Approximation of the minimal robustly positively invariant set for discrete-time LTI systems with persistent state disturbances

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***Abstract*— This paper provides a solution to the problem of computing a robustly positively invariant outer approximation of the minimal robustly positively invariant set for a discrete-time, linear, time-invariant system. It is assumed that the disturbance is additive and persistent, but bounded.**

Keywords: Set invariance, constrained control, robust control, linear systems.

I. Introduction and Notation

Set invariance plays a fundamental role in the control of constrained systems; see for instance [1], [2]. An important problem is how to compute the *minimal* robustly positively invariant (mRPI) set for a given discrete-time LTI system with additive state disturbances [3, Sect. IV]. The mRPI set is used as a target set in robust time-optimal control [4], in the design of robust predictive controllers [5] and in understanding the properties of the *maximal* robustly positively invariant set [3], [6]. The only results that allow one to compute the mRPI set exactly are given in [3, Rem. 4.2] and [4, Thm. 3], where it is assumed that the system dynamics are nilpotent. This paper presents new results that allow one to compute a robustly positively invariant, outer approximation of the mRPI set. A more detailed exposition and all proofs for the results stated in this paper can be found in [7].

The set of strictly positive integers is denoted by $\mathbb{N}_+ \triangleq \{1,2,\ldots\}$. $\|M\|_p$ and $\|v\|_p$ are the p-norms of the matrix M and vector v, respectively. The ∞-norm ball in $\mathbb{R}^n$ (hypercube) of size $r \geq 0$ is defined as $B_\infty(r) \triangleq \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq r\}$. The i'th standard basis vector $e_i \in \mathbb{R}^n$ in the Euclidean space has one as the i'th component and zero as all other components. If P and Q are subsets of $\mathbb{R}^n$, then the Minkowski (vector) sum is $P \oplus Q \triangleq \{p+q \mid p \in P,\ q \in Q\}$. The set $\bigoplus_{i=j}^{k} P_i$ is the Minkowski sum of the sets $\{P_j, \ldots, P_k\}$.

II. Problem Formulation

Consider the discrete-time, linear, time-invariant system:

$$x^+ = Ax + w, \tag{1}$$

where $x \in \mathbb{R}^n$ is the current state, x^+ is the successor state, $w \in W$ is an unknown, additive and persistent disturbance. The standing assumptions are that the matrix $A \in \mathbb{R}^{n\times n}$ is strictly stable (the spectral radius $\rho(A) < 1$) and that the set W is a convex, compact subset in $\mathbb{R}^n$ containing the origin in its interior.

Definition 1: $\Omega \subset \mathbb{R}^n$ is a *robustly positively invariant* (RPI) set of (1) if $Ax + w \in \Omega$ for all $x \in \Omega$ and all $w \in W$.

Definition 2: The *minimal* robustly positively invariant (mRPI) set F_∞ of (1) is the set in $\mathbb{R}^n$ that is contained in every closed RPI set of (1).

It is possible to show [3, Sect. IV] that the mRPI set F_∞ exists, is compact, contains the origin in its interior and is given by $F_\infty = \bigoplus_{i=0}^{\infty} A^i W$. Since F_∞ is a Minkowski sum of infinitely many terms, it is generally impossible to obtain an explicit characterization of it. However, as noted in [3, Rem. 4.2], it is possible to show that if there exist an integer $s \in \mathbb{N}_+$ and a scalar $\alpha \in [0,1)$ such that $A^s = \alpha I$, then $F_\infty = (1-\alpha)^{-1} \bigoplus_{i=0}^{s-1} A^i W$. It therefore follows trivially [4, Thm. 3] that if A is nilpotent with index s ($A^s = 0$), then $F_\infty = \bigoplus_{i=0}^{s-1} A^i W$.

In this paper, we relax the assumption that there exists an $s \in \mathbb{N}_+$ and a scalar $\alpha \in [0,1)$ such that $A^s = \alpha I$. Since we can no longer compute F_∞ exactly, we address the problem of computing an RPI set $F(\alpha, s)$ that contains the mRPI set F_∞. We conclude with some remarks on computational issues if W is a polytope given by a finite set of affine inequalities.

III. Main Results

Proposition 1: [6] If the integer $s \in \mathbb{N}_+$ and scalar $\alpha \in [0,1)$ satisfy

$$A^s W \subseteq \alpha W, \tag{2}$$

then

$$F(\alpha, s) \triangleq (1-\alpha)^{-1} \bigoplus_{i=0}^{s-1} A^i W$$

is a convex, compact, RPI set of (1) containing F_∞.

Clearly, $F(\alpha_0, s) \subset F(\alpha_1, s) \Leftrightarrow \alpha_0 < \alpha_1$ for a given s. Note also that if A is not nilpotent, then $F(\alpha, s_0) \subset F(\alpha, s_1) \Leftrightarrow s_0 < s_1$ for a given α. These observations motivate the following discussion, which explains how one can obtain a better approximation of the mRPI set F_∞, given an initial pair (α, s).

Let

$$s^o(\alpha) \triangleq \inf_{s \in \mathbb{N}_+} \{s \mid A^s W \subseteq \alpha W\}, \tag{3a}$$

$$\alpha^o(s) \triangleq \inf_{\alpha \in [0,1)} \{\alpha \mid A^s W \subseteq \alpha W\} \tag{3b}$$

be the smallest values of s and α such that (2) holds for a given α and s, respectively. Clearly, $\alpha^0(s) \to 0$ as $s \to \infty$. Note that $s^0(\alpha) \to \infty$ as $\alpha \to 0$ if and only if A is not nilpotent. However, since A is strictly stable and W is a compact set containing the origin in its interior, the infimum in (3a) is guaranteed to exist and be contained in $\mathbb{N}_+$ for any choice of $\alpha \in (0,1)$. The infimum in (3b) is also guaranteed to exist and be contained in $[0,1)$ if s is sufficiently large.

By a process of iteration, one can use the above definitions and results to compute a pair (α,s) such that $F(\alpha,s)$ is a sufficiently good RPI, outer approximation of F_∞. For example, by starting with $s = 1$, one can increment s until there exists an $\alpha \in [0,1)$ such that (2) holds. If necessary, one can increase s until $F(s,\alpha^o(s))$ is sufficiently small. Alternatively, one can take an initial value for α, compute $s^* \triangleq s^o(\alpha)$, proceed to compute $\alpha^* \triangleq \alpha^o(s^*)$ and test whether $F(\alpha^*,s^*)$ is small enough. It is clear that this iteration results in $F_\infty \subseteq F(\alpha^*,s^*) \subseteq F(\alpha,s^*) \subseteq F(\alpha,s)$. If $F(\alpha^*,s^*)$ is not small enough, then this procedure could be restarted by decreasing α. Of course, any other iteration can be implemented until a fixed point is reached or a sufficiently small $F(\alpha,s)$ has been obtained.

Because of the iterative nature of computing a suitable $F(\alpha,s)$ and the fact that $s^o(\alpha)$ may be large, it is desirable to have upper bounds on $s^o(\alpha)$ and the volume of $F(\alpha,s)$ that are easy to compute:

Proposition 2: Let $\beta_{\text{in}} \triangleq \max_{\beta\geq 0}\{\beta \mid B_\infty(\beta) \subseteq W\}$ and $\beta_{\text{out}} \triangleq \min_{\beta\geq 0}\{\beta \mid W \subseteq B_\infty(\beta)\}$. Let A be diagonizable with $A = V\Lambda V^{-1}$, where Λ is a diagonal matrix of the eigenvalues of A, and $\rho(A) \in (0,1)$. If $s \in \mathbb{N}_+$ and $\alpha \in (0,1)$ satisfy

$$s \geq \ln[\alpha\beta_{\text{in}}/(\beta_{\text{out}}\|V\|_\infty\|V^{-1}\|_\infty)]/\ln\rho(A), \qquad (4)$$

then $F(\alpha,s)$ is a convex, compact, RPI set of (1) containing F_∞. Furthermore, the set $F(\alpha,s)$ is contained in the ∞-norm ball (hypercube) $B_\infty(\eta)$, where

$$\eta \triangleq \beta_{\text{out}}\|V\|_\infty\|V^{-1}\|_\infty(1-\rho(A)^s)/[(1-\alpha)(1-\rho(A))].$$

Clearly, any s satisfying (4) is a (possibly conservative) upper bound for $s^o(\alpha)$ and η could be used to obtain a (possibly conservative) upper bound on the size of $F(\alpha,s)$.

IV. Computational Results if W is a Polytope

Before proceeding, recall that the *support function* [3] of a set $Z \subset \mathbb{R}^m$, evaluated at $a \in \mathbb{R}^m$, is $h_Z(a) \triangleq \sup_{z\in Z} a^T z$. Clearly, if Z is a polytope given by a finite set of affine inequalities, then $h_Z(a)$ is finite and can be computed by solving an LP. Recall also that if W is a polytope, then testing whether (2) holds can be implemented by evaluating the support function of W at a finite number of points [2], [3]. The set $F(\alpha,s)$ can then be computed using standard algorithms for computing the Minkowski sum of polytopes.

This section therefore considers the case when the set W is a polytope given by $W \triangleq \left\{w \in \mathbb{R}^n \;\middle|\; f_i^T w \leq g_i,\ i \in \mathscr{I}\right\}$, where $f_i \in \mathbb{R}^n$, $g_i \in \mathbb{R}$ and $\mathscr{I}$ is a finite index set. It is easy to show that (2) holds if and only if $h_W((A^s)^T f_i) \leq \alpha g_i$ for all $i \in \mathscr{I}$. This observation implies that $s^o(\alpha)$ and $\alpha^o(s)$ can be computed efficiently by solving a finite number of suitably-defined LPs. For example, recall that W contains the origin in its interior if and only if $g_i > 0$ for all $i \in \mathscr{I}$. It then follows that $\alpha^o(s) = \max_{i\in\mathscr{I}} h_W((A^s)^T f_i)/g_i$.

In a similar fashion as above, it is also easy to check whether the set $F(\alpha,s)$ (and hence F_∞) is contained in a given polyhedron $X \triangleq \left\{x \in \mathbb{R}^n \;\middle|\; c_j^T x \leq d_j,\ j \in \mathscr{J}\right\}$, where $c_j \in \mathbb{R}^n$, $d_j \in \mathbb{R}$ and $\mathscr{J}$ is a finite index set, *without having to compute* $F(\alpha,s)$ *explicitly*. This is because the inclusion $F(\alpha,s) \subseteq X$ holds if and only if $h_{\mathscr{W}}((1-\alpha)^{-1}[A^0 \ \cdots \ A^{s-1}]^T c_j) \leq d_j$ for all $j \in \mathscr{J}$, where $\mathscr{W} \triangleq W^s \triangleq W \times \cdots \times W$. Proceeding in a similar fashion, it is possible to show that $\eta^o(\alpha,s) \triangleq \min_{\eta\geq 0}\{\eta \mid F(\alpha,s) \subseteq B_\infty(\eta)\} = \max_{i\in\{1,\ldots,n\}} h_{\mathscr{W}}(\pm(1-\alpha)^{-1}[A^0 \ \cdots \ A^{s-1}]^T e_i)$ is the size of the smallest ∞-norm ball (hypercube) containing $F(\alpha,s)$, hence $\eta^o(\alpha,s)$ can be computed by solving $2n$ LPs.

We conclude this paper by referring back to Proposition 2. It is easy to show [8, Prop. 2] that $h_{B_\infty(\beta)}(f_i) = \beta\|f_i\|_1$, hence $\beta_{\text{in}} = \min_{i\in\mathscr{I}} g_i/\|f_i\|_1$. Note also that one can compute β_{out} by solving $2n$ LPs, since $\beta_{\text{out}} = \max_{i\in\{1,\ldots,n\}} h_W(\pm e_i)$.

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