# Invariant approximations of robustly positively invariant sets for constrained linear discrete-time systems subject to bounded disturbances

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**Abstract**

This paper provides results on invariant approximations of robustly positively invariant sets for a discrete-time, linear, time-invariant system subject to state constraints. Two important sets, the minimal and the maximal robustly positively invariant sets and their approximations are investigated. Novel procedures for the computation of invariant approximations to these sets are presented. It is assumed that the disturbance is bounded, persistent and acts additively on the state and that the constraints on the state and disturbance are polyhedral.

**Keywords:** Set invariance, invariant approximations, constrained control, robust control, linear systems.

## 1 Introduction

The theory of set invariance plays a fundamental role in the control of constrained systems and has been a subject of research by many authors — see for instance [Bla99, Ker00] and the references therein. Two important issues are the calculation of the minimal robustly positively invariant (mRPI) set and the maximal robustly positively invariant (MRPI) set.

The mRPI set is used as a target set in robust time-optimal control [MS97], in the design of robust predictive controllers [ML01, LCRM04, KM03, KMss] and in understanding the properties of the *maximal* robustly positively invariant set [KG98, Kou02]. The MRPI set has been used extensively

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in the design of reference governors [GK95], in the regulation problem in model predictive control and the calculation of the region of feasibility in model predictive control problems [Ker00, CRZ01].

Despite this wide use of the mRPI and MRPI sets, there are still unresolved issues. For the case of the mRPI set, there exists no method for the exact computation of the mRPI set, except those given in [Las93, Sect. 3.3], [MS97, Thm. 3] and [SM98, Sect. II.B], where it is assumed that the closed-loop system dynamics are nilpotent. In [Kou02, RKKM03] this assumption is relaxed and a method for computing a robustly positively invariant approximation of the mRPI set is investigated and a solution is obtained for a specific case. In a significant attempt to calculate the MRPI set in [KG98], an algorithm was given and a condition was established for which the algorithm explicitly calculates the MRPI set. However, when this condition is not satisfied, the algorithm may fail to calculate the MRPI set in finite time.

It is the purpose of this paper to provide methods for computation of invariant approximations of the minimal and the maximal robustly positively invariant sets. This paper presents the methods for computation of a robustly positively invariant outer approximation of the minimal robustly positively invariant set as well as the computation of a so-called robustly positively invariant $\varepsilon$-outer approximation of the mRPI set.

Furthermore, a new recursive algorithm that calculates (approximates) the maximal robustly positively invariant set when it is compact (non-compact), will be presented. This is achieved by computing a sequence of robustly positively invariant sets. Moreover, we will discuss a number of useful *a priori* efficient tests and computations of upper bounds relevant to the proposed algorithms.

This paper is organized as follows. Section 2 is concerned with the definitions of the mRPI and MRPI sets and the problem formulation. Section 3 deals with the problem of calculating a robustly positively invariant (RPI) approximation of the mRPI set for systems with disturbance inputs that are bounded. In Section 4 we address issues concerning the calculation of the MRPI set and the approximation of the MRPI set. Computational algorithms and some efficient upper bound estimates are discussed and given in Section 5. A few illustrative examples are provided in Section 6. Finally, Section 7 presents conclusions.

**Notation:** Let $\mathbb{N} \triangleq \{0, 1, 2, \ldots\}$ be the set of non-negative integers, $\mathbb{N}_+ \triangleq \{1, 2, \ldots\}$ the set of integers greater than 0, the set of integers $\mathbb{N}_{[a,b]} \triangleq \{a, a+1, \ldots, b-1, b\}$, where $0 \leq a \leq b$. Given a vector $v \in \mathbb{R}^n$ and matrix $M \in \mathbb{R}^{m \times n}$, $\|v\|_p$ is the vector $p$-norm and $\|M\|_p$ is the induced matrix $p$-norm. If $M$ is a square matrix, then $\rho(M)$ is the spectral radius of $M$. Let $\mathbb{B}_p^n(r) \triangleq \{x \in \mathbb{R}^n \mid \|x\|_p \leq r\}$ be a $p$-norm ball in $\mathbb{R}^n$, where $r \geq 0$. Given two sets $\mathcal{U}$ and $\mathcal{V}$, such that $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^n$, the Minkowski (vector) sum is defined by $\mathcal{U} \oplus \mathcal{V} \triangleq \{u + v \mid u \in \mathcal{U},\ v \in \mathcal{V}\}$, and the Pontryagin difference as $\mathcal{U} \ominus \mathcal{V} \triangleq \{u \in \mathbb{R}^n \mid u + v \in \mathcal{U},\ \forall v \in \mathcal{V}\} = \{u \in \mathbb{R}^n \mid u \oplus \mathcal{V} \subseteq \mathcal{U}\}$. Given the collection of sets $\{\mathcal{U}_i \subset \mathbb{R}^n \mid i \in \mathbb{N}_{[a,b]}\}$, where $a \leq b$, we denote $\bigoplus_{i=a}^{b} \mathcal{U}_i \triangleq \mathcal{U}_a \oplus \mathcal{U}_{a+1} \oplus \ldots \oplus \mathcal{U}_b$. Let $v(\cdot) \triangleq \{v(0), v(1), \ldots\}$ denote an infinite sequence of variables, where $v(k)$ is the $k$'th element in the sequence. The set $\mathcal{M}_{\mathcal{V}} \triangleq \{v(\cdot) \mid v(k) \in \mathcal{V},\ \forall k \in \mathbb{N}\}$ is the set of all infinite sequences whose elements take on values in $\mathcal{V} \subseteq \mathbb{R}^n$ (equivalently $\mathcal{M}_{\mathcal{V}}$ is the set of all maps $v : \mathbb{N} \to \mathcal{V}$). We use $\lceil x \rceil$ to denote the smallest integer greater than or equal to $x$.

## 2 Preliminary Definitions and Results

We consider the following autonomous discrete-time, linear, time-invariant (DLTI) system:

$$x^+ = Ax + w, \tag{1}$$

where $x \in \mathbb{R}^n$ is the current state, $x^+$ is the successor state and $w \in \mathbb{R}^n$ is an unknown disturbance. We make the standing assumption that $A \in \mathbb{R}^{n\times n}$ is a strictly stable matrix (all the eigenvalues of $A$ are strictly inside the unit disk). The disturbance $w$ is persistent, but contained in a convex and compact (i.e. closed and bounded) set $W \subset \mathbb{R}^n$, which contains the origin.

If the initial state is $x$ at time 0 (note that since the system is time-invariant, the current time can always be taken to be zero), then we denote by $\phi(k, x, w(\cdot))$ the solution to (1) at time instant $k$, given the infinite disturbance sequence $w(\cdot) \triangleq \{w(0), w(1), \ldots\}$.

### 2.1 The minimal and maximal robustly positively invariant sets

The motivation for this paper is that often one would like to determine whether the state trajectory of the system will be contained in a set $X \subset \mathbb{R}^n$, given any allowable disturbance sequence. For this purpose, we present the following definition:

**Definition 1 (RPI set).** [Bla99] The set $\Omega \subset \mathbb{R}^n$ is a *robustly positively invariant* (RPI) set of (1) if $Ax + w \in \Omega$ for all $x \in \Omega$ and all $w \in W$.

*Remark* 1. It is useful to note that, by definition, $\Omega$ is RPI if and only if $A\Omega \oplus W \subseteq \Omega$. Note also that $\Omega$ is RPI if and only if $A\Omega \subseteq \Omega \ominus W$.

**Definition 2 (Constraint-admissible set).** The set $\Omega \subset \mathbb{R}^n$ is a *constraint-admissible* set if it is contained in $X \subset \mathbb{R}^n$.

*Remark* 2. Clearly, the set $\Omega$ is a *constraint-admissible*, RPI set if it is contained in $X$ and $\Omega$ is RPI.

The existence of RPI sets is very important for the satisfaction of constraints. It is well-known [Bla99] that the solution of the system will satisfy $\phi(k, x, w(\cdot)) \in X$ for all time $k \in \mathbb{N}$ and all allowable disturbance sequences $w(\cdot) \in \mathcal{M}_W$ *if and only if* there exists a constraint-admissible, RPI set $\Omega$ and the initial state $x$ is in $\Omega$.

An important set in the analysis and synthesis of controllers for constrained systems is the minimal RPI set:

**Definition 3 (mRPI set).** The *minimal robustly positively invariant* (mRPI) set $F_\infty$ of (1) is the RPI set of (1) that is contained in every closed, RPI set of (1).

The properties of the mRPI set $F_\infty$ are well-known. It is possible to show [KG98, Sect. IV] that the mRPI set $F_\infty$ exists, is unique, compact and contains the origin. It is also easy to show that the zero initial condition response of (1) is bounded in $F_\infty$, i.e. $\phi(k, 0, w(\cdot)) \in F_\infty$ for all $w(\cdot) \in \mathcal{M}_W$ and all $k \in \mathbb{N}$. It therefore follows, from the linearity and asymptotic stability of system (1), that

$F_\infty$ is the limit set of all trajectories of (1). In particular, $F_\infty$ is the smallest closed set in $\mathbb{R}^n$ that has the following property: given any $r > 0$ and $\varepsilon > 0$, there exists a $\bar{k} \in \mathbb{N}$ such that if $x \in \mathbb{B}_p^n(r)$, then the solution of (1) satisfies $\phi(k, x, w(\cdot)) \in F_\infty \oplus \mathbb{B}_p^n(\varepsilon)$ for all $w(\cdot) \in \mathcal{M}_W$ and all $k \geq \bar{k}$.

Another important set in the analysis and synthesis of controllers for constrained systems is the maximal RPI set:

**Definition 4 (MRPI set).** The *maximal robustly positively invariant* (MRPI) set $O_\infty$ of (1) is the constraint-admissible, RPI set of (1) that contains every constraint-admissible, RPI set of (1).

The properties of the MRPI set $O_\infty$ are well-known and the reader is referred to [KG98] for a detailed study of this set. The MRPI set, if it is non-empty, is unique. It is also good to know that if $X$ is compact and convex, then $O_\infty$ is also compact and convex.

One of the reasons for our interest in the mRPI set $F_\infty$ stems from the following well-known fact, which relates the mRPI set $F_\infty$ to the MRPI set $O_\infty$:

**Proposition 1 (Existence of the MRPI set).** *[KG98] The following statements are equivalent:*

- *The MRPI set $O_\infty$ is non-empty.*
- $F_\infty \subseteq X$.
- $X \ominus F_\infty$ *contains the origin.*

*Remark* 3. A sufficient condition for checking whether $O_\infty$ is non-empty is given in [KG98, Rem. 6.6]. Without going into details, [KG98, Rem. 6.6] proposes to compute an inner approximation of $X \ominus F_\infty$ and testing whether or not the origin is in the interior of this approximation. The results in this paper can also be used to compute an inner approximation of $X \ominus F_\infty$. However, the advantage of the results in this paper is that they allow one to specify an *a priori* level of accuracy for the approximation. As a consequence, one can directly quantify the level of conservativeness in case the test for non-emptiness of $O_\infty$ fails. This is not possible with the procedure proposed in [KG98, Rem. 6.6].

## 2.2 Sequences and approximations of sets

This paper is concerned with finding invariant approximations of the minimal and maximal RPI sets of (1). A useful measure for determining whether one set is a good approximation of another set, is the well-known Hausdorff metric:

**Definition 5 (Hausdorff metric).** If $\Omega$ and $\Phi$ are two non-empty, compact sets in $\mathbb{R}^n$, then the *Hausdorff metric* is defined as

$$d_H^p(\Omega, \Phi) \triangleq \max \left\{ \sup_{\omega \in \Phi} d(\omega, \Omega),\ \sup_{\phi \in \Omega} d(\phi, \Phi) \right\}, \tag{2}$$

where

$$d(z, \mathcal{Z}) \triangleq \inf_{y \in \mathcal{Z}} \|z - y\|_p. \tag{3}$$

*Remark* 4. Clearly, $\Omega = \Phi$ if and only if $d_H^p(\Omega, \Phi) = 0$. It is also useful to note that $d_H^p(\Omega, \Phi)$ is the size of the smallest norm-ball that can be added to $\Omega$ in order to cover $\Phi$ and vice versa, i.e.

$$d_H^p(\Omega, \Phi) = \inf \left\{ \varepsilon \geq 0 \;\middle|\; \Phi \subseteq \Omega \oplus \mathbb{B}_p^n(\varepsilon) \text{ and } \Omega \subseteq \Phi \oplus \mathbb{B}_p^n(\varepsilon) \right\}. \tag{4}$$

Given this last observation, we will use the Hausdorff metric to talk about convergence of a sequence of compact sets:

**Definition 6 (Limit of a sequence of sets).** An infinite sequence of non-empty, compact sets $\{\Omega_1, \Omega_2 \ldots\}$, where each $\Omega_i \subset \mathbb{R}^n$, is said to converge to a non-empty, compact set $\Omega \subset \mathbb{R}^n$ if $d_H^p(\Omega, \Omega_i) \to 0$ as $i \to \infty$.

**Definition 7 (Increasing/decreasing sequences of sets).** A sequence of non-empty sets $\{\Omega_1, \Omega_2, \ldots\}$, where each $\Omega_i \subset \mathbb{R}^n$, is *decreasing* if $\Omega_{i+1} \subseteq \Omega_i$ for all $i \in \mathbb{N}_+$. Similarly, the sequence of sets is *increasing* if $\Omega_{i+1} \supseteq \Omega_i$ for all $i \in \mathbb{N}_+$.

In the sequel, we will be generating sequences of outer and inner approximations of the minimal and maximal RPI sets, respectively. This motivates the following definition:

**Definition 8 ($\varepsilon$-approximations).** Given a scalar $\varepsilon \geq 0$, the set $\Phi \subset \mathbb{R}^n$ is said to be an $\varepsilon$-outer approximation to the set $\Omega \subset \mathbb{R}^n$ if $\Omega \subseteq \Phi \subseteq \Omega \oplus \mathbb{B}_p^n(\varepsilon)$. The set $\Psi \subset \mathbb{R}^n$ is said to be an $\varepsilon$-inner approximation of the set $\Omega \subset \mathbb{R}^n$ if $\Psi \subseteq \Omega \subseteq \Psi \oplus \mathbb{B}_p^n(\varepsilon)$.

## 3 Approximations of the minimal robustly positively invariant set

We now turn our attention to methods for computing $F_\infty$. If we were to define the (convex and compact) set $F_s$ as

$$F_s \triangleq \bigoplus_{i=0}^{s-1} A^i W, \tag{5}$$

then it is possible to show [KG98, Sect. IV] that $F_s \subseteq F_\infty$ and that $F_s \to F_\infty$ as $s \to \infty$, i.e. for every $\varepsilon > 0$, there exists an $s \in \mathbb{N}$ such that $F_\infty \subseteq F_s \oplus \mathbb{B}_p^n(\varepsilon)$. Clearly, $F_\infty$ is then given by

$$F_\infty = \bigoplus_{i=0}^{\infty} A^i W. \tag{6}$$

Since $F_\infty$ is a Minkowski sum of infinitely many terms, it is generally impossible to obtain an explicit characterization of it. However, as noted in [Las93, Sect. 3.3] and [KG98, Rem. 4.2], it is possible to show that if there exist an integer $s \in \mathbb{N}_+$ and a scalar $\alpha \in [0, 1)$ such that $A^s = \alpha I$, then $F_\infty = (1-\alpha)^{-1} \bigoplus_{i=0}^{s-1} A^i W$. It therefore follows trivially [MS97, Thm. 3] that if $A$ is nilpotent with index $s$ ($A^s = 0$), then $F_\infty = \bigoplus_{i=0}^{s-1} A^i W$.

In this section, we relax the assumption that there exists an $s \in \mathbb{N}_+$ and a scalar $\alpha \in [0, 1)$ such that $A^s = \alpha I$. Since we can no longer compute $F_\infty$ exactly, we address the problem of computing an RPI, outer approximation of the mRPI set $F_\infty$.

We will first continue to address the problem of computing an RPI set that contains the mRPI set $F_\infty$. This will be achieved by scaling $F_s$ by a suitable amount. We will also address the problem of how to compute an RPI, $\varepsilon$-outer approximation of the mRPI set $F_\infty$, by computing the reachable set of states of a given RPI set.

## 3.1 Scaling $F_s$

### 3.1.1 The origin is in the interior of $W$

Our first main result is as follows:

**Theorem 1.** *[Kou02] If $0 \in \operatorname{int}(W)$, then there exists a finite integer $s \in \mathbb{N}_+$ and a scalar $\alpha \in [0, 1)$ that satisfies*

$$A^s W \subseteq \alpha W. \tag{7}$$

*Furthermore, if (7) is satisfied, then*

$$F(\alpha, s) \triangleq (1-\alpha)^{-1} F_s \tag{8}$$

*is a convex, compact, RPI set of (1). Furthermore, $0 \in \operatorname{int}(F(\alpha, s))$ and $F_\infty \subseteq F(\alpha, s)$.*

*Proof.* Existence of an $s \in \mathbb{N}_+$ and an $\alpha \in [0, 1)$ that satisfies (7) follows from the fact that the origin is in the *interior* of $W$ and that $A$ is strictly stable.

Convexity and compactness of $F(\alpha, s)$ follows directly from the fact that $F_s$ (and hence $F(\alpha, s)$) is the Minkowski sum of a finite set of convex and compact sets.

Let $G(\alpha, j, k) \triangleq (1-\alpha)^{-1} \bigoplus_{i=j}^{k} A^i W$. It follows that

$$\begin{aligned}
AG(\alpha, 0, s-1) \oplus W &= G(\alpha, 1, s) \oplus W && \text{(9a)}\\
&= (1-\alpha)^{-1} A^s W \oplus G(\alpha, 1, s-1) \oplus W && \text{(9b)}\\
&\subseteq (1-\alpha)^{-1} \alpha W \oplus W \oplus G(\alpha, 1, s-1) && \text{(9c)}\\
&= [(1-\alpha)^{-1}\alpha + 1] W \oplus G(\alpha, 1, s-1) && \text{(9d)}\\
&= (1-\alpha)^{-1} W \oplus G(\alpha, 1, s-1) && \text{(9e)}\\
&= G(\alpha, 0, s-1). && \text{(9f)}
\end{aligned}$$

In going from (9b) to (9c) we have used the fact that $P \subseteq Q \Rightarrow P \oplus R \subseteq Q \oplus R$ for arbitrary sets $P \subset \mathbb{R}^n$, $Q \subset \mathbb{R}^n$ and $R \subset \mathbb{R}^n$.

Since $F(\alpha, s) = G(\alpha, 0, s-1)$, it follows that $AF(\alpha, s) \oplus W \subseteq F(\alpha, s)$ holds, hence $F(\alpha, s)$ is RPI. It follows trivially from the definition of the mRPI set that $F(\alpha, s)$ contains $F_\infty$. Note also that $0 \in \operatorname{int}(F_\infty)$ if $0 \in \operatorname{int}(W)$.

□

Note that

$$F(\alpha_0, s) \subset F(\alpha_1, s) \Leftrightarrow \alpha_0 < \alpha_1. \tag{10}$$

Furthermore, if $A$ is not nilpotent, then

$$F(\alpha, s_0) \subset F(\alpha, s_1) \Leftrightarrow s_0 < s_1. \tag{11}$$

Clearly, based on these observations, one can obtain a better approximation of the mRPI set $F_\infty$, given an initial pair $(\alpha, s)$. Let

$$s^{\mathrm{o}}(\alpha) \triangleq \inf \{ s \in \mathbb{N}_+ \mid A^s W \subseteq \alpha W \}, \tag{12a}$$

$$\alpha^{\mathrm{o}}(s) \triangleq \inf \{ \alpha \in [0,1) \mid A^s W \subseteq \alpha W \} \tag{12b}$$

be the smallest values of $s$ and $\alpha$ such that (7) holds for a given $\alpha$ and $s$, respectively.

*Remark* 5. The infimum in (12a) exists for any choice of $\alpha \in (0,1)$; $s^{\mathrm{o}}(0)$ is finite if and only if $A$ is nilpotent. Note that $s^{\mathrm{o}}(\alpha) \to \infty$ as $\alpha \searrow 0$ if and only if $A$ is not nilpotent. The infimum in (12b) is also guaranteed to exist if $s$ is sufficiently large. Note that there exists a finite $s$ such that $\alpha^{\mathrm{o}}(s) = 0$ if and only if $A$ is nilpotent. However, if $A$ is not nilpotent, then $\alpha^{\mathrm{o}}(s) \searrow 0$ as $s \to \infty$.

By a process of iteration one can use the above definitions and results to compute a pair $(\alpha, s)$ such that $F(\alpha, s)$ is a sufficiently good RPI, outer approximation of $F_\infty$.

For example, by starting with $s = 1$, one can increment $s$ until there exists an $\alpha \in [0,1)$ such that (7) holds. One can then compute $\alpha^{\mathrm{o}}(s)$ and use $F(\alpha^{\mathrm{o}}(s), s)$ as an RPI aproximation to $F_\infty$. If necessary, one can increase $s$ until $F(\alpha^{\mathrm{o}}(s), s)$ is a sufficiently close approximation of $F_\infty$.

Alternatively, one can take an initial value for $\alpha$, compute $s^* \triangleq s^{\mathrm{o}}(\alpha)$, proceed to compute $\alpha^* \triangleq \alpha^{\mathrm{o}}(s^*)$ and test whether $F(\alpha^*, s^*)$ is small enough. It is clear that this iteration results in

$$F_\infty \subseteq F(\alpha^*, s^*) \subseteq F(\alpha, s^*) \subseteq F(\alpha, s). \tag{13}$$

If $F(\alpha^*, s^*)$ is not a good enough approximation of $F_\infty$, then this procedure could be restarted by choosing a smaller value for $\alpha$.

Of course, any other variation to the above can be implemented until a fixed point is reached or $F(\alpha, s)$ is deemed to be a sufficiently close approximation of $F_\infty$. The above observations allow one to strengthen Theorem 1, but before proceeding we need the following:

**Lemma 1.** *If $\Phi$ is a convex and compact set in $\mathbb{R}^n$ containing the origin and $\alpha \in [0,1)$, then $d_H^p(\Phi, (1-\alpha)^{-1}\Phi) \leq \alpha(1-\alpha)^{-1}M$, where $M \triangleq \sup_{z \in \Phi} \|z\|$, and $d_H^p(\Phi, (1-\alpha)^{-1}\Phi) \to 0$ as $\alpha \searrow 0$.*

*Proof.* Since $\alpha \in [0,1)$ and $0 \in \Phi$, it follows that $\Phi \subseteq (1-\alpha)^{-1}\Phi$ so that

$$\begin{aligned} d_H^p(\Phi, (1-\alpha)^{-1}\Phi) &= \sup \left\{ d(\Phi, x) \;\middle|\; x \in (1-\alpha)^{-1}\Phi \right\} \\ &= \sup \left\{ \inf_{y \in \Phi} \|y - x\| \;\middle|\; x \in (1-\alpha)^{-1}\Phi \right\} \\ &= \sup_{z \in \Phi} \inf_{y \in \Phi} \|y - (1-\alpha^{-1})z\| \\ &\leq \sup_{z \in \Phi} \|z - (1-\alpha^{-1})z\| \\ &= ((1-\alpha)^{-1} - 1)M = \alpha(1-\alpha)^{-1}M, \end{aligned}$$

where $M \triangleq \sup_{z \in \Phi} \|z\|$.

Hence, $d_H^p(\Phi, (1-\alpha)^{-1}\Phi) \leq \alpha(1-\alpha)^{-1}M$ and therefore $d_H^p(\Phi, (1-\alpha)^{-1}\Phi) \to 0$ as $\alpha \searrow 0$. ☐

We recall that $\{F_s\}$ is Cauchy [KG98, Sect. IV] so that $M_\infty \triangleq \lim_{s\to\infty} \sup_{z \in F_s} \|z\|$ is finite and since $F_s \subseteq F_\infty$, $\forall s \in \mathbb{N}$ we have that $M_s \triangleq \sup_{z \in F_s} \|z\| \leq M_\infty$ is finite for all $s \in \mathbb{N}$. This fact and the above Lemma allows one to state the following:

**Theorem 2.** *If* $0 \in \operatorname{int}(W)$*, then*

*(i)* $F(\alpha^o(s), s) \to F_\infty$ *as* $s \to \infty$ *and*

*(ii)* $F(\alpha, s^o(\alpha)) \to F_\infty$ *as* $\alpha \searrow 0$.

*Proof.* (i) Since $\alpha^o(s) \searrow 0$ as $s \to \infty$ and $d_H^p(F_s, F(\alpha^o(s), s)) \leq \alpha^o(s)(1-\alpha^o(s))^{-1}M(s)$, where $M(s) \triangleq \sup_{z \in F_s} \|z\|$ is finite for each $s$, it follows that $d_H^p(F_s, F(\alpha^o(s), s)) \searrow 0$ as $s \to \infty$. However, since $F(\alpha^o(s), s) \supseteq F_\infty \supseteq F_s$ for each $s$ and $F_s \to F_\infty$ as $s \to \infty$, it follows that $F(\alpha^o(s), s) \to F_\infty$ as $s \to \infty$.

(ii) Since $d_H^p(F_{s^o(\alpha)}, F(\alpha, s^o(\alpha))) \leq \alpha(1-\alpha)^{-1}M(\alpha)$, where $M(\alpha) \triangleq \sup_{z \in F_{s^o(\alpha)}} \|z\|$ is finite for each $s^o(\alpha)$ it follows that $d_H^p(F_{s^o(\alpha)}, F(\alpha, s^o(\alpha))) \searrow 0$ as $\alpha \searrow 0$. However, since $F(\alpha, s^o(\alpha)) \supseteq F_\infty \supseteq F_{s^o(\alpha)}$ for each $s^o(\alpha)$ and $F_{s^o(\alpha)} \to F_\infty$ as $\alpha \searrow 0$, since $s^o(\alpha) \to \infty$, it follows that $F(\alpha, s^o(\alpha)) \to F_\infty$ as $\alpha \searrow 0$.

☐

*Remark* 6. If $A$ is nilpotent with index $\tilde{s}$ then $\alpha^o(s) = 0$ for all $s \geq \tilde{s}$. Since $F_{\tilde{s}} = F_\infty$ it follows that $F(\alpha^o(s), s) = F_\infty$ for all $s \geq \tilde{s}$, hence $F(\alpha^o(s), s) \to F_\infty$ as $s \to \infty$. A similar argument shows that $F(\alpha, s^o(\alpha)) \to F_\infty$ as $\alpha \searrow 0$, since $F_{\tilde{s}} = F_\infty$ and $\alpha = 0$ for a finite $\tilde{s}$ so that $s^o(0) = \tilde{s}$.

Clearly, the case when the origin is in the interior of $W$ does not pose any problems with regards the existence of an $\alpha \in [0,1)$ and a finite $s \in \mathbb{N}_+$ that satisfy (7), provided one bear in mind whether or not $A$ is nilpotent.

### 3.1.2 The origin is in the relative interior of $W$

The results in the previous section can be extended to a more general case, when the interior of $W$ is empty, but the origin is in the relative interior of $W$.

Let the disturbance set now be given by

$$W \triangleq ED \tag{14}$$

where the matrix $E \in \mathbb{R}^{n\times l}$ and the set $D \subset \mathbb{R}^l$ is a convex, compact set containing the origin in its interior. Clearly, $W$ is convex and compact and the origin is in the *relative interior* of $W$. However, if $\operatorname{rank}(E) < n$, then the *interior* of $W$ is empty.

We will now attempt to calculate an RPI, outer-approximation of the mRPI set $F_\infty$ under the above, relaxed assumptions:

**Theorem 3.** *Let $0 \in \operatorname{int}(D)$ and $W \triangleq ED$, with $E \in \mathbb{R}^{n\times l}$. There exist positive integers $p$, $r$ and $s$ and a scalar $\alpha \in [0, 1)$ such that*

$$A^s ED \subseteq \alpha F_p \text{ and } A^r F_p \subseteq \alpha F_p. \tag{15}$$

*Furthermore, if (15) is satisfied, then*

$$\overline{F}(\alpha, p, r, s) \triangleq F_s \oplus \alpha(1-\alpha)^{-1} \bigoplus_{i=0}^{r-1} A^i F_p \tag{16}$$

*is a convex, compact, RPI set of (1), containing $F_\infty$.*

*Proof.* See Appendix A for the proof. □

*Remark* 7. If $ED$ contains the origin in its interior, then by letting $p = 1$, we get that

$$A^s ED \subseteq \alpha ED \text{ and } A^r ED \subseteq \alpha ED, \tag{17}$$

which for $s = r$ becomes condition (7). The set $\overline{F}(\alpha, 1, r, s) = F(\alpha, s)$ is then given by (8) and the case when $W$ contains the origin in its interior is recovered.

In practice, one often assumes disturbances on each of the states, hence it is quite often the case that the origin is indeed contained in the interior of $W$. Because of this and the fact that testing the conditions in (15) is a lot more complicated than testing (7), we will not consider the case when the interior of $W$ is empty in any further detail.

## 3.2 Computing the reachable set of an RPI set

We will now consider the case when we compute the reachable set of an RPI set. Before proceeding, we define the following:

**Definition 9 (Reachable set).** Given the non-empty set $\Omega \subset \mathbb{R}^n$, the *$N$-step reachable set*, where $N \in \mathbb{N}_+$, is defined as

$$\operatorname{Reach}_N(\Omega) \triangleq \{\phi(N, x, w(\cdot)) \mid x \in \Omega,\ w(\cdot) \in \mathcal{M}_W\}. \tag{18}$$

The set of states reachable from $\Omega$ in 0 steps is defined as $\operatorname{Reach}_0(\Omega) \triangleq \Omega$.

*Remark* 8. It is easy to show that

$$\text{Reach}_N(\Omega) = A\,\text{Reach}_{N-1}(\Omega) \oplus W \tag{19}$$

and

$$\text{Reach}_N(\Omega) = A^N\Omega \oplus F_N \tag{20}$$

for all $N \in \mathbb{N}_+$.

*Remark* 9. If $\Omega$ is closed, then $\text{Reach}_N(\Omega)$ is also closed because the linear map of a closed set is a closed set and the Minkowski sum of a finite number of closed sets is a closed set. Similarly, $\text{Reach}_N(\Omega)$ is bounded (compact) if $\Omega$ is bounded (compact).

Recalling that $F_\infty$ is the limit set of all trajectories of (1), it follows that $\text{Reach}_N(\Omega) \to F_\infty$ in the Hausdorff metric as $s \to \infty$ for any non-empty set $\Omega$. In particular:

**Lemma 2.** *If $\Omega$ is a compact set in $\mathbb{R}^n$ and $\varepsilon > 0$, then there exists an integer $N \in \mathbb{N}$ such that*

$$A^N\Omega \subseteq \mathbb{B}_p^n(\varepsilon). \tag{21}$$

*If* (21) *holds and $F_\infty \subseteq \Omega$, then* $\text{Reach}_N(\Omega)$ *is a compact, $\varepsilon$-outer approximation of $F_\infty$.*

*Proof.* Existence of an $N \in \mathbb{N}$ that satisfies (21) follows from the fact that $\Omega$ is compact and that $A$ is strictly stable. The proof is completed by recalling (20) and the fact that $F_N \subseteq F_\infty \subseteq \Omega$, hence $F_\infty \subseteq \text{Reach}_N(\Omega)$ for all $N \in \mathbb{N}_+$. □

**Lemma 3.** *If $\Omega$ is a closed, RPI set, then* $\text{Reach}_N(\Omega)$ *is a closed, RPI set and* $\text{Reach}_{N+1}(\Omega) \subseteq \text{Reach}_N(\Omega)$ *for all $N \in \mathbb{N}$. In other words,* $\{\Omega, \text{Reach}_1(\Omega), \text{Reach}_2(\Omega), \ldots\}$ *is a decreasing sequence of closed, RPI sets.*

*Proof.* The proof is by induction. Let $\text{Reach}_N(\Omega)$ be a closed, RPI set. This implies that

$$A\,\text{Reach}_N(\Omega) \oplus W \subseteq \text{Reach}_N(\Omega). \tag{22}$$

The fact that $\text{Reach}_{N+1}(\Omega) \subseteq \text{Reach}_N(\Omega)$ follows from (19).

Note also that

$$\begin{aligned} A\,\text{Reach}_{N+1}(\Omega) \oplus W &= A(A\,\text{Reach}_N(\Omega) \oplus W) \oplus W && \text{(23a)}\\ &\subseteq A\,\text{Reach}_N(\Omega) \oplus W && \text{(23b)}\\ &= \text{Reach}_{N+1}(\Omega). && \text{(23c)} \end{aligned}$$

This proves that $\text{Reach}_{N+1}(\Omega)$ is RPI.

The proof is completed by checking, in a similar fashion as above, that $\text{Reach}_1(\Omega) \subseteq \Omega$ and that $\text{Reach}_1(\Omega)$ is RPI. □

We can now state the main result of this section:

**Theorem 4.** *If $\Omega$ is a compact, RPI set, then there exists an $N \in \mathbb{N}$ that satisfies* (21)*, in which case* $\text{Reach}_N(\Omega)$ *is a compact, RPI, $\varepsilon$-outer approximation of $F_\infty$.*

*Proof.* The result follows from Lemmas 2 and 3 by recalling that $F_\infty$ is contained in all closed, RPI sets. □

**Corollary 1.** *If $\Omega$ is a closed, RPI set such that $F_\infty \subseteq \Omega$ and $\text{Reach}_N(\Omega) = \text{Reach}_{N+1}(\Omega)$ for some $N \in \mathbb{N}$, then $F_\infty = \text{Reach}_N(\Omega)$.*

*Proof.* Suppose that $\text{Reach}_N(\Omega) = \text{Reach}_{N+1}(\Omega)$. It follows from (19) that $\text{Reach}_N(\Omega) = \text{Reach}_{N+k}(\Omega)$ for all $k \in \mathbb{N}$. From (20) it follows that $\text{Reach}_{N+k}(\Omega) = A^k \,\text{Reach}_N(\Omega) \oplus F_k$ so that $\text{Reach}_{N+k}(\Omega) \to F_\infty$ as $k \to \infty$, which proves claim that $\text{Reach}_N(\Omega) = F_\infty$ if $\text{Reach}_N(\Omega) = \text{Reach}_{N+1}(\Omega)$. □

**Corollary 2.** *Let $F_\infty \subset \text{int}(X)$ and $\varepsilon > 0$ be small enough such that $F_\infty \oplus \mathbb{B}_p^n(\varepsilon) \subseteq X$. If the conditions of Theorem 1 hold and $N$ satisfies* (21) *with $\Omega \triangleq F(\alpha, s)$, then $\text{Reach}_N(F(\alpha, s))$ is a compact, constraint-admissible, RPI, $\varepsilon$-outer approximation of $F_\infty$.*

*Remark* 10. Clearly, any set obtained using the results in [Bla94] or the $O_\infty$ obtained by replacing $X$ with a sufficiently large, compact subset of $X$, are also suitable candidates for $\Omega$ in Theorem 4.

# 4 The maximal robustly positively invariant MRPI set

Before proceeding, we need to define the following:

**Definition 10 (Predecessor set).** Given the non-empty set $\Omega \subset \mathbb{R}^n$, the *$N$-step predecessor set* $\text{Pre}_N(\Omega)$, where $N \in \mathbb{N}_+$, is defined as

$$\text{Pre}_N(\Omega) \triangleq \left\{ x \in X \;\middle|\; \phi(N, x, w(\cdot)) \in \Omega,\ \phi(k, x, w(\cdot)) \in X,\ \forall k \in \mathbb{N}_{[0,N-1]},\ \forall w(\cdot) \in \mathcal{M}_W \right\}. \quad (24)$$

The *predecessor set* is defined as $\text{Pre}(\Omega) \triangleq \text{Pre}_1(\Omega)$ and $\text{Pre}_0(\Omega) \triangleq \Omega$.

It follows immediately that

$$\text{Pre}(\Omega) = \{x \in X \mid Ax + w \in \Omega,\ \forall w \in W\} = \{x \in X \mid Ax \in \Omega \ominus W\} \quad (25)$$

and

$$\text{Pre}_N(\Omega) = \text{Pre}(\text{Pre}_{N-1}(\Omega))) \quad (26)$$

for all $N \in \mathbb{N}_+$.

It is well-known [Bla99, KG98] that the *maximal* robustly positively invariant (MRPI) is the set of all initial states in $X$ for which the evolution of the system remains in $X$, i.e.

$$O_\infty = \{x \in X \mid \phi(k, x, w(\cdot)) \in X,\ \forall k \in \mathbb{N}_+,\ \forall w(\cdot) \in \mathcal{M}_W\}. \quad (27)$$

Let $O_t$ to be the set of all initial states in $X$ for which the evolution of the system remains in $X$ for $t$ steps, i.e.

$$O_t \triangleq \left\{x \in X \;\middle|\; \phi(k, x, w(\cdot)) \in X,\ \forall k \in \mathbb{N}_{[0,t]},\ \forall w(\cdot) \in \mathcal{M}_W \right\} \tag{28a}$$
$$= \text{Pre}_t(X). \tag{28b}$$

Note that $O_t \subseteq O_{t-1}$ for all $t \in \mathbb{N}_+$, i.e. $\{X, O_1, O_2, \ldots\}$ is a decreasing sequence of sets.

*Remark* 11. Given a non-empty set $\Omega$ in $\mathbb{R}^n$, it follows immediately from the definition of $O_t$ that $\Omega \subseteq O_t$ if and only if $\text{Reach}_k(\Omega) \subseteq X$ for all $k \in \mathbb{N}_{[0,t]}$.

It is well-known that $O_\infty$ can be calculated from the recursion

$$O_0 = X, \quad O_t = \text{Pre}(O_{t-1}),\ \forall t \in \mathbb{N}_+ \tag{29a}$$

and that the MRPI set is then given by

$$O_\infty = \bigcap_{t=0}^{\infty} O_t. \tag{29b}$$

Clearly, it is very difficult to calculate $O_\infty$ from (29b). However, if there exists a finite index $t \in \mathbb{N}$ such that $O_\infty = O_t$, then $O_\infty$ is said to be *finitely determined*, i.e. it can be calculated in a finite number of steps.

## 4.1 On the determinedness index of $O_\infty$

A necessary and sufficient condition for the *finite determination* of $O_\infty$ is that $O_t = O_{t+1}$ holds for some finite $t \in \mathbb{N}$. The smallest index $t$ such that $O_t = O_{t+1}$ is called the *determinedness index*, and will be denoted by $t^*$. As shown in [KG98], $O_\infty$ is finitely determined if there exists an $\ell \in \mathbb{N}$ such that $O_\ell$ is compact. We will present here a result that allows one to compute an upper bound on the determinedness index $t^*$ of $O_\infty$.

We present a number of results, which can be interpreted as restatements of results in [KG98]. However, the emphasis here is different, because we are interested in computing *a priori* whether or not $O_\infty$ is finitely determined. The results stated below allow one to do this.

**Theorem 5.** *Given any $O_\ell$, if $t \in \mathbb{N}$ satisfies*

$$\text{Reach}_{t+\ell+1}(O_\ell) \subseteq O_\ell \tag{30}$$

*then $O_{t+\ell} = O_{t+\ell+1}$. If $O_\ell$ is compact and $F_\infty \subseteq \text{int}(O_\ell)$, then there exists a finite $t$ such that* (30) *holds.*

*Alternatively, if $\Omega$ is any set such that $F_\infty \subseteq \Omega \subseteq O_\ell$ and*

$$A^{t+\ell+1} O_\ell \subseteq O_\ell \ominus \Omega, \tag{31}$$

*then $O_{t+\ell} = O_{t+\ell+1}$. If $O_\ell$ is compact and $\Omega \subseteq \text{int}(O_\ell)$, then there exists a finite $t$ such that* (31) *holds.*

*In other words, the determinedness index $t^*$ of the MRPI set $O_\infty$ is less than or equal to $t+\ell$ if* (30) *or* (31) *holds.*

*Proof.* Recalling that $O_{t+\ell} \subseteq O_\ell$, it follows that

$$A^{t+\ell+1}O_{t+\ell} \subseteq A^{t+\ell+1}O_\ell, \tag{32}$$

hence

$$A^{t+\ell+1}O_{t+\ell} \oplus F_{t+\ell+1} \subseteq A^{t+\ell+1}O_\ell \oplus F_{t+\ell+1}. \tag{33}$$

From (20) it follows that

$$\text{Reach}_{t+\ell+1}(O_{t+\ell}) \subseteq \text{Reach}_{t+\ell+1}(O_\ell). \tag{34}$$

If (30) holds, then

$$\text{Reach}_{t+\ell+1}(O_{t+\ell}) \subseteq O_\ell \subseteq X. \tag{35}$$

Recalling Remark 11, this result implies that $O_{t+\ell} \subseteq O_{t+\ell+1}$. However, since $O_{t+\ell} \supseteq O_{t+\ell+1}$ is always true, it follows that $O_{t+\ell} = O_{t+\ell+1}$.

The existence of a finite $t$ such that (30) holds follows from Lemma 2.

For the second part of the statement, recall that $(P \ominus Q) \oplus Q \subseteq P$ for any two sets $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^n$. If (31) is satisfied, then

$$\begin{aligned} A^{t+\ell+1}O_\ell \oplus F_{i+\ell+1} &\subseteq (O_\ell \ominus \Omega) \oplus F_{i+\ell+1} && \text{(36a)}\\ &\subseteq (O_\ell \ominus \Omega) \oplus \Omega && \text{(36b)}\\ &\subseteq O_\ell, && \text{(36c)} \end{aligned}$$

hence (35) is satisfied.

The existence of a finite $t$ such that (31) holds follows from the first part of Lemma 2. This is because $0 \in \Omega$ and $\Omega \subseteq \text{int}(O_\ell)$, hence $0 \in \text{int}(O_\ell \ominus \Omega)$. This implies that there exists an $\varepsilon > 0$ such that $\mathbb{B}^n_p(\varepsilon) \subseteq O_\ell \ominus \Omega$. □

**Corollary 3.** *If $t \in \mathbb{N}$ satisfies*

$$\text{Reach}_{t+1}(X) \subseteq X, \tag{37}$$

*then $O_t = O_{t+1}$. If, in addition, $X$ is compact and $F_\infty \subseteq \text{int}(X)$, then there exists a finite $t$ such that* (37) *holds.*

*Alternatively, if $\Omega$ is any set such that $F_\infty \subseteq \Omega \subseteq X$ and*

$$A^{t+1}X \subseteq X \ominus \Omega, \tag{38}$$

*then $O_t = O_{t+1}$. If, in addition, $X$ is compact and $\Omega \subseteq \operatorname{int}(X)$, then there exists a finite $t$ such that* (38) *holds.*

*In other words, the determinedness index $t^*$ of the MRPI set $O_\infty$ is less than or equal to $t$ if* (37) *or* (38) *holds.*

The results in previous sections can be applied here. For example, let the conditions in Theorem 1 hold. If $F(\alpha, s) \subseteq \operatorname{int}(O_\ell)$, $F(\alpha, s) \subseteq \operatorname{int}(X)$ or $F(\alpha, s) \subseteq \Omega$, then $F_\infty \subseteq \operatorname{int}(O_\ell)$, $F_\infty \subseteq \operatorname{int}(X)$ or $F_\infty \subseteq \Omega$, respectively. Of course, one could let $\Omega \triangleq F(\alpha, s)$ if the first two conditions are satisfied.

As another example, let the conditions in Corollary 2 hold. If $\operatorname{Reach}_N(F(\alpha, s)) \subseteq \operatorname{int}(O_\ell)$, $\operatorname{Reach}_N(F(\alpha, s)) \subseteq \operatorname{int}(X)$ or $\operatorname{Reach}_N(F(\alpha, s)) \subseteq \Omega$, then $F_\infty \subseteq \operatorname{int}(O_\ell)$, $F_\infty \subseteq \operatorname{int}(X)$ or $F_\infty \subseteq \Omega$, respectively. Of course, one could let $\Omega \triangleq \operatorname{Reach}_N(F(\alpha, s))$ if the first two conditions are satisfied.

In many cases, it is not possible to guarantee that the assumptions in this section hold. It is then important to find an alternative way to compute an RPI approximation of the set $O_\infty$. This problem is addressed in the following section.

## 4.2 Inner approximation of the MRPI set

We will consider the computation of the predecessor sets of an RPI set. Before proceeding, recall the following result, which is a special case of a procedure suggested in [Ker00, Sect. 3.2] for improving on an inner approximation of the MRPI set:

**Proposition 2.** *If $\Omega$ is a constraint-admissible, RPI set, then $\operatorname{Pre}_N(\Omega)$ is a constraint-admissible, RPI set and $\operatorname{Pre}_{N+1}(\Omega) \supseteq \operatorname{Pre}_N(\Omega)$ for all $N \in \mathbb{N}_+$. In other words, $\{\Omega, \operatorname{Pre}_1(\Omega), \operatorname{Pre}_2(\Omega), \ldots\}$ is an increasing sequence of constraint-admissible, RPI sets.*

*Remark* 12. Clearly, if $\Omega$ is a constraint-admissible, RPI set, then $\operatorname{Pre}_N(\Omega) \subseteq O_\infty$ for all $N \in \mathbb{N}_+$.

For the sake of completeness, we also recall the following result, which is a special case of [Bla94, Prop. 2.1]:

**Proposition 3.** *Let $\Omega$ be a convex, RPI set containing the origin. If the scalar $\mu \geq 1$, then $\mu\Omega$ is also a convex, RPI set containing the origin.*

We now present the first main result of this section, of which the proof follows immediately from Propositions 2 and 3:

**Theorem 6.** *If $\Omega$ is a convex, constraint-admissible, RPI set containing the origin and*

$$\mu^{\mathrm{o}} \triangleq \sup \{\mu \in [1, \infty) \mid \mu\Omega \subseteq X\}, \tag{39}$$

*then $\{\Omega, \mu^{\mathrm{o}}\Omega, \operatorname{Pre}_1(\mu^{\mathrm{o}}\Omega), \operatorname{Pre}_2(\mu^{\mathrm{o}}\Omega), \ldots\}$ is an increasing sequence of constraint-admissible, RPI sets.*

**Definition 11 (Maximal stabilizable set).** Given any constraint-admissible, RPI set $\Omega$ that contains the mRPI set $F_\infty$ in its interior, we define the *maximal stabilizable set* $S_\infty(\Omega)$ as

$$S_\infty(\Omega) \triangleq \bigcup_{M=0}^{\infty} \operatorname{Pre}_M(\Omega). \tag{40}$$

Clearly, since $F_\infty$ is the limit set of all trajectories of system (1), $S_\infty(\Omega)$ is all initial states in $X$ such that, given any allowable disturbance sequence, the solution of the system will be in $X$ for all time, enter $\Omega$ in some finite time and remain in $\Omega$ thereafter, while converging to $F_\infty$. The proof of the second main result of this section follows immediately from recognizing this fact and is therefore omitted:

**Theorem 7.** *Let $\Omega$ be a constraint-admissible, RPI set containing $F_\infty$ in its interior.*

*(i) If there exists an $M \in \mathbb{N}_+$ such that $\text{Pre}_M(\Omega) = \text{Pre}_{M+1}(\Omega)$, then $O_\infty = S_\infty(\Omega) = \text{Pre}_M(\Omega)$.*

*(ii) If $O_\ell$ is compact for some $\ell \in \mathbb{N}$, then there exists a finite $M \in \mathbb{N}_+$ such that $O_\infty = S_\infty(\Omega) = \text{Pre}_M(\Omega)$.*

The results in previous sections can be applied in Theorems 6 and 7. For example, let the conditions of Theorem 1 hold. If $F(\alpha, s) \subseteq X$ and $\mu^\text{o}$ is defined as in (39) with $\Omega \triangleq F(\alpha, s)$, then the sequence of sets $\{F(\alpha, s), \mu^\text{o} F(\alpha, s), \text{Pre}_1(\mu^\text{o} F(\alpha, s)), \text{Pre}_2(\mu^\text{o} F(\alpha, s)), \ldots\}$ is an increasing sequence of constraint-admissible, RPI sets. Alternatively, one could use $\Omega \triangleq \text{Reach}_N(F(\alpha, s))$ if the conditions of Corollary 2 are satisfied. Clearly, any set obtained using the results in [Bla94] or the $O_\infty$ obtained by replacing $X$ with a sufficiently large, compact subset of $X$, are also suitable candidates for $\Omega$ in Theorems 6 and 7.

# 5 Efficient computations and *a priori* upper bounds

This section will present results that allow for the development of efficient tests and computations of *a priori* upper bounds for the conditions presented in (7), (21), (30), (31), (37) and (38). Results will also be given that allow for the efficient computation of $s^\text{o}(\alpha)$ and $\alpha^\text{o}(s)$ in (12) and $\mu^\text{o}$ in (39).

Note that if all the sets mentioned in this paper are polyhedra or polytopes (bounded polyhedra), then efficient computations are possible. Computations are also much simpler if the sets contain the origin in their interiors. As such, we will assume throughout this section $W$, $X$ and $\Omega$, where appropriate, are polyhedra that contain the origin in their interiors.

If $X$, $\Omega$ and $W$ are polyhedra/polytopes, then the computation of the Minkowski sum, Pontryagin difference, linear maps and inverses of linear maps can be done by using standard software for manipulating polytopes. These packages therefore allow one to compute, for example, $F_s$, $F(\alpha, s)$, $\text{Reach}_N(\Omega)$, $\text{Pre}_N(\Omega)$, $O_t$, $O_\infty$, etc.

However, often we are not interested in the explicit computation of these sets, but only whether the conditions presented in (7), (21), (30), (31), (37) and (38) are satisfied or whether $\text{Reach}_N(F(\alpha, s)) \subseteq \Omega$, where $\Omega$ is any polyhedron. This is the case we will mainly be addressing in this section. For this purpose, we recall the following definition:

**Definition 12 (Support function).** The *support function* of a set $\Pi \subset \mathbb{R}^n$, evaluated at $z \in \mathbb{R}^n$, is defined as

$$h(\Pi, z) \triangleq \sup_{\pi \in \Pi} z^T \pi. \tag{41}$$

Our main interest in the support function is the well-known fact that the support function of a set allows one to write equivalent conditions for the set to be a subset of another. In particular:

**Proposition 4.** *Let* $\Pi$ *be a non-empty set in* $\mathbb{R}^n$ *and the polyhedron*

$$\Psi = \left\{ \psi \in \mathbb{R}^n \;\middle|\; f_i^T \psi \leq g_i,\ i \in \mathcal{I} \right\}, \tag{42}$$

*where* $f_i \in \mathbb{R}^n$, $g_i \in \mathbb{R}$ *and* $\mathcal{I}$ *is a finite index set.*

*(i)* $\Pi \subseteq \Psi$ *if and only if* $h(\Pi, f_i) \leq g_i$ *for all* $i \in \mathcal{I}$.

*(ii)* $\Pi \subseteq \operatorname{int}(\Psi)$ *if and only if* $h(\Pi, f_i) < g_i$ *for all* $i \in \mathcal{I}$.

The following result allows one to compute the support function of a set that is the Minkowski sum of a finite sequence of linear maps of non-empty, compact sets.

**Proposition 5.** *Let each matrix* $L_k \in \mathbb{R}^{n\times m}$ *and each* $\Phi_k$ *be a non-empty, compact set in* $\mathbb{R}^m$ *for all* $k \in \{1, \ldots, K\}$. *If*

$$\Pi = \bigoplus_{k=1}^{K} L_k \Phi_k, \tag{43}$$

*then*

$$h(\Pi, z) = \sum_{k=1}^{K} \max_{\phi \in \Phi_k} (z^T L_k)\phi. \tag{44}$$

*Furthermore, if* $\Phi_k = \mathbb{B}_\infty^m(1)$, *then*

$$\max_{\phi \in \Phi_k} (z^T L_k)\phi = \|L_k^T z\|_1. \tag{45}$$

*Proof.* The result follows immediately from the fact that if $\pi \triangleq \pi_1 + \cdots + \pi_k$, where each $\pi_k \in L_k\Phi_k$, then $h(\Pi, z) = \max\left\{ z^T\pi \mid \pi \in \Pi \right\} = \max\left\{ z^T(\pi_1 + \cdots + \pi_K) \mid \pi_k \in L_k\Phi_k,\ k = 1, \ldots, K \right\} = \sum_{k=1}^{K} \max\left\{ z^T\pi_k \mid \pi_k \in L_k\Phi_k \right\}$. The last equality follows because of the fact that the constraints on $\pi_k$ are independent of the constraint on $\pi_l$ for all $k \neq l$. Noting that $\max\left\{ z^T\pi_k \mid \pi_k \in L_k\Phi_k \right\} = \max\left\{ z^T L_k\phi_k \mid \phi_k \in \Phi_k \right\}$, it follows that (44) holds. The fact that (45) holds can be proven in a similar manner [KM03, Prop. 2]. □

*Remark* 13. Clearly, if all the $\Phi_k$ in the above result are polytopes, then the computation of the value of the support function in (44) can be done by solving $K$ LPs. However, it is extremely useful to note that if any $\Phi_k$ is a hypercube ($\infty$-norm ball), then the value of the support function of $\Pi$ can be computed much faster by evaluating the explicit expression in (45).

Note that, by a straightforward application of (20) and Propositions 4 and 5, it follows that (7), (21), (30), (31), (37) and (38) can be checked efficiently, without having to compute the respective linear maps or Minkowski sums. The same is true when testing whether $\text{Reach}_N(F(\alpha, s)) \subseteq \Omega$, where $\Omega$ is any polyhedron. Clearly, $s^o(\alpha)$, $\alpha^o(s)$ and $\mu^o$ can also be computed by solving a finite number of LPs.

Before proceeding, note that one can compute the size of the smallest hypercube containing $\text{Reach}_N(F(\alpha, s))$ by solving a finite number of LPs, without having to compute $\text{Reach}_N(F(\alpha, s))$ explicitly. This claim is justified by the following discussion.

**Definition 13 (Smallest/largest hypercube).** Let $\Psi$ be a non-empty, compact set in $\mathbb{R}^n$ containing the origin. The size of the largest hypercube in $\Psi$ is defined as

$$\beta_{\text{in}}(\Psi) \triangleq \max\{r \geq 0 \mid \mathbb{B}^n_\infty(r) \subseteq \Psi\} \tag{46}$$

and the size of the smallest hypercube containing $\Psi$ is defined as

$$\beta_{\text{out}}(\Psi) \triangleq \min\{r \geq 0 \mid \Psi \subseteq \mathbb{B}^n_\infty(r)\}. \tag{47}$$

The next result follows from a straightforward application of Propositions 4 and 5:

**Proposition 6.** *Let $\Psi$ be a non-empty, compact set in $\mathbb{R}^n$ containing the origin.*

*(i) The smallest hypercube containing $\Psi$ is given by*

$$\beta_{\text{out}}(\Psi) = \max_{j\in\{1,\ldots,n\}} \max\{h(\Psi, e_j), h(\Psi, -e_j)\}, \tag{48}$$

*where $e_j$ denotes the j'th standard basis vector in $\mathbb{R}^n$.*

*(ii) If $\Psi$ is a polytope given as in* (42)*, then*

$$\beta_{\text{in}}(\Psi) = \min_{i\in\mathcal{I}} \frac{g_i}{\|f_i\|_1}. \tag{49}$$

*Remark* 14. The above result implies that if $\Psi$ is a polytope, then $\beta_{\text{in}}(\Psi)$ is easily computed by evaluating the explicit expression in (49). However, the computation of $\beta_{\text{out}}(\Psi)$ is slightly more involved, since it involves comparing the solutions of a number of LPs in (48).

In other words, the size of the smallest $\infty$-norm ball (hypercube) containing $F(\alpha, s)$ can be computed by solving a finite number of LPs because

$$\beta_{\text{out}}(F(\alpha,s)) = \max_{j\in\{1,\ldots,n\}} h(W^s, \pm(1-\alpha)^{-1}[A^0 \ \cdots \ A^{s-1}]^T e_j) \tag{50a}$$

$$= (1-\alpha)^{-1} \max_{j\in\{1,\ldots,n\}} \max\left\{\sum_{i=0}^{s-1} h(W, (A^i)^T e_j), \sum_{i=0}^{s-1} h(W, -(A^i)^T e_j)\right\}, \tag{50b}$$

where $W^s \triangleq W \times \cdots \times W$. Clearly, if $W$ is the linear map of a hypercube, then no LPs are necessary; one can use (45) to compute the explicit expression of the support functions in (50).

## 5.1 *A priori* upper bounds if $A$ is diagonizable

Many of the conditions in the previous sections, such as (7), (21), (31), and (38) have the specific form

$$A^i\Pi \subseteq \Psi. \tag{51}$$

This section shows how one can efficiently obtain *a priori* upper bounds on

$$i^o(A, \Pi, \Psi) \triangleq \inf\left\{i \in \mathbb{N} \;\middle|\; A^i\Pi \subseteq \Psi\right\}, \tag{52}$$

which is the smallest $i$ such that (51) holds.

We first present the following result:

**Lemma 4.** *Let $\Pi$ and $\Psi$ be two non-empty polytopes in $\mathbb{R}^n$ containing the origin and the matrix $L \in \mathbb{R}^{n\times n}$.*

*Let $\beta_{\text{in}}(\Psi)$ be the size of the largest hypercube in $\Psi$ and $\beta_{\text{out}}(\Pi)$ be the size of the smallest hypercube containing $\Pi$.*

*(i) If $L\mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi)) \subseteq \mathbb{B}^n_\infty(\beta_{\text{in}}(\Psi))$, then $L\Pi \subseteq \Psi$.*

*(ii) If $\|L\|_\infty \leq \beta_{\text{in}}(\Psi)/\beta_{\text{out}}(\Pi)$, then $L\Pi \subseteq \Psi$.*

*Proof.* (i) Note that $\Pi \subseteq \mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi))$ so that $L\Pi \subseteq L\mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi))$.

Since $\mathbb{B}^n_\infty(\beta_{\text{in}}(\Psi)) \subseteq \Psi$, if $L\mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi)) \subseteq \mathbb{B}^n_\infty(\beta_{\text{in}}(\Psi))$, then $L\mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi)) \subseteq \Psi$.

Since $L\Pi \subseteq L\mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi))$, $L\Pi \subseteq \Psi$ as claimed.

(ii) Note that for any $x \in L\mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi))$ we have $\|x\|_\infty \leq \|L\|_\infty\beta_{\text{out}}(\Pi)$ so that $L\mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi)) \subseteq \{x \mid \|x\|_\infty \leq \|L\|_\infty\beta_{\text{out}}(\Pi)\}$.

If $\|L\|_\infty \leq \beta_{\text{in}}(\Psi)/\beta_{\text{out}}(\Pi)$ it follows that $L\mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi)) \subseteq \{x \mid \|x\|_\infty \leq \beta_{\text{in}}(\Psi)\}$ so that $L\mathbb{B}^n_\infty(\beta_{\text{out}}(\Pi)) \subseteq \mathbb{B}^n_\infty(\beta_{\text{in}}(\Psi))$, as claimed.

□

The previous result turns out to be very useful in providing an upper bound on $i^{\text{o}}(A,\Pi,\Psi)$:

**Proposition 7.** *Let $\Pi$ and $\Psi$ be two non-empty polytopes in $\mathbb{R}^n$ containing the origin and the matrix $L \in \mathbb{R}^{n\times n}$.*

*Let $\beta_{\text{in}}(\Psi)$ be the size of the largest hypercube in $\Psi$ and $\beta_{\text{out}}(\Pi)$ be the size of the smallest hypercube containing $\Pi$.*

*Let $A$ be diagonizable with $A = V\Lambda V^{-1}$, where $\Lambda$ is a diagonal matrix of the eigenvalues of $A$ and the spectral radius $\rho(A) \in (0,1)$.*

*It follows that*

$$i^{\text{o}}(A,\Pi,\Psi) \leq \left\lceil \ln\left(\beta_{\text{in}}(\Psi)/\left(\beta_{\text{out}}(\Pi)\|V\|_\infty\|V^{-1}\|_\infty\right)\right)/\ln\rho(A)\right\rceil . \tag{53}$$

*Proof.* From Lemma 4 it follows that (51) is satisfied if

$$\|A^i\|_\infty \leq \beta_{\text{in}}(\Psi)/\beta_{\text{out}}(\Pi). \tag{54}$$

From the basic properties of operator norms it follows that

$$\|A^i\|_\infty = \|V\Lambda^i V^{-1}\|_\infty \tag{55a}$$

$$\leq \|V\|_\infty\|\Lambda^i\|_\infty V^{-1}\|_\infty \tag{55b}$$

$$= \|V\|_\infty\rho(A)^i\|V^{-1}\|_\infty \tag{55c}$$

| $A$ | $\begin{bmatrix} 0.28 & 0.02 \\ -0.72 & 0.02 \end{bmatrix}$ | $\begin{bmatrix} 0.44 & -0.24 \\ -0.56 & -0.24 \end{bmatrix}$ | $\begin{bmatrix} -0.17 & -0.03 \\ -1.17 & -0.03 \end{bmatrix}$ | $\begin{bmatrix} 0.98 & 0.72 \\ -0.02 & 0.72 \end{bmatrix}$ |
|---|---|---|---|---|
| $\rho(A)$ | 0.2 | 0.6 | 0.3 | 0.9 |
| $\lambda_i,\ i = 1, 2$ | (0.1,0.2) | (-0.4,0.6) | (-0.3,0.1) | (0.8,0.9) |
| $s^* \triangleq s^o(\alpha)$ | 4 | 7 | 4 | 50 |
| $\alpha^* \triangleq \alpha^o(s^o(\alpha))$ | 0.0119 | 0.0304 | 0.0261 | 0.0463 |
| $\bar{s}$ | 4 | 8 | 5 | 56 |
| $\alpha^o(\bar{s})$ | 0.0119 | 0.0181 | 0.0079 | 0.0246 |

Table 1: Data for $2^{\text{nd}}$ order examples with $\alpha \triangleq 0.05$

The proof is completed by multiplying (54) with $\|V\|_\infty$ and $\|V^{-1}\|_\infty$ and solving for $i$. $\square$

The above result shows that the upper bound on $i^o(A, \Pi, \Psi)$ depends on the magnitudes of the eigenvalues (in particular, the spectral radius) and the eigenvectors of $A$.

Proposition 7 is particularly useful in obtaining upper bounds on the power of the integer on the left hand side in (7), (21), (31) and (38). For example, an upper bound on $s^o(\alpha)$ is easily obtained. By applying Proposition 7 with $\Pi = W$ and $\Psi = \alpha W$, it follows that

$$s^o(\alpha) \leq \left\lceil \ln\left(\alpha\beta_{\text{in}}(W)/\left(\beta_{\text{out}}(W)\|V\|_\infty\|V^{-1}\|_\infty\right)\right)/\ln\rho(A)\right\rceil . \tag{56}$$

In order to save space, the details for upper bounds on the other conditions are not given. It is hopefully clear how one could proceed.

## 6 Examples

In order to illustrate our results on invariant approximations of the minimal robustly positively invariant set (i.e. $F(\alpha, s)$), we consider four second order systems, with various values of spectral radii:

$$x^+ = Ax + w \tag{57}$$

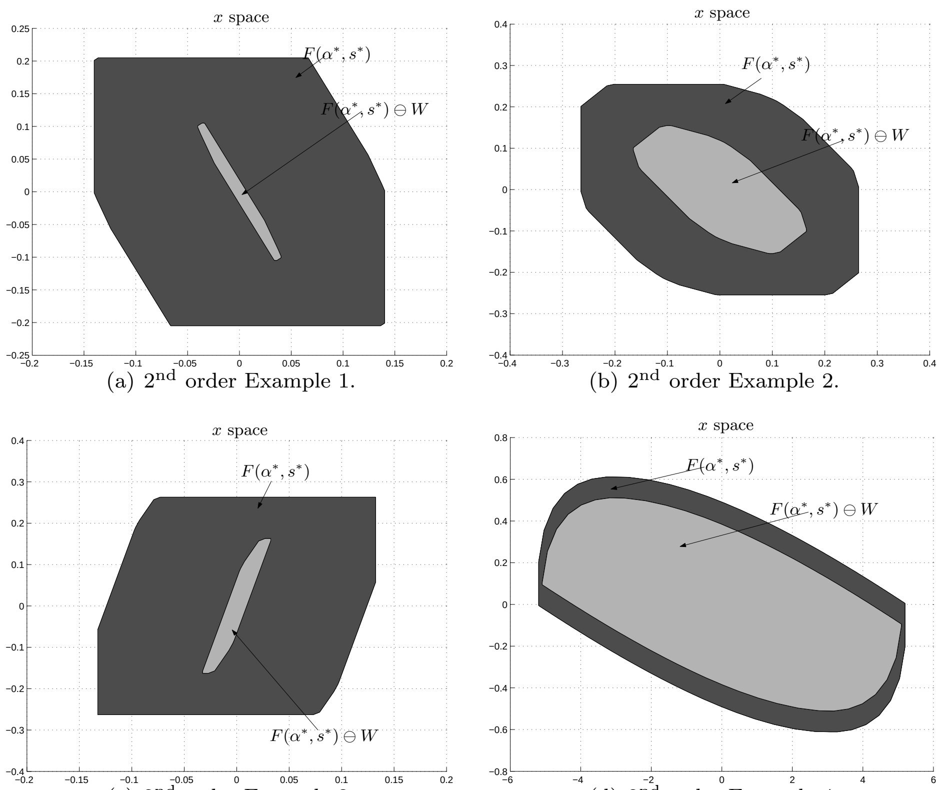
with additive disturbance:

$$W \triangleq \left\{w \in \mathbb{R}^2 \mid \|w\|_\infty \leq 0.1\right\}. \tag{58}$$

The dynamics, eigenvalues and particular values of $s^* \triangleq s^o(\alpha)$, $\alpha^* \triangleq \alpha^o(s^o(\alpha))$, $\bar{s}$ and $\alpha^o(\bar{s})$ are reported in Table 1, where $\bar{s}$ is the upper bound on $s^o(\alpha)$ obtained from (56). The initial value of $\alpha$ was chosen to be 0.05.

The invariant sets $F(\alpha^*, s^*)$ together with $F(\alpha^*, s^*) \ominus W$ are shown in Figure 1. The dynamics were obtained by applying four various state feedback control laws to a second order double integrator example in order to illustrate the influence of the eigenvectors and eigenvalues on the geometry of the set $F(\alpha^*, s^*)$.

The reachable sets for the third example are shown in Figure 2. The initial invariant set $\Omega$ is the




(a) $2^{\text{nd}}$ order Example 1. (b) $2^{\text{nd}}$ order Example 2.

(c) $2^{\text{nd}}$ order Example 3. (d) $2^{\text{nd}}$ order Example 4.

Figure 1: Invariant Approximations of $F_\infty$: Sets $F(\alpha^*, s^*)$ and $F(\alpha^*, s^*) \ominus W$

| $\rho(A)$ | $s^* \triangleq s^o(\alpha)$ | $\alpha^* \triangleq \alpha^o(s^o(\alpha))$ | $\bar{s}$ | $\alpha^o(\bar{s})$ |
|---|---|---|---|---|
| 0.2 | 9 | 0.08395 | 13 | $7.93 \cdot 10^{-5}$ |

Table 2: Data for $10^{\text{th}}$ order example with $\alpha \triangleq 0.1$

maximal robustly positively invariant set contained in a polytope $X$, i.e. $\Omega \triangleq O_\infty$, where

$$X \triangleq \{x \in \mathbb{R}^2 \mid -10 \leq x_2 \leq 10,\ -0.7506x_1 - 0.6608x_2 \leq 0.6415,\ 0.7506x_1 + 0.6608x_2 \leq 0.6415\}.$$

It is possible to say that $\text{Reach}_{14}(\Omega) \approx F_\infty$ with an accuracy of $8 \cdot 10^{-8}$, in other words $\text{Reach}_{14}(\Omega)$ is an $\varepsilon$-outer approximation of $F_\infty$, where $\varepsilon = 8 \cdot 10^{-8}$ was computed by using (21). The sets $F(\alpha^*, s^*)$ and $\text{Reach}_{14}(\Omega)$ for the third example are shown in Figure 3.

In order to demonstrate that our result can be applied to higher order systems, a $10^{\text{th}}$ order system is considered. The values of $\rho(A)$, $s^o(\alpha)$, $\alpha^o(s^o(\alpha))$, $\bar{s}$ and $\alpha^o(\bar{s})$, where $\bar{s}$ is the upper bound on $s^o(\alpha)$ obtained by (56), are reported in Table 2 and the matrix $A$ is given in Appendix B. The disturbance was bounded in the hypercube $W \triangleq \left\{w \in \mathbb{R}^{10} \mid \|w\|_\infty \leq 0.1\right\}$. The initial value of $\alpha$ was chosen to be 0.1.

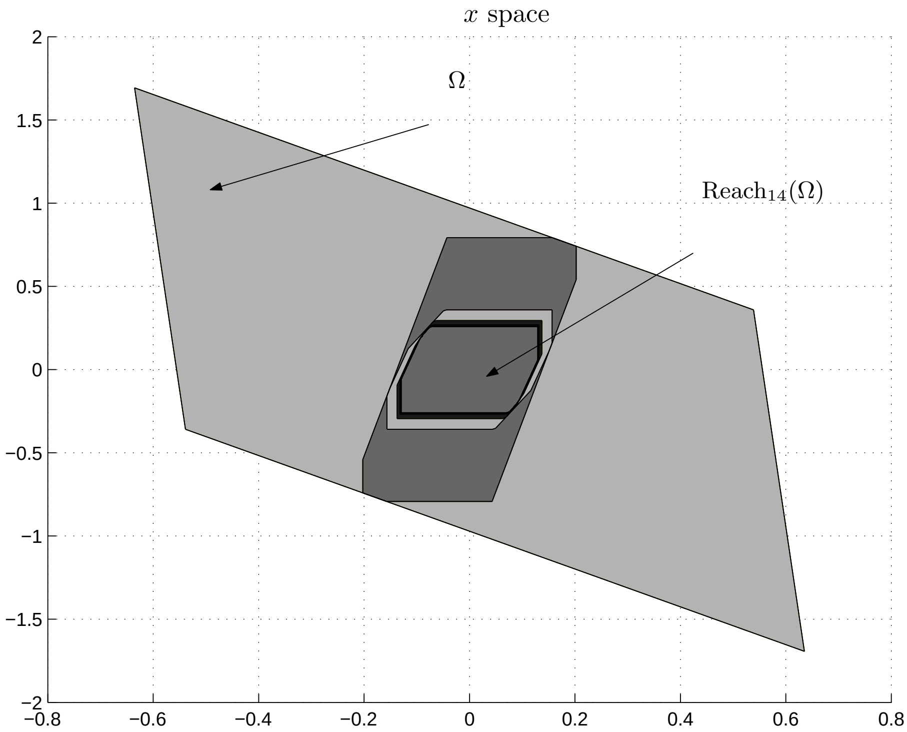



Figure 2: Reach sets of $\Omega \triangleq O_\infty$ for third example

# 7 Conclusions

This paper presented new insights regarding the robustly positively invariant sets for linear systems. It was shown how to compute invariant, outer approximations of the minimal robustly positively invariant sets. An algorithm for the computation of the maximal robustly invariant set or its approximation was also presented. This algorithm improves on existing algorithms, since it involves the computation of a sequence of robustly positively invariant sets. Hence, the computational results are useful at any iteration of the algorithm. Furthermore, a number of useful *a-priori* bounds and efficient tests were given. The presented results enable robust control of constrained linear discrete time systems subject to constraints and additive but bounded disturbances.

# Appendix

# A Proof of Theorem 3

It is obvious that there exist integers $p$ and $\bar{n} \leq n$ such that for all $j \geq p$,

$$\text{rank}\left[E\ AE\ \ldots\ A^{j-1}E\right] = \bar{n}. \tag{59}$$

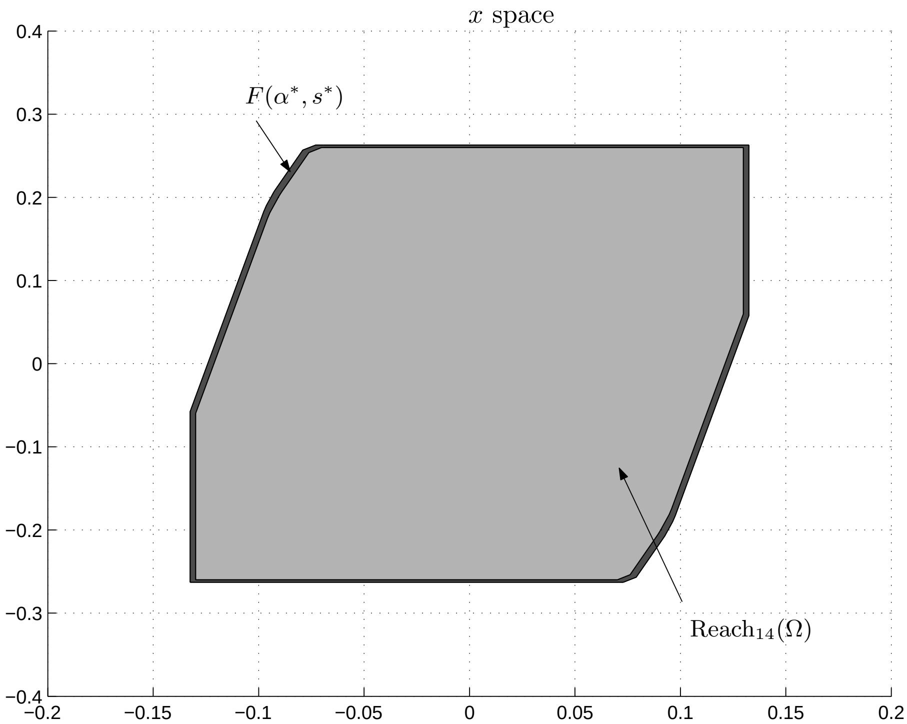



Figure 3: $\mathrm{Reach}_{14}(\Omega)$ vs $F(\alpha^*, s^*)$ for third example with $\Omega \triangleq O_\infty$

The set

$$\mathrm{C}(A, E) \triangleq \mathrm{range}([E\ AE\ \ldots\ A^{p-1}E]) \tag{60}$$

is then an $\bar{n}$-dimensional subspace of $\mathbb{R}^n$ spanned by $\bar{n}$ linearly independent columns of the matrix $[E\ AE\ \ldots\ A^{p-1}E]$, which can be chosen arbitrarily. For any $j \geq p$ and any set of vectors $d_0, \ldots, d_{j-2}, d_{j-1} \in \mathbb{R}^l$ it follows that $Ed_{j-1} + AEd_{j-2} + \cdots + A^{j-1}Ed_0 \in \mathrm{C}(A, E)$. Clearly, this implies that

$$F_\infty \subseteq \mathrm{C}(A, E) \text{ and } F_j \subseteq \mathrm{C}(A, E),\ \forall j \geq p. \tag{61}$$

Moreover, $A^iW = A^iED \subseteq \mathrm{C}(A, E)$ for all $i \in \mathbb{N}_0$. The reader should also note that, since (59) holds, $F_\infty$ and $F_j$, with $j \geq p$, are bounded, $\bar{n}$-dimensional sets.

By recalling (15) and the fact that $P \subseteq Q \Rightarrow P \oplus R \subseteq Q \oplus R$, it follows that

$$
\begin{aligned}
A\overline{F}(\alpha, p, r, s) \oplus ED &= A\left(F_s \oplus \alpha(1-\alpha)^{-1} \bigoplus_{i=0}^{r-1} A^i F_p\right) \oplus ED && \text{(62a)}\\
&= \left(\bigoplus_{i=1}^{s} A^i ED\right) \oplus \left(\alpha(1-\alpha)^{-1} \bigoplus_{i=1}^{r} A^i F_p\right) \oplus ED && \text{(62b)}\\
&= ED \oplus \left(\bigoplus_{i=1}^{s} A^i ED\right) \oplus \left(\alpha(1-\alpha)^{-1} \bigoplus_{i=1}^{r} A^i F_p\right) && \text{(62c)}\\
&= F_s \oplus A^s ED \oplus \left(\alpha(1-\alpha)^{-1} \bigoplus_{i=1}^{r-1} A^i F_p\right) \oplus \alpha(1-\alpha)^{-1} A^r F_p && \text{(62d)}\\
&= F_s \oplus A^s ED \oplus \alpha(1-\alpha)^{-1} A^r F_p \oplus \left(\alpha(1-\alpha)^{-1} \bigoplus_{i=1}^{r-1} A^i F_p\right) && \text{(62e)}\\
&\subseteq F_s \oplus \alpha F_p \oplus \alpha^2(1-\alpha)^{-1} F_p \oplus \left(\alpha(1-\alpha)^{-1} \bigoplus_{i=1}^{r-1} A^i F_p\right) && \text{(62f)}\\
&= F_s \oplus \left(\alpha + \alpha^2(1-\alpha)^{-1}\right) F_p \oplus \left(\alpha(1-\alpha)^{-1} \bigoplus_{i=1}^{r-1} A^i F_p\right) && \text{(62g)}\\
&= F_s \oplus \alpha(1-\alpha)^{-1} F_p \oplus \left(\alpha(1-\alpha)^{-1} \bigoplus_{i=1}^{r-1} A^i F_p\right) && \text{(62h)}\\
&= F_s \oplus \alpha(1-\alpha)^{-1} \bigoplus_{i=0}^{r-1} A^i F_p. && \text{(62i)}
\end{aligned}
$$

Hence, $A\overline{F}(\alpha, p, r, s) \oplus ED \subseteq \overline{F}(\alpha, p, r, s)$ and the set $F(\alpha, p, r, s)$ is an RPI set.

Convexity and compactness follows immediately from the properties of the Minkowski sum. Since $\overline{F}(\alpha, p, r, s)$ is closed and RPI, it follows immediately from the definition that $F_\infty \subseteq \overline{F}(\alpha, p, r, s)$.

# B Matrix $A$ for the $10^{th}$ order system

$$
A = \begin{bmatrix}
-0.2713 & 0.3004 & 0.1460 & 0.1592 & -0.2634 & 0.1013 & -0.5457 & -1.0030 & -0.3074 & 0.1131\\
-0.0750 & 0.1632 & 0.1152 & -0.3785 & -1.1337 & 0.1022 & -0.2252 & -1.3581 & -0.2761 & 0.3572\\
0.0630 & 0.1329 & 0.6675 & -0.0420 & 0.6569 & 0.4865 & 0.1887 & 0.3698 & 0.3771 & 0.0639\\
0.0054 & -0.3648 & -0.1315 & -0.3111 & -0.3509 & -0.4195 & 0.1430 & -0.3664 & 0.0041 & -0.2103\\
-0.0989 & 0.0182 & -0.0338 & 0.6819 & 0.6055 & 0.3177 & 0.1566 & 0.6116 & 0.3225 & 0.2249\\
-0.1405 & -0.2986 & -0.5272 & -0.2254 & -0.4732 & -0.3210 & -0.4723 & -0.6953 & -1.2377 & 0.0627\\
-0.0691 & 0.5372 & -0.1464 & -0.2886 & -0.5583 & -0.1757 & 0.0958 & -0.4458 & -0.4504 & 0.7228\\
0.0210 & -0.1330 & 0.0150 & -0.2470 & 0.2097 & -0.1566 & -0.0648 & -0.1555 & -0.2068 & -0.2772\\
0.2066 & 0.0050 & 0.3944 & -0.2396 & -0.4478 & 0.4977 & -0.5521 & -0.0216 & -0.2452 & -0.2350\\
-0.4705 & -0.0676 & -0.0053 & -0.3805 & -0.4381 & 0.4012 & -0.2391 & -0.7415 & 0.2096 & -0.2979
\end{bmatrix}
$$

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