Invariant approximations of the minimal robust positively invariant set

S.V. Raković, E.C. Kerrigan, K.I. Kouramas and D.Q. Mayne

***Abstract*— This paper provides results on approximating the minimal robust positively invariant (mRPI) set (also known as the 0-reachable set) of an asymptotically stable discrete-time linear time-invariant system. It is assumed that the disturbance is bounded, persistent and acts additively on the state and that the constraints on the disturbance are polyhedral. Results are given that allow for the computation of a robust positively invariant, outer approximation of the mRPI set. Conditions are also given that allow one to *a priori* specify the accuracy of this approximation.**

***Index Terms*— Set invariance, invariant approximations, robust control, linear systems.**

I. Introduction

Set invariance plays a fundamental role in control [1]. The focus of this paper is on the *minimal robust positively invariant (mRPI) set*, also often referred to as the 0*-reachable set* [2], i.e. the set of states that can be reached from the origin under a bounded state disturbance. The mRPI set is important for performance analysis and synthesis of controllers for uncertain systems [1, Sects. 6.4–6.5] and for computing the *maximal* robust positively invariant (MRPI) set [3]. Set invariance is fundamental in the synthesis of reference governors [4], [5] and predictive controllers [6]–[9] with guaranteed invariance, stability and convergence properties. The mRPI set is also a suitable target set in robust time-optimal control [10]–[13] and plays an integral part in a novel robust predictive control method, recently proposed in [14].

Despite the wide-spread use of the mRPI set in control, there are still a number of unresolved issues. As pointed out in [1, Sects. 6.4–6.5] and the survey paper [2], one of the more important outstanding problems is how to compute an *exact* representation of the mRPI set. To the best of our knowledge, the only results that allow for the exact computation of the mRPI set are given in [13, Thm. 3] and [15, Sect. 3.3], where the restrictive assumption is made that the system dynamics are nilpotent.

For the more general case, where the dynamics are not nilpotent, it is only possible to compute an *approximation* to the mRPI set and the reader is referred to [1, Sects. 6.4–6.5] and [2] for a review of methods on how this can be achieved. However, though these methods allow for the approximation of the mRPI set, they do not allow for the computation of an *invariant* approximation to the mRPI set. Since reference governors, predictive controllers and time-optimal controllers use invariant sets, it is important that the approximation of the mRPI set be invariant. The approximation methods reviewed in [1, Sects. 6.4–6.5] and [2] are clearly inadequate for our purpose. Hence, the aim of this paper is to provide a solution to this problem by providing a number of new results that allow for the computation of a *robust positively invariant* approximation of the mRPI set. We also give results that allow one to specify *a priori* an upper bound on the error of this approximation.

This paper is organized as follows. §II is concerned with definitions, existing results and the problem formulation. §III deals with the problem of calculating a robust positively invariant (RPI) approximation of the mRPI set for linear systems with bounded state disturbances. §IV shows how the results can be implemented efficiently if the disturbance set is a polytope; an illustrative example is also provided. Finally, §V presents some conclusions. In order to keep the presentation as transparent as possible, all proofs are given in the appendix. A more detailed exposition and extension of the results in this paper can be found in the technical report [16].

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NOTATION: Let $\mathbb{N} \triangleq \{0, 1, 2, \ldots\}$ be the set of non-negative integers and $\mathbb{N}_+ \triangleq \{1, 2, \ldots\}$. Let $\mathbb{B}_p^n(r) \triangleq \{x \in \mathbb{R}^n \mid \|x\|_p \le r\}$ be a p-norm ball in $\mathbb{R}^n$, where $r \ge 0$. Given two sets $\mathcal{U}$ and $\mathcal{V}$, such that $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^n$, the Minkowski (vector) sum is defined by $\mathcal{U} \oplus \mathcal{V} \triangleq \{u + v \mid u \in \mathcal{U},\ v \in \mathcal{V}\}$, $\operatorname{int}(\mathcal{U})$ denotes the interior of $\mathcal{U}$. Given the sequence of sets $\{\mathcal{U}_i \subset \mathbb{R}^n\}_{i=a}^b$, we denote $\bigoplus_{i=a}^b \mathcal{U}_i \triangleq \mathcal{U}_a \oplus \cdots \oplus \mathcal{U}_b$. The set $\mathcal{M}_W \triangleq \{w(\cdot) \mid w(k) \in W,\ \forall k \in \mathbb{N}\}$ is the set of all infinite sequences whose elements take on values in $W \subset \mathbb{R}^n$ (equivalently, $\mathcal{M}_W$ is the set of all maps $w : \mathbb{N} \to W$).

II. Preliminaries and Existing Results

We consider the following autonomous discrete-time linear time-invariant (DLTI) system:

$$x^+ = Ax + w, \tag{1}$$

where $x \in \mathbb{R}^n$ is the current state, x^+ is the successor state and $w \in \mathbb{R}^n$ is an unknown disturbance. We make the standing assumption that $A \in \mathbb{R}^{n\times n}$ is a strictly stable matrix (all the eigenvalues of A are strictly inside the unit disk). The disturbance w is contained in a convex and compact set $W \subset \mathbb{R}^n$ that contains the origin. Since the system is time-invariant, current time can always be taken to be zero. We denote by $\phi(k, x, w(\cdot))$ the solution to (1) at time instant k, given that the initial state (at time 0) is x and the infinite disturbance sequence is $w(\cdot) \triangleq \{w(0), w(1), \ldots\}$.

First we recall the following well-known definition [1]:

Definition 1 (RPI set): The set $\Omega \subset \mathbb{R}^n$ is a *robust positively invariant* (RPI) set of (1) if $Ax + w \in \Omega$ for all $x \in \Omega$ and all $w \in W$, i.e. if and only if $A\Omega \oplus W \subseteq \Omega$.

Given a set X, the solution satisfies $\phi(k, x, w(\cdot)) \in X$ at all time instants $k \in \mathbb{N}$ and for all allowable disturbance sequences $w(\cdot) \in \mathcal{M}_W$ *if and only if* there exists an RPI set Ω that is contained in X and the initial state x is in Ω [1].

Definition 2 (Minimal RPI set): The *minimal robust positively invariant (mRPI) set* F_∞ of (1) is the RPI set in $\mathbb{R}^n$ that is contained in every closed RPI set of (1).

It is possible to show [3, Sect. IV] that the mRPI set F_∞ exists, is unique, compact and contains the origin and that the zero initial condition response of (1) is bounded by F_∞, i.e. $\phi(k, 0, w(\cdot)) \in F_\infty$ for all $w(\cdot) \in \mathcal{M}_W$ and all $k \in \mathbb{N}$. It follows, from linearity and asymptotic stability of (1), that F_∞ is the limit set of all trajectories of (1).

In order to quantify a "good" approximation, we introduce:

Definition 3 (ε-approximations): Given a scalar $\varepsilon > 0$ and a set $\Omega \subset \mathbb{R}^n$, the set $\Phi \subset \mathbb{R}^n$ is an *outer ε-approximation* of Ω if $\Omega \subseteq \Phi \subseteq \Omega \oplus \mathbb{B}_p^n(\varepsilon)$ and it is an *inner ε-approximation* of Ω if $\Phi \subseteq \Omega \subseteq \Phi \oplus \mathbb{B}_p^n(\varepsilon)$.

For all $s \in \mathbb{N}_+$, let the (convex and compact) set F_s be defined by:

$$F_s \triangleq \bigoplus_{i=0}^{s-1} A^i W, \quad F_0 \triangleq \{0\}. \tag{2}$$

Each F_s is contained in F_∞ and if A is strictly stable, then $F_s \to F_\infty$ as $s \to \infty$ [3, Sect. IV], i.e. for every $\varepsilon > 0$, there exists an $s \in \mathbb{N}$

such that F_s is an inner ε-approximation of F_∞. Clearly, the mRPI set F_∞ is given by

$$F_\infty = \bigoplus_{i=0}^{\infty} A^i W. \tag{3}$$

It is generally impossible to obtain an explicit characterization of F_∞ using (3) [2]. However, as noted in [15, Sect. 3.3] and [3, Rem. 4.2], if there exist an integer $s \in \mathbb{N}_+$ and a scalar $\alpha \in [0,1)$ such that $A^s = \alpha I$, then $F_\infty = (1-\alpha)^{-1} F_s$. It follows [13, Thm. 3] that if A is nilpotent with index s ($A^s = 0$), then $F_\infty = F_s$. If A is strictly stable, but not nilpotent, then there does not exist a finite s such that $F_s = F_\infty$. As a consequence, none of the sets in the sequence $\{F_s \mid s \in \mathbb{N}\}$ are RPI sets and it is impossible to compute an RPI, inner approximation of the mRPI set F_∞.

III. Approximations of the Minimal robust Positively Invariant Set

In this section, we address the problem of computing an RPI, *outer* approximation of the mRPI set F_∞ when A is not nilpotent. We also address the problem of computing an RPI, outer ε-approximation of the mRPI set F_∞ for a given $\varepsilon > 0$.

Before proceeding, we make a clear distinction between the results reported in the recent conference paper [17] and this paper. By applying the standard algorithm of [3], the authors of [17] propose to compute the *maximal* robust positively invariant set (MRPI) contained in $(1+\varepsilon)F_s$, for a given $\varepsilon > 0$ and $s \in \mathbb{N}$. This set, if non-empty, is an RPI, outer approximation of the mRPI set F_∞. For a given $\varepsilon > 0$, the algorithm is based on incrementing the integer s until the MRPI set contained in $(1+\varepsilon)F_s$ is non-empty. This recursive calculation is necessary, since the authors clearly state in [17, Rem. 6] that they do not have a criterion for the *a priori* determination of the integer s such that the MRPI set contained in $(1+\varepsilon)F_s$ is non-empty. In contrast to this method, we propose to compute an RPI, outer approximation of the mRPI set F_∞ by first computing a sufficiently large s, computing F_s and scaling the latter by a suitable amount. The proposed method does not rely on the computation of MRPI sets and thus is simpler and probably more efficient than the procedure reported in [17]. Our first result is:

Theorem 1 (RPI set): [18] If $0 \in \operatorname{int}(W)$, then there exists a finite integer $s \in \mathbb{N}_+$ and a scalar $\alpha \in [0,1)$ that satisfies

$$A^s W \subseteq \alpha W. \tag{4}$$

If (4) is satisfied, then

$$F(\alpha, s) \triangleq (1-\alpha)^{-1} F_s \tag{5}$$

is a convex, compact, RPI set of (1). Furthermore, $0 \in \operatorname{int}(F(\alpha,s))$ and $F_\infty \subseteq F(\alpha, s)$.

It is easy to develop and implement an algorithm based on Theorem 1. If W is a polytope, standard "off-the-shelf" optimization and computational geometry software may be used (See §IV).

Clearly $F(\alpha,s)$, as defined above, is an RPI, outer approximation of the mRPI set F_∞. However, the former could be a very poor approximation of the latter. We therefore proceed to address the question as to whether, in the limit, $F(\alpha,s)$ tends to the true mRPI set F_∞ if we choose s sufficiently large and/or choose α sufficiently small. For this purpose, let

$$s^{o}(\alpha) \triangleq \min\{s \in \mathbb{N}_+ \mid A^s W \subseteq \alpha W\}, \tag{6a}$$

$$\alpha^{o}(s) \triangleq \min\{\alpha \in \mathbb{R} \mid A^s W \subseteq \alpha W\} \tag{6b}$$

be the smallest values of s and α such that (4) holds for a given α and s, respectively. The minimum in (6a) exists for any choice of $\alpha \in (0,1)$ and $\alpha^{o}(s) \in [0,1)$ only if s is sufficiently large. The computation of $s^{o}(\alpha)$ and $\alpha^{o}(s)$ is easy if W is a polytope given by a set of affine inequality constraints (See §IV). Our next result is:

Theorem 2 (Limiting behavior of the RPI approximation): If $0 \in \operatorname{int}(W)$, then

(i) $F(\alpha^{o}(s), s) \to F_\infty$ as $s \to \infty$ and
(ii) $F(\alpha, s^{o}(\alpha)) \to F_\infty$ as $\alpha \searrow 0$.

Theorem 1 provides a way for the computation of an RPI, outer approximation of F_∞ and Theorem 2 establishes the limiting behavior of this approximation. However, for a given pair (α,s) that satisfies (4), it is not immediately obvious whether or not $F(\alpha,s)$ is a good approximation of the mRPI set F_∞. Given a pair (α,s) satisfying the conditions of Theorem 1, it can be shown (along similar lines as in the proof of Theorem 3 in Appendix III) that if

$$\varepsilon \geq \alpha(1-\alpha)^{-1} \max_{x \in F_s} \|x\|_p = \alpha(1-\alpha)^{-1} \min_{\gamma} \{\gamma \mid F_s \subseteq \mathbb{B}_p^n(\gamma)\} \tag{7}$$

then $F_\infty \subseteq F(\alpha,s) \subseteq F_\infty \oplus \mathbb{B}_p^n(\varepsilon)$. In other words, $F(\alpha,s)$ is an RPI, outer ε-approximation of F_∞ if ε satisfies (7).

Though this observation allows one to determine *a posteriori* whether or not $F(\alpha,s)$ is a good approximation of F_∞, it is perhaps more useful to have a result that allows one to determine *a priori* how large s and/or how small α need to be in order for $F(\alpha,s)$ to be a sufficiently accurate approximation of F_∞. The following result establishes that this is possible:

Theorem 3 (Error bound): If $0 \in \operatorname{int}(W)$, then for all $\varepsilon > 0$, there exist an $\alpha \in [0,1)$ and an associated integer $s \in \mathbb{N}_+$ such that (4) and

$$\alpha(1-\alpha)^{-1} F_s \subseteq \mathbb{B}_p^n(\varepsilon) \tag{8}$$

hold. Furthermore, if (4) and (8) are satisfied, then $F(\alpha,s)$ is an RPI, outer ε-approximation of the mRPI set F_∞.

Remark 1: Note that (7) and (8) are equivalent. If W is a polytope and $p = \infty$, then it is not necessary to compute F_s in order to check whether (8) holds (see §IV).

It is straightforward to develop a conceptual algorithm based on Theorem 3. Note that (4) provides a lower bound on α such that $F(\alpha,s)$ is guaranteed to be RPI and contain F_∞. In addition, the conditions (7) and (8) give an upper bound on α such that $F(\alpha,s)$ is guaranteed to be an outer ε-approximation of F_∞. The reader is referred to Algorithm 1 in §IV for more details.

A whole collection of RPI, outer ε-approximations of the mRPI set F_∞ can be computed; the complexity of $F(\alpha,s)$ is highly dependent on the eigenstructure of A and the description of W. However, for a given error bound ε, it is usually a good idea to find the smallest value of the integer s for which there exists an $\alpha \in [0,1)$ such that (4) and (8) hold. This is because, for a given α, a lower value of s generally results in a lower complexity for the description of $F(\alpha,s)$. In contrast, for a given s, the value of α does not affect the complexity of $F(\alpha,s)$.

Remark 2 (Origin is in the relative interior of W): The results in this section can be extended to the more general case when the *interior* of W is empty, but the *relative interior* of W contains the origin (see [16]).

IV. Efficient Computation if W is a Polytope

This section presents results that allow for the efficient computation of *a priori* upper bounds for the conditions presented in (4) and (8) to hold. In particular, results are given that allow one to test whether or not F_s is contained in a given polyhedron X *without having to compute F_s explicitly*. The interested reader is referred to [1], [3] and [16] for information on the methods used to derive the results in this section.

The *support function* [3] of a set $W \subset \mathbb{R}^m$, evaluated at $a \in \mathbb{R}^m$, is defined as

$$h_W(a) \triangleq \sup_{w \in W} a^T w. \tag{9}$$

If W is a polytope (bounded and closed polyhedron), then $h_W(a)$ is finite. Furthermore, if W is described by a finite set of affine inequality constraints, then $h_W(a)$ can be computed by solving a linear program (LP). Testing whether (4) and (8) hold can be implemented by evaluating the support function of W at a finite number of points [3], or by solving a single Phase I LP [1, Lem. 4.1]. The set F_s (and hence $F(\alpha, s)$) is easily computed using standard computational geometry software for computing the Minkowski sum of polytopes, such as [19] and [20].

Remark 3: If $W \triangleq \{Ed + c \mid \|d\|_\infty \le \eta\}$, where $E \in \mathbb{R}^{n \times n}$ and $c \in \mathbb{R}^n$, then $h_W(a) = \eta \|E^T a\|_1 + a^T c$.

In order to be as general as possible, we will consider the case when W is in the form $W \triangleq \{w \in \mathbb{R}^n \mid f_i^T w \le g_i,\ i \in \mathcal{I}\}$, where $f_i \in \mathbb{R}^n$, $g_i \in \mathbb{R}$ and $\mathcal{I}$ is a finite index set. Following a standard procedure [3] it is possible to show that

$$A^s W \subseteq \alpha W \iff h_W((A^s)^T f_i) \le \alpha g_i,\ \forall i \in \mathcal{I}. \tag{10}$$

This observation allows for efficient checking of whether or not (4) is satisfied. Hence, it permits the efficient computation of $s^o(\alpha)$ and $\alpha^o(s)$. For example, recall that W contains the origin in its interior if and only if $g_i > 0$ for all $i \in \mathcal{I}$. It follows that

$$\alpha^o(s) = \max_{i \in \mathcal{I}} h_W((A^s)^T f_i)/g_i. \tag{11}$$

It is also possible to check whether the set F_s is contained in a given polyhedron $X \triangleq \{x \in \mathbb{R}^n \mid c_j^T x \le d_j,\ j \in \mathcal{J}\}$, where $c_j \in \mathbb{R}^n$, $d_j \in \mathbb{R}$ and $\mathcal{J}$ is a finite index set, *without computing F_s explicitly*:

$$F_s \subseteq X \iff \sum_{i=0}^{s-1} h_W((A^i)^T c_j) \le d_j,\ \forall j \in \mathcal{J}. \tag{12}$$

Thus, $F(\alpha, s) \subseteq X \Leftrightarrow F_s \subseteq (1-\alpha)X \Leftrightarrow \sum_{i=0}^{s-1} h_W((A^i)^T c_j) \le (1-\alpha)d_j,\ \forall j \in \mathcal{J}$.

One can also use the support function to compute *a priori* an error bound on the approximation $F(\alpha, s)$ if the ∞-norm is used to define the error bound, i.e. $p = \infty$ in (8). Proceeding in a similar fashion as above, it is possible to show that

$$\begin{aligned} M(s) &\triangleq \min_{\gamma} \{\gamma \mid F_s \subseteq \mathbb{B}_\infty^n(\gamma)\} \\ &= \max_{j \in \{1,\dots,n\}} \left\{ \sum_{i=0}^{s-1} h_W((A^i)^T e_j),\quad \sum_{i=0}^{s-1} h_W(-(A^i)^T e_j) \right\}, \end{aligned} \tag{13}$$

where e_j is the j^{th} standard basis vector in $\mathbb{R}^n$. If $\alpha \in (0,1)$, then (8) is equivalent to $F_s \subseteq \alpha^{-1}(1-\alpha)B_p^n(\varepsilon)$. Hence, if $p = \infty$ in (8), straightforward algebraic manipulation yields

$$\alpha(1-\alpha)^{-1} F_s \subseteq B_\infty^n(\varepsilon) \iff \alpha \le \varepsilon/(\varepsilon + M(s)). \tag{14}$$

Clearly, (11) is an easily-computed lower bound and (14) is an easily-computed upper bound on α such that $F(\alpha, s)$ is an RPI, outer ε-approximation of the mRPI set F_∞. We are now in a position to put together a prototype algorithm for computing an RPI, outer ε-approximation of F_∞ if the ∞-norm is used to bound the error. These steps are outlined in Algorithm 1. In order to reduce computational effort, note that in step 5 of Algorithm 1 it is not necessary to compute $\sum_{i=0}^{s-2} h_W((A^i)^T e_j)$ and $\sum_{i=0}^{s-2} h_W(-(A^i)^T e_j)$ at each iteration. These sums would have been computed at previous iterations. All that is needed is to update these sums by computing and adding $h_W((A^{s-1})^T e_j)$ and $h_W(-(A^{s-1})^T e_j)$, respectively.

Algorithm 1 Computation of an RPI, outer ε-approximation of the mRPI set F_∞

Require: A, W and $\varepsilon > 0$
Ensure: $F(\alpha, s)$ such that $F_\infty \subseteq F(\alpha, s) \subseteq F_\infty \oplus \mathbb{B}_\infty^n(\varepsilon)$

1: Choose any $s \in \mathbb{N}$ (ideally, set $s \leftarrow 0$).
2: **repeat**
3: Increment s by one.
4: Compute $\alpha^o(s)$ as in (11) and set $\alpha \leftarrow \alpha^o(s)$.
5: Compute $M(s)$ as in (13).
6: **until** $\alpha \le \varepsilon/(\varepsilon + M(s))$
7: Compute F_s as the Minkowski sum (2) and scale it to give $F(\alpha, s) \triangleq (1-\alpha)^{-1} F_s$.

Our illustrative example is a double integrator:

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + w \tag{15}$$

with the additive disturbance $W \triangleq \{w \in \mathbb{R}^2 \mid \|w\|_\infty \le 1\}$ and $u = -[1.17\ 1.03]x$. The sets F_s, for $s = 1, 2, \dots, 10$, for the example are shown in Figure 1 together with the set $F(1.9 \cdot 10^{-5}, 10)$ for which it was required that $\varepsilon = 5 \cdot 10^{-5}$ (see (8)); it is clear that the sequence $\{F_s\}$ is a monotonically non-decreasing sequence and it converges to F_∞ and that $F(1.9 \cdot 10^{-5}, 10)$ is a sufficiently good approximation of F_∞, i.e. $F(1.9 \cdot 10^{-5}, 10)$ satisfies that $F_\infty \subseteq F(1.9 \cdot 10^{-5}, 10) \subseteq F_\infty \oplus \mathbb{B}_\infty^2(5 \cdot 10^{-5})$.

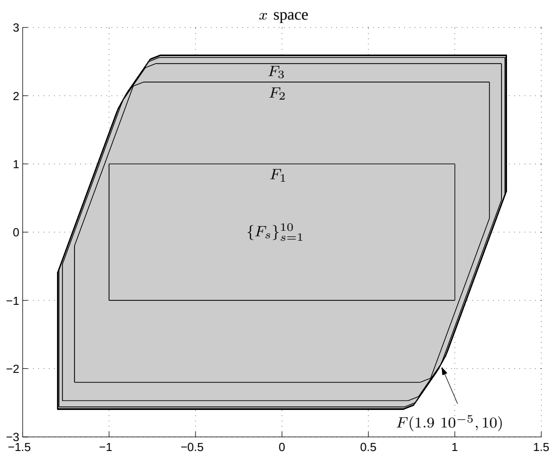

Fig. 1. Approximations of F_∞: Sets F_s and $F(\alpha, s)$

V. Conclusions

The reported novel results complement existing results and permit the efficient computation of an RPI, outer approximation of the minimal robust positively invariant set and allow one to specify *a priori* the accuracy of the approximation. The presented results can be exploited in the design of robust reference governors, predictive controllers and time-optimal controllers for constrained, linear discrete time systems subject to additive, but bounded disturbances.

Appendix I
Proof of Theorem 1

Existence of an $s \in \mathbb{N}_+$ and an $\alpha \in [0, 1)$ that satisfies (4) follows from the fact that the origin is in the *interior* of W and that A is strictly stable. Convexity and compactness of $F(\alpha, s)$ follows directly from the fact that F_s (and hence $F(\alpha, s)$) is the Minkowski sum of a finite set of convex, compact sets. Let $G(\alpha, j, k) \triangleq (1-\alpha)^{-1} \bigoplus_{i=j}^{k} A^i W$. It follows that

$$AG(\alpha, 0, s-1) \oplus W = G(\alpha, 1, s) \oplus W \tag{16a}$$
$$= (1-\alpha)^{-1} A^s W \oplus G(\alpha, 1, s-1) \oplus W \tag{16b}$$
$$\subseteq (1-\alpha)^{-1} \alpha W \oplus W \oplus G(\alpha, 1, s-1) \tag{16c}$$
$$= [(1-\alpha)^{-1}\alpha + 1] W \oplus G(\alpha, 1, s-1) \tag{16d}$$
$$= (1-\alpha)^{-1} W \oplus G(\alpha, 1, s-1) = G(\alpha, 0, s-1). \tag{16e}$$

In going from (16b) to (16c) we have used the fact that $P \subseteq Q \Rightarrow P \oplus R \subseteq Q \oplus R$ for arbitrary sets $P \subset \mathbb{R}^n$, $Q \subset \mathbb{R}^n$ and $R \subset \mathbb{R}^n$. Since $F(\alpha, s) = G(\alpha, 0, s-1)$, it follows that $AF(\alpha, s) \oplus W \subseteq F(\alpha, s)$ holds, hence $F(\alpha, s)$ is RPI. It follows trivially from the definition of the mRPI set that $F(\alpha, s)$ contains F_∞. Note also that $0 \in \mathrm{int}(F_\infty)$ if $0 \in \mathrm{int}(W)$.

APPENDIX II
PROOF OF THEOREM 2

In order to talk about limits of sets, we recall the definition of the Haussdorff metric:

Definition 4 (Hausdorff metric): If Ω and Φ are two non-empty, compact sets in $\mathbb{R}^n$, then the *Hausdorff metric* is defined as

$$\delta(\Omega, \Phi) \triangleq \max \left\{ \sup_{\omega \in \Phi} d(\omega, \Omega), \ \sup_{\phi \in \Omega} d(\phi, \Phi) \right\}, \tag{17}$$

where $d(z, \mathcal{Z}) \triangleq \inf_{y \in \mathcal{Z}} \|z - y\|_p$.

We also need the following intermediate result [16]:

Lemma 1: If Φ is a convex and compact set in $\mathbb{R}^n$ containing the origin and $\alpha \in [0, 1)$, then $\delta(\Phi, (1-\alpha)^{-1}\Phi) \leq \alpha(1-\alpha)^{-1} M$, where $M \triangleq \sup_{z \in \Phi} \|z\|_p$ is finite.

Recall that the sequence $\{F_s\}_{s=0}^{\infty}$ is Cauchy [3, Sect. IV] so that $M_\infty \triangleq \lim_{s \to \infty} \sup_{z \in F_s} \|z\|_p$ is finite. Since $F_s \subseteq F_\infty$, $\forall s \in \mathbb{N}$, we have that $M(s) \triangleq \sup_{z \in F_s} \|z\|_p \leq M_\infty < \infty$ for all $s \in \mathbb{N}$.

We can now proceed with the proof of Theorem 2:

(i) It follows from Lemma 1 that $\delta(F_s, F(\alpha^o(s), s)) = \delta(F_s, (1-\alpha^o(s))^{-1} F_s) \leq \alpha^o(s)(1-\alpha^o(s))^{-1} M(s)$, where $M(s) \leq M_\infty < \infty$ for all $s \in \mathbb{N}$. Since $\alpha^o(s) \searrow 0$ as $s \to \infty$, we get that $\delta(F_s, F(\alpha^o(s), s)) \to 0$ as $s \to \infty$. However, since $F(\alpha^o(s), s) \supseteq F_\infty \supseteq F_s$ for all $s \in \mathbb{N}$ and $F_s \to F_\infty$ as $s \to \infty$, we conclude that $F(\alpha^o(s), s) \to F_\infty$ as $s \to \infty$.

(ii) It follows from Lemma 1 that $\delta(F_{s^o(\alpha)}, F(\alpha, s^o(\alpha))) = \delta(F_{s^o(\alpha)}, (1-\alpha)^{-1} F_{s^o(\alpha)}) \leq \alpha(1-\alpha)^{-1} M(s^o(\alpha))$, where $M(s^o(\alpha)) \leq M_\infty < \infty$ for all $\alpha \in (0, 1)$, hence $\delta(F_{s^o(\alpha)}, F(\alpha, s^o(\alpha))) \to 0$ as $\alpha \searrow 0$. Note that $s^o(\alpha) \to \infty$ as $\alpha \searrow 0$. Since $F(\alpha, s^o(\alpha)) \supseteq F_\infty \supseteq F_{s^o(\alpha)}$ for all $\alpha \in (0, 1)$ and $F_{s^o(\alpha)} \to F_\infty$ as $\alpha \searrow 0$, we conclude that $F(\alpha, s^o(\alpha)) \to F_\infty$ as $\alpha \searrow 0$.

APPENDIX III
PROOF OF THEOREM 3

We refer to the proof of Theorem 2 for the definition of M_∞. Let $\varepsilon > 0$ and recall that $0 < M_\infty < \infty$ and $F_s \subseteq F_\infty$ for all $s \in \mathbb{N}$. Since F_s and F_∞ are convex and contain the origin, it follows that $\alpha(1-\alpha)^{-1} F_s \subseteq \alpha(1-\alpha)^{-1} F_\infty$ for any $s \in \mathbb{N}$ and $\alpha \in [0, 1)$. Note that the inclusion $\alpha(1-\alpha)^{-1} F_\infty \subseteq \mathbb{B}_p^n(\varepsilon)$ is true if $\alpha(1-\alpha)^{-1} M_\infty \leq \varepsilon$ or, equivalently, if $\alpha \leq \varepsilon(\varepsilon + M_\infty)^{-1}$. Hence, (8) is true for any $s \in \mathbb{N}$ and $\alpha \in [0, \bar{\alpha}]$, where $\bar{\alpha} \triangleq \varepsilon(\varepsilon + M_\infty)^{-1} \in (0, 1)$. Clearly, (4) is also true if we choose $\alpha \in (0, \bar{\alpha}]$ and $s = s^o(\alpha)$. This establishes the existence of a suitable couple (α, s) such that (4) and (8) hold simultaneously.

Let (α, s) be such that (4) and (8) are true. Since $F(\alpha, s) = (1-\alpha)^{-1} F_s$ is a convex and compact set that contains the origin, $F(\alpha, s) = (1-\alpha)^{-1} F_s = (1 + \alpha(1-\alpha)^{-1}) F_s = F_s \oplus \alpha(1-\alpha)^{-1} F_s$. Since $F_s \subseteq F_\infty \subseteq F(\alpha, s) \subseteq F_s \oplus \mathbb{B}_p^n(\varepsilon) \subseteq F_\infty \oplus \mathbb{B}_p^n(\varepsilon)$, it follows that $F(\alpha, s)$ is an RPI, outer ε-approximation of the mRPI set F_∞.

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