_k} (z^T L_k) \phi = \|L_k^T z\|_1. \tag{15}$$

Proof. The result follows immediately from the fact that if $\pi \triangleq \pi_1 + \cdots + \pi_k$, where each $\pi_k \in L_k \Phi_k$, then $h(\Pi, z) = \max\left\{ z^T \pi \mid \pi \in \Pi \right\} = \max\left\{ z^T(\pi_1 + \cdots + \pi_K) \mid \pi_k \in L_k \Phi_k, \; k = 1, \dots, K \right\} =$

$\sum_{k=1}^{K} \max \left\{ z^T \pi_k \mid \pi_k \in L_k \Phi_k \right\}$. The last equality follows because of the fact that the constraints on π_k are independent of the constraint on π_l for all $k \neq l$. Noting that $\max \left\{ z^T \pi_k \mid \pi_k \in L_k \Phi_k \right\} = \max \left\{ z^T L_k \phi_k \mid \phi_k \in \Phi_k \right\}$, it follows that (14) holds. The fact that (15) holds can be proven in a similar manner [KM03, Prop. 2]. □

Remark 4. Clearly, if all the Φ_k in the above result are polytopes, then the computation of the value of the support function in (14) can be done by solving K LPs. However, it is extremely useful to note that if any Φ_k is a hypercube (∞-norm ball), then the value of the support function of Π can be computed much faster by evaluating the explicit expression in (15).

Note that, by a straightforward application of Propositions 1 and 2, it follows that (4) can be checked efficiently, without having to compute the respective linear maps or Minkowski sums. The same is true when testing whether $F(\alpha, s) \subseteq X$, where X is any polyhedron. Clearly, $s^{\mathrm{o}}(\alpha)$ and $\alpha^{\mathrm{o}}(s)$ can also be computed by solving a finite number of LPs.

Before proceeding, note that one can compute the size of the smallest hypercube containing $F(\alpha, s)$ by solving a finite number of LPs, without having to compute $F(\alpha, s)$ explicitly. This claim will now be justified.

Definition 5 (Smallest/largest hypercube). Let Ψ be a non-empty, compact set in $\mathbb{R}^n$ containing the origin. The size of the largest hypercube in Ψ is defined as

$$\beta_{\text{in}}(\Psi) \triangleq \max \left\{ r \geq 0 \mid \mathbb{B}_\infty^n(r) \subseteq \Psi \right\} \tag{16}$$

and the size of the smallest hypercube containing Ψ is defined as

$$\beta_{\text{out}}(\Psi) \triangleq \min \left\{ r \geq 0 \mid \Psi \subseteq \mathbb{B}_\infty^n(r) \right\}. \tag{17}$$

The next result follows from a straightforward application of Propositions 1 and 2:

Proposition 3. *Let Ψ be a non-empty, compact set in $\mathbb{R}^n$ containing the origin.*

(i) The smallest hypercube containing Ψ is given by

$$\beta_{\text{out}}(\Psi) = \max_{j \in \{1, \ldots, n\}} \max\{h(\Psi, e_j), h(\Psi, -e_j)\}, \tag{18}$$

where e_j denotes the j'th standard basis vector in $\mathbb{R}^n$.

(ii) If Ψ is a polytope given as in (12)*, then*

$$\beta_{\text{in}}(\Psi) = \min_{i \in \mathcal{I}} \frac{g_i}{\|f_i\|_1}. \tag{19}$$

Remark 5. The above result implies that if Ψ is a polytope, then $\beta_{\text{in}}(\Psi)$ is easily computed by evaluating the explicit expression in (19). However, the computation of $\beta_{\text{out}}(\Psi)$ is slightly more involved, since it involves comparing the solutions of a number of LPs in (18).

In other words, the size of the smallest ∞-norm ball (hypercube) containing $F(\alpha, s)$ can be computed by solving a finite number of LPs because

$$\beta_{\text{out}}(F(\alpha,s)) = \max_{j\in\{1,\dots,n\}} h(W^s, \pm(1-\alpha)^{-1}[A^0 \ \cdots \ A^{s-1}]^T e_j) \tag{20a}$$

$$= (1-\alpha)^{-1} \max_{j\in\{1,\dots,n\}} \max\left\{\sum_{i=0}^{s-1} h(W,(A^i)^T e_j), \sum_{i=0}^{s-1} h(W,-(A^i)^T e_j)\right\}. \tag{20b}$$

Clearly, if W is the linear map of a hypercube, then no LPs are necessary; one can use (15) to compute the explicit expression of the support functions in (20).

4.1 *A priori* upper bounds if A is diagonizable

The condition in the previous section, i.e. (4), has the specific form

$$A^i\Pi \subseteq \Psi. \tag{21}$$

This section shows how one can efficiently obtain *a priori* upper bounds on

$$i^{\text{o}}(A,\Pi,\Psi) \triangleq \inf\left\{i\in\mathbb{N} \;\middle|\; A^i\Pi\subseteq\Psi\right\}, \tag{22}$$

which is the smallest i such that (21) holds.

We first present the following result:

Lemma 1. *Let Π and Ψ be two non-empty polytopes in $\mathbb{R}^n$ containing the origin and the matrix $L\in\mathbb{R}^{n\times n}$.*

Let $\beta_{\text{in}}(\Psi)$ be the size of the largest hypercube in Ψ and $\beta_{\text{out}}(\Pi)$ be the size of the smallest hypercube containing Π.

(i) If $L\mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi)) \subseteq \mathbb{B}^n_\infty(\beta_{\text{in}}(\Psi))$, then $L\Pi\subseteq\Psi$.

(ii) If $\|L\|_\infty \le \beta_{\text{in}}(\Psi)/\beta_{\text{out}}(\Pi)$, then $L\Pi\subseteq\Psi$.

Proof. (i) Note that $\Pi\subseteq\mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi))$ so that $L\Pi\subseteq L\mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi))$.

Since $\mathbb{B}^n_\infty(\beta_{\text{in}}(\Psi))\subseteq\Psi$, if $L\mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi)) \subseteq \mathbb{B}^n_\infty(\beta_{\text{in}}(\Psi))$, then $L\mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi))\subseteq\Psi$.

Since $L\Pi\subseteq L\mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi))$, $L\Pi\subseteq\Psi$ as claimed.

(ii) Note that for any $x\in L\mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi))$ we have $\|x\|_\infty\le\|L\|_\infty\beta_{\text{out}}(\Pi)$ so that $L\mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi))\subseteq \{x \mid \|x\|_\infty\le\|L\|_\infty\beta_{\text{out}}(\Pi)\}$.

If $\|L\|_\infty \le \beta_{\text{in}}(\Psi)/\beta_{\text{out}}(\Pi)$ it follows that $L\mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi))\subseteq\{x \mid \|x\|_\infty\le\beta_{\text{in}}(\Psi)\}$ so that $L\mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi))\subseteq\mathbb{B}^n_\infty(\beta_{\text{in}}(\Psi))$, as claimed.

□

The previous result turns out to be very useful in providing an upper bound on $i^{\text{o}}(A,\Pi,\Psi)$:

Proposition 4. *Let Π and Ψ be two non-empty polytopes in $\mathbb{R}^n$ containing the origin in their interiors. Let $\beta_{\text{in}}(\Psi)$ be the size of the largest hypercube in Ψ and $\beta_{\text{out}}(\Pi)$ be the size of the smallest hypercube containing Π.*

If A is diagonizable with $A = V\Lambda V^{-1}$, where Λ is a diagonal matrix of the eigenvalues of A and the spectral radius $\rho(A) \in (0,1)$, then

$$i^{\circ}(A,\Pi,\Psi) \leq \left\lceil \ln\left(\beta_{\text{in}}(\Psi)/\left(\beta_{\text{out}}(\Pi)\|V\|_{\infty}\|V^{-1}\|_{\infty}\right)\right)/\ln\rho(A)\right\rceil . \tag{23}$$

Proof. From Lemma 1 it follows that (21) is satisfied if

$$\|A^i\|_{\infty} \leq \beta_{\text{in}}(\Psi)/\beta_{\text{out}}(\Pi). \tag{24}$$

From the basic properties of operator norms it follows that

$$\|A^i\|_{\infty} = \|V\Lambda^i V^{-1}\|_{\infty} \tag{25a}$$

$$\leq \|V\|_{\infty}\|\Lambda^i\|_{\infty}V^{-1}\|_{\infty} \tag{25b}$$

$$= \|V\|_{\infty}\rho(A)^i\|V^{-1}\|_{\infty} \tag{25c}$$

The proof is completed by multiplying (24) with $\|V\|_{\infty}$ and $\|V^{-1}\|_{\infty}$ and solving for i. □

The above result shows that the upper bound on $i^{\circ}(A,\Pi,\Psi)$ depends on the magnitudes of the eigenvalues (in particular, the spectral radius) and the eigenvectors of A.

Proposition 4 is particularly useful in obtaining upper bounds on the power of the integer on the left hand side in (4) For instance, an upper bound on $s^{\circ}(\alpha)$ is easily obtained. By applying Proposition 4 with $\Pi = W$ and $\Psi = \alpha W$, it follows that

$$s^{\circ}(\alpha) \leq \left\lceil \ln\left(\alpha\beta_{\text{in}}(W)/\left(\beta_{\text{out}}(W)\|V\|_{\infty}\|V^{-1}\|_{\infty}\right)\right)/\ln\rho(A)\right\rceil . \tag{26}$$

The next result shows that an appropriate estimate, possibly conservative, of the space that is occupied by the set $F(\alpha, s)$ can be obtain by means of a hypercube, in the sense of the ∞-norm:

Proposition 5. *If A is diagonizable as in Proposition 4, then the set $F(\alpha, s)$ is contained in the hypercube $\mathbb{B}^n_{\infty}(\eta)$, where η is given by*

$$\eta = \beta_{\text{out}}(W)\left(1-\alpha\right)^{-1}\left(1-\rho\left(A\right)\right)^{-1}\left(1-\rho\left(A\right)^s\right)\|V\|_{\infty}\|V^{-1}\|_{\infty}. \tag{27}$$

Proof. See Appendix. □

A	$\begin{bmatrix} 0.28 & 0.02 \\ -0.72 & 0.02 \end{bmatrix}$	$\begin{bmatrix} 0.44 & -0.24 \\ -0.56 & -0.24 \end{bmatrix}$	$\begin{bmatrix} -0.17 & -0.03 \\ -1.17 & -0.03 \end{bmatrix}$	$\begin{bmatrix} 0.98 & 0.72 \\ -0.02 & 0.72 \end{bmatrix}$
$\rho(A)$	0.2	0.6	0.3	0.9
$\lambda_i,\ i = 1, 2$	(0.1,0.2)	(-0.4,0.6)	(-0.3,0.1)	(0.8,0.9)
$s^{\text{o}}(\alpha)$	4	7	4	50
$\alpha^{\text{o}}(s^{\text{o}}(\alpha))$	0.0119	0.0304	0.0261	0.0463
$\bar{s}$	4	8	5	56
$\alpha^{\text{o}}(\bar{s})$	0.0119	0.0181	0.0079	0.0246

Table 1: Data for 2D examples

$\rho(A)$	$s^{\text{o}}(\alpha)$	$\alpha^{o}(s^{\text{o}}(\alpha))$	$\bar{s}$	$\alpha^{\text{o}}(\bar{s})$
0.2	9	0.08395	13	$7.93 \cdot 10^{-5}$

Table 2: Data for 10D example

5 Examples

In order to illustrate our results on invariant approximations of the minimal robustly positively invariant set, i.e. $F(\alpha, s)$ we consider four 2-D systems, with various values of spectral radii:

$$x^{+} = Ax + w \tag{28}$$

with additive disturbance:

$$W \triangleq \left\{ w \in \mathbb{R}^2 \ \mid \ \|w\|_\infty \leq 0.1 \right\}. \tag{29}$$

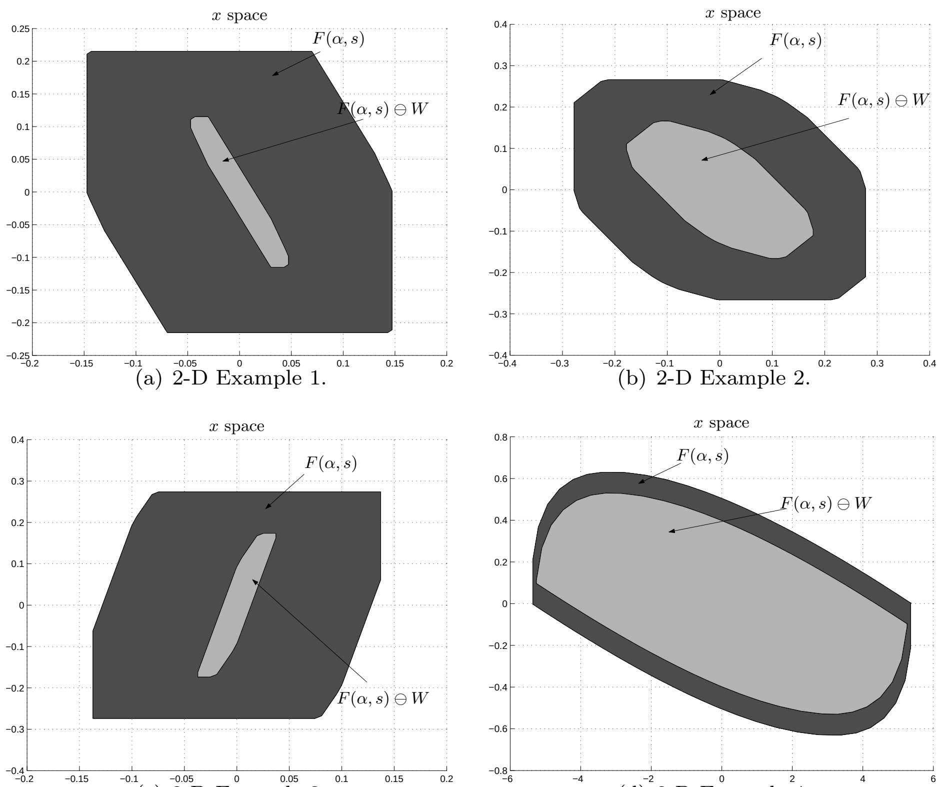
The dynamics, eigenvalues and particular values of $s^{\text{o}}(\alpha)$, $\alpha^{\text{o}}(s^{\text{o}}(\alpha))$, $\bar{s}$ and $\alpha^{\text{o}}(\bar{s})$ are reported in Table 1, where $\bar{s}$ is the upper bound on $s^{\text{o}}(\alpha)$ obtained by (26). The initial value of α was chosen to be 0.05.

The invariant sets $F(\alpha, s)$ together with $F(\alpha, s) \ominus W$ are shown in Figure 1. The dynamics were obtained by applying four various state feedback control laws to second order double integrator example in order to illustrate influence of the eigenvectors and eigenvalues on the geometric position of the set $F(\alpha, s)$.

In order to demonstrate that our result can be applied to higher order systems, a 10^{th} order system is considered. The values of $\rho(A)$, $s^{\text{o}}(\alpha)$, $\alpha^{\text{o}}(s^{\text{o}}(\alpha))$, $\bar{s}$ and $\alpha^{\text{o}}(\bar{s})$, where $\bar{s}$ is upper bound on $s^{\text{o}}(\alpha)$ obtained by (26), are reported in Table 2 and the matrix A is given in Appendix B. The disturbance was bounded in the hypercube $W \triangleq \left\{ w \in \mathbb{R}^{10} \ \mid \ \|w\|_\infty \leq 0.1 \right\}$. The initial value of α was chosen to be 0.1.

6 Conclusions and Future Research

This paper presented new results regarding the minimal robustly positively invariant set for linear systems. It was shown how to compute an invariant, outer approximation of the minimal robustly positively invariant set. Furthermore, a number of useful *a-priori* bounds and efficient tests were



(a) 2-D Example 1. (b) 2-D Example 2. (c) 2-D Example 3. (d) 2-D Example 4.

Figure 1: Invariant Approximations - Sets $F(\alpha, s)$ and $F(\alpha, s) \ominus W$

given.

Future research will extended the presented results on the outer approximation of the minimal robustly positively invariant set to a more general case, when the interior of W is empty, but the origin is in the relative interior of W. Also, it is possible to compute an RPI, outer approximation of the mRPI set F_∞, by computing the reachable set of states of a given RPI set.

Moreover, the presented results also enable us to devise an alternative algorithm for computing the maximal robustly positively invariant set or an approximation of it. Such an algorithm can be based on the computation of a sequence of robustly positively invariant sets.

Appendix

A Proof of Proposition 5

Proof. We need to show that $F(\alpha, s) \subseteq \mathbb{B}^n_\infty(\eta)$. Firstly, note that for any $x \in A^i W$, $i \in \mathbb{N}_{[0,s-1]}$, we have

$$\|x\|_\infty \leq \|A^i\|_\infty \beta_{\text{out}}(W) = \|V\Lambda^i V^{-1}\|_\infty \beta_{\text{out}}(W) \leq \rho(A)^i \|V\|_\infty \|V^{-1}\|_\infty \beta_{\text{out}}(W). \tag{30}$$

Hence we conclude that

$$A^i W \subseteq \{x \mid \|x\|_\infty \leq \beta_{\text{out}}(W)\rho(A)^i \|V\|_\infty \|V^{-1}\|_\infty\}, \quad \forall i \in \mathbb{N}_{[0,s-1]}. \tag{31}$$

From the fact that $\bigoplus_{i\in\mathbb{N}_{[0,s-1]}} P_i \subseteq \bigoplus_{i\in\mathbb{N}_{[0,s-1]}} Q_i$ if $P_i \subseteq Q_i$ for all $i \in \mathbb{N}_{[0,s-1]}$, and the fact that $\bigoplus_{i\in\mathbb{N}_{[0,s-1]}} \mathbb{B}^n_\infty(r_i) = \mathbb{B}^n_\infty(\sum_{i\in\mathbb{N}_{[0,s-1]}} r_i)$, it follows that

$$\bigoplus_{i=0}^{s-1} A^i W \subseteq \bigoplus_{i=0}^{s-1} \{x \mid \|x\|_\infty \leq \beta_{\text{out}}(W)\rho(A)^i \|V\|_\infty \|V^{-1}\|_\infty\} \tag{32a}$$

$$= \left\{ x \;\middle|\; \|x\|_\infty \leq \beta_{\text{out}}(W) \sum_{i=0}^{s-1} \rho(A)^i \|V\|_\infty \|V^{-1}\|_\infty \right\}. \tag{32b}$$

Recalling that $\sum_{i=0}^{s-1} \rho(A)^i = (1-\rho(A))^{-1}(1-\rho(A)^s)$, we have

$$\bigoplus_{i=0}^{s-1} A^i W \subseteq \{x \mid \|x\|_\infty \leq \beta_{\text{out}}(W)(1-\rho(A))^{-1}(1-\rho(A)^s)\|V\|_\infty \|V^{-1}\|_\infty\} \tag{33}$$

and hence

$$(1-\alpha)^{-1} \bigoplus_{i=0}^{s-1} A^i W \subseteq \{x \mid \|x\|_\infty \leq \beta_{\text{out}}(W)(1-\alpha)^{-1}(1-\rho(A))^{-1}(1-\rho(A)^s)\|V\|_\infty \|V^{-1}\|_\infty\} \tag{34}$$

We conclude that $F(\alpha, s) \subseteq \{x \mid \|x\|_\infty \leq \eta\}$, where η is given in (27). □

B Matrix A for the 10^{th} order system

$$A = \begin{bmatrix}
-0.2713 & 0.3004 & 0.1460 & 0.1592 & -0.2634 & 0.1013 & -0.5457 & -1.0030 & -0.3074 & 0.1131 \\
-0.0750 & 0.1632 & 0.1152 & -0.3785 & -1.1337 & 0.1022 & -0.2252 & -1.3581 & -0.2761 & 0.3572 \\
0.0630 & 0.1329 & 0.6675 & -0.0420 & 0.6569 & 0.4865 & 0.1887 & 0.3698 & 0.3771 & 0.0639 \\
0.0054 & -0.3648 & -0.1315 & -0.3111 & -0.3509 & -0.4195 & 0.1430 & -0.3664 & 0.0041 & -0.2103 \\
-0.0989 & 0.0182 & -0.0338 & 0.6819 & 0.6055 & 0.3177 & 0.1566 & 0.6116 & 0.3225 & 0.2249 \\
-0.1405 & -0.2986 & -0.5272 & -0.2254 & -0.4732 & -0.3210 & -0.4723 & -0.6953 & -1.2377 & 0.0627 \\
-0.0691 & 0.5372 & -0.1464 & -0.2886 & -0.5583 & -0.1757 & 0.0958 & -0.4458 & -0.4504 & 0.7228 \\
0.0210 & -0.1330 & 0.0150 & -0.2470 & 0.2097 & -0.1566 & -0.0648 & -0.1555 & -0.2068 & -0.2772 \\
0.2066 & 0.0050 & 0.3944 & -0.2396 & -0.4478 & 0.4977 & -0.5521 & -0.0216 & -0.2452 & -0.2350 \\
-0.4705 & -0.0676 & -0.0053 & -0.3805 & -0.4381 & 0.4012 & -0.2391 & -0.7415 & 0.2096 & -0.2979
\end{bmatrix}$$

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