Reachability computations for constrained discrete-time systems with state- and input-dependent disturbances

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***Abstract*— This paper presents new results that permit the computation of the set of states that can be robustly steered, using state feedback, to a given target set in a finite number of steps. It is assumed that the system is discrete-time, nonlinear, time-invariant and subject to mixed constraints on the state and input. A persistent disturbance, dependent on the current state and input, acts on the system. The results in this paper generalize previously published results that are not able to address state-input dependent disturbances. The application of the results to the computation of the maximal robustly controlled invariant set is briefly discussed. It is shown how polyhedral algebra, linear programming and computational geometry may be employed for set computations relevant to the analysis of linear and piecewise affine systems with additive state disturbances. Some simple examples are given to demonstrate that convexity of the robustly controllable sets cannot be guaranteed even if all relevant sets are convex and the system is linear.**

Keywords: Constrained control, robust control, nonlinear systems, piecewise affine systems, set invariance, controllability, uncertain systems.

I. Introduction

The problems of controllability to a target set and computation of robustly controlled invariant sets for systems subject to constraints and persistent, unmeasured disturbances have been the subject of study for many authors [1], [2], [3], [4], [5], [6], [7], [8]. Though many papers have results that can be applied to a large class of nonlinear discrete-time systems, most authors assume that the disturbance is not dependent on the state and input. The paper [4] appears to be the only previously published paper that addresses state-dependent disturbances directly. In [5] a general framework is introduced for systems with mixed state and input constraints subject to state-input dependent disturbances, but the specific results obtained for the computation of the set of states from which the system can be controlled to a target set are restricted to the case when disturbances are independent of the state and input. This paper extends the results of [4], [5], [6] to the case where the disturbance is dependent on the state and input. Furthermore, results are given for linear and piecewise affine systems that show how polyhedral algebra, linear programming and computational geometry may be employed to perform the relevant set computations.

The need for a framework that can deal with state-input dependent disturbances was briefly motivated in [5]. Disturbances that are dependent on the state and/or input frequently arise in practice when trying to model systems with physical constraints. For example, consider the nonlinear (piecewise affine) system

$$x^+ = Ax + B\text{sat}(u + E_u w) + E_x w \tag{1}$$

which is subject to a bounded disturbance $w \in \mathscr{W}$. The function $\text{sat}(\cdot)$ models physical saturation limits on the input. Assuming that these saturation limits are symmetric and have unit magnitude, an equivalent way of modelling (1) is to treat it as linear system with input-dependent disturbances, i.e. letting

$$x^+ = Ax + Bu + BE_u w + E_x w, \tag{2a}$$

where the control is constrained to satisfy $u \in \mathscr{U}$, where

$$\mathscr{U} \triangleq \{u \mid ||u||_\infty \leq 1\}, \tag{2b}$$

and the input-dependent disturbance $w \in \mathscr{W}(u)$ satisfies

$$\mathscr{W}(u) \triangleq \{w \mid \|u + E_u w\|_\infty \leq 1 \text{ and } w \in \mathscr{W}\}. \tag{2c}$$

State-input dependent disturbances arise in practice when the uncertainty associated with a model is greater in some regions of the state-input space than in other regions. For example, a model obtained by linearizing a nonlinear model is obviously more accurate near the point at which the model was linearized. A state-input dependent disturbance model permits less conservative results to be obtained than can be obtained with a model in which the disturbance is assumed to be independent of the state and input.

Another example where uncertainty may be modelled as a state-input dependent disturbance arises if there is parametric uncertainty present in the model. The reader is referred to [9], [10] to see how reachability computations can be carried out for this type of uncertainty.

This paper is organized as follows. Section II presents the main results of this paper and Section III briefly discusses how the results in Section II can be used to iteratively compute the set of states that can be steered to a target set in a finite number of steps, as well as how one could compute the maximal robustly controlled invariant set. In order to validate the results presented in this paper, Section IV presents a few simple numerical examples. The main contributions of this paper are summarized in Section V. The appendix contains definitions for the continuity of set-valued maps and results which allow one to compute the set difference of (possibly non-convex) polygons.

A more detailed exposition, together with proofs for all results stated in this paper, may be found in [11].

NOTATION AND DEFINITIONS: The set difference of $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$ is $A \setminus B \triangleq \{x \in A \mid x \notin B\} = A \cap B^c$, where B^c is the complement of B. 2^A is the set of subsets of A. Given a set $\Omega \subset C \times D$, the projection of Ω onto C is defined as $\mathrm{Proj}_C(\Omega) \triangleq \{c \in C \mid \exists d \in D \text{ such that } (c,d) \in \Omega\}$. A *polyhedron* is the (convex) intersection of a finite number of open and/or closed half-spaces and a *polygon* is the (possibly non-convex) union of a finite number of polyhedra.

II. THE ONE-STEP ROBUSTLY CONTROLLABLE SET

Section II-A gives the main results of the paper, which are then specialized in Section II-B for the case when the disturbance is dependent only on the state or input or when the system does not have a control input. Section II-C shows that the set of states robustly controllable to the target set is a polygon if the system is linear, affine or piecewise affine, the target set is a polygon and all relevant constraint sets are polygons.

A. General Case

Let $X = \mathbb{R}^n$ denote the state space, $U = \mathbb{R}^m$ the input space and $W = \mathbb{R}^p$ the disturbance space. Consider the nonlinear, time-invariant, discrete-time system

$$x^+ = f(x,u,w), \tag{3}$$

where $x \in X$ is the current state (assumed to be measured), x^+ is the successor state, $u \in U$ is the input, and $w \in W$ is an unmeasured, persistent disturbance that is dependent on the current state and input:

$$w \in \mathscr{W}(x,u) \subset W. \tag{4}$$

The state and input are required to satisfy the constraints

$$(x,u) \in \mathscr{Y} \subset X \times U. \tag{5}$$

The constraint $(x,u) \in \mathscr{Y}$ defines the state-dependent set of admissible inputs

$$\mathscr{U}(x) \triangleq \{u \mid (x,u) \in \mathscr{Y}\} \tag{6}$$

as well as the set of admissible states

$$\mathscr{X} \triangleq \{x \mid \exists u \text{ s.t.} (x,u) \in \mathscr{Y}\} = \{x \mid \mathscr{U}(x) \neq \emptyset\}. \tag{7}$$

In order to have a well-defined problem, we assume the following:

A1. For all $(x,u) \in \mathscr{Y}$, $\mathscr{W}(x,u) \neq \emptyset$ and $\mathscr{W}(\cdot)$ is bounded on bounded sets.

Given a set $\Omega \subseteq \mathscr{X}$, this section shows how the one-step robustly controllable set (the set of states $\mathrm{Pre}(\Omega)$ for which there exists an admissible input such that, for all allowable disturbances, the successor state is in Ω) may be computed. The set $\mathrm{Pre}(\Omega)$ is defined by

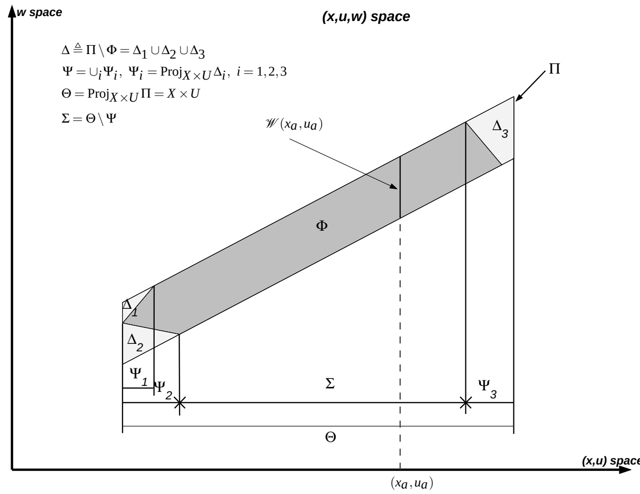

Fig. 1. Graphical illustration of Theorem 1

$$\mathrm{Pre}(\Omega) \triangleq \{x \mid \exists u \in \mathscr{U}(x) \text{ s.t.} f(x,u,\mathscr{W}(x,u)) \subseteq \Omega\}. \tag{8}$$

Remark 1: If $(x,u) \in \mathscr{Y} \Leftrightarrow x \in \mathscr{X}$ and $u \in \mathscr{U}$, then $\mathrm{Pre}(\Omega) \triangleq \{x \in \mathscr{X} \mid \exists u \in \mathscr{U} \text{ s.t.} f(x,u,\mathscr{W}(x,u)) \subseteq \Omega\}$.

Let the sets Σ, Π and Φ be defined, respectively, by

$$\Sigma \triangleq \{(x,u) \in \mathscr{Y} \mid f(x,u,\mathscr{W}(x,u)) \subseteq \Omega\}, \tag{9a}$$
$$\Pi \triangleq \{(x,u,w) \mid (x,u) \in \mathscr{Y} \text{ and } w \in \mathscr{W}(x,u)\}, \tag{9b}$$
$$\Phi \triangleq f^{-1}(\Omega) \triangleq \{(x,u,w) \mid f(x,u,w) \in \Omega\}. \tag{9c}$$

Theorem 1 (Main result): Suppose **A1** holds. The set of states that are robustly controllable to Ω is

$$\mathrm{Pre}(\Omega) = \mathrm{Proj}_X(\Sigma), \tag{10a}$$

where

$$\Sigma = \mathrm{Proj}_{X \times U}(\Pi) \setminus \mathrm{Proj}_{X \times U}(\Pi \setminus \Phi). \tag{10b}$$

Note that the set Σ defined in (9a) is equal to $\mathrm{Proj}_{X \times U}(\Pi) \setminus \mathrm{Proj}_{X \times U}(\Pi \setminus \Phi)$, as stated in (10b). A graphical illustration of Theorem 1 is given in Figure 1, where the set $\mathscr{W}(x_a,u_a)$ for a point $(x_a,u_a) \in \mathscr{Y}$ is also shown.

Theorem 2: Suppose **A1** holds, $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ is continuous and $\mathscr{W} : \mathbb{R}^r \to 2^{\mathbb{R}^p}$, $r \triangleq n + m$, is continuous and bounded on bounded sets. If Ω is closed, then $\mathrm{Pre}(\Omega)$ is closed.

B. Special Cases

Consider first the simpler case when the disturbance constraint set is a function of x only, i.e. the disturbance w

satisfies $w \in \mathscr{W}(x)$. The definitions of Σ and Π in (9a) and (9b), respectively, and $\text{Pre}(\Omega)$ become

$$\Sigma \triangleq \{(x,u) \in \mathscr{Y} \mid f(x,u,\mathscr{W}(x)) \subseteq \Omega\}, \tag{11a}$$

$$\Pi \triangleq \{(x,u,w) \mid (x,u) \in \mathscr{Y} \text{ and } w \in \mathscr{W}(x)\}, \tag{11b}$$

$$\text{Pre}(\Omega) \triangleq \{x \mid \exists u \in \mathscr{U}(x) \text{ s.t.} f(x,u,\mathscr{W}(x)) \subseteq \Omega\}. \tag{11c}$$

Theorem 1 remains true with these changes and it covers the case studied in [4]. A similar modification is needed if the disturbance constraint set is a function of u only, i.e. the disturbance w satisfies $w \in \mathscr{W}(u)$.

Remark 2: If the disturbance is independent of the state and input, Theorem 1 provides a method for computing the one-step robustly controllable set and is an alternative to the method in [5], [6], [10], where it is proposed to compute the so-called *Pontryagin difference*. Obviously, both methods will result in the same set. The difference between the two methods is that Theorem 1 relies on projection whereas the method in [5], [6], [10] does not. It is not easy to determine *a priori* which method would be more efficient. The computational requirements depend very much on the specifics of the problem and the computational tools that are available.

Next, consider the case when f is a function of (x,w) only, i.e. the system has no input u. In this case, the constraint $(x,u) \in \mathscr{Y}$ is replaced by $x \in \mathscr{X} \subset X$ and assumption **A1** is replaced by:

A1': For all $x \in \mathscr{X}$, $\mathscr{W}(x) \neq \emptyset$ and $\mathscr{W}(\cdot)$ is bounded on bounded set.

Also, in this case the definitions of Σ, Π and Φ in Theorem 1, and $\text{Pre}(\Omega)$ are replaced by

$$\Sigma \triangleq \{x \in \mathscr{X} \mid f(x,w) \in \Omega,\ \forall w \in \mathscr{W}(x)\}, \tag{12a}$$

$$\Pi \triangleq \{(x,w) \mid x \in \mathscr{X} \text{ and } w \in \mathscr{W}(x)\}, \tag{12b}$$

$$\Phi \triangleq f^{-1}(\Omega) \triangleq \{(x,w) \mid f(x,w) \in \Omega\}, \tag{12c}$$

and

$$\text{Pre}(\Omega) \triangleq \{x \in \mathscr{X} \mid f(x,\mathscr{W}(x)) \subseteq \Omega\}. \tag{12d}$$

Thus, $\text{Pre}(\Omega)$ is now the set of admissible states such that the successor state lies in Ω for all $w \in \mathscr{W}(x)$. In this case, the conclusion of Theorem 1 becomes

$$\text{Pre}(\Omega) = \Sigma = \text{Proj}_X(\Pi) \setminus \text{Proj}_X(\Pi \setminus \Phi). \tag{13}$$

This special case requires less computational effort since operations are performed in lower dimensional spaces and only two projection operations are needed.

C. Linear and Piecewise Affine $f(\cdot)$ with Additive State Disturbances

Consider the system defined in (3) with

$$f(x,u,w) \triangleq A_q x + B_q u + E_q w + c_q \text{ if } (x,u,w) \in P_q. \tag{14}$$

The sets $\{P_q \mid q \in Q\}$, where Q has finite cardinality, are polyhedra and constitute a polyhedral partition of Π, i.e. $\Pi \triangleq \bigcup_{q \in Q} P_q$ and the sets P_q have non-intersecting interiors. It is assumed that $f(\cdot)$ is continuous on the interior of Π. For all $q \in Q$, the matrices $A_q \in \mathbb{R}^{n\times n}$, $B_q \in \mathbb{R}^{n\times m}$, $E_q \in \mathbb{R}^{n\times p}$ and vector $c_q \in \mathbb{R}^n$.

Note that if $\Omega := \bigcup_{j\in J} \Omega_j$, where $\{\Omega_j \mid j \in J\}$ is a finite set of polyhedra, then Φ in (9c) is given by

$$\Phi = \bigcup_{(j,q)\in J\times Q} \left\{(x,u,w) \in P_q \;\middle|\; A_q x + B_q u + E_q w + c_q \in \Omega_j\right\}.$$

Since $\left\{(x,u,w) \in P_q \;\middle|\; A_q x + B_q u + E_q w + c_q \in \Omega_j\right\}$ is a polyhedron, it follows that Φ is the union of a finite set of polyhedra, hence Φ is a polygon.

The Appendix contains new results that allow one to compute the set difference between two (possibly non-convex) polygons. The projection of the set difference is then equal to the union of the projections of the individual polyhedra that constitute the set difference. The projection of each individual polyhedron can be computed via Fourier-Motzkin elimination [12] or via enumeration and projection of its vertices, followed by a convex hull computation [13]; see also [14], [15] for alternative projection methods.

We can now state the following result:

Theorem 3 (Piecewise affine systems): Suppose assumption **A1** holds. If the system is given by (14) and Π and Ω are polygons, then the robustly controllable set $\text{Pre}(\Omega)$, as given in (8) and (10a), is a polygon.

Remark 3: Clearly, Theorem 3 holds if the system is linear or affine (i.e. Q has cardinality 1). It is interesting to observe that, even if Ω and Π are both convex sets and $f(\cdot)$ is linear, there is no guarantee that $\text{Pre}(\Omega)$ is convex. This is demonstrated in Section IV via a numerical example.

III. The i-step Robustly Controllable Set and Robustly Controlled Invariant Sets

Consider the general case (Section II-A). For any integer i, let X_i denote the *i-step (robustly controllable) set* to Ω, i.e. X_i is the set of states that can be steered, by a time-varying state feedback control law, to the target set Ω in i steps, for all allowable disturbance sequences while satisfying, at all times, the constraint $(x,u) \in \mathscr{Y}$. As is well-known [5], [6], [7], the sequence of sets $\{X_i\}_{i=0}^{\infty}$ may be calculated recursively as follows:

$$X_{i+1} = \text{Pre}(X_i), \tag{15a}$$

$$X_0 = \Omega. \tag{15b}$$

Before giving the next result, recall that a set $\mathscr{S}$ is *robustly controlled invariant* if and only if for any $x \in \mathscr{S}$, there exists a $u \in \mathscr{U}(x)$ such that $f(x,u,w) \in \mathscr{S}$ for all $w \in \mathscr{W}(x,u)$, i.e. $\mathscr{S}$ is robustly controlled invariant if and only if $\mathscr{S} \subseteq \text{Pre}(\mathscr{S})$ [3], [10]. Recall also that the *maximal* robustly controlled invariant set C_∞ in $\mathscr{X}$ is the union of all robustly controlled invariant sets contained in $\mathscr{X}$.

Theorem 4: Suppose **A1** holds:

(i) If the system is piecewise affine (defined by (14)) and if the sets Ω and Π are polygons, then each i-step set X_i, $i \in \{0,1,\ldots\}$, is a polygon.

(ii) If $X_j \subseteq X_{j+1}$ for some $j \in \{0,1,\ldots\}$, then each set X_i, $i \in \{j, j+1,\ldots\}$, is robustly controlled invariant.

(iii) If the set Ω is robustly controlled invariant, then each set X_i, $i \in \{0,1,\ldots\}$, is robustly controlled invariant.

(iv) If $\Omega \triangleq \mathscr{X}$ and $X_j = X_{j+1}$ for some $j \in \{0,1,\ldots\}$, then each set X_i, $i \in \{j, j+1,\ldots\}$, is equal to the maximal robustly controlled invariant set C_∞ contained in $\mathscr{X}$.

Remark 4: Note that, if $\Omega \neq \mathscr{X}$ and Ω is robustly controlled invariant, then the maximal robustly controllable set X_∞ to Ω ($X_\infty = \bigcup_{i=0}^{\infty} X_i$, where $X_0 \triangleq \Omega$) is, in general, *not* equal to the maximal robustly controlled invariant set C_∞ in $\mathscr{X}$ ($C_\infty = \bigcap_{i=0}^{\infty} X_i$, where $X_0 \triangleq \mathscr{X}$).

Remark 5: As in Section II-B, if the system has no input u, i.e. if f is a function only of (x,w), then with the appropriate modifications to definitions, Theorem 4 still holds, but with 'robustly controlled invariant' replaced with 'robustly positively invariant'.

IV. Numerical Examples

In order to illustrate our results we consider the following scalar system:

$$x^+ = x + u + w, \tag{16}$$

which is subject to the constraints

$$(x,u) \in \mathscr{X} \times \mathscr{U}, \tag{17}$$

where the state constraints $\mathscr{X} \triangleq \{x \in \mathbb{R} \mid -5 \leq x \leq 20\}$ and the input constraints $\mathscr{U} \triangleq \{u \in \mathbb{R} \mid -2 \leq u \leq 2\}$. The state-dependent disturbance satisfies:

$$w \in \mathscr{W}(x) \Leftrightarrow (x,w) \in \Delta \triangleq \Delta_1 \cup \Delta_2, \tag{18}$$

where the (convex) sets Δ_1 and Δ_2 are shown in Figure 2. The (robustly controlled invariant) target set is $X_0 = \Omega = \{x \mid -0.6 \leq x \leq 0.6\}$.

The sequence of i-step sets is computed by using the results of Theorem 1 and some of the sets are: $X_1 = \{x \mid -0.7 \leq x \leq 0.7\}$, $X_2 = \{x \mid -0.9 \leq x \leq 0.9\}$, $X_3 = \{x \mid -1.3 \leq x \leq 1.3\}$, $X_4 = \{x \mid -2.0468 \leq x \leq 2.0468\}$, $\ldots$, $X_8 = \{x \mid -4.5793 \leq x \leq 4.5793\}$, $X_9 = \{x \mid -5 \leq x \leq 5.1131\}$, $X_{10} = \{x \mid -5 \leq x \leq 5.6123\}$, $\ldots$, $X_{49} = \{x \mid -5 \leq x \leq 12.2759\}$, $X_{50} = \{x \mid -5 \leq x \leq 12.3099\}$. The set X_∞ of all states that can be steered to the target set, while satisfying state and control constraints, for all allowable disturbance sequences, is: $X_\infty = \{x \mid -5 \leq x \leq 12.7999\}$. The sets Σ_i for $i = 1,2,3,4$ are also shown in Figure 3.

To illustrate the fact that the i-step sets can be non-convex even if $\mathscr{X}$, $\mathscr{U}$, Ω and the graph of $\mathscr{W}(x)$ are convex, consider the same example. This time the state-dependent disturbance satisfies:

$$w \in \mathscr{W}(x) \Leftrightarrow (x,w) \in \Delta, \tag{19}$$

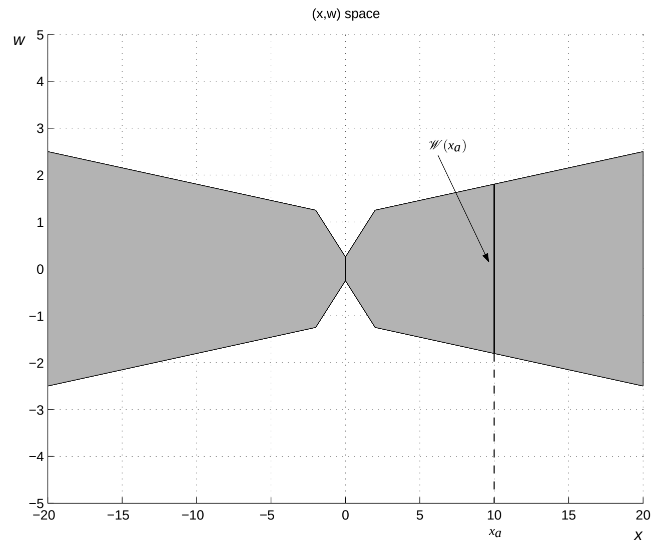

Fig. 2. Graph of $\mathscr{W}$

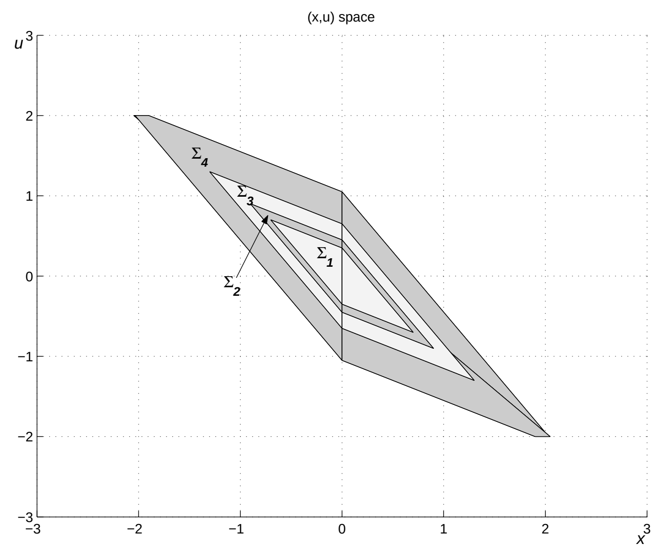

Fig. 3. Sets Σ_i for $i = 1,2,3,4$

where the sets Δ and Σ are shown in Figure 4. If the target set is $X_0 = \Omega = \{x \mid -2.5 \leq x \leq 2.5\}$, the one-step set is $X_1 = \{x \mid -3.75 \leq x \leq -0.8333\} \cup \{x \mid 0.8333 \leq x \leq 3.75\}$.

Even if Ω is a robustly controlled invariant set, the convexity of each i-step set cannot be guaranteed. This can be illustrated by considering the example above with $\mathscr{X} = \{x \mid -5 \leq x \leq 4\}$, $w \in \mathscr{W}(x) \Leftrightarrow (x,w) \in \Delta$, the set Δ shown in Figure 5, and the robustly controlled invariant target set $X_0 = \Omega = \{x \mid -2.5 \leq x \leq 2.5\}$. In this case, the one-step robustly controlled invariant set is $X_1 = \{x \mid -3.75 \leq x \leq 2.5\} \cup \{x \mid 3.5455 \leq x \leq 4\}$. The set Σ is also shown in Figure 5.

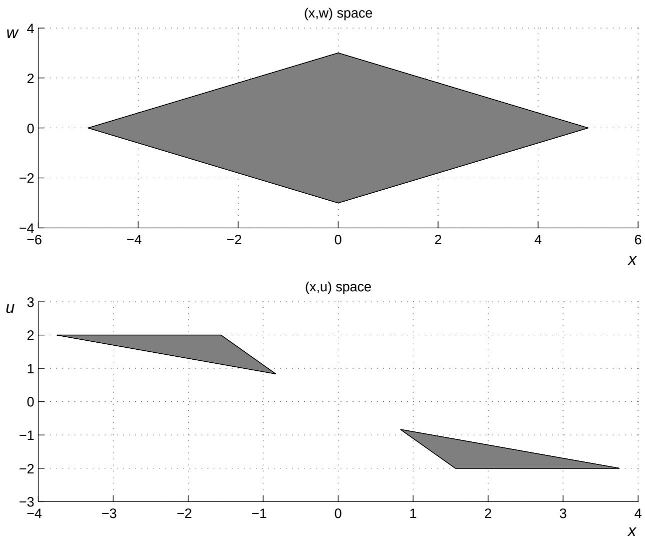

Fig. 4. Graph of $\mathscr{W}$ (top) and the set Σ (bottom)

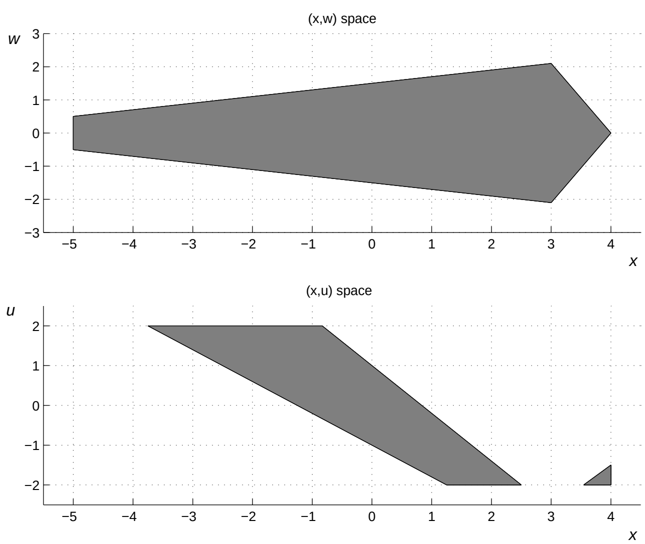

Fig. 5. Graph of $\mathscr{W}$ (top) and the set Σ (bottom)

V. Conclusions

The main result of this paper (Theorem 1) showed how one can obtain $\text{Pre}(\Omega)$, the set of states that can be robustly steered to Ω, via the computation of a sequence of set differences and projections. It was then shown in Theorem 3 that if Ω and the relevant constraint sets are polygons (i.e. they are given by the unions of finite sets of convex polyhedra) and the system is linear or piecewise affine, then $\text{Pre}(\Omega)$ is also a polygon and can be computed using standard computational geometry software. In particular, new results were given in Appendix which allow one to compute the set difference for (possibly non-convex) polygons by solving a finite number of LPs. It was then shown in Section III how $\text{Pre}(\cdot)$ can be used to recursively compute the i-step set, i.e. the set of states which can be robustly steered to a given target set in i steps, as well as the maximal robustly controlled invariant set. Finally, some simple examples were given which show that, even if the system is linear, the respective constraint sets are convex and the target set is robustly controlled invariant, convexity of the i-step sets cannot be guaranteed.

VI. Acknowledgements

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VII. Appendix

Set-valued Functions

The definitions of inner and outer semi-continuity employed below are due to Rockafellar and Wets [16]; for Definitions 1–3 see [17]. In what follows, $B(z,\rho) \triangleq \{z \mid \|z\| \leq \rho\}$.

Definition 1: A set-valued map $F : \mathbb{R}^r \to 2^{\mathbb{R}^n}$ is outer semi-continuous (o.s.c.) at $\hat{z}$ if $F(\hat{z})$ is closed and, for every compact set S such that $F(\hat{z}) \cap S = \emptyset$, there exists a $\rho > 0$ such that $F(z) \cap S = \emptyset$ for all $z \in B(\hat{z},\rho)$. A set-valued map $F : \mathbb{R}^r \to 2^{\mathbb{R}^n}$ is o.s.c. if it is o.s.c. at every $z \in \mathbb{R}^r$.

Definition 2: A set-valued map $F : \mathbb{R}^r \to 2^{\mathbb{R}^n}$ is inner semi-continuous (i.s.c.) at $\hat{z}$ if $F(\hat{z})$ is closed and, for every open set S such that $F(\hat{z}) \cap S \neq \emptyset$, there exists a $\rho > 0$ such that $F(z) \cap S \neq \emptyset$ for all $z \in B(\hat{z},\rho)$. A set-valued map $F : \mathbb{R}^r \to 2^{\mathbb{R}^n}$ is i.s.c. if it is i.s.c. at every $z \in \mathbb{R}^r$.

Definition 3: A set-valued map $F : \mathbb{R}^r \to 2^{\mathbb{R}^n}$ is continuous if it is both o.s.c. and i.s.c.

Set Difference of Polygons

In the following, $\mathbb{N}_n \triangleq \{1,2,\dots,n\}$.

The first result, which is adapted from [18, Thm. 3], allows one to compute the set difference of two polyhedra:

Proposition 1 (Set difference for polyhedra): Let $A \subset \mathbb{R}^n$ and $B \triangleq \{x \in \mathbb{R}^n \mid c_i'x \leq d_i,\ i = 1,\dots,r\}$ be non-empty polyhedra, where all the $c_i \in \mathbb{R}^n$ and $d_i \in \mathbb{R}$. If

$$S_1 \triangleq \left\{x \in A \;\middle|\; c_1'x > d_1\right\}, \text{ and} \tag{20a}$$

$$S_i \triangleq \left\{x \in A \;\middle|\; c_i'x > d_i,\ c_j'x \leq d_j,\ \forall j \in \mathbb{N}_{i-1}\right\}, \tag{20b}$$

for $i = 2,\dots,r$, then $A \setminus B = \bigcup_{i=1}^r S_i$ is a polygon. Furthermore, $\{S_i \neq \emptyset \mid i \in \mathbb{N}_r\}$ is a partition of $A \setminus B$.

In practice, computation can be reduced by checking whether $A \cap B$ is empty or whether $A \subseteq B$ before actually computing $A \setminus B$; if $A \cap B = \emptyset$, then $A \setminus B = A$ and if $A \subseteq B$, then $A \setminus B = \emptyset$. Using an extended version of Farkas' Lemma [3, Lem. 4.1], [10, Lem. 3.1] checking whether one polyhedron is contained in another amounts to solving a single linear program (LP). Alternatively, one can solve a finite number of smaller LPs to check for set inclusion [10, Prop. 3.4].

Once $A \setminus B$ has been computed, the memory requirements can be reduced by removing all empty S_i and removing any redundant inequalities describing the non-empty S_i. Checking whether a polyhedron is non-empty can be done by solving a single LP. Removing redundant inequalities can be done by solving a finite number of LPs [10, App. B]. As a result, it is a good idea to determine first whether an S_i is non-empty or not before removing redundant inequalities.

The second result shows how the set difference of a polygon and a polyhedron may be computed:

Proposition 2 (Set difference of polygon and polyhedron): Let $C \triangleq \bigcup_{j=1}^{p} C_j$ be a polygon, where all the C_j, $j \in \mathbb{N}_p$, are non-empty polyhedra. If A is a non-empty polyhedron, then

$$C \setminus A = \bigcup_{j=1}^{p} (C_j \setminus A) \tag{21}$$

is a polygon.

Note that if $\{C_j \mid j \in \mathbb{N}_p\}$ is a partition of C and $C \setminus A \neq \emptyset$, then $\{C_j \setminus A \neq \emptyset \mid j \in \mathbb{N}_p\}$ is a partition of $C \setminus A$ if Proposition 1 is used to compute each polygon $C_j \setminus A$, $j \in \mathbb{N}_p$.

The last result shows how the set difference of two polygons may be computed:

Proposition 3 (Set difference of polygons): Let the sets $C \triangleq \bigcup_{j=1}^{p} C_j$ and $D \triangleq \bigcup_{k=1}^{q} D_k$ be polygons, where all the C_j, $j \in \mathbb{N}_p$, and D_k, $k \in \mathbb{N}_q$, are non-empty polyhedra. If

$$E_0 \triangleq C, \tag{22a}$$

$$E_k \triangleq E_{k-1} \setminus D_k, \quad k \in \mathbb{N}_q, \tag{22b}$$

then $C \setminus D = E_q$ is a polygon.

Note that each polygon $E_{k-1} \setminus D_k$, $k \in \mathbb{N}_q$, can be computed using Proposition 2. It follows that if $\{C_j \mid j \in \mathbb{N}_p\}$ is a partition of C and $C \setminus D \neq \emptyset$, then the sets that define E_q form a partition of $C \setminus D$ if Propositions 1 and 2 were used to compute all the E_k, $k \in \mathbb{N}_q$.

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