Properties of a new parameterization for the control of constrained systems with disturbances

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***Abstract*— This paper is concerned with the application and analysis of a recent result in the literature on robust optimization to the control of linear discrete-time systems, which are subject to unknown state disturbances and mixed constraints on the state and input. By parameterizing the control input sequence as an affine function of the disturbance sequence, it can be shown that a certain class of robust finite horizon control problems can be solved in a computationally tractable fashion, provided the constraint and the disturbance sets are polytopic. The main contribution of the paper is to show that this parameterization includes the class of affine time-varying state feedback control laws. The paper also shows how this parameterization can be used to efficiently synthesize receding horizon control laws that are robustly invariant.**

I. INTRODUCTION

The problem of finding a nonlinear state feedback control law, which guarantees that a set of state and input constraints are satisfied for all time, despite the presence of a persistent state disturbance, has been the subject of study for many authors [1]–[7]. However, the problem is that all of these solutions are computationally prohibitive (even for problems of 'moderate' size) or can be shown to be computationally intractable (the complexity of the solution can be shown to be an exponential function of the problem data). As a consequence, a number of researchers have proposed compromise solutions [8]–[11], which, though not able to guarantee the same level of performance or region of attraction, is computationally tractable.

Recently, a new parameterization for solving so-called *robust optimization* problems was proposed in [12] and [13]. The authors proposed that, instead of solving for a general, nonlinear function that guarantees that the constraints in the optimization problem are met for all values of the uncertainty, one could aim to find an *affine* function of the uncertainty. They proceeded to show that, if the uncertainty set is a polyhedron and the constraints in the robust optimization problem are affine, then an affine function of the uncertainty can be found by solving a single, computationally tractable LP. They also demonstrated, via an example, how their results can be successfully applied to an inventory control problem.

The same affine parameterization was later used in [14, Chap. 7] and [15] to approximate a class of so-called *feedback* min-max finite horizon control problems [1], [5]–[7]. It was shown in [14] and [15], via numerical examples, that the parameterization of [12] and [13] leads to a significant improvement over schemes such as *open-loop* min-max model predictive control [6, Sect. 4.5] and those proposed in [9]–[11], where a sequence of *perturbations* to a stabilizing control law is sought.

Motivated by the very promising results reported in [12]–[15], the aim of this paper is to make a first step towards a detailed, theoretical understanding of the geometric and system-theoretic properties of the parameterization proposed by [12] and [13], with the goal of ultimately using this parametrization in efficiently synthesizing robustly invariant and stabilizing receding horizon control (RHC) laws.

This paper is organized as follows: Section II briefly introduces the control problem that will be considered in this paper and some standing assumptions are introduced. Section III proceeds to review the parameterization proposed in [12] and [13], within the context of finding a solution to a certain robust finite horizon control problem.

Section IV contains the main contribution of this paper. Theorem 1 shows that the set of states for which the parameterization in Section III is feasible, contains the set of states for which one can find an affine time-varying state feedback control policy such that for all allowable values of the disturbance, the constraints are satisfied over a finite horizon.

Further new results are given in Section V. It is shown that, provided the target/terminal constraint set is robustly invariant, one can guarantee certain geometric and system-theoretic properties of a number of control policies based on the parameterization proposed in Section III. Theorem 2 shows that the size of the set of states for which a control policy can be defined, increases with an increase in horizon length. Theorem 3 shows that one can design an RHC law that is guaranteed to be robustly invariant.

Section VI discusses the computational complexity of the parameterization reviewed in Section III. Most of the points discussed in Section VI can be found in [12]–[15] in one form or another and this section is therefore mainly included for completeness. The key point to note from Section VI is that finding a solution to the finite horizon control problems discussed in Sections III and V is a convex optimization problem, where the number of decision variables and constraints is a *polynomial* function of the problem data. In particular, it is shown that, provided the disturbance is an affine map of a hypercube, one need only solve a Phase I LP of size $O(N^2)$, where N is the length of

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the control horizon.

The paper concludes in Section VII and briefly discusses directions for further research. The reader is also referred to [16] and [17] for extensions of the results in this paper. Some numerical examples and results on minimum-time control are given in [17] and [16] shows how the parameterization discussed in this paper can be used to efficiently solve finite horizon min-max problems, where the cost is quadratic and the cost is negatively weighted, as in H_∞ control.

II. PROBLEM DESCRIPTION

Consider the following discrete-time LTI system:

$$x^+ = Ax + Bu + w, \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, x^+ is the successor state, $u \in \mathbb{R}^m$ is the control input and $w \in \mathbb{R}^n$ is the disturbance. The actual values of the state, input and disturbance at a time instant k are denoted by $x(k)$, $u(k)$ and $w(k)$, respectively; where it is clear from the context, x, u and w will be used to denote the current value of the state, input and disturbance.

It is assumed that (A,B) is stabilizable and that at each sample instant a measurement of the state is available. It is further assumed that the current and future values of the disturbance are unknown and that the disturbance is persistent, but contained in a convex and compact set W, which contains the origin.

Since the disturbance is persistent, it is not possible to drive the state of the system to the origin. Instead, the aim will be to drive the state of the system to a target/terminal constraint set X_f, given by

$$X_f := \{x \in \mathbb{R}^n \mid Yx \leq z\}, \tag{2}$$

where the matrix $Y \in \mathbb{R}^{r\times n}$ and the vector $z \in \mathbb{R}^r$; r is the number of affine inequality constraints that define X_f. It is assumed that X_f contains the origin in its interior.

The system is subject to mixed constraints on the state and input:

$$\mathscr{Y} := \{(x,u) \in \mathbb{R}^n \times \mathbb{R}^m \mid Cx + Du \leq b\}, \tag{3}$$

where the matrices $C \in \mathbb{R}^{s\times n}$, $D \in \mathbb{R}^{s\times m}$ and the vector $b \in \mathbb{R}^s$; s is the number of affine inequality constraints that define $\mathscr{Y}$. It is assumed that $\mathscr{Y}$ contains the origin in its interior. An additional design goal is to guarantee that the state and input of the closed-loop system satisfy $\mathscr{Y}$ for all time and for all allowable disturbance sequences.

The final standing assumption is that a state feedback gain matrix $K \in \mathbb{R}^{m\times n}$ is given, such that $A+BK$ is strictly stable (the eigenvalues of $A+BK$ are strictly inside the unit disk).

NOTATION: $A \otimes B$ is the Kronecker product of matrices A and B. Given an integer n, I_n is the $n \times n$ identity matrix and $\mathbf{1}_n$ is a column vector of n ones.

III. AN AFFINE PARAMETERIZATION OF THE CONTROL INPUT SEQUENCE

Let N be a positive integer and the vectors $\mathbf{v} \in \mathbb{R}^{mN}$ and $\mathbf{w} \in \mathbb{R}^{nN}$ be defined as

$$\mathbf{v} := \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}, \quad \mathbf{w} := \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix}, \tag{4}$$

where the vectors $v_i \in \mathbb{R}^m$ and $w_i \in \mathbb{R}^n$ for all $i \in \{0,\dots,N-1\}$.

Let the set $\mathscr{W} := W^N := W \times \cdots \times W$.

We define the *strictly block lower triangular* matrix $\mathbf{M} := [M_{i,j}] \in \mathbb{R}^{mN\times nN}$, where the matrices $M_{i,j} \in \mathbb{R}^{m\times n}$ for all $i \in \{0,\dots,N-1\}$ and $j \in \{0,\dots,N-1\}$ and $M_{i,j} := 0$ for all $j \in \{i,\dots,N-1\}$. In other words,

$$\mathbf{M} := \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ M_{1,0} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{N-2,0} & M_{N-2,1} & \cdots & 0 & 0 \\ M_{N-1,0} & M_{N-1,1} & \cdots & M_{N-1,N-2} & 0 \end{bmatrix}. \tag{5}$$

This constraint on $\mathbf{M}$ is assumed throughout the rest of this paper.

The variable ψ is defined as the pair

$$\psi := (\mathbf{v}, \mathbf{M}). \tag{6}$$

Using the same affine parameterization of the control input sequence proposed in [12], [13], we use the current value of the state x to define the set of admissible ψ, which will be used to define a number of different feedback policies, as:

$$\mathbf{\Psi}_N(x) := \left\{ \psi \;\middle|\; \begin{array}{r} \mathbf{v},\ \mathbf{w} \text{ satisfies (4)},\ \mathbf{M} \text{ satisfies (5)}, \\ x_{i+1} = Ax_i + Bu_i + w_i,\ x_0 = x, \\ u_i = v_i + \sum_{j=0}^{i-1} M_{i,j} w_j, \\ (x_i,u_i) \in \mathscr{Y},\ x_N \in X_f, \\ \forall i \in \{0,\dots,N-1\},\ \forall \mathbf{w} \in \mathscr{W} \end{array} \right\}. \tag{7}$$

Note that the predicted value of the input u_i at a time instant i steps into the future, is an affine function of the disturbance sequence $\{w_0,\dots,w_{i-1}\}$; because the state is measured at each sample instant, the values in this disturbance sequence will be known at a time instant i steps into the future. The strictly block lower triangular constraint on $\mathbf{M}$ in (5) can therefore be seen to be a *causality constraint* on u_i, which ensures that the input u_i is not a function of the (as yet unknown) disturbance sequence $\{w_i,\dots,w_{N-1}\}$.

Given any $\psi \in \mathbf{\Psi}_N(x(0))$ and the stabilizing state feedback gain $K \in \mathbb{R}^{m\times n}$, one can now define the following *time-varying* feedback policy:

$$u(k) = \begin{cases} v_k + \sum_{j=0}^{k-1} M_{i,j} w(j) & \text{if } k \in \{0,\dots,N-1\} \\ Kx(k) & \text{if } k \in \{N, N+1, \dots\} \end{cases} \tag{8}$$

Clearly, (8) is a causal feedback policy that is dependent not only on the current state, but also on past values of the state and input; since measurements of the state are available and past inputs are known, $w(j)$ in (8) is given by

$$w(j) = x(j+1) - Ax(j) - Bu(j), \quad \forall j \in \{0,\dots,N-1\}. \tag{9}$$

Before proceeding to analyze the properties of (8) and other feedback policies, let the set X_N^{ψ} denote the set of states for which there exists an admissible ψ:

$$X_N^{\psi} := \{x \in \mathbb{R}^n \mid \Psi_N(x) \neq \emptyset\}. \tag{10}$$

IV. HOW TO MATCH AN AFFINE TIME-VARYING FEEDBACK LAW

Let the variable θ be defined as the tuple

$$\theta := (L_0, g_0, L_1, g_1, \dots, L_{N-1}, g_{N-1}), \tag{11}$$

where the matrix $L_i \in \mathbb{R}^{m\times n}$ and vector $g_i \in \mathbb{R}^m$ for all $i \in \{0,\dots,N-1\}$.

Consider now the set of admissible θ:

$$\Theta_N(x) := \left\{ \theta \;\middle|\; \begin{array}{r} \theta \text{ satisfies (11)},\ \mathbf{w} \text{ satisfies (4)}, \\ x_{i+1} = Ax_i + Bu_i + w_i,\ x_0 = x, \\ u_i = L_i x_i + g_i, \\ (x_i, u_i) \in \mathscr{Y},\ x_N \in X_f \\ \forall i \in \{0,\dots,N-1\},\ \forall \mathbf{w} \in \mathscr{W} \end{array} \right\} \tag{12}$$

The set of states for which there exist an admissible θ is defined as:

$$X_N^{\theta} := \{x \in \mathbb{R}^n \mid \Theta_N(x) \neq \emptyset\}. \tag{13}$$

Given a stabilizing state feedback gain $K \in \mathbb{R}^{m\times n}$ and a $\theta \in \Theta_N(x(0))$, one can define the following affine time-varying (ATV) state feedback policy:

$$u(k) = \begin{cases} L_k x(k) + g_k & \text{if } k \in \{0,\dots,N-1\} \\ Kx(k) & \text{if } k \in \{N, N+1, \dots\} \end{cases} \tag{14}$$

The main result of this paper states that the set of initial states X_N^{θ}, for which an ATV feedback policy of the form (14) can be defined, is contained inside X_N^{ψ}, the set of initial states for which a feedback policy of the form (8) can be defined:

Theorem 1 (Main result). *X_N^{ψ} contains X_N^{θ}.*

Proof. Let $x \in X_N^{\theta}$. One can easily verify that given a $\theta \in \Theta_N(x)$ and $\mathbf{w} \in \mathscr{W}$, it follows that for all $i \in \{1,\dots,N\}$,

$$x_i = S_i x + \sum_{j=1}^{i-1} T_{i,j}\left(Bg_{i-1-j} + w_{i-1-j}\right) + Bg_{i-1} + w_{i-1}, \tag{15}$$

where $S_i := \prod_{j=0}^{i-1}(A + BL_j)$ and $T_{i,j} := \prod_{l=1}^{j}(A + BL_{i-l})$, $j = 1,\dots,i-1$.

Since $u_i = L_i x_i + g_i$ for all $i \in \{0,\dots,N-1\}$, it follows that

$$u_i = L_i S_i x + \sum_{j=1}^{i-1} L_i T_{i,j}\left(Bg_{i-1-j} + w_{i-1-j}\right) + L_i B g_{i-1} + L_i w_{i-1} + g_i. \tag{16}$$

It is easy to check that (16) is equal to

$$u_i = v_i + \sum_{j=0}^{i-1} M_{i,i-1-j} w_{i-1-j}, \quad \forall i \in \{0,\dots,N-1\} \tag{17}$$

if $v_0 := L_0 x + g_0$ and for all $i \in \{1,\dots,N-1\}$,

$$v_i := L_i S_i x + \sum_{j=1}^{i-1} L_i T_{i,j} B g_{i-1-j} + L_i B g_{i-1} + g_i \tag{18}$$

and

$$M_{i,i-1-j} := \begin{cases} L_i & \text{if } j = 0 \\ L_i T_{i,j} & \text{if } j \in \{1,\dots,i-1\} \end{cases} \tag{19}$$

It follows from the definition of $\Theta(x)$ that for all $i \in \{0,\dots,N-1\}$ and $\mathbf{w} \in \mathscr{W}$, $(x_i, u_i) \in \mathscr{Y}$ and $x_N \in X_f$. Given the above definitions, if $(\mathbf{v}, \mathbf{M})$ is defined as in (4) and (5), then $(\mathbf{v}, \mathbf{M}) \in \Psi_N(x)$, hence $x \in X_N^{\psi}$.

□

It is interesting to note that the proof of Theorem 1 implies that if, for a given initial state $x(0)$, one can find an ATV feedback policy of the form (14) such that for all allowable disturbance sequences of length N, the state will be in X_f in exactly N steps while satisfying the constraints $\mathscr{Y}$ over a horizon of length N, then one can find a $\psi \in \Psi_N(x(0))$ in order to define a time-varying feedback policy of the form (8), which will result in *exactly the same control input sequence* as the one that would result from implementing (14).

We conclude this section by pointing out that, at present, there does not exist an efficient algorithm for finding a $\theta \in \Theta_N(x)$. However, as will be shown in Section VI, finding a $\psi \in \Psi_N(x)$ is computationally tractable if W is a polytope (closed and bounded polyhedron) or the affine map of a hypercube. As a consequence of Theorem 1, the results in Section VI and the lack of an efficient method for finding a $\theta \in \Theta_N(x)$, we will only consider feedback policies that can be defined from the parameterization proposed in Section III.

V. GEOMETRIC AND INVARIANCE PROPERTIES

For this section, we introduce the following assumption:
A1: The set X_f is contained inside X_K, which is given by

$$X_K := \{x \in \mathbb{R}^n \mid (x, Kx) \in \mathscr{Y}\} = \{x \mid (C + DK)x \leq b\}, \tag{20}$$

and X_f is robustly positively invariant [4, Def. 2.2] for the closed-loop system $x^+ = (A + BK)x + w$, i.e.

$$(A + BK)x + w \in X_f, \quad \forall x \in X_f,\ \forall w \in W. \tag{21}$$

Remark 1. Under some additional, mild technical assumptions, it is easy to compute an X_f that satisfies **A1** if W is a polytope. For example, [18] gives results for computing the *maximal* robustly positively invariant set in X_K and [19] gives some new results that allow one to compute a robustly positively invariant *outer* approximation to the *minimal* robustly positively invariant set in X_K. See also [3] for results on computing a robustly positively invariant *inner* approximation to the *maximal* robustly positively invariant set in X_K. For results on computing an X_f of a given complexity, which satisfies **A1**, see [11].

The next result follows immediately from the above:

Proposition 1. *Let* ***A1*** *hold, the initial state* $x(0) \in X_N^\psi$ *and* $\psi \in \Psi_N(x(0))$. *For all allowable infinite disturbance sequences, the state of system* (1), *in closed-loop with the feedback policy* (8), *enters* X_f *in* N *steps or less and is in* X_f *for all* $k \in \{N, N+1, \ldots\}$. *Furthermore, the constraints in* (3) *are satisfied for all time and for all allowable infinite disturbance sequences.*

The following result gives a sufficient condition under which one can guarantee that an increase in the horizon length N does not result in a decrease in the size of X_N^ψ:

Theorem 2 (Size of X_N^ψ). *If* ***A1*** *holds, then the following set inclusion holds:*

$$X_f \subseteq X_1^\psi \subseteq \cdots \subseteq X_{N-1}^\psi \subseteq X_N^\psi \subseteq X_{N+1}^\psi \subseteq \cdots, \tag{22}$$

where each X_i^ψ *is defined as in* (10) *with* $N = i$.

Proof. The proof is by induction. Let $x \in X_N^\psi$, $(\mathbf{v}, \mathbf{M}) \in \Psi_N(x)$ and $\mathbf{w} \in \mathscr{W}$. It is easy to verify that

$$x_N = A^N x + \sum_{i=0}^{N-1} \Bigg(A^i B v_{N-1-i} + \left(A^i + \sum_{j=0}^{i-1} A^j B M_{N-1-j,N-1-i} \right) w_{N-1-i} \Bigg). \tag{23}$$

Let

$$v_N := K A^N x + \sum_{i=0}^{N-1} K A^i B v_{N-1-i} \tag{24}$$

and for all $i \in \{0, \ldots, N-1\}$, let

$$M_{N,N-1-i} := K A^i + \sum_{j=0}^{i-1} K A^j B M_{N-1-j,N-1-i}. \tag{25}$$

From these definitions, one can check that the terminal control law

$$u_N := v_N + \sum_{j=0}^{N-1} M_{N,j} w_j = v_N + \sum_{i=0}^{N-1} M_{N,N-1-i} w_{N-1-i} = K x_N.$$

From the definition of $\Psi_N(x)$, recall that $x_N \in X_f$. Note also that that since $X_f \subseteq X_K$, it follows that $(x_N, u_N) \in \mathscr{Y}$. Since X_f is robustly positively invariant for the closed-loop system $x^+ = (A + BK)x + w$, it follows that

$$x_{N+1} = A x_N + B u_N + w_N \in X_f, \quad \forall w_N \in W. \tag{26}$$

By putting all of the above together and letting the vector $\overline{\mathbf{v}} \in \mathbb{R}^{m(N+1)}$ be defined as

$$\overline{\mathbf{v}} := \begin{bmatrix} \mathbf{v} \\ v_N \end{bmatrix} \tag{27}$$

and the matrix $\overline{\mathbf{M}} \in \mathbb{R}^{m(N+1) \times n(N+1)}$ be defined as

$$\overline{\mathbf{M}} := \begin{bmatrix} \mathbf{M} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ M_{N,0} & M_{N,1} & \cdots & M_{N,N-1} & 0 \end{bmatrix}, \tag{28}$$

it follows from the definition of $\Psi_{N+1}(x)$ that $(\overline{\mathbf{v}}, \overline{\mathbf{M}}) \in \Psi_{N+1}(x)$, hence $x \in X_{N+1}^\psi$. The proof is completed by verifying, in a similar manner, that $X_f \subseteq X_1^\psi \subseteq X_2^\psi$. □

We now consider what happens when $\Psi_N(x)$ is used to design a *time-invariant* receding horizon control law. Consider the *set-valued* receding horizon control (RHC) law $\kappa_N : X_N^\psi \to 2^{\mathbb{R}^m}$ ($2^{\mathbb{R}^m}$ is the set of all subsets of $\mathbb{R}^m$), which is defined by considering only the first portion of a $\mathbf{v}$ for which there exists an $\mathbf{M}$ such that $(\mathbf{v}, \mathbf{M}) \in \Psi_N(x)$:

$$\kappa_N(x) := \{ u \in \mathbb{R}^m \mid \exists (\mathbf{v}, \mathbf{M}) \in \Psi_N(x) \text{ s.t. } u = [I_m \; 0]\mathbf{v} \}. \tag{29}$$

The following result implies that if the initial state is in X_N^ψ, then all trajectories of (1) in closed-loop with the RHC policy $u \in \kappa_N(x)$ will remain in X_N^ψ for all time and for all allowable disturbance sequences:

Theorem 3 (Robust invariance of RHC laws). *If* ***A1*** *holds, then the set* X_N^ψ *is robustly positively invariant for system* (1) *in closed-loop with the RHC law* (29), *i.e. if* $x \in X_N^\psi$, *then*

$$Ax + Bu + w \in X_N^\psi, \quad \forall u \in \kappa_N(x), \; \forall w \in W. \tag{30}$$

Furthermore, the constraints (3) *are satisfied for all time and for all allowable infinite disturbance sequences.*

Proof. The method of proof very closely parallels that of Theorem 2 and the same definitions are assumed. However, rather than showing that an appended version of $(\mathbf{v}, \mathbf{M})$ is admissible, one proceeds by showing that a "shifted" version of $(\mathbf{v}, \mathbf{M})$ is admissible at the next time instant. For this purpose, we introduce the following variables:

$$\tilde{\mathbf{v}} := \begin{bmatrix} v_1 + M_{1,0} w \\ \vdots \\ v_{N-1} + M_{N-1,0} w \\ v_N + M_{N,0} w \end{bmatrix} \tag{31}$$

and

$$\tilde{\mathbf{M}} := \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ M_{2,1} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{N-1,1} & M_{N-1,2} & \cdots & 0 & 0 \\ M_{N,1} & M_{N,2} & \cdots & M_{N,N-1} & 0 \end{bmatrix}. \tag{32}$$

Using similar arguments as in proving Theorem 2, but taking care with notation, one can now show that if $x \in X_N^\psi$, $u \in \kappa_N(x)$ and $w \in W$, then $(\tilde{\mathbf{v}}, \tilde{\mathbf{M}}) \in \Psi_N(Ax+Bu+w)$, hence $Ax+Bu+w \in X_N^\psi$. □

Remark 2. In this paper, we do not consider the important problem of how to synthesize an RHC law such that the closed-loop system is robustly stable and robust convergence to X_f is guaranteed. However, it is hopefully clear from the proofs of Theorems 2 and 3 that we can always choose the so-called 'terminal control law' (as used in the RHC literature) to be $u = Kx$. Hence, one can use well-known results, surveyed in [5] and [6], to synthesize robustly stabilizing RHC controllers, using the parameterization discussed in this paper. For example, the reader is referred to [16] for some initial results on how the parameterization in this paper may be used to efficiently solve finite horizon H_∞ problems. Alternatively, the results in this paper and those in [14] and [15] may be extended and combined with the results in [20] to efficiently synthesize min-max RHC laws with robust stability guarantees. The results in [10] may also be generalized to allow one to efficiently compute RHC laws with input-to-state stability (ISS) guarantees, if some so-called 'nominal/expected cost' is minimized, rather than a worst-case cost.

VI. COMPUTING AN ADMISSIBLE ψ

It is straightforward to find matrices $F \in \mathbb{R}^{q\times mN}$, $G \in \mathbb{R}^{q\times nN}$, $H \in \mathbb{R}^{q\times n}$ and a vector $c \in \mathbb{R}^q$, where $q := sN + r$, such that one can rewrite $\Psi_N(x)$ in (7) as

$$\Psi_N(x) = \left\{ \psi \;\middle|\; \begin{array}{r} \mathbf{M} \text{ satisfies (5)}, \\ F\mathbf{v} + (F\mathbf{M}+G)\mathbf{w} \le c + Hx,\ \forall \mathbf{w} \in \mathscr{W} \end{array} \right\}. \tag{33}$$

It is well-known (see, e.g. [1]–[5], [8]–[15], [18]) that one can eliminate the quantifier in (33) by noting that

$$\Psi_N(x) = \left\{ \psi \;\middle|\; \begin{array}{r} \mathbf{M} \text{ satisfies (5)}, \\ F\mathbf{v} + \max_{\mathbf{w}\in\mathscr{W}} (F\mathbf{M}+G)\mathbf{w} \le c + Hx \end{array} \right\}, \tag{34}$$

where the maximization in $\max_{\mathbf{w}\in\mathscr{W}} (F\mathbf{M}+G)\mathbf{w}$ is row-wise. It follows immediately that $\Psi_N(x)$ is a convex set.

If W is a polytope (closed and bounded polyhedron) given by a finite set of affine inequalities, then it is easy to check whether a given ψ is in $\Psi_N(x)$ by solving the q LPs that define $\max_{\mathbf{w}\in\mathscr{W}} (F\mathbf{M}+G)\mathbf{w}$ and checking the constraints in (34). Conversely, one can find a pair $\psi \in \Psi_N(x)$ in a computationally tractable way by solving Phase I of a single LP by writing down the dual of each of the LPs defining $\max_{\mathbf{w}\in\mathscr{W}} (F\mathbf{M}+G)\mathbf{w}$. The reader is referred to [12, Thm. 3.2] and [13, Thm 4.2] for details as to how this can be done.

However, in this paper we will not consider the general case when W is an arbitrary polytope. Instead, we will consider the special case when W is known to be the affine map of a hypercube. This is because, in many practical applications, W is nearly always assumed to be the affine map of a hypercube (for example, when upper and lower bounds on the components of the disturbance are known and the disturbance acts on the state in an affine manner). This observation leads to a significant reduction in computational effort, compared to the case of treating W as an arbitrary polytope.

To see why this is the case, note that if W is the affine translation of a hypercube, i.e. if

$$W := \{ Ed + f \mid \|d\|_\infty \le \eta \} \tag{35}$$

where the matrix $E \in \mathbb{R}^{n\times l}$, the vectors $f \in \mathbb{R}^n$, $d \in \mathbb{R}^l$ and η is a positive scalar, then

$$\mathscr{W} := \{ J\mathbf{d} + g \mid \|\mathbf{d}\|_\infty \le \eta \}, \tag{36}$$

where the matrix $J := I_N \otimes E$, the vectors $g := \mathbf{1}_N \otimes f$, $\mathbf{d} \in \mathbb{R}^t$ and the integer $t := lN$. It follows that

$$\max_{\mathbf{w}\in\mathscr{W}} (F\mathbf{M}+G)\mathbf{w} = \max_{\mathbf{d}} \{ (F\mathbf{M}+G)(J\mathbf{d}+g) \mid \|\mathbf{d}\|_\infty \le \eta \} \tag{37a}$$

$$= \max_{\mathbf{d}} \{ (F\mathbf{M}J+GJ)\mathbf{d} + (F\mathbf{M}+G)g \mid \|\mathbf{d}\|_\infty \le \eta \} \tag{37b}$$

$$= \max_{\mathbf{d}} \{ (F\mathbf{M}J+GJ)\mathbf{d} \mid \|\mathbf{d}\|_\infty \le \eta \} + (F\mathbf{M}+G)g \tag{37c}$$

$$= \eta\, \mathrm{abs}(F\mathbf{M}J+GJ)\mathbf{1}_t + (F\mathbf{M}+G)g, \tag{37d}$$

where the components of the matrix $\mathrm{abs}(F\mathbf{M}J+GJ)$ are the absolute values of the corresponding components of the matrix $F\mathbf{M}J+GJ$. Hence,

$$\Psi_N(x) = \left\{ \psi \;\middle|\; \begin{array}{r} \mathbf{M} \text{ satisfies (5)}, \\ F\mathbf{v} + \eta\, \mathrm{abs}(F\mathbf{M}J+GJ)\mathbf{1}_t \\ + (F\mathbf{M}+G)g \le c + Hx \end{array} \right\}. \tag{38}$$

Remark 3. Note that $\mathrm{abs}(F\mathbf{M}J+GJ)\mathbf{1}_t$ is a vector formed from the 1-norms of the rows of $F\mathbf{M}J+GJ$. In going from (37c) to (37d) we have used the well-known fact that $\max_{\mathbf{d}} \left\{ a^T\mathbf{d} \mid \|\mathbf{d}\|_\infty \le \eta \right\} = \eta \|a\|_1$ for any vector $a \in \mathbb{R}^t$ and scalar η (see, for example, [10, Prop. 2] or [14, Thm. 3.1]).

If $\Psi_N(x)$ is given as in (38), then it is easy to *check* whether a given pair ψ is in $\Psi_N(x)$ by computing $\mathrm{abs}(F\mathbf{M}J+GJ)\mathbf{1}_t$ and checking whether the constraints in (38) are satisfied. However, we will now make an important observation that allows one to *compute* a $\psi \in \Psi_N(x)$, given the current state x. It follows immediately from (38) that

$$\Psi_N(x) = \left\{ \psi \;\middle|\; \begin{array}{r} \mathbf{M} \text{ satisfies (5)},\ \exists \Lambda \in \mathbb{R}^{q\times t} \text{ such that} \\ F\mathbf{v} + \eta\Lambda\mathbf{1}_t + (F\mathbf{M}+G)g \le c + Hx, \\ \mathrm{abs}(F\mathbf{M}J+GJ) \le \Lambda \end{array} \right\}$$

$$= \left\{ \psi \;\middle|\; \begin{array}{r} \mathbf{M} \text{ satisfies (5)},\ \exists \Lambda \in \mathbb{R}^{q\times t} \text{ such that} \\ F\mathbf{v} + \eta\Lambda\mathbf{1}_t + (F\mathbf{M}+G)g \le c + Hx, \\ -\Lambda \le F\mathbf{M}J+GJ \le \Lambda \end{array} \right\},$$

where the matrix and vector inequalities are componentwise.

Remark 4. Note that $\Psi_N(x)$ is the projection of the polyhedron

$$\mathscr{C}_N(x) := \left\{ (\psi, \Lambda) \;\middle|\; \begin{array}{r} \mathbf{M} \text{ satisfies (5)}, \\ F\mathbf{v} + \eta\Lambda\mathbf{1}_t + (F\mathbf{M}+G)g \leq c + Hx, \\ -\Lambda \leq F\mathbf{M}J + GJ \leq \Lambda \end{array} \right\} \tag{40}$$

onto a subspace, hence $\Psi_N(x)$ is also a polyhedron.

The key point to note here is the following: if the number of constraints in (3) is $s = O(m+n)$ and $l = O(m+n)$ in (35) (this is nearly always the case in practice), then the dimension of $\mathscr{C}_N(x)$ is bounded by $O((m+n)^2N^2 + r(m+n)N)$ and the number of constraints that define $\mathscr{C}_N(x)$ in (40) is also bounded by $O((m+n)^2N^2 + r(m+n)N)$. This implies that the problem of finding a pair $(\mathbf{v}, \mathbf{M}) \in \Psi_N(x)$ is computationally tractable.

For example, finding a $\psi \in \Psi_N(x)$ is easily done by solving the following Phase I LP, in which γ is a scalar:

$$(\psi^*(x), \Lambda^*(x), \gamma^*(x)) := \underset{(\psi,\Lambda,\gamma)}{\arg\inf}\ \gamma \tag{41a}$$

subject to (5) and

$$F\mathbf{v} + \eta\Lambda\mathbf{1}_t + (F\mathbf{M}+G)g \leq c + Hx + \mathbf{1}_q\gamma, \tag{41b}$$

$$-\Lambda \leq F\mathbf{M}J + GJ \leq \Lambda. \tag{41c}$$

Clearly, $\Psi_N(x)$ is non-empty and $\psi^*(x) \in \Psi_N(x)$ if and only if $\gamma^*(x) \leq 0$.

It is easy to find an initial feasible point to (41) by choosing any $\mathbf{M}$ that satisfies (5), followed by choosing a Λ sufficiently large enough to satisfy (41c) and finally, choosing any $\mathbf{v}$ and a sufficiently large γ such that (41b) is satisfied. Once initialized with a feasible point, the LP solver can proceed with minimizing the cost until $\gamma \leq 0$. The reader is referred to [17] for a discussion on how to efficiently translate (41) into a form suitable to be passed to a standard LP solver.

VII. Conclusions and Further Research

Though the affine parameterization defined in Section III was shown to be useful for efficiently implementing control laws with guaranteed system-theoretic properties such as robust invariance and robust convergence to a target set, there are still a number of issues that need to be addressed.

It was proven in Section IV that the set of states, for which the parameterization in Section III is feasible, contains the set of states for which an affine time-varying policy exists. It still remains to be determined whether there exist examples for which the inclusion in Theorem 1 is strict or whether it is always satisfied with equality.

Section V showed that the parameterization in Section III can be used to construct receding horizon control laws such that the region of feasibility is robustly invariant for the closed-loop system. The focus of current research is in extending these results to guaranteeing robust convergence and stability of the target set, as well as ensuring offset-free control if the disturbance tends to a non-zero limit.

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