Stability of Switching Systems: Initialization and Realization

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Abstract

In the design of switching control systems, the analysis of transient signals is of utmost importance. Each time a control transfer takes place, the resulting transient response may degrade performance. When switching takes place rapidly, interaction between switching transients may cause instability even when each of the 'component' loops are stable taken separately.

We examine the issues of controller realization and controller initialization in the context of switching systems. Our objective is to minimize the performance degradation caused by transient signals at controller transitions, while guaranteeing stability under arbitrary switching.

Some theoretical tools are needed to analyze such systems, where the states are permitted to change discontinuously at mode switches. We consider a general Lyapunov function approach to analyze the stability of 'reset switching systems', and use it to devise some LMI methods for synthesizing stabilizing reset schemes.

1 Introduction

When we design ideal linear controllers (without switching), the realization of controllers is a relatively peripheral issue. Similarly the initialization of the controller rarely merits much thought (zero is good enough most of the time). The reason is that, once initial transients have died down, only the input-output transfer functions matter. Furthermore, if the plant state is unknown at the initial time, it may be impossible to compute optimal controller initial states in any case.

In a controller switching context, the issues of realization and initialization are crucialy important. At every controller transition, new transient signals are introduced which are directly related both to the controller realizations and the controller states at switching times. Such transient signals can degrade performance or even cause instability.

It is not difficult to construct examples of switching systems where each component system is stable, yet switching may result in unstable trajectories (see for example [3]). We can also construct such examples in a controller/plant framework. That is, we may switch between stabilizing controllers for a single (linear) plant in such a way that the trajectories become unstable (see example 3.1).

2 Controller Initialization

Suppose we have a family of stabilizing controllers (with given realizations) for a particular linear plant. If we switch between these controllers, what is the correct initial state when we switch to a new controller? Naive approaches such as resetting to zero each time, or having a continuous common controller state (where the controllers all have the same order) may result in very poor performance or, in the worst case, instability.

We will introduce a system of controller resets, where the new controller at each transition is initialized by a function of the plant state (either measured or observed) at that time. We do so in order to minimize (in some sense) the initial state transient introduced at each switch, while guaranteeing stability.

2.1 Single switch

Consider first of all a single switch to a controller K (from another controller, or from manual control) at some time t. If we have a measurement or observation of the plant state $x_G(t)$, then it is a straightforward matter to minimize the initial state transients (in the finite or infinite horizon) with respect to the controller state $x_K(t)$ according to some weighted cost function (see [6] for more details).

For example, suppose the closed loop state space equations can be written

$$\begin{bmatrix} x_G \\ x_K \end{bmatrix} = A_1 x_G + A_2 x_K + Bu$$
$$y = C_1 x_G + C_2 x_K,$$

where y is a generalized output that may include the plant input.

The the minimum initial state transient component of y in the interval $[t, \infty)$ occurs when the function

$$V(t) = \int_t^\infty y^T(\tau) y(\tau) d\tau.$$

achieves a minimum, assuming $u = 0$.

The optimal controller state is then

$$x_K(t) = -P_{22}^{-1} P_{21} x_G(t),$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} > 0$$

is the solution to the Lyapunov equation

$$A^T P + PA = -C^T C,$$

$$A = [A_1 \quad A_2], \quad \text{and} \quad C = [C_1 \quad C_2].$$

That is, the solution is achieved at the minimum of the function

$$V(t) = x^T(t)Px(t)$$

with respect to the controller state $x_K(t)$ (which is a Lyapunov function for the system if $C^T C > 0$).

Similar solutions can also be obtained by optimizing over weighted signals.

2.2 Reset switching systems

If we now consider the arbitrary switching case, stability is clearly a major concern. Since we have seen that the use of controller resets for switching can improve performance, the next question is whether a sensible choice of controller resets can stabilize an otherwise (potentially) unstable switching system.

For the stability analysis we introduce some Lyapunov function results for *reset switching systems*, where the state of the system is permitted to change discontinuously when the system switches.

Consider the family of linear vector fields

$$\dot{x}(t) = A_i x(t), \qquad i \in I, \quad x \in \mathbb{R}^n, \tag{1}$$

where I is some index set (typically discrete valued).

Now define a piecewise constant *switching signal* $\sigma(t)$

$$\sigma(t) = i_k \quad t_k \le t < t_{k+1}, \quad i_k \in I \tag{2}$$

for some sequence of switching times $\{t_k\}$ and indices $\{i_k\}$ ($k \in \mathbb{Z}^+$). We assume that $t_k < t_{k+1}$ and $i_k \neq i_{k+1}$ for all k.

We define a linear *reset switching system* by the equations

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)} x(t), \\ \sigma(t) &= i_k, \qquad \text{for } t_k \le t < t_{k+1}, \quad i_k \in I, \quad k \in \mathbb{Z}^+, \\ x(t_k^+) &= G_{i_{k-1}, i_k} x(t_k^-). \end{aligned} \tag{3}$$

The linear functions $G_{i,j}$ for $i, j \in I$ are *reset relations* between the discrete states i and j.

Note that when each of the reset relations $G_{i,j}$ are identity, the continuous state is constrained to be continuous across switching times. Such systems are extensively analyzed in the literature. See for example [1, 3]. We will use the term *simple switching system* to distinguish such systems from those where reset relations are applied.

We shall choose an index set I such that the family of matrices A_i forms a compact set.

We will denote by S the set of all admissible switching signals σ. We will assume in general that the signals in S are non-zeno: that is, there are at most finitely many transitions in any finite time interval.

Theorem 2.1. *The reset switching system* (3) *is uniformly asymptotically stable for all admissible switching signals $\sigma \in S$ if and only if there exist a family of functions $V_i : \mathbb{R}^n \to \mathbb{R}$ with the following properties:*

- *V_i are positive definite, decrescent and radially unbounded*
- *V_i are continuous and convex*
- *There exist constants c_i such that*

$$\lim_{\Delta t \to 0^+} \left(\frac{V_i(e^{A_i t}x) - V(x)}{\Delta t} \right) \leq -c_i \|x\|^2$$

- *$V_j(G_{i,j}x) \leq V_i(x)$ for all $x \in \mathbb{R}^n$, and $i, j \in I$.*

Proof. (if)

Choose an admissible switching signal $\sigma \in S$. Then, the switching system with resets for this particular signal can be considered to be a linear time varying system with at most finitely many state discontinuities in any finite interval (by the restrictions on S).

If functions V_i exist, satisfying the theorem conditions, then we can construct a time-varying function

$$V_{\sigma(t)}(x(t)) = V_i(x(t)) \text{ when } \sigma(t) = i.$$

Since the functions V_i are decrescent and radially unbounded, we can find a_i and b_i such that

$$a_i \|x\|^2 < V_i(x) < b_i \|x\|^2 .$$

By the third condition on V_i, we know that the V_i are strictly decreasing on trajectories of A_i, and we have the bound

$$\lim_{\Delta t \to 0^+} \left(\frac{V_i(e^{A_i \Delta t}x) - V(x)}{\Delta t} \right) \leq -c_i \|x\|^2 .$$

Furthermore, since $V_j(G_{i,j}x) \leq V_i(x)$, we know that the function $V_{\sigma(t)}(x(t))$ is non-increasing at the (non-zeno) switching times.

Let $a = \inf_i a_i$, $b = \sup_i b_i$, and $c = \inf_i c_i$. By the compactness of the family A_i, a, b and c must be positive and finite. Hence we can write bounds on the function $V_{\sigma(t)}(x(t))$

$$a \left\| x(t) \right\|^2 < V_{\sigma(t)}(x(t)) < b \left\| x(t) \right\|^2 ,$$

and

$$\lim_{\Delta t \to 0^+} \left(\frac{V_{\sigma(t)}(e^{A_i \Delta t} x(t)) - V(x(t))}{\Delta t} \right) \leq -c \left\| x(t) \right\|^2 .$$

Now if we let the initial state be x_0 at time $t = 0$, we have the bound

$$V_{\sigma(t)}(x(t)) < b x_0^2 e^{-\lambda t},$$

where $\lambda = c/a$, and hence

$$\left\| x \right\|^2 < \frac{b x_0^2}{a} e^{-\lambda t}.$$

Therefore, the point $x = 0$ is a uniformly asymptotically stable equilibrium of the switching system with resets.

(only if)

The point $x = 0$ is a uniformly asymptotically stable equillibrium of the switching system with resets. Let $\phi_{\sigma(t)}(t, x_0, t_0)$ denote the state of the switching system at time t given initial conditions x_0 at time t_0 and a particular switching signal σ. Since the set S is time invariant, we may assume without loss of generality that $t_0 = 0$. Since both the vector fields and reset relations are linear, we can write the trajectory for a given initial condition as follows:

$$\phi_{\sigma(t)}(t, x_0, 0) = \Phi_{\sigma(t)}(t) x_0,$$

where $\Phi_{\sigma(t)}(t)$ is a 'composite' state transition matrix defined by

$$\Phi_{\sigma(t)}(t) = e^{A_{i_k}(t - t_k)} G_{i_{k-1}, i_k} \dots e^{A_{i_1}(t_2 - t_1)} G_{i_0, i_1} e^{A_{i_0} t_1}$$

when $t_k < t < t_{k+1}$.

Now let us define the functions V_i as follows

$$\begin{aligned} V_i(x) &= \sup_{\sigma(0)=i} \int_0^\infty \left\| \phi_{\sigma(t)}(t, x, 0) \right\|^2 dt \\ &= \sup_{\sigma(0)=i} x^T \left(\int_0^\infty \Phi_{\sigma(t)}^T(t) \Phi_{\sigma(t)}(t) dt \right) x \end{aligned}$$

That is, $V_i(x)$ is the supremum of the two-norm of trajectories beginning at state x with dynamics i. The integrals exist and are bounded since the equilibrium is asymptotically stable (and hence exponentially stable).

Let

$$Q(\sigma) = \int_0^\infty \Phi_{\sigma(t)}^T(t)\Phi_{\sigma(t)}(t)dt$$

for some σ, and let the set of all such $Q(\sigma)$ with $\sigma(0) = i$ be

$$\mathcal{Q}_i = \{Q(\sigma) : \sigma \in S \text{ with } \sigma(0) = i\}.$$

Now denote the closure of $\mathcal{Q}_i$ by $\bar{\mathcal{Q}}_i$. $\mathcal{Q}_i$ is bounded by the exponential stability of the system, so $\bar{\mathcal{Q}}_i$ is compact. Therefore, we can write

$$V_i(x) = \max_Q \{x^T Q x : Q \in \bar{\mathcal{Q}}_i\}.$$

Each function $x^T Q x$ is a continuous map from $\bar{\mathcal{Q}}_i \times \mathbb{R}^n$ to $\mathbb{R}$, so the maximum must be continuous (however, it is not necessarily differentiable).

We can show that the functions V_i are convex as follows. Let

$$x_\mu = \mu x_1 + (1-\mu)x_0$$

for $\mu \in [0,1]$. Since positive definite quadratic forms are convex, we have for $Q \in \bar{\mathcal{Q}}_i$,

$$x_\mu^T Q x_\mu \le \mu x_1^T Q x_1 + (1-\mu) x_0^T Q x_0.$$

Taking the maximum over $\bar{\mathcal{Q}}_i$, we have

$$V_i(x_\mu) \le \mu V_i(x_1) + (1-\mu)V_i(x_0),$$

hence the functions V_i are convex. Since they are continuous and convex, the V_i are also Lipschitz continuous.

Now we show that the functions V_i must be strictly decreasing on trajectories of the i'th vector field. We can see this as follows:

$$V_i(x) = \sup_{\sigma(0)=i} x^T \left(\int_0^\infty \Phi_{\sigma(t)}^T(t)\Phi_{\sigma(t)}(t)dt \right) x.$$

The supremum must include all switching signals which have $\sigma(t) = i$ for $0 < t < \tau$, so we have

$$\begin{aligned} V_i(x) &\le \int_0^\tau \left\| e^{A_i t} x \right\|^2 dt + \sup_{\sigma(\tau)=i} (e^{A_i t}x)^T \left(\int_\tau^\infty \Phi_{\sigma(t)}^T(t)\Phi_{\sigma(t)}(t)dt \right) e^{A_i t} x \\ &= \int_0^\tau \left\| e^{A_i t} x \right\|^2 dt + V(e^{A_i t}x). \end{aligned}$$

By taking τ small, we have

$$\lim_{\tau \to 0} \frac{V_i(x) - V(e^{A_i t} x)}{\tau} \leq \frac{\|x\|^2}{2}.$$

So V_i is strictly decreasing on the i'th vector field. Note that since V_i is in general a quasi-quadratic function, it is not necessarily continuously differentiable.

From the definition of V_i it is clear that

$$V_j(G_{i,j} x) \leq V_i(x),$$

since the supremum

$$\sup_{\sigma(0)=i} \int_0^\infty \left\| \phi_{\sigma(t)}(t, x, 0) \right\|^2 dt$$

clearly includes all those switching trajectories which begin with an almost immediate switch from i to j.

□

The theorem effectively states that stability of the reset switching system depends upon the existence of a family of Lyapunov functions for the separate vector fields such that at any switch on the switching system, it is guaranteed that the value of the 'new' Lyapunov function after the switch will be no larger than the value of the 'old' function prior to the switch. The functions are not necessarily differentiable everywhere, and so are of similar form to those considered by Molchanov in [5].

Dayawansa and Martin [1] proved that a simple switching system is stable for all switching signals if and only if there exists a common Lyapunov function for the component systems. Our theorem can be considered an extension of that theorem to reset switching systems, and a similar construction is used to prove existence.

The functions V_i are not necessarily quadratic. Indeed, Dayawansa gives an example of a stable two component simple switching system for which no quadratic common Lyapunov function exists. It still however makes sense to first consider quadratic functions in attempting to prove stability of a switching system. We can write the quadratic version of the theorem as the following sufficient condition.

Corollary 2.2. *The reset switching system* (3) *is uniformly asymptotically stable for all admissible switching signals* $\sigma \in S$ *if there exist a family of matrices* $P_i > 0$ *with the following properties:*

- $A_i^T P_i + P_i A_i < 0$
- $G_{i,j}^T P_j G_{i,j} - P_i \leq 0$ *for all* $i, j \in I$.

Proof. The sufficiency part of theorem 2.1 is clearly satisfied if quadratic functions V_i exist which satisfy the conditions. That is, let

$$V_i(x) = x^T P_i x.$$

Then, V_i is positive definite, decrescent and radially unbounded when $P_i > 0$. V_i is strictly decreasing on trajectories of the i'th vector field when $A_i^T P_i + P_i A_i < 0$, and the condition

$$V_j(G_{i,j}x) \leq V_i(x)$$

is satisfied for all $x \in \mathbb{R}^n$ if and only if

$$G_{i,j}^T P_j G_{i,j} - P_i \leq 0.$$

□

2.3 Plant/controller structure

Now we consider a class of resets with a particular structure. We are primarily interested in systems where the component vector fields are made up of plant/controller closed loops. The reset relations we consider then are such that the plant state remains constant across switching boundaries, and the controller state only is reset.

Specifically, we consider a family of N controllers K_i in a switching arrangement such that at each instant one of K_i are in feedback with the plant G.

If G and K have the following state-space representations

$$G = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right] \qquad K_i = \left[\begin{array}{c|c} A_{Ki} & B_{Ki} \\ \hline C_{Ki} & D_{Ki} \end{array}\right], \tag{4}$$

then the closed loop matrices A_i can be written

$$\begin{aligned} A_i &= \begin{bmatrix} A_i(1,1) & A_i(1,2) \\ A_i(2,1) & A_i(2,2) \end{bmatrix} \\ &:= \begin{bmatrix} A_G + B_G D_{Ki} C_G & B_G C_{Ki} \\ -B_{Ki} C_G & A_{Ki} + B_{Ki} D_G C_{Ki} \end{bmatrix}. \end{aligned} \tag{5}$$

The plant state is x_G with dimension n_G, and the controllers K_i have states x_{Ki} with dimensions n_K. For simplicity we restrict consideration to controllers of the same dimension, however the results do in fact hold in general for controllers of different dimensions with relatively straightforward modifications.

We define the current controller state to be

$$x_K(t) = x_{Ki}(t) \text{ when } \sigma(t) = i,$$

and the state of the closed loop system is

$$x = \begin{bmatrix} x_G \\ x_K \end{bmatrix}.$$

Suppose the resets are such that the plant state is continuous, and the controller state is a linear function of plant state. That is, we restrict the matrices $G_{i,j}$ to the form

$$G_{i,j} = \begin{bmatrix} I & 0 \\ X_{i,j} & 0 \end{bmatrix} \tag{6}$$

where $X_{i,j} \in \mathbb{R}^{n_K \times n_G}$.

We now make an important observation.

Remark 2.1. Consider the reset switching system (3), with reset matrices with structure given in (6). If theorem 2.1 is satisfied, then the Lyapunov functions V_i must satisfy the condition

$$\underset{x_K}{\text{argmin}} V_i \left(\begin{bmatrix} x_G \\ x_K \end{bmatrix} \right) = X_{j,i} x_G.$$

Put another way, any family of such resets for which stability is guaranteed must minimize some Lyapunov functions for the respective subsystems.

A further consequence of this observation is that if $G_{i,j}$ are stabilizing resets of the form (6) and the arguments

$$\underset{x_K}{\text{argmin}} V_i \left(\begin{bmatrix} x_G \\ x_K \end{bmatrix} \right)$$

are unique, then the matrices $X_{i,j}$ and hence the $G_{i,j}$ can only depend on the index of the new dynamics j. This makes sense, since the future behaviour of the system is not dependent on the previous values of the switching signal. We will write $X_{i,j} = X_j$, and $G_{i,j} = G_j$ subsequently when appropriate.

Now consider the *potentially stabilizing* resets G_i of the form

$$G_i = \begin{bmatrix} I & 0 \\ X_i & 0 \end{bmatrix}, \tag{7}$$

where $X_i x_G = \underset{x_K}{\text{argmin}} V_i \left(\begin{bmatrix} x_G \\ x_K \end{bmatrix} \right)$, and V_i is a Lyapunov function for the i'th subsystem.

Theorem 2.3. *Consider the reset switching system* (3)*, with reset matrices with structure given in* (7)*. The system is asymptotically stable for all switching signals $\sigma \in S$ if and only if there exists a family of functions $V_i : \mathbb{R}^n \to \mathbb{R}$ with the following properties:*

- *V_i are positive definite, decrescent and radially unbounded*
- *V_i are continuous, with continuous partial dervatives*
- *There exist constants c_i such that*

$$\lim_{\Delta t \to 0^+} \left(\frac{V_i(e^{A_i t}x) - V(x)}{\Delta t} \right) \leq -c_i \left\| x \right\|^2$$

- *V_i are such that*

$$X_i x_G = \underset{x_K}{\operatorname{argmin}} V_i \left(\begin{bmatrix} x_G \\ x_K \end{bmatrix} \right)$$

  for all $x_G \in \mathbb{R}^{n_G}$

- $$V_j \left(\begin{bmatrix} x_G \\ X_j x_G \end{bmatrix} \right) = V_i \left(\begin{bmatrix} x_G \\ X_i x_G \end{bmatrix} \right)$$

  for all $x_G \in \mathbb{R}^{n_G}$ and $i, j \in I$

Proof. (if)

Since $X_i x_G$ minimizes V_i with respect to x_K, we can guarantee that

$$V_j \left(\begin{bmatrix} x_G \\ X_j x_G \end{bmatrix} \right) \leq V_i \left(\begin{bmatrix} x_G \\ x_K \end{bmatrix} \right)$$

when x_K is permitted to vary.

(only if)

Consider the case when the loop i is operating, the plant state x_G is in the set $\partial\Omega_i(k)$, and the controller state is such that $x_K = X_i x_G$ (the minimum of V_i is achieved at x_K). That is,

$$V_i \left(\begin{bmatrix} x_G \\ x_K \end{bmatrix} \right) = k.$$

Now let the loop switch from i to j. From the results of theorem 2.1, we know that the switching system can only be asymptotically stable if

$$V_j \left(\begin{bmatrix} x_G \\ X_j x_G \end{bmatrix} \right) \leq V_i(x)$$

for all $x \in \mathbb{R}^n$, and $i, j \in I$. Thus, it must be true for the specific case when $x_K = X_i x_G$. So we require

$$V_j\left(\begin{bmatrix} x_G \\ X_j x_G \end{bmatrix}\right) \leq V_i\left(\begin{bmatrix} x_G \\ X_i x_G \end{bmatrix}\right).$$

The converse is also true, so we must have

$$V_j\left(\begin{bmatrix} x_G \\ X_j x_G \end{bmatrix}\right) = V_i\left(\begin{bmatrix} x_G \\ X_i x_G \end{bmatrix}\right)$$

for all x_G. □

This theorem says that for reset switching system (of the form (7)) to be asymptotically stable for all switching signals, there must exist Lyapunov functions V_i, such that the level curves have the same projection into the plant subspace.

We now have quite strict conditions which must be met if a reset switching system is to be stable for all admissible signals σ. It is a relatively straightforward matter to test the condition for quadratic Lyapunov functions.

An immediate consequence of the previous theorem is that if the plant is first order, and the family of resets X_i are equivalent to the minimization of quadratic Lyapunov functions for the i'th loop, then stability is automatically guaranteed.

For plants of more than first order, theorem 2.3 is difficult to satisfy for given resets. It does, however lead to a good method for synthesizing stabilizing resets for given systems.

2.4 Reset synthesis for stability

For a set of given controllers, we may ask the question of whether a family of reset relations exist which guarantee asymptotic stability for all switching signals.

We shall call such a family of resets a *stabilizing* family of reset relations.

It is a relatively straightforward matter Computationally, we may easily to perform computations on

While a general search for Lyapunov functions which satisfy theorem 2.3 is a difficult problem, it is relatively straightforward to find quadratic Lyapunov functions, and the corresponding stabilizing resets if they exist.

The aim is to find a set of positive definite matrices

$$P_i = \begin{bmatrix} P_i(1,1) & P_i(1,2) \\ P_i(2,1) & P_i(2,2) \end{bmatrix}$$

such that

$$P_i(1,1) - P_i(1,2)P_i(2,2)^{-1}P_i(2,1) \\ = P_j(1,1) - P_j(1,2)P_j(2,2)^{-1}P_j(2,1)$$

for all $j \neq i$, and that the Lyapunov inequalities

$$A_i^T P_i + P_i A_i < 0$$

are satisfied for all i.

Using Schur complements, we can now form an equivalent problem in terms of matrices Q_i where $Q_i = P_i^{-1}$.

Define

$$\Delta = P_i(1,1) - P_i(1,2)P_i(2,2)^{-1}P_i(2,1).$$

Δ can be thought of as the inverse of the $(1,1)$ block of the inverse of P_i, so the equivalent problem is to find positive definite matrices

$$Q_i = \begin{bmatrix} \Delta^{-1} & Q_i(1,2) \\ Q_i(1,2)^T & Q_i(2,2) \end{bmatrix}$$

satisfying

$$Q_i A_i^T + A_i Q_i < 0$$

Then the required reset relations are

$$x_K = -P_i(2,2)^{-1}P_i(2,1)x_G,$$

where $P_i = Q_i^{-1}$.

Theorem 2.4. *Consider the continuous-time linear plant G, and N controllers K_i defined according to* (4)*, and let the closed loop matrices*

$$A_i = \begin{bmatrix} A_i(1,1) & A_i(1,2) \\ A_i(2,1) & A_i(2,2) \end{bmatrix}$$

be defined according to eqquation (5)*.*

There exists a stabilizing family of reset relations when there exist matrices Δ,
$Q_i(1,2) \in \mathbb{R}^{n_G \times n_K}$ and $Q_i(2,2) \in \mathbb{R}^{n_{Ki} \times n_K}$ for each $i = \{1, \ldots, N\}$ such that the following system of LMIs is satisfied:

$$\begin{bmatrix} \Phi_i(1,1) & \Phi_i(1,2) \\ \Phi_i(2,1) & \Phi_i(2,2) \end{bmatrix} < 0 \qquad (8)$$

where

$$\begin{aligned}\Phi_i(1,1) &= \Delta^{-1}A_i(1,1)^T + Q_i(1,2)A_i(1,2)^T \\ &\quad + A_i(1,1)\Delta^{-1} + A_i(1,2)Q_i(1,2)^T, \\ \Phi_i(1,2) &= \Delta^{-1}A_i(2,1)^T + Q_i(1,2)A_i(2,2)^T \\ &\quad + A_i(1,1)Q_i(1,2) + A_i(1,2)Q_i(2,2), \\ \Phi_i(2,1) &= Q_i(1,2)^T A_i(1,1)^T + Q_i(2,2)A_i(1,2)^T \\ &\quad + A_i(2,1)\Delta^{-1} + A_i(2,2)Q_i(1,2)^T, \\ \Phi_i(2,2) &= Q_i(1,2)^T A_i(2,1)^T + Q_i(2,2)A_i(2,2)^T \\ &\quad + A_i(2,1)Q_i(1,2) + A_i(2,2)Q_i(2,2).\end{aligned}$$

The reset relations guaranteeing stability are

$$x_K = -P_i(2,2)^{-1}P_i(2,1),$$

where

$$P_i = \begin{bmatrix} P_i(1,1) & P_i(1,2) \\ P_i(2,1) & P_i(2,2) \end{bmatrix} = \begin{bmatrix} \Delta^{-1} & Q_i(1,2) \\ Q_i(1,2)^T & Q_i(2,2) \end{bmatrix}^{-1}.$$

Proof. We prove the theorem by attempting to find quadratic functions $V_i(x) = x^T P_i x$, and the corresponding resets $X_i = -P_i(2,2)^{-1}P_i(2,1)$ which satisfy theorem 2.3. Since we consider only quadratic functions, the necessary and sufficient condition becomes only sufficient.

The LMI conditions (8) are simply an expanded version of the Lyapunov inequalities

$$Q_i A_i^T + A_i Q_i < 0$$

where

$$Q_i = \begin{bmatrix} \Delta^{-1} & Q_i(1,2) \\ Q_i(1,2)^T & Q_i(2,2) \end{bmatrix},$$

and

$$A_i = \begin{bmatrix} A_i(1,1) & A_i(1,2) \\ A_i(2,1) & A_i(2,2) \end{bmatrix}.$$

Define $P_i = Q_i^{-1}$. Then the matrices P_i satisfy the Lyapunov inequalitites

$$A_i^T P_i + P_i A_i < 0,$$

and also,

$$P_i(1,1) - P_i(1,2)P_i(2,2)^{-1}P_i(2,1) = \Delta^{-1}$$

for each i. Hence the Lyapunov functions

$$V_i(x) = x^T P_i x$$

satisfy theorem 2.3, and asymptotic stability of the reset switching system is proved with the corresponding resets

$$X_i = -P_i(2,2)^{-1}P_i(2,1).$$

□

For the discrete-time case, we require the following lemma.

Lemma 2.5. *Let $P \in \mathbb{R}^{n\times n}$ be a positive definite matrix, and $A \in \mathbb{R}^{n\times n}$ any real valued matrix. Then the inequality*

$$A^TPA - P < 0$$

holds if and only if the inequality

$$AP^{-1}A^T - P^{-1} < 0$$

also holds.

Proof. Consider the following matrix, decomposed in two alternative ways:

$$\begin{bmatrix} P^{-1} & A \\ A^T & P \end{bmatrix} = \begin{bmatrix} I & 0 \\ A^TP & I \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & P - A^TPA \end{bmatrix} \begin{bmatrix} I & PA \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} I & AP^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} P^{-1} - AP^{-1}A^T & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} I & 0 \\ P^{-1}A^T & I \end{bmatrix}$$

Since both P and P^{-1} are positive definite, then

$$P - A^TPA > 0 \iff P^{-1} - AP^{-1}A^T > 0$$

□

Theorem 2.6. *Consider the discrete-time linear plant G, and N controllers K_i defined according to* (4), *and let the closed loop matrices*

$$A_i = \begin{bmatrix} A_i(1,1) & A_i(1,2) \\ A_i(2,1) & A_i(2,2) \end{bmatrix}$$

be defined according to eqquation (5).

There exists a reset relation for which the switching system is stable for any switching sequence when there exist matrices Δ, $Q_i(1,2) \in \mathbb{R}^{n_G\times n_K}$ and $Q_i(2,2) \in \mathbb{R}^{n_K\times n_K}$ for each $i = \{1,\ldots,N\}$ such that the following system of LMIs is satisfied:

$$\begin{bmatrix} \Phi_i(1,1) & \Phi_i(1,2) \\ \Phi_i(2,1) & \Phi_i(2,2) \end{bmatrix} < 0 \tag{9}$$

where

$$\Phi_1 = A_i(1,1)^T \Delta^{-1} A_i(1,1) + A_i(2,1)^T Q_i(1,2)^T A_i(1,1) + A_i(1,1)^T Q_i(1,2) A_i(2,1) + A_i(2,1)^T Q_i(2,2) A_i(2,1) - \Delta^{-1}$$
$$\Phi_2 = A_i(1,1)^T \Delta^{-1} A_i(1,2) + A_i(2,1)^T Q_i(1,2)^T A_i(1,2) + A_i(1,1)^T Q_i(1,2) A_i(2,2) + A_i(2,1)^T Q_i(2,2) A_i(2,2) - Q_i(1,2)$$
$$\Phi_3 = A_i(1,2)^T \Delta^{-1} A_i(1,1) + A_i(2,2)^T Q_i(1,2)^T A_i(1,1) + A_i(1,2)^T Q_i(1,2) A_i(2,1) + A_i(2,2)^T Q_i(2,2) A_i(2,1) - Q_i(1,2)^T$$
$$\Phi_4 = A_i(1,2)^T \Delta^{-1} A_i(1,2) + A_i(2,2)^T Q_i(1,2)^T A_i(1,2) + A_i(1,2)^T Q_i(1,2) A_i(2,2) + A_i(2,2)^T Q_i(2,2) A_i(2,2) - Q_i(2,2)$$

The reset relations guaranteeing stability are

$$x_K = -P_i(2,2)^{-1} P_i(2,1),$$

where

$$P_i = \begin{bmatrix} P_i(1,1) & P_i(1,2) \\ P_i(2,1) & P_i(2,2) \end{bmatrix} = \begin{bmatrix} \Delta^{-1} & Q_i(1,2) \\ Q_i(1,2)^T & Q_i(2,2) \end{bmatrix}^{-1}.$$

Proof. The LMI conditions (9) are simply an expanded version of the Lyapunov inequalities

$$A_i Q_i A_i^T - Q_i < 0$$

where

$$Q_i = \begin{bmatrix} \Delta^{-1} & Q_i(1,2) \\ Q_i(1,2)^T & Q_i(2,2) \end{bmatrix},$$

and

$$A_i = \begin{bmatrix} A_i(1,1) & A_i(1,2) \\ A_i(2,1) & A_i(2,2) \end{bmatrix}.$$

Define $P_i = Q_i^{-1}$. Then from lemma 2.5, the matrices P_i satisfy the Lyapunov inequalitites

$$A_i^T P_i A_i - P_i < 0,$$

and also

$$P_i(1,1) - P_i(1,2) P_i(2,2)^{-1} P_i(2,1) = \Delta^{-1}$$

for each i. Hence the Lyapunov functions

$$V_i(x) = x^T P_i x$$

satisfy theorem 2.3, and asymptotic stability of the reset switching system is proved with the corresponding resets

$$X_i = -P_i(2,2)^{-1} P_i(2,1).$$

□

It is not always possible to find controller resets which will guarantee stability under arbitrary switching. This is trivially shown by constructing an example of a dynamic plant with two static gain controllers, but where no common Lyapunov function exists. Since the controllers have no state, a reset cannot help! We shall see however that we can always construct a non-minimal realization of the controllers such that stabilizing resets do exist.

The reset results so far depend on precise knowledge of the plant state. In fact the results hold if we reset based on observed plant states, as long as the observer converges (that is, the system is observable).

3 Controller realization

Recent work by Hespanha and Morse [2] considers the problem of selection of appropriate realizations for a family of stabilizing controllers for a particular process. They have shown that it is possible to choose realizations for families of stabilizing controllers such that the (simple) switching system is stable under arbitrary switching. The scheme uses an internal model control arrangement, where the realized controller contains a model of both the plant and the desired closed loop.

We can also realize controllers to guarantee stability by implementing the controllers in a particular coprime factor form.

Suppose we have a plant G, and a set of stabilizing controllers K_i. We may choose a right coprime factorization of the plant $G = NM^{-1}$, and left coprime factorizations of the controllers $K_i = V_i^{-1}U_i$, such that for each i the bezout identity

$$V_iM + U_iN = I$$

is satisfied . Furthermore given any Q such that $Q, Q^{-1} \in \mathscr{RH}_\infty$, the factorizations $G = \tilde{N}\tilde{M}^{-1}$, and $K_i = \tilde{V}_i^{-1}\tilde{U}_i$ also satisfy the bezout identities

$$\tilde{V}_i\tilde{M} + \tilde{U}_i\tilde{N} = I,$$

where $\tilde{N} = NQ, \tilde{M} = MQ, \tilde{U}_i = Q^{-1}U_i$, and $\tilde{V}_i = Q^{-1}V_i$.

A particular choice of Q for a controller factorization can also be thought of as a particular choice for the plant factorization (via Q), or *vice versa*. In the switching controller case, this is true provided that all of the controllers have the same choice of Q.

Now consider the coprime factor switching arrangement in figure 1. The switching connection is such that $u(t) = \hat{u}_{\sigma(t)}$, where $\sigma(t)$ is the switching signal governing the controller selection. The signals u, v, and w are common

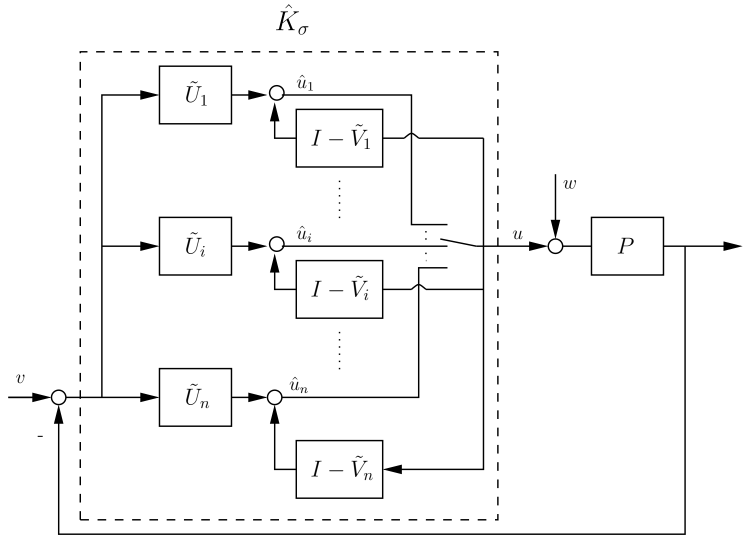


Figure 1: Switching arrangement

to the loops. We can think of this system as a plant P in a feedback loop with the augmented controller $\hat{K}_\sigma$.

Note that for each loop i, we have

$$\begin{aligned}
\hat{u}_i &= (I-\tilde{V}_i)u - \tilde{U}_i P u - \tilde{U}_i P w + \tilde{U}_i v \\
&= (I-\tilde{V}_i - \tilde{U}_i \tilde{N}\tilde{M}^{-1})u - \tilde{U}_i P w + \tilde{U}_i v \\
&= (M - \tilde{V}_i M - \tilde{U}_i \tilde{N})\tilde{M}^{-1} u - \tilde{U}_i P w + \tilde{U}_i v \\
&= (I - \tilde{M}^{-1})u - \tilde{U}_i P w + \tilde{U}_i v.
\end{aligned}$$

Since $u = \hat{u}_\sigma$, we can write

$$\begin{aligned}
u &= (I - \tilde{M}^{-1})u - \tilde{U}_\sigma P w + \tilde{U}_\sigma v \\
&= -\tilde{M}\tilde{U}_\sigma P w + \tilde{M}\tilde{U}_\sigma v \\
&= -\tilde{M}(\tilde{U}_\sigma \tilde{N})\tilde{M}^{-1} w + \tilde{M}\tilde{U}_\sigma v \\
&= -\tilde{M}(I - \tilde{V}_\sigma \tilde{M})\tilde{M}^{-1} w + \tilde{M}\tilde{U}_\sigma v \\
&= -(I - \tilde{M}\tilde{V}_\sigma)w + \tilde{M}\tilde{U}_\sigma v,
\end{aligned}$$

and

$$\begin{aligned}
y &= P(u+w) \\
&= \tilde{N}\tilde{M}^{-1}((-(I - \tilde{M}\tilde{V}_\sigma)w + \tilde{M}\tilde{U}_\sigma v) + w) \\
&= \tilde{N}\tilde{V}_\sigma w + \tilde{N}\tilde{U}_\sigma v.
\end{aligned}$$

We assume that the signals w and v are bounded with bounded two norm, and we know all of the coprime factors are stable. Then the signals $\tilde{V}_\sigma w$, $\tilde{V}_\sigma v$, $\tilde{U}_\sigma w$, and $\tilde{U}_\sigma v$ will all be bounded with bounded two norm. Hence u and y are bounded with bounded two norm, and the switching system is stable for all admissible switching sequences.

We can write these closed-loop relationships in the compact form

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} -(I - \tilde{M}\tilde{V}_\sigma) & \tilde{M}\tilde{U}_\sigma \\ \tilde{N}\tilde{V}_\sigma & \tilde{N}\tilde{U}_\sigma \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix}.$$

The stability of this switching system is guaranteed since $\tilde{M}$, $\tilde{N}$, and each $\tilde{U}_i$ and $\tilde{V}_i$ are stable. Note that the states of the controllers evolve identically irrespective of which controller is active.

This structure is similar to that employed in the work of Miyamoto and Vinnicombe [4] for controllers subject to saturation. In that case, Q may be computed via an $\mathscr{H}_\infty$ optimization without reference to the controller. Hence the same Q may be used to guarantee stability in the switching case.

We may combine the results on controller realization and initialization. The addition of a reset arrangement to a system of controllers realized for stability can result in a substantial performance improvement as the following example shows.

Example 3.1. Take a second order lightly damped plant

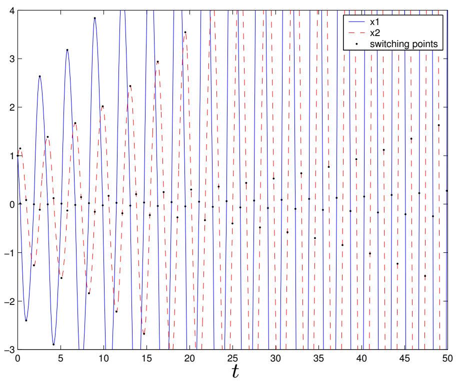
$$P(s) = \frac{1}{s^2 + 0.2s + 1}$$

implemented in controller canonical form

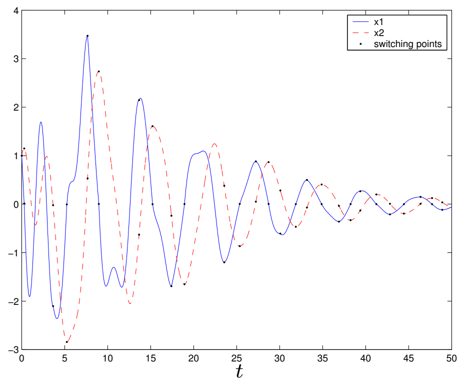
$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -0.2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \end{aligned}$$

and two static stabilizing feedback gains $k_1 = 2$, and $k_2 = 4$. The closed loop equations formed by setting $u = k_1(r - y)$, and $u = k_2(r - y)$ (where r is some reference) are respectively

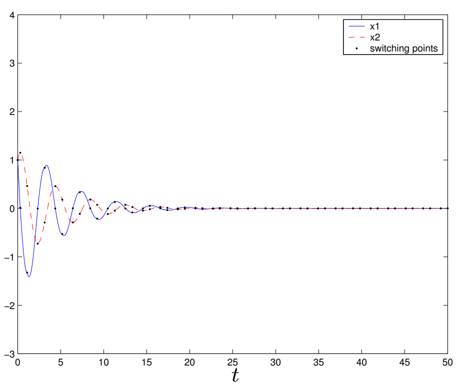
$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -0.2 & -3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \end{aligned}$$



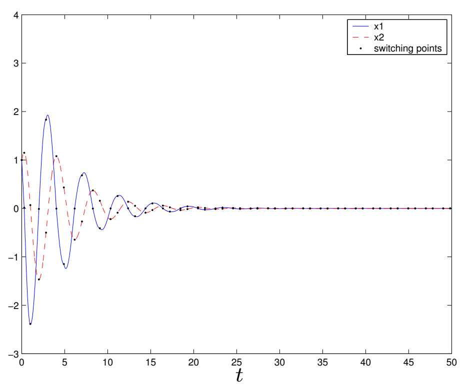
(a) Unstable trajectory



(b) Coprime factor realization



(c) Coprime factor realizations (Lyapunov based reset full state knowledge)



(d) Coprime factor realizations (Lyapunov based reset, state observer)

Figure 2:

and

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-0.2 & -5\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix} r$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}.$$

We shall refer to the respective state-space matrices as A_1, A_2, B and C. It is reasonably straightforward to show that while both A_1 and A_2 have eigenvalues in the left half plane, they do not share a common quadratic Lyapunov function.

The switching system defined by

$$\begin{aligned}\dot{x} &= A_{\sigma(t)}x + Bu\\ y &= Cx\end{aligned}$$

is therefore not guaranteed to be stable for all switching signals $\sigma(t)$. Indeed, we can construct a destabilizing signal by switching from k_1 to k_2 when x_2^2 is a maximum (for that loop), and from k_2 to k_1 when x_1^2 is a maximum. This produces the unstable state trajectories shown in figure 2(a) from an initial state of $x_1 = x_2 = 1$, and zero reference.

Since the controller is static, we obviously cannot improve stability by resetting controller states! We can, however implement the controllers in a non-minimal form, for which stability can be guaranteed. We use here the coprime factor approach.

When we implement these controllers in the arrangement of figure 1 using the same initial condition and switching criterion as before (the non-minimal are initialized to zero), we obtain the stable trajectory shown in figure 2(b). Note however, that the performance is poor and the states take over 50 seconds to converge.

We now apply the results of theorem 8 to the loops formed by these non-minimal controllers. We find that there exist as expected, Lyapunov functions of the respective closed loops with common projection into plant-space. Hence we can find a stabilizing controller reset. This results in the stable trajectory shown in figure 2(c). Note the performance improvement obtained by using the extra freedom in the controller states at the switching times.

Since the reset scheme as applied for figure 2(c) requires full state knowledge, it is not quite a fair comparison with the (non-reset) coprime factor scheme. Therefore, we also implement the results using a plant state observer. The results, shown in figure 2(d) show that while performance is slightly worse than the full-state knowledge case, it is still substantially better than the other schemes.

4 Conclusions

We have introduced a new Lyapunov stability theorem which allows us to analyze stability of switching systems where the state is permitted to reset at switching times. This primarily allows us to examine resets of the controller in controller switching systems.

The theorem has a number of important consequences. Principally, it leads us to a method for synthesizing reset rules for a given switching system, which then guarantee stability under arbitrary switching.

This approach may also be combined with methods for realizing controllers such that stability may be guaranteed for arbitrary switching, and performance substantially improved.

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