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Abstract

This paper is concerned with the application and analysis of a recent result in the literature on robust optimization to the control of linear discrete-time systems, which are subject to unknown state disturbances and mixed constraints on the state and input. By parameterizing the control input sequence as an affine function of the disturbance sequence, it is shown that a certain class of finite horizon min-max control problems is convex and that the number of variables and constraints grows polynomially with the problem size. It is assumed that the constraint and the disturbance sets are polyhedral and that the cost is a suitably-chosen quadratic, in which the disturbance is negatively weighted as in H_∞ control.

Keywords: Constrained control, robust optimization, optimal control, robust control, receding horizon control, predictive control.

1 Introduction

Consider the following discrete-time LTI system:

$$x^+ = Ax + Bu + w, \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, x^+ is the successor state, $u \in \mathbb{R}^m$ is the control input and $w \in \mathbb{R}^n$ is the disturbance. The actual values of the state, input and disturbance at a

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time instant k are denoted by $x(k)$, $u(k)$ and $w(k)$, respectively; where it is clear from the context, x, u and w will be used to denote the current or initial state, input and disturbance. It is assumed that (A, B) is stabilizable and that at each sample instant a measurement of the state is available. The current and future values of the disturbance are unknown and the disturbance is persistent, but contained in a polytope (bounded polyhedron) W. Without loss of generality and in order to simplify notation (see [1, 2] for ways of generalizing the results in this paper), we assume that W is a hypercube:

$$W := \{w \in \mathbb{R}^n \mid \|w\|_\infty \leq \eta\}. \tag{2}$$

The system is subject to polyhedral, mixed constraints on the state and input:

$$\mathcal{Y} := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid Cx + Du \leq b\}, \tag{3}$$

where the matrices $C \in \mathbb{R}^{s\times n}$, $D \in \mathbb{R}^{s\times m}$ and the vector $b \in \mathbb{R}^s$; s is the number of affine inequality constraints in (3).

For a given initial state, a time-varying control policy is to be designed, which guarantees that for all disturbance sequences of a length N, the state and input of the closed-loop system is in $\mathcal{Y}$ over the horizon $k = 0, \dots, N-1$. The state is required to be in a target/terminal constraint set X_f at the end of the horizon ($k = N$), where X_f is a polyhedron given by

$$X_f := \{x \in \mathbb{R}^n \mid Yx \leq z\}, \tag{4}$$

where the matrix $Y \in \mathbb{R}^{r\times n}$ and the vector $z \in \mathbb{R}^r$; r is the number of affine inequality constraints that define X_f.

Notation: $\mathbf{1}$ is an appropriately-size column vector of ones. If A and B are matrices, then $\mathrm{abs}(A)$ is a matrix of the absolute values of the corresponding components of A, $B \succ 0$ denotes that B is positive definite and $A \leq B$ is used to denote component-wise inequality.

2 An affine parameterization of the control input sequence

Let N be a positive integer and the vectors $\mathbf{v} \in \mathbb{R}^{mN}$ and $\mathbf{w} \in \mathbb{R}^{nN}$ be defined as

$$\mathbf{v} := \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}, \quad \mathbf{w} := \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix}, \tag{5}$$

where the vectors $v_i \in \mathbb{R}^m$ and $w_i \in \mathbb{R}^n$ for all $i \in \{0, \dots, N-1\}$.

Let the set $\mathcal{W} := W^N := W \times \cdots \times W$.

We define the *strictly block lower triangular* matrix $\mathbf{M} := [M_{i,j}] \in \mathbb{R}^{mN \times nN}$, where the matrices $M_{i,j} \in \mathbb{R}^{m \times n}$ for all $i \in \{0, \ldots, N-1\}$ and $j \in \{0, \ldots, N-1\}$ and $M_{i,j} := 0$ for all $j \in \{i, \ldots, N-1\}$. In other words,

$$\mathbf{M} := \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ M_{1,0} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{N-2,0} & M_{N-2,1} & \cdots & 0 & 0 \\ M_{N-1,0} & M_{N-1,1} & \cdots & M_{N-1,N-2} & 0 \end{bmatrix}. \tag{6}$$

This constraint on $\mathbf{M}$ is assumed throughout the rest of this paper.

The variable ψ is defined as the pair

$$\psi := (\mathbf{v}, \mathbf{M}). \tag{7}$$

Using the same affine parameterization of the control input sequence originally proposed in [1], let the current value of the state x define the set of admissible ψ, which will be used to define a feedback policy, as:

$$\Psi_N(x) := \left\{ \psi \;\middle|\; \begin{array}{r} \mathbf{v},\ \mathbf{w} \text{ satisfies (5)},\ \mathbf{M} \text{ satisfies (6)}, \\ x_{i+1} = Ax_i + Bu_i + w_i,\ x_0 = x, \\ u_i = v_i + \sum_{j=0}^{i-1} M_{i,j} w_j, \\ (x_i, u_i) \in \mathcal{Y},\ x_N \in X_f, \\ \forall i \in \{0, \ldots, N-1\},\ \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}. \tag{8}$$

Remark 1. The reader is referred to [1–3] for a discussion on advantages and system-theoretic properties of the above parameterization, compared to the case if $\mathbf{M} = 0$, as in open-loop finite horizon control.

By eliminating x_i and u_i from (8), it is easy to find matrices $F \in \mathbb{R}^{q \times mN}$, $G \in \mathbb{R}^{q \times nN}$, $L \in \mathbb{R}^{q \times n}$ and a vector $c \in \mathbb{R}^q$, where $q := sN + r$, such that one can rewrite $\Psi_N(x)$ in (8) as

$$\Psi_N(x) = \left\{ \psi \;\middle|\; F\mathbf{v} + (F\mathbf{M} + G)\mathbf{w} \le c + Lx,\ \forall \mathbf{w} \in \mathcal{W} \right\} \tag{9a}$$

$$= \left\{ \psi \;\middle|\; F\mathbf{v} + \eta \operatorname{abs}(F\mathbf{M} + G)\mathbf{1} \le c + Lx \right\}. \tag{9b}$$

Note that $\operatorname{abs}(F\mathbf{M}+G)\mathbf{1}$ is a vector formed from the 1-norms of the rows of $F\mathbf{M}+G$. In going from (9a) to (9b) we have used the well-known fact (see, for example, [1–3]) that $a^T\mathbf{w} \le d$ for all $\mathbf{w} \in \mathcal{W}$ if and only if $\max_{\mathbf{w}} \left\{ a^T\mathbf{w} \mid \|\mathbf{w}\|_\infty \le \eta \right\} = \eta\|a\|_1 \le d$, where a is any vector in $\mathbb{R}^{nN}$ and d is any scalar.

It follows immediately from (9b) that $\psi \in \Psi_N(x)$ if and only if there exists a matrix $\Lambda \in \mathbb{R}^{q\times nN}$ such that

$$F\mathbf{v} + \eta\Lambda\mathbf{1} \le c + Lx \tag{10a}$$

$$-\Lambda \le F\mathbf{M} + G \le \Lambda. \tag{10b}$$

3 A min-max problem with a quadratic cost

Consider now the following finite horizon quadratic cost, as encountered in the literature on H_∞ control:

$$J_N(x,\gamma,\psi,\mathbf{w}) := x_N P x_N + \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i - \gamma^2 w_i^T w_i \tag{11}$$

where $x_0 = x$, $x_{i+1} = Ax_i + Bu_i + w_i$ and $u_i = v_i + \sum_{j=0}^{i-1} M_{i,j}w_j$ for all $i \in \{0,\dots,N-1\}$. The matrices P, Q and R are positive definite and γ is a positive scalar.

As in (9a), one can eliminate x_i and u_i in (11) to get matrices H_{xx}, $H_{x\mathbf{u}}$, $H_{x\mathbf{w}}$, $H_{\mathbf{uu}}$, $H_{\mathbf{uw}}$, $H_{\mathbf{ww}}$ of suitable dimensions such that

$$\begin{aligned} J_N(x,\gamma,\psi,\mathbf{w}) = x^T H_{xx} x + 2x^T H_{x\mathbf{u}}\mathbf{v} + \mathbf{v}^T H_{\mathbf{uu}}\mathbf{v} \\ + 2x^T(H_{x\mathbf{u}}\mathbf{M} + H_{x\mathbf{w}})\mathbf{w} + 2\mathbf{v}^T(H_{\mathbf{uu}}\mathbf{M} + H_{\mathbf{uw}})\mathbf{w} \\ - \mathbf{w}^T(\gamma^2 I - H_{\mathbf{ww}} - 2\mathbf{M}^T H_{\mathbf{uw}} - \mathbf{M}^T H_{\mathbf{uu}}\mathbf{M})\mathbf{w}, \end{aligned} \tag{12}$$

where H_{xx} and $H_{\mathbf{uu}}$ are positive definite and $H_{\mathbf{ww}}$ is positive semi-definite.

It is easy to show that $J_N(x,\gamma,\psi,\mathbf{w})$ is a convex function in ψ. To see why this is the case, note that it is sufficient to show that

$$f(\psi,\mathbf{w}) := \mathbf{v}^T H_{\mathbf{uu}}\mathbf{v} + 2\mathbf{v}^T H_{\mathbf{uu}}\mathbf{M}\mathbf{w} + \mathbf{w}^T\mathbf{M}^T H_{\mathbf{uu}}\mathbf{M}\mathbf{w} \tag{13}$$

is convex in ψ. Consider the function $g(\mathbf{u}) := \mathbf{u}H_{\mathbf{uu}}\mathbf{u}$, which is convex in $\mathbf{u}$. Since $f(\psi,\mathbf{w}) = g(\mathbf{v} + \mathbf{M}\mathbf{w})$ and recalling that convexity of a function is preserved under an affine map, it follows that $f(\psi,\mathbf{w})$ is convex in ψ.

Since the pointwise supremum of an arbitrary, infinite set of convex functions is convex, it follows that

$$V_N(x,\gamma,\psi) := \max_{\mathbf{w}\in\mathcal{W}} J_N(x,\gamma,\psi,\mathbf{w}) \tag{14}$$

is a *convex* function in ψ.

Note also that γ can be chosen sufficiently large such that

$$\gamma^2 I - H_{\mathbf{ww}} - \mathbf{M}^T H_{\mathbf{uw}} - H_{\mathbf{uw}}^T \mathbf{M} - \mathbf{M}^T H_{\mathbf{uu}} \mathbf{M} \succ 0. \tag{15}$$

Clearly, if (15) is satisfied, then $J_N(x, \gamma, \psi, \mathbf{w})$ is a *strictly concave* function in $\mathbf{w}$. This implies that $V_N(x, \gamma, \psi)$ can be computed by defining and solving a *tractable, strictly convex* quadratic programming (QP) problem.

Note that the number of variables and constraints in (10) is polynomial in N, n, m, r and s. Observe also that (15) is a quadratic matrix inequality (QMI) that, by Schur complement, can be converted to a linear matrix inequality (LMI) in $\mathbf{M}$ and γ^2. This implies that, for a given initial state $x = x(0)$, a sufficiently large γ and an admissible ψ can be found by solving an LMI defined from (6), (10) and (15).

We can now state the min-max problem that is of interest to us. For a given initial state $x = x(0)$ and γ, let

$$V_N^*(x, \gamma) := \min_{(\psi, \Lambda)} \left\{ V_N(x, \gamma, \psi) \;\middle|\; (\psi, \Lambda) \text{ satisfy (6), (10) and (15)} \right\}. \tag{16}$$

Recalling from the above that $V_N(x, \gamma, \psi)$ can be calculated efficiently by solving a tractable QP, it follows that one can compute $V_N^*(x, \gamma)$ efficiently using standard tools from convex optimization, such as cutting plane and interior-point methods.

4 Finite ℓ_2 gain

As a final, motivating point for this paper, let $\psi^*(x, \gamma)$ be a minimizer of the problem in (16) for the initial state $x = x(0)$ and a *time-varying* control policy be given by

$$u(k) = v_k^*(x, \gamma) + \sum_{j=0}^{k-1} M_{k,j}^*(x, \gamma) w(j), \quad \forall k \in \{0, \dots, N-1\}. \tag{17}$$

Note that (17) is a causal feedback policy that is dependent on the current state and past values of the state and input; since measurements of the state are available and past inputs are known, $w(j)$ in (17) is given by

$$w(j) = x(j+1) - Ax(j) - Bu(j), \quad \forall j \in \{0, \dots, N-1\}.$$

It follows from the optimality of $\psi^*(x, \gamma)$ that if the disturbance sequence $\{w(0), \dots, w(N-1)\}$ takes on values in W and the input sequence $\{u(0), \dots, u(N-1)\}$ is defined as in (17), then one has the following finite ℓ_2 gain property:

$$\sum_{k=0}^{N-1} x(k)^T Q(k) + u(k)^T R u(k) + x(N)^T P(N) \leq \gamma^2 \sum_{k=0}^{N-1} w(k)^T w(k) + V_N^*(x, \gamma). \tag{18}$$

Furthermore, $(x(k), u(k)) \in \mathcal{Y}$ for all $k \in \{0, \ldots, N-1\}$ and $x(N) \in X_f$

Remark 2. Further research may involve extending the results in this paper to H_∞ receding horizon control [4, Sect. 4.7]. The reader is referred to [2] for some initial results on the robust invariance of receding horizon controllers that are based on the parameterization in Section 2.

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