How to match an affine time-varying feedback law: Properties of a new parameterization for the control of constrained systems with disturbances

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Abstract

This paper is concerned with the application and analysis of a recent result in the literature on robust optimization to the control of linear discrete-time systems, which are subject to unknown state disturbances and mixed constraints on the state and input. By parameterizing the control input sequence as an affine function of the disturbance sequence, it can be shown that a certain class of robust finite horizon control problems can be solved in a computationally tractable fashion, provided the constraint and the disturbance sets are polytopic. The main contribution of the paper, as the title suggests, is to show that this parameterization includes the class of affine time-varying state feedback control laws. The paper also shows how this parameterization can be used to efficiently synthesize receding horizon and minimum-time control laws that are robustly invariant. Two small numerical examples are also presented that highlight some of the strengths and limitations of the proposed parameterization.

Keywords: Constrained control, robust optimization, optimal control, robust control, receding horizon control, predictive control.

1 Introduction

The problem of finding a nonlinear state feedback control law, which guarantees that a set of state and input constraints are satisfied for all time, despite the presence of a persistent state disturbance, has been the subject of study for many authors [1–12]. However, the problem is that the solutions offered to date are computationally expensive or intractable.

As a consequence, many researchers have proposed compromise solutions, which, though not able to guarantee the same level of performance, is computationally tractable [13–16].

Recently, a new parameterization for solving so-called *robust optimization* problems was proposed in [17, 18]. The authors proposed that, instead of solving for a general, nonlinear function that guarantees that the constraints in the optimization problem are met for all values of the uncertainty, one could aim to find an *affine* function of the uncertainty. They proceeded to show that, if the uncertainty set is a polyhedron and the constraints in the robust optimization problem are affine, then an affine function of the uncertainty can be found by solving a single, computationally tractable LP. They also demonstrated, via an example, how their results can be applied to an inventory control problem.

The same affine parameterization was later used in [19, Chap. 7] and [20] to approximate a class of so-called *feedback* min-max finite horizon control problems [1, 7–10, 12]. It was also shown in [19,20], via numerical examples, that this parameterization leads to an improvement over schemes such as *open-loop* min-max model predictive control [10, Sect. 4.5] and those proposed in [14–16], where a sequence of *perturbations* to a stabilizing control law is sought.

Within the context of synthesizing robust control laws for discrete-time LTI systems, which are subject to unknown state disturbances and mixed constraints on the state and input, this paper makes a contribution by presenting a number of new results regarding the geometric and system-theoretic properties of the parameterization proposed by [17, 18].

This paper is organized as follows: Section 2 briefly introduces the control problem that will be considered in this paper and some standing assumptions are introduced. Section 3 proceeds to review the parameterization proposed in [17, 18] within the context of finding a solution to a certain robust finite horizon control problem.

Section 4 contains the main contribution of this paper. Theorem 1 shows that the set of states for which the parameterization in Section 3 is feasible, contains the set of states for which one can find an affine time-varying state feedback control policy such that for all allowable values of the disturbance, the constraints are satisfied over a finite horizon.

Further new results are given in Section 5. It is shown that, provided the target/terminal constraint set is robustly invariant, one can guarantee certain geometric and system-theoretic properties of a number of control policies based on the parameterization proposed in Section 3. Theorem 2 shows that the size of the set of states for which a control policy can be defined, increases with an increase in horizon length. Theorem 3 shows that one can design a receding horizon control (RHC) law that is guaranteed to be robustly invariant. Theorem 4 shows that one can synthesize a time-invariant minimum-time control law that is robustly invariant and guarantees robust convergence to the target set.

Section 6 discusses the computational complexity of the parameterization reviewed in Section 3. Most of the points discussed in Section 6 can be found in [17–20] in one form or another and this section is therefore mainly included for completeness. The key point to note from Section 6 is that the complexity of finding solutions to the finite horizon control problems discussed in Sections 3 and 5 is computationally tractable. In particular, it is shown that, provided the disturbance is an affine map of a hypercube, one need only solve a Phase I LP of size $O(N^2)$, where N is the length of the control horizon.

Section 7 contains two simple, but illustrative examples. The examples not only validate some of the theoretical results presented in this paper, but also aim to highlight some of the advantages and limitations of the parameterization discussed in Section 3, compared to other approaches available in the literature.

The paper concludes in Section 8 and briefly discusses directions for further research.

2 Problem setup

Consider the following discrete-time LTI system:

$$x^+ = Ax + Bu + w, \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, x^+ is the successor state, $u \in \mathbb{R}^m$ is the control input and $w \in \mathbb{R}^n$ is the disturbance. The actual values of the state, input and disturbance at a time instant k are denoted by $x(k)$, $u(k)$ and $w(k)$, respectively; where it is clear from the context, x, u and w will be used to denote the current value of the state, input and disturbance.

It is assumed that (A, B) is stabilizable and that at each sample instant a measurement of the state is available. It is further assumed that the current and future values of the disturbance are unknown and that the disturbance is persistent, but contained in a convex and compact set W, which contains the origin.

Since the disturbance is persistent, it is not possible to drive the state of the system to the origin. Instead, the aim will be to drive the state of the system to a target/terminal constraint set X_f, given by

$$X_f := \{x \in \mathbb{R}^n \mid Yx \leq z\}, \tag{2}$$

where the matrix $Y \in \mathbb{R}^{r \times n}$ and the vector $z \in \mathbb{R}^r$; r is the number of affine inequality constraints that define X_f. It is assumed that X_f contains the origin in its interior.

The system is subject to mixed constraints on the state and input:

$$\mathcal{Y} := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid Cx + Du \leq b\}, \tag{3}$$

where the matrices $C \in \mathbb{R}^{s \times n}$, $D \in \mathbb{R}^{s \times m}$ and the vector $b \in \mathbb{R}^s$; s is the number of affine inequality constraints that define $\mathcal{Y}$. It is assumed that $\mathcal{Y}$ contains the origin in its interior. An additional design goal is to guarantee that the state and input of the closed-loop system satisfy $\mathcal{Y}$ for all time and for all allowable disturbance sequences.

The final standing assumption is that a state feedback gain matrix $K \in \mathbb{R}^{m \times n}$ is given, such that $A + BK$ is strictly stable (the eigenvalues of $A + BK$ are strictly inside the unit disk).

NOTATION: $A \otimes B$ is the Kronecker product of matrices A and B and $\mathrm{vec}(A)$ denotes the vector formed by stacking the columns of matrix A into one long vector. Given an integer n, I_n is the $n \times n$ identity matrix and $\mathbf{1}_n$ is a column vector of n ones.

3 An affine parameterization of the control input sequence

Let N be a positive integer and the vectors $\mathbf{v} \in \mathbb{R}^{mN}$ and $\mathbf{w} \in \mathbb{R}^{nN}$ be defined as

$$\mathbf{v} := \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}, \quad \mathbf{w} := \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix}, \tag{4}$$

where the vectors $v_i \in \mathbb{R}^m$ and $w_i \in \mathbb{R}^n$ for all $i \in \{0, \dots, N-1\}$.

Let the set $\mathcal{W} := W^N := W \times \cdots \times W$.

We define the *strictly block lower triangular* matrix $\mathbf{M} := [M_{i,j}] \in \mathbb{R}^{mN \times nN}$, where the matrices $M_{i,j} \in \mathbb{R}^{m \times n}$ for all $i \in \{0, \dots, N-1\}$ and $j \in \{0, \dots, N-1\}$ and $M_{i,j} := 0$ for all $j \in \{i, \dots, N-1\}$. In other words,

$$\mathbf{M} := \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ M_{1,0} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{N-2,0} & M_{N-2,1} & \cdots & 0 & 0 \\ M_{N-1,0} & M_{N-1,1} & \cdots & M_{N-1,N-2} & 0 \end{bmatrix}. \tag{5}$$

This constraint on $\mathbf{M}$ is assumed throughout the rest of this paper.

The variable ψ is defined as the pair

$$\psi := (\mathbf{v}, \mathbf{M}). \tag{6}$$

Using the same affine parameterization of the control input sequence proposed in [17, 18], we use the current value of the state x to define the set of admissible ψ, which will be used to define a number of different feedback policies, as:

$$\Psi_N(x) := \left\{ \psi \;\middle|\; \begin{array}{r} \mathbf{v},\ \mathbf{w} \text{ satisfies (4)},\ \mathbf{M} \text{ satisfies (5)}, \\ x_{i+1} = Ax_i + Bu_i + w_i,\ x_0 = x, \\ u_i = v_i + \sum_{j=0}^{i-1} M_{i,j} w_j, \\ (x_i, u_i) \in \mathcal{Y},\ x_N \in X_f, \\ \forall i \in \{0, \dots, N-1\},\ \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}. \tag{7}$$

Note that the predicted value of the input u_i at a time instant i steps into the future, is an affine function of the disturbance sequence $\{w_0, \dots, w_{i-1}\}$; because the state is measured at each sample instant, the values in this disturbance sequence will be known at a time instant i steps into the future. The strictly block lower triangular constraint on $\mathbf{M}$ in (5) can therefore be seen to be a *causality constraint* on u_i, which ensures that the input u_i is not a function of the (as yet unknown) disturbance sequence $\{w_i, \dots, w_{N-1}\}$.

Given any $\psi \in \Psi_N(x(0))$ and the stabilizing state feedback gain $K \in \mathbb{R}^{m \times n}$, one can now define the following *time-varying* feedback policy:

$$u(k) = \begin{cases} v_k + \sum_{j=0}^{k-1} M_{i,j} w(j) & \text{if } k \in \{0, \dots, N-1\} \\ Kx(k) & \text{if } k \in \{N, N+1, \dots\} \end{cases} \tag{8}$$

Clearly, (8) is a causal feedback policy that is dependent not only on the current state, but also on past values of the state and input; since measurements of the state are available and past inputs are known, $w(j)$ in (8) is given by

$$w(j) = x(j+1) - Ax(j) - Bu(j), \quad \forall j \in \{0, \dots, N-1\}. \tag{9}$$

Before proceeding to analyze the properties of (8) and other feedback policies, let the set X_N^{ψ} denote the set of states for which there exists an admissible ψ:

$$X_N^{\psi} := \{x \in \mathbb{R}^n \mid \Psi_N(x) \neq \emptyset\}. \tag{10}$$

4 How to match an affine time-varying feedback law

Let the variable θ be defined as the tuple

$$\theta := (L_0, g_0, L_1, g_1, \dots, L_{N-1}, g_{N-1}), \tag{11}$$

where the matrix $L_i \in \mathbb{R}^{m\times n}$ and vector $g_i \in \mathbb{R}^m$ for all $i \in \{0,\dots,N-1\}$.

Consider now the set of admissible θ:

$$\Theta_N(x) := \left\{ \theta \;\middle|\; \begin{array}{r} \theta \text{ satisfies (11)},\ \mathbf{w} \text{ satisfies (4)}, \\ x_{i+1} = Ax_i + Bu_i + w_i,\ x_0 = x, \\ u_i = L_i x_i + g_i, \\ (x_i, u_i) \in \mathcal{Y},\ x_N \in X_f \\ \forall i \in \{0,\dots,N-1\},\ \forall \mathbf{w} \in \mathcal{W} \end{array} \right\} \tag{12}$$

The set of states for which there exist an admissible θ is defined as:

$$X_N^\theta := \{x \in \mathbb{R}^n \mid \Theta_N(x) \neq \emptyset\}. \tag{13}$$

Given a stabilizing state feedback gain $K \in \mathbb{R}^{m\times n}$ and a $\theta \in \Theta(x(0))$, one can define the following affine time-varying (ATV) state feedback policy:

$$u(k) = \begin{cases} L_k x(k) + g_k & \text{if } k \in \{0,\dots,N-1\} \\ Kx(k) & \text{if } k \in \{N, N+1, \dots\} \end{cases} \tag{14}$$

The main result of this paper states that the set of initial states X_N^θ, for which an ATV feedback policy of the form (14) can be defined, is contained inside X_N^ψ, the set of initial states for which a feedback policy of the form (8) can be defined:

Theorem 1 (Main result). *X_N^ψ contains X_N^θ.*

Proof. Let $x \in X_N^\theta$. One can easily verify that given a $\theta \in \Theta_N(x)$ and $\mathbf{w} \in \mathcal{W}$, it follows that for all $i \in \{1,\dots,N\}$,

$$x_i = S_i x + \sum_{j=1}^{i-1} T_{i,j}\left(Bg_{i-1-j} + w_{i-1-j}\right) + Bg_{i-1} + w_{i-1}, \tag{15}$$

where $S_i := \prod_{j=0}^{i-1}(A + BL_j)$ and $T_{i,j} := \prod_{l=1}^{j}(A + BL_{i-l})$, $j = 1,\dots,i-1$.

Since $u_i = L_i x_i + g_i$ for all $i \in \{0,\dots,N-1\}$, it follows that

$$u_i = L_i S_i x + \sum_{j=1}^{i-1} L_i T_{i,j}\left(Bg_{i-1-j} + w_{i-1-j}\right) + L_i B g_{i-1} + L_i w_{i-1} + g_i. \tag{16}$$

It is easy to check that (16) is equal to

$$u_i = v_i + \sum_{j=0}^{i-1} M_{i,i-1-j} w_{i-1-j}, \quad \forall i \in \{0, \ldots, N-1\} \tag{17}$$

if $v_0 := L_0 x + g_0$ and for all $i \in \{1, \ldots, N-1\}$,

$$v_i := L_i S_i x + \sum_{j=1}^{i-1} L_i T_{i,j} B g_{i-1-j} + L_i B g_{i-1} + g_i \tag{18}$$

and

$$M_{i,i-1-j} := \begin{cases} L_i & \text{if } j = 0 \\ L_i T_{i,j} & \text{if } j \in \{1, \ldots, i-1\} \end{cases} \tag{19}$$

It follows from the definition of $\Theta(x)$ that for all $i \in \{0, \ldots, N-1\}$ and $\mathbf{w} \in \mathcal{W}$, $(x_i, u_i) \in \mathcal{Y}$ and $x_N \in X_f$. Given the above definitions, if $(\mathbf{v}, \mathbf{M})$ is defined as in (4) and (5), then $(\mathbf{v}, \mathbf{M}) \in \Psi_N(x)$, hence $x \in X_N^{\psi}$.

□

Theorem 1 is an interesting and surprising result. The proof of Theorem 1 implies that if, for a given initial state $x(0)$, one can find an ATV feedback policy of the form (14) such that for all allowable disturbance sequences of length N, the state will be in X_f in exactly N steps while satisfying the constraints $\mathcal{Y}$ over a horizon of length N, then one can find a $\psi \in \Psi_N(x(0))$ in order to define a time-varying feedback policy of the form (8), which will result in *exactly the same control input sequence* as the one that would result from implementing (14).

We conclude this section by pointing out that, at present, there does not exist an efficient algorithm for finding a $\theta \in \Theta_N(x)$. However, as will be shown in Section 6, finding a $\psi \in \Psi_N(x)$ is computationally tractable if W is a polytope (closed and bounded polyhedron) or the affine map of a hypercube. As a consequence of Theorem 1, the results in Section 6 and the lack of an efficient method for finding a $\theta \in \Theta_N(x)$, we will only consider feedback policies that can be defined from the parameterization proposed in Section 3.

5 Geometric and invariance properties

For this section, we introduce the following assumption:
A1: The set X_f is contained inside X_K, which is given by

$$X_K := \{x \in \mathbb{R}^n \mid (x, Kx) \in \mathcal{Y}\} = \{x \mid (C + DK)x \leq b\}, \tag{20}$$

and X_f is robustly positively invariant [5, Def. 2.2] for the closed-loop system $x^+ = (A + BK)x + w$, i.e.

$$(A + BK)x + w \in X_f, \quad \forall x \in X_f,\ \forall w \in W. \tag{21}$$

Remark 1. Under some additional, mild technical assumptions, it is easy to compute an X_f that satisfies **A1** if W is a polytope. For example, [21] gives results for computing the *maximal* robustly positively invariant set in X_K and [22] gives some new results that allow one to compute a robustly positively invariant *outer* approximation to the *minimal* robustly positively invariant set in X_K. See also [4] for results on computing a robustly positively invariant *inner* approximation to the *maximal* robustly positively invariant set in X_K. For results on computing an X_f of a given complexity, which satisfies **A1**, see [16].

The next result follows immediately from the above:

Proposition 1. *Let **A1** hold, the initial state $x(0) \in X_N^\psi$ and $\psi \in \Psi_N(x(0))$. For all allowable infinite disturbance sequences, the state of system* (1)*, in closed-loop with the feedback policy* (8)*, enters X_f in N steps or less and is in X_f for all $k \in \{N, N+1, \ldots\}$. Furthermore, the constraints in* (3) *are satisfied for all time and for all allowable infinite disturbance sequences.*

5.1 On the size of X_N^ψ as N increases

The following result gives a sufficient condition under which one can guarantee that an increase in the horizon length N does not result in a decrease in the size of X_N^ψ:

Theorem 2 (Size of X_N^ψ). *If **A1** holds, then the following set inclusion holds:*

$$X_f \subseteq X_1^\psi \subseteq \cdots \subseteq X_{N-1}^\psi \subseteq X_N^\psi \subseteq X_{N+1}^\psi \subseteq \cdots, \tag{22}$$

where each X_i^ψ is defined as in (10) *with $N = i$.*

Proof. The proof is by induction. Let $x \in X_N^\psi$, $(\mathbf{v}, \mathbf{M}) \in \Psi_N(x)$ and $\mathbf{w} \in \mathcal{W}$. It is easy to verify that

$$x_N = A^N x + \sum_{i=0}^{N-1} \left(A^i B v_{N-1-i} + \left(A^i + \sum_{j=0}^{i-1} A^j B M_{N-1-j,N-1-i} \right) w_{N-1-i} \right). \tag{23}$$

Let

$$v_N := KA^N x + \sum_{i=0}^{N-1} KA^i B v_{N-1-i} \tag{24}$$

and for all $i \in \{0, \dots, N-1\}$, let

$$M_{N,N-1-i} := KA^i + \sum_{j=0}^{i-1} KA^j BM_{N-1-j,N-1-i}. \tag{25}$$

From these definitions, one can check that

$$u_N := v_N + \sum_{j=0}^{N-1} M_{N,j} w_j = v_N + \sum_{i=0}^{N-1} M_{N,N-1-i} w_{N-1-i} = Kx_N. \tag{26}$$

From the definition of $\Psi_N(x)$, recall that $x_N \in X_f$. Note also that that since $X_f \subseteq X_K$, it follows that $(x_N, u_N) \in \mathcal{Y}$. Since X_f is robustly positively invariant for the closed-loop system $x^+ = (A+BK)x + w$, it follows that

$$x_{N+1} = Ax_N + Bu_N + w_N \in X_f, \quad \forall w_N \in W. \tag{27}$$

By putting all of the above together and letting the vector $\overline{\mathbf{v}} \in \mathbb{R}^{m(N+1)}$ be defined as

$$\overline{\mathbf{v}} := \begin{bmatrix} \mathbf{v} \\ v_N \end{bmatrix} \tag{28}$$

and the matrix $\overline{\mathbf{M}} \in \mathbb{R}^{m(N+1) \times n(N+1)}$ be defined as

$$\overline{\mathbf{M}} := \begin{bmatrix} \mathbf{M} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ M_{N,0} & M_{N,1} & \cdots & M_{N,N-1} & 0 \end{bmatrix}, \tag{29}$$

it follows from the definition of $\Psi_{N+1}(x)$ that $(\overline{\mathbf{v}}, \overline{\mathbf{M}}) \in \Psi_{N+1}(x)$, hence $x \in X_{N+1}^{\psi}$. The proof is completed by verifying, in a similar manner, that $X_f \subseteq X_1^{\psi} \subseteq X_2^{\psi}$. $\square$

5.2 Robust invariance of receding horizon control laws

We now consider what happens when $\Psi_N(x)$ is used to design a *time-invariant* receding horizon control law. Consider the *set-valued* receding horizon control (RHC) law $\kappa_N : X_N^{\psi} \to 2^{\mathbb{R}^m}$ ($2^{\mathbb{R}^m}$ is the set of all subsets of $\mathbb{R}^m$), which is defined by considering only the first portion of a $\mathbf{v}$ for which there exists an $\mathbf{M}$ such that $(\mathbf{v}, \mathbf{M}) \in \Psi_N(x)$:

$$\kappa_N(x) := \{u \in \mathbb{R}^m \mid \exists (\mathbf{v}, \mathbf{M}) \in \Psi_N(x) \text{ s.t. } u = [I_m \ 0]\mathbf{v}\}. \tag{30}$$

The following result implies that if the initial state is in X_N^{ψ}, then all trajectories of (1) in closed-loop with the RHC policy $u \in \kappa_N(x)$ will remain in X_N^{ψ} for all time and for all allowable disturbance sequences:

Theorem 3 (Robust invariance of RHC laws). *If **A1** holds, then the set X_N^{ψ} is robustly positively invariant for system* (1) *in closed-loop with the RHC law* (30)*, i.e. if $x \in X_N^{\psi}$, then*

$$Ax + Bu + w \in X_N^{\psi}, \quad \forall u \in \kappa_N(x),\ \forall w \in W. \tag{31}$$

Furthermore, the constraints (3) *are satisfied for all time and for all allowable infinite disturbance sequences.*

Proof. The method of proof very closely parallels that of Theorem 2 and the same definitions are assumed. However, rather than showing that an appended version of $(\mathbf{v}, \mathbf{M})$ is admissible, one proceeds by showing that a "shifted" version of $(\mathbf{v}, \mathbf{M})$ is admissible at the next time instant. For this purpose, we introduce the following variables:

$$\tilde{\mathbf{v}} := \begin{bmatrix} v_1 + M_{1,0}w \\ \vdots \\ v_{N-1} + M_{N-1,0}w \\ v_N + M_{N,0}w \end{bmatrix} \tag{32}$$

and

$$\tilde{\mathbf{M}} := \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ M_{2,1} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{N-1,1} & M_{N-1,2} & \cdots & 0 & 0 \\ M_{N,1} & M_{N,2} & \cdots & M_{N,N-1} & 0 \end{bmatrix}. \tag{33}$$

Using similar arguments as in proving Theorem 2, but taking care with notation, one can now show that if $x \in X_N^{\psi}$, $u \in \kappa_N(x)$ and $w \in W$, then $(\tilde{\mathbf{v}}, \tilde{\mathbf{M}}) \in \Psi_N(Ax + Bu + w)$, hence $Ax + Bu + w \in X_N^{\psi}$. □

Remark 2. In this paper, we will not consider the important problem of how to synthesize an RHC law such that the closed-loop system is robustly stable and robust convergence to X_f is guaranteed. However, we will mention here that it is possible to extend the results in [15] to efficiently compute an RHC law such that the closed-loop system is input-to-state stable (ISS).

5.3 Robust invariance and robust convergence of minimum-time control

We conclude this section by showing how the parameterization in Section 3 can be used to define a robust minimum-time control law.

Given a maximum horizon length $N_{\max}$ and the set $\mathcal{N} := \{1, \ldots, N_{\max}\}$, let

$$N^*(x) := \min_{N} \{N \in \mathcal{N} \mid \Psi_N(x) \neq \emptyset\}. \tag{34}$$

Consider the following *time-invariant* set-valued control law $\kappa : \mathcal{X} \to 2^{\mathbb{R}^m}$, where $\mathcal{X} := X_f \cup \left(\cup_{N \in \mathcal{N}} X_N^\psi\right)$ and $\kappa_{N^*(x)}(x)$ is given by (30):

$$\kappa(x) := \begin{cases} \kappa_{N^*(x)}(x) & \text{if } x \notin X_f \\ Kx & \text{if } x \in X_f \end{cases} \tag{35}$$

We can now state the last result of this section:

Theorem 4 (Minimum-time control). *If **A1** holds, then the set $\mathcal{X} = X_{N_{\max}}^\psi$ is robustly positively invariant for system* (1) *in closed-loop with the minimum-time control law* (35)*, i.e. if $x \in \mathcal{X}$, then*

$$Ax + Bu + w \in X_N^\psi, \quad \forall u \in \kappa(x), \ \forall w \in W. \tag{36}$$

Furthermore, the constraints (3) *are satisfied for all time and for all allowable disturbance sequences. The state of the closed-loop system enters X_f in N steps or less and, once inside, remains inside for all time and for all allowable disturbance sequences.*

Proof. The proof closely parallels that of Theorems 2 and 3. However, this time one has to show that a "truncated" version of $(\mathbf{v}, \mathbf{M})$ is feasible at the next time instant. Let $N \in \mathcal{N}$, $x \in X_N^\psi$ and $(\mathbf{v}, \mathbf{M}) \in \Psi_N(x)$. By considering the same definitions as in Theorems 2 and 3, it is possible to show that if $\underline{\mathbf{v}} := [I_{m(N-1)} \ 0]\tilde{\mathbf{v}}$ and $\underline{\mathbf{M}} := [I_{m(N-1)} \ 0]\tilde{\mathbf{M}}$, then $(\underline{\mathbf{v}}, \underline{\mathbf{M}}) \in \Psi_{N-1}(Ax + Bu + w)$ for all $u \in \kappa(x)$ and all $w \in W$. $\square$

Theorem 4 should be contrasted with Proposition 1. Whereas (8) is a *time-varying* feedback policy that is also dependent on current and past values of the state and input, (35) is a *time-invariant* feedback policy that is dependent only on the current state. Note also that (8) does not guarantee that the state of the system will enter X_f in less than N steps if this is possible, whereas (35) guarantees that the state of the system will be in X_f in less than N steps if this is possible.

Remark 3. Note that the control law defined above is not optimal in the sense of [2, 3, 6, 11], since X_N^ψ is not, in general, equal to the set of states for which an arbitrary, nonlinear, time-varying state feedback control policy exists such that for all allowable disturbance sequences of length N, the constraints (3) are satisfied over a horizon of length N and the state is in X_f in exactly N steps.

6 Finding an admissible ψ if W is the affine map of a hypercube

6.1 The set $\Psi_N(x)$ is convex

It is straightforward to find matrices $F \in \mathbb{R}^{q\times mN}$, $G \in \mathbb{R}^{q\times nN}$, $H \in \mathbb{R}^{q\times n}$ and a vector $c \in \mathbb{R}^q$, where $q := sN + r$ (for completeness, the matrices and vectors are given in the Appendix), such that one can rewrite $\Psi_N(x)$ in (7) as

$$\Psi_N(x) = \left\{ \psi \;\middle|\; \begin{array}{r} \mathbf{M} \text{ satisfies (5)}, \\ F\mathbf{v} + (F\mathbf{M} + G)\mathbf{w} \leq c + Hx, \ \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}. \tag{37}$$

It is well-known [1, 3, 4, 15–21] that one can eliminate the quantifier in (37) by noting that

$$\Psi_N(x) = \left\{ \psi \;\middle|\; \begin{array}{r} \mathbf{M} \text{ satisfies (5)}, \\ F\mathbf{v} + \max\limits_{\mathbf{w}\in\mathcal{W}}(F\mathbf{M} + G)\mathbf{w} \leq c + Hx \end{array} \right\}, \tag{38}$$

where the maximization in $\max_{\mathbf{w}\in\mathcal{W}}(F\mathbf{M} + G)\mathbf{w}$ is row-wise.

Proposition 2. *$\Psi_N(x)$ is a convex set.*

Proof. Since the pointwise supremum over an infinite set of convex functions is convex, each row of $\max_{\mathbf{w}\in\mathcal{W}}(F\mathbf{M} + G)\mathbf{w}$ is convex in $\mathbf{M}$. It follows that all the inequalites in (38) are convex, hence $\Psi_N(x)$ is convex. $\square$

If W is a polytope (closed and bounded polyhedron) given by a finite set of affine inequalities, then it is easy to check whether a given ψ is in $\Psi_N(x)$ by solving the q LPs that define $\max_{\mathbf{w}\in\mathcal{W}}(F\mathbf{M}+G)\mathbf{w}$ and checking the constraints in (38). By writing down the dual of each of the LPs defining $\max_{\mathbf{w}\in\mathcal{W}}(F\mathbf{M}+G)\mathbf{w}$, one can find a pair $\psi \in \Psi_N(x)$ in a computationally tractable way by solving Phase I of a single LP. The reader is referred to [17, Thm. 3.2] and [18, Thm 4.2] for details as to how this can be done.

6.2 Eliminating the quantifier in (37) if W is the affine map of a hypercube

In this paper we will not consider the general case when W is an arbitrary polytope. Instead, we will consider the special case when W is known to be the affine map of a hypercube. This is because, in many practical applications, W is nearly always assumed to be the affine map of a hypercube (for example, when upper and lower bounds on the components of the disturbance are known and the disturbance acts on the state in an affine manner). This observation leads to a significant reduction in computational effort, compared to the case of treating W as an arbitrary polytope.

To see why this is the case, note that if W is the affine translation of a hypercube, i.e. if

$$W := \{Ed + f \mid \|d\|_\infty \leq \eta\} \tag{39}$$

where the matrix $E \in \mathbb{R}^{n\times l}$, the vectors $f \in \mathbb{R}^n$, $d \in \mathbb{R}^l$ and η is a positive scalar, then

$$\mathcal{W} := \{J\mathbf{d} + g \mid \|\mathbf{d}\|_\infty \leq \eta\}, \tag{40}$$

where the matrix $J := I_N \otimes E$, the vectors $g := \mathbf{1}_N \otimes f$, $\mathbf{d} \in \mathbb{R}^t$ and the integer $t := lN$. It follows that

$$\begin{aligned}
\max_{\mathbf{w}\in\mathcal{W}} (F\mathbf{M} + G)\mathbf{w} &= \max_{\mathbf{d}} \{(F\mathbf{M} + G)(J\mathbf{d} + g) \mid \|\mathbf{d}\|_\infty \leq \eta\} && \text{(41a)}\\
&= \max_{\mathbf{d}} \{(F\mathbf{M}J + GJ)\mathbf{d} + (F\mathbf{M} + G)g \mid \|\mathbf{d}\|_\infty \leq \eta\} && \text{(41b)}\\
&= \max_{\mathbf{d}} \{(F\mathbf{M}J + GJ)\mathbf{d} \mid \|\mathbf{d}\|_\infty \leq \eta\} + (F\mathbf{M} + G)g && \text{(41c)}\\
&= \eta \mathrm{abs}(F\mathbf{M}J + GJ)\mathbf{1}_t + (F\mathbf{M} + G)g, && \text{(41d)}
\end{aligned}$$

where the components of the matrix $\mathrm{abs}(F\mathbf{M}J + GJ)$ are the absolute values of the corresponding components of the matrix $F\mathbf{M}J + GJ$. Hence,

$$\Psi_N(x) = \left\{ \psi \;\middle|\; \begin{array}{r} \mathbf{M} \text{ satisfies (5)}, \\ F\mathbf{v} + \eta \mathrm{abs}(F\mathbf{M}J + GJ)\mathbf{1}_t + (F\mathbf{M} + G)g \leq c + Hx \end{array} \right\}. \tag{42}$$

Remark 4. Note that $\mathrm{abs}(F\mathbf{M}J + GJ)\mathbf{1}_t$ is a vector formed from the 1-norms of the rows of $F\mathbf{M}J + GJ$. In going from (41c) to (41d) we have used the well-known fact that

$$\max_{\mathbf{d}} \left\{ a^T\mathbf{d} \mid \|\mathbf{d}\|_\infty \leq \eta \right\} = \eta \|a\|_1 \tag{43}$$

for any vector $a \in \mathbb{R}^t$ (see, for example, [15, Prop. 2] or [19, Thm. 3.1]).

If $\Psi_N(x)$ is given as in (42), then it is easy to check whether a given pair ψ is in $\Psi_N(x)$ by computing $\mathrm{abs}(F\mathbf{M}J + GJ)\mathbf{1}_t$ and checking whether the constraints in (42) are satisfied.

6.3 A computationally tractable method for finding an admissible ψ if W is the affine map of a hypercube

We can now make an important observation, which allows one to find a $\psi \in \Psi_N(x)$. It follows immediately from (42) that

$$\Psi_N(x) = \left\{ \psi \;\middle|\; \begin{array}{r} \mathbf{M} \text{ satisfies (5)}, \;\exists \Lambda \in \mathbb{R}^{q \times t} \text{ such that} \\ F\mathbf{v} + \eta \Lambda \mathbf{1}_t + (F\mathbf{M} + G)g \leq c + Hx, \\ \operatorname{abs}(F\mathbf{M}J + GJ) \leq \Lambda \end{array} \right\} \tag{44a}$$

$$= \left\{ \psi \;\middle|\; \begin{array}{r} \mathbf{M} \text{ satisfies (5)}, \;\exists \Lambda \in \mathbb{R}^{q \times t} \text{ such that} \\ F\mathbf{v} + \eta \Lambda \mathbf{1}_t + (F\mathbf{M} + G)g \leq c + Hx, \\ -\Lambda \leq F\mathbf{M}J + GJ \leq \Lambda \end{array} \right\}, \tag{44b}$$

where the matrix and vector inequalities are component-wise.

Remark 5. Note that $\Psi_N(x)$ is the projection of the polyhedron

$$\mathcal{C}_N(x) := \left\{ (\psi, \Lambda) \;\middle|\; \begin{array}{r} \mathbf{M} \text{ satisfies (5)}, \\ F\mathbf{v} + \eta \Lambda \mathbf{1}_t + (F\mathbf{M} + G)g \leq c + Hx, \\ -\Lambda \leq F\mathbf{M}J + GJ \leq \Lambda \end{array} \right\} \tag{45}$$

onto a subspace, hence $\Psi_N(x)$ is also a polyhedron.

The key point to note here is the following: if the number of constraints in (3) is $s = O(m+n)$ and $l = O(m+n)$ in (39) (this is nearly always the case in practice), then the dimension of $\mathcal{C}_N(x)$ is bounded by $O((m+n)^2N^2 + r(m+n)N)$ and the number of constraints that define $\mathcal{C}_N(x)$ in (45) is also bounded by $O((m+n)^2N^2 + r(m+n)N)$. This implies that the problem of finding a pair $(\mathbf{v}, \mathbf{M}) \in \Psi_N(x)$ is computationally tractable.

For example, finding a $\psi \in \Psi_N(x)$ is easily done by solving the following Phase I LP, in which γ is a scalar:

$$(\psi^*(x), \Lambda^*(x), \gamma^*(x)) := \arg\inf_{(\psi,\Lambda,\gamma)} \gamma \tag{46a}$$

subject to (5) and

$$F\mathbf{v} + \eta \Lambda \mathbf{1}_t + (F\mathbf{M} + G)g \leq c + Hx + \mathbf{1}_q \gamma, \tag{46b}$$

$$-\Lambda \leq F\mathbf{M}J + GJ \leq \Lambda. \tag{46c}$$

Clearly, $\Psi_N(x)$ is non-empty and $\psi^*(x) \in \Psi_N(x)$ if and only if $\gamma^*(x) \leq 0$.

Remark 6. It is easy to find an initial feasible point to (46) by choosing any $\mathbf{M}$ that satisfies (5), followed by choosing a Λ sufficiently large enough to satisfy (46c) and finally, choosing any $\mathbf{v}$ and a sufficiently large γ such that (46b) is satisfied. Once initialized with a feasible point, the LP solver can proceed with minimizing the cost until $\gamma \leq 0$.

The following observations allow one to efficiently translate (46) into a form suitable to be passed to a standard LP solver.

6.3.1 Getting (46) into a standard LP form

By applying the identities $\text{vec}(A+B) = \text{vec}(A) + \text{vec}(B)$ and $\text{vec}(ABC) = (C^T \otimes A)\text{vec}(B)$, where A, B and C are matrices of compatible dimensions, note that (46b) is equivalent to

$$F\mathbf{v} + \eta(\mathbf{1}_t^T \otimes I_q)\text{vec}(\Lambda) + (g^T \otimes F)\text{vec}(\mathbf{M}) + Gg \leq c + Hx + \mathbf{1}_q\gamma \tag{47a}$$

and that (46c) is equivalent to

$$-\text{vec}(\Lambda) \leq (J^T \otimes F)\text{vec}(\mathbf{M}) + \text{vec}(GJ) \leq \text{vec}(\Lambda). \tag{47b}$$

The decision variable in the Phase I LP (46) now becomes the vector $[\mathbf{v}^T \; \text{vec}(\mathbf{M})^T \; \text{vec}(\Lambda)^T \; \gamma]^T$.

6.3.2 Reducing the number of decision variables and constraints

By exploiting the structure and sparsity present in (46), one can reduce the number of decision variables and constraints in (46) in at least two ways:

- Due to the strictly block lower triangular constraint (5) on $\mathbf{M}$, the most immediately obvious reduction in decision variables can be achieved by removing all corresponding components of $\text{vec}(\mathbf{M})$ that are zero, as well as removing the corresponding columns of $g^T \otimes F$ and $J^T \otimes F$ in (47).
- A second reduction in decision variables and constraints can be achieved by exploiting the structure and sparsity of $F\mathbf{M}J + GJ$. On inspection of the Appendix and recalling that J is block diagonal, it can be shown that many of the components of $F\mathbf{M}J + GJ$ are zero for any choice of $\mathbf{M}$ that satisfies (5), hence many of the components of $\text{abs}(F\mathbf{M}J + GJ)$ are always zero. This implies that a large number of the components of Λ are redundant. By determining which components of $F\mathbf{M}J + GJ$ are zero for all possible choices of $\mathbf{M}$ that satisfy (5), one can remove the corresponding components of $\text{vec}(\Lambda)$, the corresponding columns of $\mathbf{1}_t^T \otimes I_q$ in (47a) and the corresponding constraints in (47b).

7 Numerical examples

In the following two examples, let X_N denote the set of states for which an arbitrary, nonlinear, time-varying state-feedback control policy exists such that for all allowable disturbance sequences of length N, the constraints (3) are satisfied over a horizon of length N and the state of the system is in X_f in exactly N steps. In other words, X_N is the region of attraction of the robust controllers defined in [1–3, 6–8, 11, 12].

Also, let $X_N^{\mathbf{v}}$ denote the set of states for which a sequence of length N of perturbations to a stabilizing control law K exists such that for all allowable disturbance sequences of length N, the constraints (3) are satisfied over a horizon of length N and the state of the system is in X_f in exactly N steps. In other words, $X_N^{\mathbf{v}}$ is the region of attraction of the robust controllers defined in [14–16].

It is easy to show that, in general, $X_N^{\mathbf{v}} \subseteq X_N^{\theta} \subseteq X_N^{\psi} \subseteq X_N$.

7.1 An example illustrating the benefits of the affine parameterization used in this paper

Consider the open-loop unstable, discrete-time LTI system

$$x^+ = 1.5x + 3u + w \tag{48}$$

and let $\mathcal{Y} := \{(x, u) \in \mathbb{R} \times \mathbb{R} \mid |u| \leq 1\}$, $X_f := \{x \in \mathbb{R} \mid |x| \leq 1\}$ and the disturbance set $W := \{w \in \mathbb{R} \mid |w| \leq 0.1\}$. For the purpose of computing $X_N^{\mathbf{v}}$, let $K := -0.8$ and note that $A + BK = -0.9$, X_f is robustly positively invariant for the closed-loop system $x^+ = (A + BK)x + w$ and $X_f \subseteq X_K$.

Figure 1 is a plot of the size of X_N, X_N^{ψ} and $X_N^{\mathbf{v}}$ for increasing values of N. As expected, since X_f is robustly positively invariant for the closed-loop system $x^+ = (A + BK)x + w$, an increase in N results in an increase in the size of all of the sets. Observe also that $X_N = X_N^{\psi}$ for all values of N, but that $X_N^{\mathbf{v}}$ is a strict subset of X_N and X_N^{ψ} for all values of $N > 1$.

The observation that $X_N = X_N^{\psi}$ for this example, though an interesting demonstration of the benefits of the parameterization used in this paper, is not very revealing as to its limitations. We therefore turn to an example with $n = 2$ and $m = 1$.

7.2 An example illustrating the limitations of the affine parameterization used in this paper

Though it is easy to find examples for which $X_N = X_N^{\psi}$ and X_N^{ψ} is a strict superset of $X_N^{\mathbf{v}}$, it is just as easy to find examples for which X_N is a strict superset of X_N^{ψ}.

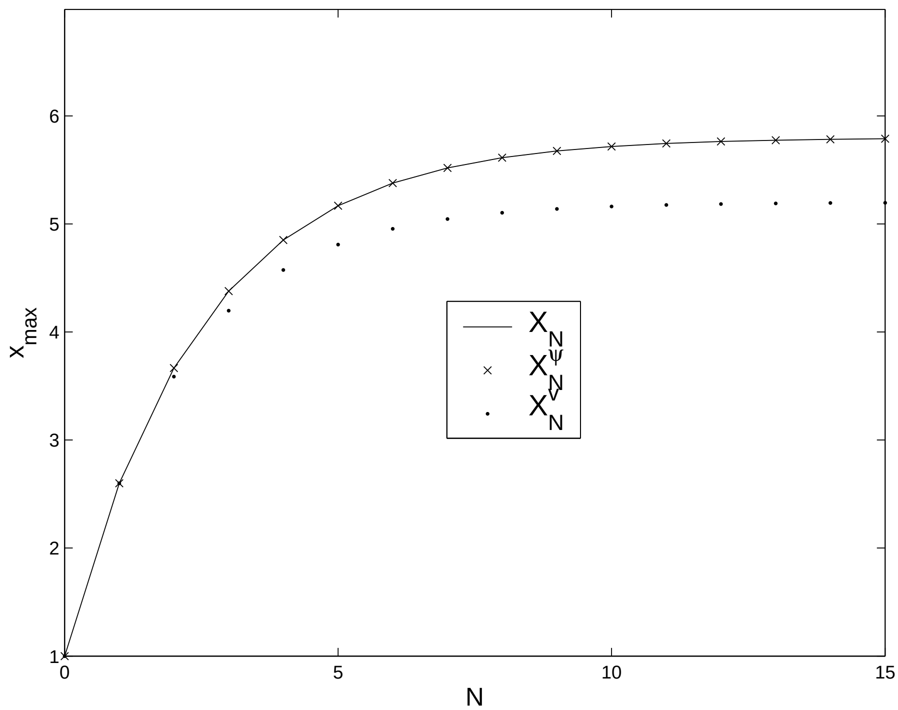

Figure 1: Plot of $x_{\max}$ versus N, where each $X_N = \{x \mid |x| \leq x_{\max}\}$, $X_N^{\psi} = \{x \mid |x| \leq x_{\max}\}$ and $X_N^{\mathbf{v}} = \{x \mid |x| \leq x_{\max}\}$

Consider the open-loop unstable, discrete-time LTI system

$$x^+ = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + w \tag{49}$$

and let $\mathcal{Y} := \{(x, u) \in \mathbb{R}^2 \times \mathbb{R} \mid |u| \leq 1\}$, $X_f := \{x \in \mathbb{R}^2 \mid \|x\|_\infty \leq 1\}$ and the disturbance set $W := \{w \in \mathbb{R}^2 \mid \|w\| \leq 0.1\}$. Note that X_f is not robustly controlled invariant [5, Def. 2.3] for (49).

By solving a Phase I LP, one can check that X_2 is non-empty, but that X_2^{ψ} and $X_2^{\mathbf{v}}$ are empty. This example therefore demonstrates that, in general, X_N^{ψ} may be a strict subset of $\neq X_N$ and that X_N^{ψ} may even be empty when X_N is not. However, recall from Theorem 2 that if **A1** holds, then one can always guarantee that X_N^{ψ} is non-empty.

8 Conclusions

Though the affine parameterization defined in Section 3 was shown to be useful for efficiently implementing control laws with guaranteed system-theoretic properties such as robust invariance and robust convergence to a target set, there are still a number of issues that need to be addressed.

It was proven in Section 4 that the set of states for which the parameterization in Section 3 is feasible, contains the set of states for which an affine time-varying policy exists. It still remains to be determined whether the inclusion in Theorem 1 is strict or whether it is satisfied with equality in general.

Section 5 showed how to construct receding horizon and minimum-time control laws with guaranteed robust invariance. In the case of minimum-time control, robust convergence to the target set can be guaranteed. However, the results in this section on the invariance of receding horizon control still need to be extended in order to guarantee robust convergence and stability of the target set, as well as guaranteeing offset-free control if the disturbance tends to a non-zero limit.

Finally, the results in Section 6 on the computational tractability could be extended to exploit any additional structure inherent in the robust finite horizon control problem, beyond the obvious simplifications mentioned in Section 6.3. It would also be interesting to see whether the class of uncertainties that can be addressed can be extended to include, for example, state- and input-dependent disturbances or parametric uncertainty in A and B.

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Appendix: Matrices used in Section 6

Let the matrices $\mathbf{A} \in \mathbb{R}^{n(N+1)\times n}$ and $\mathcal{A} \in \mathbb{R}^{n(N+1)\times nN}$ be defined as

$$\mathbf{A} := \begin{bmatrix} I_n \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad \mathcal{A} := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ I_n & 0 & \cdots & 0 \\ A & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1} & A^{N-2} & \cdots & I_n \end{bmatrix}. \tag{50}$$

We also define the matrices $\mathbf{B} \in \mathbb{R}^{n(N+1)\times mN}$, $\mathbf{C} \in \mathbb{R}^{q\times n(N+1)}$ and $\mathbf{D} \in \mathbb{R}^{q\times mN}$ as

$$\mathbf{B} := \mathcal{A}(I_N \otimes B),\ \mathbf{C} := \begin{bmatrix} I_N \otimes C & 0 \\ 0 & Y \end{bmatrix},\ \mathbf{D} := \begin{bmatrix} I_N \otimes D \\ 0 \end{bmatrix}. \tag{51}$$

It is easy (though tedious) to check that the expression in (7) is equivalent to (37) with

$$F := \mathbf{CB} + \mathbf{D},\ G := \mathbf{C}\mathcal{A},\ H := -\mathbf{CA},\ c := \begin{bmatrix} \mathbf{1}_N \otimes b \\ z \end{bmatrix}. \tag{52}$$

References

[1] A. Bemporad, F. Borrelli, and M. Morari. Min-max control of constrained uncertain discrete-time linear systems. *IEEE Transactions on Automatic Control*, 48(9):1600–1606, September 2003.

[2] D.P. Bertsekas and I.B. Rhodes. On the minimax reachability of target sets and target tubes. *Automatica*, 7:233–247, 1971.

[3] F. Blanchini. Minimum-time control for uncertain discrete-time linear systems. In *Proc. 31st IEEE Conference on Decision and Control*, Tucson, Arizona, USA, December 1992.

[4] F. Blanchini. Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions. *IEEE Trans. Automatic Control*, 39(2):428–433, 1994.

[5] F. Blanchini. Set invariance in control. *Automatica*, 35(11):1747–1767, 1999. Survey paper.

[6] J.D. Glover and F.C. Schweppe. Control of linear dynamic systems with set constrained disturbances. *IEEE Transactions on Automatic Control*, AC-16(5):411–423, October 1971.

[7] E.C. Kerrigan and J.M. Maciejowski. Robustly stable feedback min-max model predictive control. In *Proc. 2003 American Control Conference*, Denver, Colorado, USA, June 2003.

[8] E.C. Kerrigan and D.Q. Mayne. Optimal control of constrained, piecewise affine systems with bounded disturbances. In *Proc. 41st IEEE Conference on Decision and Control*, Las Vegas, Nevada, USA, December 2002.

[9] D.Q. Mayne. Control of constrained dynamic systems. *European Journal of Control*, 7:87–99, 2001. Survey paper.

[10] D.Q. Mayne, J.B. Rawlings, C.V. Rao, and P.O.M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36:789–814, 2000. Survey paper.

[11] D.Q. Mayne and W.R. Schroeder. Robust time-optimal control of constrained linear systems. *Automatica*, 33:2103–2118, 1996.

[12] P.O.M. Scokaert and D.Q. Mayne. Min-max feedback model predictive control for constrained linear systems. *IEEE Trans. Automatic Control*, 43:1136–1142, 1998.

[13] A. Bemporad. Reducing conservativeness in predictive control of constrained systems with disturbances. In *Proc. 37th IEEE Conference on Decision and Control*, Tampa, Florida, USA, December 1998.

[14] L. Chisci, J.A. Rossiter, and G. Zappa. Systems with persistent disturbances: predictive control with restricted constraints. *Automatica*, 37:1019–1028, 2001.

[15] E.C. Kerrigan and J.M. Maciejowski. On robust optimization and the optimal control of constrained linear systems with bounded state disturbances. In *Proc. European Control Conference*, Cambridge, UK, September 2003.

[16] Y.I. Lee and B. Kouvaritakis. Constrained receding horizon predictive control for systems with disturbances. *International Journal of Control*, 72(11):1027–1032, 1999.

[17] A. Ben-Tal, A. Goryashko, E. Guslitzer, and A. Nemirovski. Adjustable robust solutions of uncertain linear programs. Technical report, Technion (Israel Institute of Technology), Haifa, Israel, 2002. Downloadable from http://iew3.technion.ac.il/Labs/Opt/index.php?4.

[18] E. Guslitser. Uncertainty-immunized solutions in linear programming. Master's thesis, Technion (Israel Institute of Technology), Haifa, Israel, June 2002. Downloadable from http://iew3.technion.ac.il/Labs/Opt/index.php?4.

[19] J. Löfberg. *Minimax Approaches to Robust Model Predictive Control*. PhD thesis, Department of Electrical Engineering, Linköping University, SE-581 83, Linköping, Sweden, April 2003.

[20] J. Löfberg. Approximations of closed-loop minimax MPC. In *Proc. 42nd IEEE Conference on Decision and Control*, Maui, Hawaii, USA, December 2003.

[21] I. Kolmanovsky and E.G. Gilbert. Theory and computation of disturbance invariance sets for discrete-time linear systems. *Mathematical Problems in Engineering: Theory, Methods and Applications*, 4:317–367, 1998.

[22] S.V. Rakovic, E.C. Kerrigan, K. Kouramas, and D.Q. Mayne. Approximation of the minimal robustly positively invariant set for discrete-time LTI systems with persistent state disturbances. In *Proc. 42nd IEEE Conference on Decision and Control*, Maui, Hawaii, USA, December 2003.