Optimal control of constrained piecewise affine systems with state- and input-dependent disturbances*

Saša V. Raković†, Eric C. Kerrigan‡§ and David Q. Mayne†

Abstract

Finite horizon optimal control of piecewise affine systems with a piecewise affine (1-norm or ∞-norm) stage cost and terminal cost is considered. Provided the respective constraint sets are given as the unions of polyhedra, it is shown that the partial value functions and partial optimal control laws are piecewise affine on a polyhedral cover of the set of states that can be steered, by an admissible control policy, to a terminal set of states in a finite number of steps. Existing results only consider the case of systems without disturbances, or systems with disturbances that are independent of the state and input. This paper extends these results to the case where the disturbance is dependent on the state and input.

1 Introduction

The problems of controllability to a target set and computation of optimal control laws for systems subject to constraints and persistent, unmeasured disturbances are well-known and have been the subject of study for many authors [3–5, 9–12, 14, 15]. Though the existing results are fairly general and can be applied to a large class of nonlinear discrete-time systems, most authors assume that the disturbance is not dependent on the state and input.

The need for a framework that can deal with state- and input-dependent disturbances was briefly motivated in [9, 13]. Disturbances that are dependent on the state and/or input frequently arise in practice when trying to model systems with physical constraints. For example, disturbances on systems with hard state and/or input constraints, modelling errors due to the linearization of nonlinear systems, parametric model uncertainty and dynamic model uncertainty can all be accurately modelled by disturbances that are dependent on the state and input.

In [10] it was shown how to compute the solution to a finite horizon optimal control problem for constrained, piecewise affine systems with bounded state disturbances. However, in [10] it was assumed that the disturbance is independent of the state and input. This paper extends these results to the case where the disturbance is dependent on the state and input. In particular, specific results are given for linear and piecewise affine systems that allow the use of polyhedral algebra, parametric linear programming and computational geometry software to compute the solution to a finite horizon optimal control problem.

*Research supported by the Engineering and Physical Sciences Research Council and the Royal Academy of Engineering, UK.

†Department of Electrical and Electronic Engineering, Imperial College, London SW7 2BT, United Kingdom. Tel: +44-(0)20-7594-6295/87/81. Fax: +44-(0)20-7594-6282, E-mail: sasa.rakovic@imperial.ac.uk and d.mayne@imperial.ac.uk

‡Royal Academy of Engineering Post-doctoral Research Fellow, Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, United Kingdom. Tel: +44-(0)1223-332600. Fax: +44-(0)1223-332662. Email: erickerrigan@ieee.org

§Corresponding author.

2 Notation and Definitions

Definition 1. A *polyhedron* is the intersection of a finite number of closed and/or open halfspaces[1] and a *polytope* is a closed and bounded (hence compact) polyhedron. A *polygon* is the union of a finite number of polyhedra and is thus not necessarily convex.

Definition 2. A family of sets $\mathcal{P} := \{\mathcal{P}_i \mid i \in \mathcal{I}\}$ is a *(closed) polyhedral cover* of a (closed) polygon $\mathcal{X} \subseteq \mathbb{R}^n$ if the index set $\mathcal{I}$ is finite, each $\mathcal{P}_i$ is a (closed) polyhedron and $\mathcal{X} = \cup_{i \in \mathcal{I}} \mathcal{P}_i$.

We often use $\mathcal{P}^{\mathcal{X}}$, $\mathcal{I}^{\mathcal{X}}$ and $\mathcal{P}_i^{\mathcal{X}}$ to denote, respectively, a polyhedral cover of $\mathcal{X}$, its associated index set and the i^{th} polyhedron in the cover.

Remark 1. Note that the definition of a polyhedral cover given here does not require that each $\mathcal{P}_i$ have a non-empty interior, nor does it require that $\mathcal{P} := \{\mathcal{P}_i \mid i \in \mathcal{I}\}$ be a *partition* of $\mathcal{X}$. Note also that our use of the term *cover* is stronger than the commonly-used definition, where a *cover* is a collection of sets $\mathcal{P} := \{\mathcal{P}_i \mid i \in \mathcal{I}\}$ such that $\mathcal{X} \subseteq \cup_{i \in \mathcal{I}} \mathcal{P}_i$ — we require equality and not the weaker condition of inclusion.

Definition 3. A function $\psi : \mathcal{X} \to \mathbb{R}^n$ is said to be *piecewise affine* on a polyhedral cover $\mathcal{P} := \{\mathcal{P}_i \mid i \in \mathcal{I}\}$ of $\mathcal{X} \subseteq \mathbb{R}^m$ if it satisfies

$$\psi(x) = K_i x + k_i, \quad \forall x \in \mathcal{P}_i,\ i \in \mathcal{I},$$

for some K_i, k_i, $i \in \mathcal{I}$.

3 Detailed Problem Formulation

In this paper, we will consider the problem of controlling nonlinear discrete-time systems in the form:

$$x^+ = f(x, u, w), \tag{1}$$

where x is the current state (assumed to be measured), x^+ is the successor state, u is the input, and w is an unmeasured, persistent disturbance that is dependent on the current state and input:

$$w \in \mathcal{W}(x, u) \subset W, \tag{2}$$

where $W = \mathbb{R}^p$ denotes the disturbance space. The state and input are required to satisfy:

$$(x, u) \in \mathcal{Y} \subset X \times U, \tag{3}$$

where $X = \mathbb{R}^n$ is the state space and $U = \mathbb{R}^m$ is the input space. The constraint $(x, u) \in \mathcal{Y}$ defines the state-dependent set of admissible inputs:

$$\mathcal{U}(x) := \{u \mid (x, u) \in \mathcal{Y}\}. \tag{4}$$

3.1 Optimal Control Problem

Let $\pi := \{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$ denote a control policy (sequence of control *laws*, i.e. $\mu_i : X \to U$, $i = 0, \ldots, N-1$) over horizon N and let $\mathbf{w} := \{w_0, w_1, \ldots, w_{N-1}\}$ denote a sequence of disturbances. Also, let $\phi(i; x, \pi, \mathbf{w})$ denote the solution of (1) when the initial state is x at time 0 (note that since the system is time-invariant, we can always take the current time to be zero), the control policy is π and the disturbance sequence is $\mathbf{w}$.

If the initial state is x, the control policy is π and the disturbance sequence is $\mathbf{w}$, then the cost $V_N(x, \pi, \mathbf{w})$ is defined as

$$V_N(x, \pi, \mathbf{w}) := \sum_{i=0}^{N-1} \ell(x_i, u_i) + V_f(x_N), \tag{5}$$

[1] Note that we depart slightly from convention, where a polyhedron is defined to be the intersection of a finite number of *closed* halfspaces only.

where for all i, $x_i := \phi(i; x, \pi, \mathbf{w})$ and $u_i := \mu_i(x_i)$. The stage cost $\ell(\cdot)$ and the terminal cost $V_f(\cdot)$ are assumed to be piecewise affine[2] (ℓ_1 or ℓ_∞):

$$\ell(x,u) := \|Qx\|_p + \|Ru\|_p, \qquad p = 1, \infty \tag{6a}$$

$$V_f(x) := \|Px\|_p, \qquad p = 1, \infty \tag{6b}$$

where P, Q and R are matrices of suitable dimensions.

The optimal control problem $\mathbb{P}_N(x)$ that we will consider is

$$\mathbb{P}_N(x): \qquad V_N^0(x) := \inf_{\pi \in \Pi_N(x)} \sup_{\mathbf{w} \in \mathbf{W}} V_N(x, \pi, \mathbf{w}) \tag{7}$$

where, with some abuse of notation, $\mathbf{W} := \mathbf{W}(x, \pi)$ is the set of admissible disturbance sequences:

$$\mathbf{W}(x,\pi) := \{\mathbf{w} \mid w_i \in \mathcal{W}(\phi(i;x,\pi,\mathbf{w}), \mu_i(\phi(i;x,\pi,\mathbf{w}))),\ i = 0, 1, \ldots, N-1\}. \tag{8}$$

$\Pi_N(x)$ is the set of admissible policies, i.e. those policies that satisfy, for all $\mathbf{w} \in \mathbf{W}$, the state and control constraints (3), and the terminal constraint

$$\phi(N; x, \pi, \mathbf{w}) \in X_f. \tag{9}$$

Hence the set of admissible policies is

$$\Pi_N(x) := \{\pi \mid (\phi(i;x,\pi,\mathbf{w}), \mu_i(\phi(i;x,\pi,\mathbf{w}))) \in \mathcal{Y},\ i = 0, 1, \ldots, N-1, \\ \phi(N; x, \pi, \mathbf{w}) \in X_f,\ \forall \mathbf{w} \in \mathbf{W}(x, \pi)\}. \tag{10}$$

We let X_N denote the set of initial states for which an admissible policy exists (the domain of $V_N^0(\cdot)$, the controllability set):

$$X_N := \{x \mid \Pi_N(x) \neq \emptyset\}. \tag{11}$$

The solution to $\mathbb{P}_N(x)$, if it exists, is

$$\pi^0(x) := \left\{\mu_0^0(x), \mu_1^0(\cdot; x), \ldots, \mu_{N-1}^0(\cdot; x)\right\} := \arg\inf_{\pi \in \Pi_N(x)} \sup_{\mathbf{w} \in \mathbf{W}} V_N(x, \pi, \mathbf{w}). \tag{12}$$

Finally, in order to simplify the presentation and have a well-defined problem, we make the following standing assumptions:

A1. The system $f : \mathcal{S} \to X$ is continuous, where $\mathcal{S}$ is a closed polygon with a non-empty interior.
A2. The sets $\mathcal{Y}$ and X_f are closed polygons and contain the origin in their interiors.
A3. The set-valued map $x \mapsto \mathcal{U}(x)$ is continuous and bounded on bounded sets.
A4. For all $(x,u) \in \mathcal{Y}$, the set $\mathcal{W}(x,u) \neq \emptyset$.
A5. The set-valued map $(x,u) \mapsto \mathcal{W}(x,u)$ is continuous and bounded on bounded sets.
A6. The solution $\pi^0(x)$ to $\mathbb{P}_N(x)$ exists for all $x \in X_N$.

Remark 2. It is easy to check *a priori* whether or not **A1**–**A5** hold. However, it is not yet known whether or not **A1**–**A5** are sufficient for **A6** to be satisfied, hence why this assumption is made.

3.2 Piecewise Affine Systems

In the sequel, we will consider the system $f : \mathcal{S} \to X$ with the closed polyhedral cover

$$\mathcal{P}^{\mathcal{S}} := \left\{\mathcal{P}_i^{\mathcal{S}} \mid i \in \mathcal{I}^{\mathcal{S}}\right\} \tag{13}$$

and the piecewise affine description:

$$f(x,u,w) := A_i x + B_i u + G_i w + g_i, \quad \forall (x,u,w) \in \mathcal{P}_i^{\mathcal{S}},\ i \in \mathcal{I}^{\mathcal{S}}, \tag{14}$$

where for all $i \in \mathcal{I}^{\mathcal{S}}$, the matrices $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, $G_i \in \mathbb{R}^{n \times p}$ and vector $g_i \in \mathbb{R}^n$. For convenience, we also define the functions $f_i : \mathcal{P}_i^{\mathcal{S}} \to X$, $i \in \mathcal{I}^{\mathcal{S}}$, as

$$f_i(x,u,w) := A_i x + B_i u + G_i w + g_i. \tag{15}$$

[2]The results in this paper easily extend to more general, piecewise affine stage costs and terminal costs, such as $\ell(x,u) := \min_{\xi \in \Xi} \|Q(x-\xi)\|_p + \|R(u-Kx)\|_p$, where $p \in \{1, \infty\}$, Ξ is a closed polygon and K is a feedback gain.

Remark 3. Clearly, if $\mathcal{I}^{\mathcal{S}}$ has cardinality 1, then $f(\cdot)$ is affine (linear if $g_i = 0$). Note also that, since $f(\cdot)$ is assumed to be continuous, it follows that if $i \neq j$ and $\mathcal{P}_i^{\mathcal{S}} \cap \mathcal{P}_j^{\mathcal{S}} \neq \emptyset$, then

$$f_i(x,u,w) = f_j(x,u,w), \quad \forall (x,u,w) \in \mathcal{P}_i^{\mathcal{S}} \cap \mathcal{P}_j^{\mathcal{S}}. \tag{16}$$

To enable us to apply the results in [10] and [13], we make the following additional standing assumptions:
A7. The system $f : \mathcal{S} \to X$ is piecewise affine on a closed polyhedral cover of the polygon $\mathcal{S}$.
A8. $\Gamma := \{(x,u,w) \mid w \in \mathcal{W}(x,u)\}$ is a closed polygon.

4 Dynamic Programming Solution

Dynamic programming provides a recursive procedure for computing sequentially the partial return functions $V_j^0(\cdot)$ (defined in (7) with $N = j$), the associated set-valued control laws $\kappa_j(\cdot)$ as well as their domains (here j denotes 'time-to-go' so that $\kappa_j(\cdot) = \mu_{N-j}^0(\cdot)$ if $j \in \{1,\ldots,N-1\}$ and $\kappa_N(\cdot) = \mu_0^0(\cdot)$). The domain of $V_j^0(\cdot)$ and $\kappa_j(\cdot)$ is X_j, the set of states that can be robustly steered to the terminal set X_f in j steps or less.

Remark 4. Standard optimal control implements the time-varying policy $\pi_N^0(x) = \{\kappa_N(x), \kappa_{N-1}(\cdot), \ldots, \kappa_1(\cdot)\}$ ($u \in \kappa_{N-i}(x)$ at event (x,i), i.e. at state x, time i), whereas receding horizon control uses the time-invariant control law $\kappa_N(\cdot)$ ($u \in \kappa_N(x)$ at state x).

The solution to $\mathbb{P}_N(x)$ may be obtained as follows. For all $j \in \mathbb{N}_+ := \{1,2,\ldots\}$, j denotes "time-to-go" and problem $\mathbb{P}_j(x)$, the partial return function $V_j^0(\cdot)$ and the controllability set X_j are defined as:

$$V_j^0(x) = \inf_{u\in\mathcal{U}(x)} \sup_{w\in\mathcal{W}(x,u)} \{\ell(x,u) + V_{j-1}^0(f(x,u,w)) \mid f(x,u,\mathcal{W}(x,u)) \subseteq X_{j-1}\},\ \forall x \in X_j \tag{17a}$$

$$\kappa_j(x) = \arg\inf_{u\in\mathcal{U}(x)} \sup_{w\in\mathcal{W}(x,u)} \{\ell(x,u) + V_{j-1}^0(f(x,u,w)) \mid f(x,u,\mathcal{W}(x,u)) \subseteq X_{j-1}\},\ \forall x \in X_j \tag{17b}$$

$$X_j = \{x \mid \exists u \in \mathcal{U}(x) \text{ s.t. } f(x,u,\mathcal{W}(x,u)) \subseteq X_{j-1}\} \tag{17c}$$

with boundary conditions

$$V_0^0(x) = V_f(x), \qquad X_0 = X_f. \tag{17d}$$

The conditions $f(x,u,\mathcal{W}(x,u)) \subseteq X_{j-1}$ *and* $u \in \mathcal{U}(x)$ in (17) may be expressed as

$$(x,u) \in \Sigma_j := \{(x,u) \in \mathcal{Y} \mid f(x,u,w) \in X_{j-1} \text{ for all } w \in \mathcal{W}(x,u)\}, \tag{18}$$

in which case X_j can be interpreted as the projection of the set Σ_j onto X, i.e.

$$X_j = \{x \mid \exists u \text{ such that } (x,u) \in \Sigma_j\}. \tag{19}$$

In order to analyze $\mathbb{P}_N(x)$, it is convenient to introduce the functions $J_j^0(\cdot)$, $j = 1,2,\ldots,N-1$, defined by

$$J_j^0(x,u) := \sup_{w\in\mathcal{W}(x,u)} V_{j-1}^0(f(x,u,w)). \tag{20}$$

Note that we are interested in values of the functions $J_j^0(\cdot)$, $j = 1,2,\ldots,N-1$, and the sets Σ_j, $j = 1,2,\ldots,N-1$. The recursion equations (17a)–(17c) may therefore be rewritten as

$$J_{j-1}^0(x,u) = \sup_w \left\{V_{j-1}^0(f(x,u,w)) \mid w \in \mathcal{W}(x,u)\right\}, \quad \forall (x,u) \in \Sigma_j \tag{21a}$$

$$V_j^0(x) = \inf_u \left\{\ell(x,u) + J_{j-1}^0(x,u) \mid (x,u) \in \Sigma_j\right\}, \quad \forall x \in X_j \tag{21b}$$

$$\kappa_j(x) = \arg\inf_u \left\{\ell(x,u) + J_{j-1}^0(x,u) \mid (x,u) \in \Sigma_j\right\}, \quad \forall x \in X_j \tag{21c}$$

$$\Sigma_j = \{(x,u) \in \mathcal{Y} \mid f(x,u,\mathcal{W}(x,u)) \subseteq X_{j-1}\} \tag{21d}$$

$$X_j = \{x \mid \exists u \text{ s.t. } (x,u) \in \Sigma_j\} \tag{21e}$$

4.1 Prototype Problems

For each j, the dynamic programming recursion requires the solution of two optimization problems (21a) and (21b). These problems are instances of the two prototype problems $\mathbb{P}_{\text{sup}}$ and $\mathbb{P}_{\text{inf}}$ defined below.

$$\mathbb{P}_{\text{sup}}(z): \quad J^0(z) := \sup_w \{J(z,w) \mid (z,w) \in \Gamma\}, \quad \forall z \in \mathcal{Z} \tag{22a}$$

$$\mathbb{P}_{\text{inf}}(x): \quad V^0(x) := \inf_u \{V(x,u) \mid (x,u) \in \mathcal{Z}\}, \quad \forall x \in \mathcal{X} \tag{22b}$$

where

$$z := (x,u), \tag{23a}$$

$$\Gamma := \{(z,w) \mid w \in \mathcal{W}(z)\}, \tag{23b}$$

$$\mathcal{Z} := \{z \in \mathcal{Y} \mid f(z,\mathcal{W}(z)) \subseteq \Omega\}, \tag{23c}$$

$$\mathcal{X} := \{x \mid \exists u \text{ such that } (x,u) \in \mathcal{Z}\}. \tag{23d}$$

Thus, if we identify $J^0(z)$ with $J^0_{j-1}(z)$, $J(z,w)$ with $V^0_{j-1}(f(z,w))$ and Ω with X_{j-1}, problem $\mathbb{P}_{\text{sup}}(z)$ is identical to problem (21a). Similarly, if we identify $V^0(x)$ with $V^0_j(x)$, $V(x,u)$ with $\ell(x,u) + J^0_{j-1}(x,u)$ and $\mathcal{X}$ with X_j, problem $\mathbb{P}_{\text{inf}}(x)$ is identical with (21b). The solution to $\mathbb{P}_{\text{inf}}(x)$, if it exists, is defined as

$$\kappa(x) := \arg\inf_u \{V(x,u) \mid (x,u) \in \mathcal{Z}\}, \quad \forall x \in \mathcal{X}. \tag{24}$$

With the above, we can identify $\kappa(x)$ with $\kappa_j(x)$.

5 Solving The Prototype Problems $\mathbb{P}_{\text{sup}}$ and $\mathbb{P}_{\text{inf}}$

This section contains results that allow the computation of expressions for the functions $J^0(\cdot)$, $V^0(\cdot)$ and $\kappa(\cdot)$. As will be seen, it can easily be shown that these functions are piecewise affine and can be computed, as in [10], by solving a number of suitably-defined *parametric piecewise affine programs* (pPAPs).

5.1 Solution to a Parametric Piecewise Affine Program

In [10], the authors proposed that the solution to a pPAP can be found by comparing the solutions to a finite number of suitably-defined *parametric linear programs* (pLPs). It was later shown in [1], via a numerical example, that this approach can be considerably more efficient compared to computing the solution to a pPAP via parametric *mixed-integer* linear programming (pMILP), as originally proposed in [2]. In this section, we recall the relevant results from [10] for solving a pPAP.

First, we recall a basic result on the nature of the solution to a pLP, where the cost is a linear/affine function of the decision variable y and parameter θ and the constraints on the decision variables and parameters are given by a polytope. The reader is referred to [7] for details of a geometric algorithm for computing the solution to a pLP.

Proposition 1 (Solution to a pLP). *If*

$$\Psi^0(\theta) := \inf_y \{l'\theta + m'y + n \mid (\theta,y) \in C\}, \quad \forall \theta \in \Theta \tag{25a}$$

$$y^0(\theta) := \arg\inf_y \{l'\theta + m'y + n \mid (\theta,y) \in C\}, \quad \forall \theta \in \Theta \tag{25b}$$

where $(l,m,n) \in \mathbb{R}^{n_\theta} \times \mathbb{R}^{n_y} \times \mathbb{R}$*,* C *is a (closed) polyhedron and the (closed) polyhedron*

$$\Theta := \{\theta \mid \exists y \text{ such that } (\theta,y) \in C\}, \tag{26}$$

then $\Psi^0 : \Theta \to \mathbb{R}$ *is a convex, piecewise affine function on a (closed) polyhedral cover of* Θ*. Furthermore, provided* $y^0(\theta)$ *exists for all* $\theta \in \Theta$*, then there exists a continuous, piecewise affine function*[3] $\upsilon : \Theta \to \mathbb{R}^{n_y}$ *on a (closed) polyhedral cover of* Θ *such that* $\upsilon(\theta) \in y^0(\theta)$ *for all* $\theta \in \Theta$*.*

We now recall the following result, which characterizes the solution to a pPAP, where the cost is a piecewise affine function of the decision variables y and parameters θ and polyhedral covers are given for the constraints on the decision variables and parameters. Since the proof is constructive, it is also recalled below.

Proposition 2 (Solution to a pPAP [10]). *Let* $\Psi : D \to \mathbb{R}$*, where* D *is a (closed) polygon, be a piecewise affine function of the form*

$$\Psi(\theta, y) := l_i'\theta + m_i'y + n_i, \quad \forall (\theta, y) \in \mathcal{P}_i^D,\ i \in \mathcal{I}^D, \tag{27}$$

where $\mathcal{P}^D := \left\{\mathcal{P}_i^D \mid i \in \mathcal{I}^D\right\}$ *is a (closed) polyhedral cover of* D *and* $(l_i, m_i, n_i) \in \mathbb{R}^{n_\theta} \times \mathbb{R}^{n_y} \times \mathbb{R}$ *for all* $i \in \mathcal{I}^D$*.*

If $\mathcal{P}^C := \left\{\mathcal{P}_i^C \mid i \in \mathcal{I}^C\right\}$ *is a (closed) polyhedral cover of the (closed) polygon* C*,*

$$\Psi^0(\theta) := \inf_y \left\{\Psi(\theta, y) \mid (\theta, y) \in C\right\}, \quad \forall \theta \in \Theta \tag{28}$$

$$y^0(\theta) := \arg\inf_y \left\{\Psi(\theta, y) \mid (\theta, y) \in C\right\}, \quad \forall \theta \in \Theta \tag{29}$$

where

$$\Theta := \left\{\theta \mid \exists y \text{ such that } (\theta, y) \in C \cap D\right\}, \tag{30}$$

then Θ *is a (closed) polygon and* $\Psi^0 : \Theta \to \mathbb{R}$ *is piecewise affine on a polyhedral cover of* Θ*. Furthermore, provided* $y^0(\theta)$ *exists for all* $\theta \in \Theta$*, then there exists a function* $\upsilon : \Theta \to \mathbb{R}^{n_y}$ *that is piecewise affine on a polyhedral cover of* Θ *such that* $\upsilon(\theta) \in y^0(\theta)$ *for all* $\theta \in \Theta$*.*

Proof. For each $(i, j) \in \mathcal{I}^C \times \mathcal{I}^D$, let $\Theta_{i,j}$ be the orthogonal projection of the (closed) polyhedron $\mathcal{P}_i^C \cap \mathcal{P}_j^D$ onto the θ-space, i.e.

$$\Theta_{i,j} := \left\{\theta \mid \exists y \text{ such that } (\theta, y) \in \mathcal{P}_i^C \cap \mathcal{P}_j^D\right\}, \quad \forall (i, j) \in \mathcal{I}^C \times \mathcal{I}^D. \tag{31}$$

If $\mathcal{P}_i^C \cap \mathcal{P}_j^D$ is non-empty, then $\Theta_{i,j}$ is a (closed) polyhedron, hence $\Theta = \cup_{i,j}\Theta_{i,j}$ is a (closed) polygon.

From Proposition 1 it follows that the function $\Psi_{i,j}^0 : \Theta_{i,j} \to \mathbb{R}$, defined as

$$\Psi_{i,j}^0(\theta) := \inf_y \left\{l_j'\theta + m_j'y + n_j \mid (\theta, y) \in \mathcal{P}_i^C \cap \mathcal{P}_j^D\right\}, \quad \forall \theta \in \Theta_{i,j} \tag{32}$$

is a convex, piecewise affine function on a (closed) polyhedral cover of $\Theta_{i,j}$.

Consider now the index set

$$\mathcal{K}(\theta) := \left\{(i, j) \in \mathcal{I}^C \times \mathcal{I}^D \mid \theta \in \Theta_{i,j}\right\} \tag{33}$$

and note that for all $\theta \in \Theta$,

$$\Psi^0(\theta) = \inf_{y,i} \left\{\Psi(\theta, y) \mid (\theta, y) \in \mathcal{P}_i^C \cap D,\ i \in \mathcal{I}^C\right\} \tag{34a}$$

$$= \inf_{y,i,j} \left\{l_j'\theta + m_j'y + n_j \mid (\theta, y) \in \mathcal{P}_i^C \cap \mathcal{P}_j^D,\ (i, j) \in \mathcal{I}^C \times \mathcal{I}^D\right\} \tag{34b}$$

$$= \inf_{i,j} \left\{\Psi_{i,j}^0(\theta) \mid (i, j) \in \mathcal{K}(\theta)\right\}. \tag{34c}$$

Since $\Psi^0(\cdot)$ is the pointwise-infimum of a finite set of functions $\{\Psi_{i,j}^0(\cdot)\}$, where each $\Psi_{i,j}^0(\cdot)$ is piecewise affine over a polyhedral cover of its domain $\Theta_{i,j}$, it follows that $\Psi^0(\cdot)$ is piecewise affine on a polyhedral cover of Θ.

The claim that there exists a piecewise affine function $\upsilon : \Theta \to \mathbb{R}^{n_y}$ such that $\upsilon(\theta) \in y^0(\theta)$ for all $\theta \in \Theta$, follows from Proposition 1 and the above. ☐

[3]Note that, in general, $y^0(\theta)$ is set-valued for all $\theta \in \Theta$.

Remark 5. We would once again like to emphasize that, unlike the proof of [2, Lem. 1], the above result does *not* require the introduction of integer variables and finding the solution to a pMILP. The result in (34) is based on comparing the solutions to the finite number of pLPs defined by (32). Given the solution to each of the pLPs in (34), an explicit piecewise affine representation for $\Psi^0(\cdot)$ can easily be computed by solving a number of suitably-defined LPs (see, for example, [8, App. A]). Alternatively, one could use so-called *order-region functions* [6] to represent $\Psi^0(\cdot)$.

We now proceed to show how Proposition 2 can be applied in order to solve the prototype problems $\mathbb{P}_{\text{inf}}$ and $\mathbb{P}_{\text{sup}}$, which were defined in Section 4.1.

5.2 The Maximization Subproblem $\mathbb{P}_{\text{sup}}$

This section is concerned with solving the problem

$$\mathbb{P}_{\text{sup}}(z): \quad J^0(z) := \sup_w \{J(z,w) \mid (z,w) \in \Gamma\}, \quad \forall z \in \mathcal{Z}, \tag{35}$$

where

$$z := (x,u), \tag{36a}$$
$$\Gamma := \{(z,w) \mid w \in \mathcal{W}(z)\}, \tag{36b}$$
$$\mathcal{Z} := \{z \in \mathcal{Y} \mid f(z,\mathcal{W}(z)) \subseteq \Omega\}, \tag{36c}$$
$$J(z,w) := V^0(f(z,w)). \tag{36d}$$

Note that we need to compute $\mathcal{Z}$ before proceeding. We recall the following[4], which follows immediately from the main results in [13]:

Proposition 3. *Consider problem $\mathbb{P}_{\text{sup}}$. If **A1**–**A8** hold and Ω is a (closed) polygon, then $\mathcal{Z}$ is a (closed) polygon.*

Let $\mathcal{P}^{\mathcal{S}}$, $\mathcal{P}^{\Gamma}$ and $\mathcal{P}^{\Omega}$ be polyhedral covers for the polygons $\mathcal{S}$, Γ and Ω, respectively. Also, let $V^0(\cdot)$ be piecewise affine over the polyhedral cover $\mathcal{P}^{\Omega}$ of Ω, i.e.

$$V^0(x) := a_i'x + b_i, \quad \forall x \in \mathcal{P}_i^{\Omega},\ i \in \mathcal{I}^{\Omega} \tag{37}$$

where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for all $i \in \mathcal{I}^{\Omega}$.

We would now like to define the function $\Psi : D \to \mathbb{R}$ so that we can apply Proposition 2 in order to characterize the solution to problem $\mathbb{P}_{\text{sup}}$. Clearly, if $\Psi(\theta,y) := -J(\theta,y)$ with the parameter $\theta := z$ and decision variable $y := w$, then $J^0(z) = -\Psi^0(z)$. Note also that $D := f^{-1}(\Omega)$.

Hence, for each $(i,j) \in \mathcal{I}^{\Omega} \times \mathcal{I}^{\mathcal{S}}$, we define the set

$$D_{i,j} := \left\{(z,w) \in \mathcal{P}_j^{\mathcal{S}} \mid f_j(z,w) \in \mathcal{P}_i^{\Omega}\right\} \tag{38}$$

so that $\mathcal{P}^D := \left\{D_{i,j} \neq \emptyset \mid (i,j) \in \mathcal{I}^{\Omega} \times \mathcal{I}^{\mathcal{S}}\right\}$ is a polyhedral cover of D. It then follows that $J(\cdot)$ is piecewise affine over D with

$$J(z,w) = a_i' f_j(z,w) + b_i, \quad \forall (z,w) \in D_{i,j},\ (i,j) \in \mathcal{I}^{\Omega} \times \mathcal{I}^{\mathcal{S}}. \tag{39}$$

By letting $C := \{(z,w) \in \Gamma \mid z \in \mathcal{Z}\}$ and noting that $C \subseteq D$ and $\Theta = \mathcal{Z}$, the next result follows immediately from Propositions 2 and 3:

Corollary 1. *Consider problem $\mathbb{P}_{\text{sup}}$. If **A1**–**A8** hold and $V^0(\cdot)$ is a piecewise affine function over a polyhedral cover of the (closed) polygon Ω, then $J^0(\cdot)$ is a piecewise affine function over a polyhedral cover of the (closed) polygon $\mathcal{Z}$.*

[4] In order to keep the presentation in this paper short, the reader is referred to [13] for a constructive proof.

5.3 The Minimization Subproblem $\mathbb{P}_{\text{inf}}$

This section is concerned with solving the problem

$$\mathbb{P}_{\text{inf}}(x): \quad V^0(x) := \inf_u \{V(x,u) \mid (x,u) \in \mathcal{Z}\}, \quad \forall x \in \mathcal{X}, \tag{40a}$$

$$\mathcal{X} := \{x \mid \exists u \text{ such that } (x,u) \in \mathcal{Z}\}, \tag{40b}$$

$$V(x,u) := \ell(x,u) + J^0(x,u). \tag{40c}$$

where $J^0(\cdot)$ is piecewise affine over a polyhedral cover $\mathcal{P}^{\mathcal{Z}}$ of the polygon $\mathcal{Z}$, i.e.

$$J^0(x,u) := c_i'x + d_i'u + e_i, \quad \forall (x,u) \in \mathcal{P}_i^{\mathcal{Z}}, \ i \in \mathcal{I}^{\Omega} \tag{41}$$

where $c_i \in \mathbb{R}^n$, $d_i \in \mathbb{R}^m$ and $e_i \in \mathbb{R}$ for all $i \in \mathcal{I}^{\mathcal{Z}}$.

We would now like to define the function $\Psi : D \to \mathbb{R}$ so that we can apply Proposition 2 in order to characterize the solution to problem $\mathbb{P}_{\text{inf}}$.

First, note that $D := \mathcal{Z}$ and consider the case when $p = 1$. Following a well-known procedure, we get that

$$V^0(x) := \inf_u \{V(x,u) \mid (x,u) \in \mathcal{Z}\} = \inf_u \left\{\ell(x,u) + J^0(x,u) \mid (x,u) \in \mathcal{Z}\right\} \tag{42a}$$

$$= \inf_u \left\{\|Qx\|_1 + \|Ru\|_1 + J^0(x,u) \mid (x,u) \in \mathcal{Z}\right\} \tag{42b}$$

$$= \inf_{u,\alpha,\beta} \left\{\mathbf{1}'\alpha + \mathbf{1}'\beta + J^0(x,u) \mid -\alpha \le Qx \le \alpha, \ -\beta \le Ru \le \beta, \ (x,u) \in \mathcal{Z}\right\} \tag{42c}$$

$$= \inf_{u,\alpha,\beta} \{\Psi(\theta,y) \mid -\alpha \le Qx \le \alpha, \ -\beta \le Ru \le \beta, \ (x,u) \in \mathcal{Z}, \ \theta := x, \ y := (u,\alpha,\beta)\} \tag{42d}$$

where

$$\Psi(\theta,y) := \mathbf{1}'\alpha + \mathbf{1}'\beta + J^0(x,u), \quad \theta := x, \ y := (u,\alpha,\beta), \tag{43}$$

$\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}^m$ and $\mathbf{1}$ is a column vector of ones of appropriate length. Clearly, $V^0(x) = \Psi^0(x)$.

If $p = \infty$, then we proceed in a similar fashion by noting that

$$V^0(x) := \inf_u \{V(x,u) \mid (x,u) \in \mathcal{Z}\} = \inf_u \left\{\ell(x,u) + J^0(x,u) \mid (x,u) \in \mathcal{Z}\right\} \tag{44a}$$

$$= \inf_u \left\{\|Qx\|_\infty + \|Ru\|_\infty + J^0(x,u) \mid (x,u) \in \mathcal{Z}\right\} \tag{44b}$$

$$= \inf_{u,\alpha,\beta} \left\{\alpha + \beta + J^0(x,u) \mid -\mathbf{1}\alpha \le Qx \le \mathbf{1}\alpha, \ -\mathbf{1}\beta \le Ru \le \mathbf{1}\beta, \ (x,u) \in \mathcal{Z}\right\} \tag{44c}$$

$$= \inf_{u,\alpha,\beta} \{\Psi(\theta,y) \mid -\alpha \le Qx \le \alpha, \ -\beta \le Ru \le \beta, \ (x,u) \in \mathcal{Z}, \ \theta := x, \ y := (u,\alpha,\beta)\} \tag{44d}$$

where

$$\Psi(\theta,y) := \alpha + \beta + J^0(x,u), \quad \theta := x, \ y := (u,\alpha,\beta) \tag{45}$$

$\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. Again, it follows that $V^0(x) = \Psi^0(x)$.

By letting

$$C := \begin{cases} \{(x,u,\alpha,\beta) \mid -\alpha \le Qx \le \alpha, \ -\beta \le Ru \le \beta, \ (x,u) \in \mathcal{Z}\} & \text{if } p = 1 \\ \{(x,u,\alpha,\beta) \mid -\mathbf{1}\alpha \le Qx \le \mathbf{1}\alpha, \ -\mathbf{1}\beta \le Ru \le \mathbf{1}\beta, \ (x,u) \in \mathcal{Z}\} & \text{if } p = \infty \end{cases} \tag{46}$$

and noting that $\Theta = \mathcal{X}$, the next result follows immediately from Propositions 2 and 3:

Corollary 2. *Consider problem $\mathbb{P}_{\text{inf}}$ with the stage cost $\ell(\cdot)$ given by* (6a)*. If **A1**–**A8** hold and $J^0(\cdot)$ is a piecewise affine function over a polyhedral cover of the (closed) polygon $\mathcal{Z}$, then $\mathcal{X}$ is a (closed) polygon and $V^0(\cdot)$ is a piecewise affine function over a polyhedral cover of $\mathcal{X}$. Furthermore, there exists a function $\upsilon : \mathcal{X} \to U$ that is piecewise affine on a polyhedral cover of $\mathcal{X}$ such that $\upsilon(x) \in \kappa(x)$ for all $x \in \mathcal{X}$.*

5.4 The Optimal Control Problems $\mathbb{P}_j$, $j = 1, \ldots, N$

By combining Corollaries 1 and 2, we can now state our main result, which follows by induction:

Theorem 1. *Consider the problems $\mathbb{P}_j$, $j = 1, \ldots, N$, with the stage cost $\ell(\cdot)$ and terminal cost $V_f(\cdot)$ given by (6). If **A1**–**A8** hold, then, for each $j \in \{1, \ldots, N\}$, the sets Σ_j and X_j are closed polygons and the value functions $J^0_{j-1}(\cdot)$ and $V^0_j(\cdot)$ are piecewise affine on, respectively, polyhedral covers of Σ_j and X_j. Furthermore, for each $j \in \{1, \ldots, N\}$, there exists a function $\upsilon_j : X_j \to U$ that is piecewise affine on a polyhedral cover of X_j such that $\upsilon_j(x) \in \kappa_j(x)$ for all $x \in X_j$.*

Remark 6. Note that, in order to initialize the DP recursion, one may have to compute an explicit, piecewise affine expression for $V^0_0(x) := \|Px\|_p$ if $p = 1$ or $p = \infty$.

6 Conclusions

This paper has extended previous results on optimal control of constrained piecewise affine systems by permitting the disturbance constraint set to be dependent on the state and input; previous work dealt with the case when the disturbance constaint set was constant. It was shown that that the value function and optimal control law are piecewise affine on the so-called N-step controllability set, which is a closed polygon.

References

[1] M. Baotić, F. J. Christophersen, and M. Morari. A a new algorithm for constrained finite time optimal control of hybrid systems with a linear performance index. In *Proc. 2003 European Control Conference*, Cambridge, UK, September 2003.

[2] A. Bemporad, F. Borrelli, and M. Morari. Robust model predictive control: Piecewise linear explicit solution. In *Proc. 2001 European Control Conference*, Porto, Portugal, September 2001.

[3] D. P. Bertsekas and I. B. Rhodes. Sufficiently informative functions and the minimax feedback control of uncertain dynamic systems. *IEEE Trans. Automatic Control*, 18:117–123, 1973.

[4] D.P. Bertsekas and I.B. Rhodes. On the minimax reachability of target sets and target tubes. *Automatica*, 7:233–247, 1971.

[5] F. Blanchini. Set invariance in control. *Automatica*, 35(11):1747–1767, 1999. Survey paper.

[6] F. Borrelli, M. Baotic, A. Bemporad, and M. Morari. An efficient algorithm for computing the state feedback optimal control law for discrete time hybrid systems. In *Proc. American Control Conference*, pages 4717–4722, Denver, Colorado, USA, June 2003.

[7] F. Borrelli, A. Bemporad, and M. Morari. Geometric algorithm for multiparametric linear programming. *Journal of Optimization Theory and Applications*, 118(3):515–540, September 2003.

[8] V. Dua and E. N. Pistikopoulos. An algorithm for the solution of multiparametric mixed integer linear programming problems. *Annals of Operations Research*, 99:123–139, 2000.

[9] E.C. Kerrigan, J. Lygeros, and J.M. Maciejowski. A geometric approach to reachability computations for constrained discrete-time systems. In *Proc. 15th IFAC World Congress on Automatic Control*, Barcelona, Spain, July 2002.

[10] E.C. Kerrigan and D.Q. Mayne. Optimal control of constrained, piecewise affine systems with bounded disturbances. In *Proc. 41st IEEE Conference on Decision and Control*, Las Vegas, Nevada, USA, December 2002.

[11] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36:789–814, 2000. Survey paper.

[12] D.Q. Mayne. Control of constrained dynamic systems. *European Journal of Control*, 7:87–99, 2001. Survey paper.

[13] S. V. Raković, E. C. Kerrigan, and D. Q. Mayne. Reachability computations for constrained discrete-time systems with state- and input-dependent disturbances. In *Proc. 42nd IEEE Conference on Decision and Control*, Maui, Hawaii, USA, December 2003.

[14] R. Vidal, S. Schaffert, O. Shakernia, J. Lygeros, and S. Sastry. Decidable and semi-decidable controller synthesis for classes of discrete time hybrid systems. In *Proc. 40th IEEE Conference on Decision and Control*, Orlando, Florida, USA, December 2001.

[15] H. S. Witsenhausen. A minimax control problem for sampled linear systems. *IEEE Trans. Automatic Control*, 13(1):5–21, February 1968.