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Virtual Reference Feedback Tuning: a direct method for the design of feedback controllers

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Abstract. This paper considers the problem of designing a controller for an unknown plant based on input/output measurements. The new design method we propose is direct (no model identification of the plant is needed) and can be applied using a single set of data generated by the plant, with no need for specific experiments nor iterations. It is shown that the method searches for the global optimum of the design criterion and that, in the case of restricted complexity controller design, the achieved controller is a good approximation of the restricted complexity global optimal controller. A simulation example shows the effectiveness of the method.

1 Introduction

In many practical control applications, a mathematical description of the plant is not available, and the controller has to be designed on the basis of measurements. This problem has attracted the attention of control engineers since the forties with the pioneering work by Ziegler and Nichols [24], which focuses on the design of industrial PID controllers. After [24], many more techniques started to appear, partly as modifications and extensions of the Ziegler and Nichols method, partly as developments in new directions (see e.g. [2, 8, 3, 15, 1]). The main characteristic of these techniques is that they can be easily implemented: simple experiments on the plant are performed and some pre-specified rule is applied to the corresponding outcome.

This paper describes a new controller tuning method called *Virtual Reference Feedback Tuning* (VRFT). Similarly to the above mentioned tuning techniques, VRFT only requires a single experiment on the plant.

Problem formulation

It is assumed that the plant is a *linear* SISO discrete-time dynamical system described by the rational transfer function $P(z)$. Such a transfer function is *unknown* and a set of I/O data, collected during an experiment on the plant, is available for design purposes.

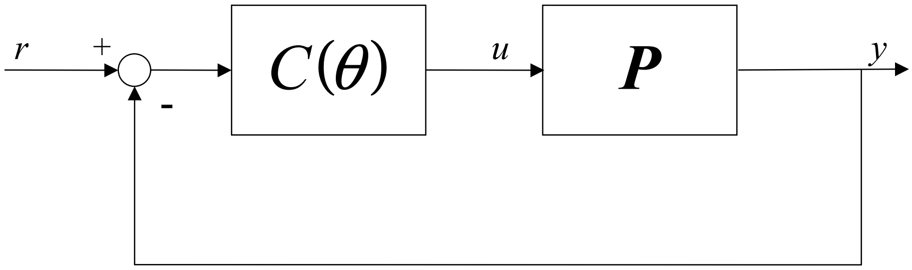


Figure 1: The control system

The control specifications are assigned via a reference model $M(z)$. This describes the desired transfer function of the closed-loop system (see Fig.(1)). Attention is restricted to controllers which linearly depend on the parameter vector, namely the controller class $\{C(z;\theta)\}$ takes the form $C(z;\theta) = \beta^T(z)\theta$, where $\beta(z) = [\beta_1(z)\ \beta_2(z) \cdots \beta_n(z)]^T$ is a known vector of linear discrete-time transfer functions, and $\theta = [\vartheta_1\ \vartheta_2 \cdots \vartheta_n]^T$ is the n-dimensional vector of parameters.
The control objective is the minimization of the following model-reference criterion:

$$J_{MR}(\theta) = \left\| \left(\frac{P(z)C(z;\theta)}{1+P(z)C(z;\theta)} - M(z) \right) W(z) \right\|_2^2, \tag{1}$$

where $W(z)$ is a weighting function chosen by the user.

Motivation and Original Contributions

VRFT builds on a general idea for controller selection originally proposed in [6]. In [6] only the bare idea was set up, with no concern for fundamental implementation and performance issues. The aim of this paper is to develop a complete ready-to-use technique for the controller design. This will be done by addressing two crucial issues:

- the design of a *pre-filter* of the data, in order to generate a controller that minimizes (1);
- the treatment of data affected by *noise*.

Comparison with other methods

By comparing VRFT with the tuning rules presented in [24, 2, 8, 3, 15, 1], it can be noted that the control problem addressed by VRFT is a model reference control problem, where the user can specify his control objectives by a suitable selection of a reference model $M(z)$. This is a nontrivial advantage over existing methods for the tuning of industrial controllers, where the control specifications are given empirically, or assigned in a very simple and limited fashion. Moreover, the range of applicability of VRFT is not restricted to PID controllers.
It is worth noticing that VRFT has some similarities with some iterative schemes for the controller design of unknown plants recently developed in the literature (see e.g. [20, 5, 11, 9, 21, 23, 4, 10]). Among them, the innovative Iterative Feedback Tuning (IFT) method proposed by Hjalmarsson and coauthors ([11, 10]) shares with VRFT the characteristic of being *direct*; namely both VRFT and IFT do not make use of any intermediate model construction step, and point directly to the

controller selection. Using a very smart idea for reconstructing the cost criterion gradient from data, IFT performs a fine tuning of the controller through an iterative procedure. In contrast, VRFT is not iterative and its main characteristic is its ease of use. On the other hand, VRFT is suboptimal for restricted controller classes. As we can see, IFT and VRFT are in a sense complementary methods with their own area of applicability.

Outline of the paper

The structure of the paper is as follows. In Section 2 the Virtual Reference idea is introduced. Starting from this basic idea, the VRFT technique is developed by addressing two main issues: the problem of selecting a suitable pre-filter in order to obtain a controller that minimizes (1) (Section 3), and the problem of dealing with noise (Section 4). A simulation example ends the paper.

2 The Virtual Reference Framework

In this section the Virtual Reference framework, as it was originally introduced in [6], is briefly recalled. A similar idea, though in a special setting, has also been used in the context of control with neural networks, see [17]. The Virtual Reference approach has been applied in a nonlinear setting in [18, 19, 16, 7].

The basic idea

Suppose that a controller $C(z;\theta)$ results in a closed-loop system whose transfer function is $M(z)$. Then, if the closed-loop system is fed by *any* reference signal $r(t)$, its output equals $M(z)r(t)$. Hence, a necessary condition for the closed-loop system to have the same transfer function as the reference model is that the output of the two systems is the same for a *given* $\bar{r}(t)$.

Standard model reference design methods try to impose such a necessary condition by first selecting a reference $\bar{r}(t)$ and then by choosing $C(z;\theta)$ such that the condition is satisfied. However, for a general selection of $\bar{r}(t)$, the above task is difficult to accomplish if a model of the plant is not available. The basic idea of the Virtual Reference approach is to perform a wise selection of $\bar{r}(t)$ so that the determination of the controller becomes easy.

Suppose that we have in our hands two files collected from the plant, one containing u measurements and the other one the corresponding output y (how these files have been generated is immaterial for the discussion to come. Suppose, however, that the system is noise-free. This is for ease of explanation and the noisy case will be dealt with in Section 4 of this paper). Given the measured $y(t)$, consider a reference $\bar{r}(t)$ such that $M(z)\bar{r}(t) = y(t)$, where $M(z)$ is the desired reference model for the closed-loop system we wish to design. Such a reference is called "virtual" because it was not used to generate $y(t)$. Notice that $y(t)$ is the desired output of the closed-loop system when the reference signal is $\bar{r}(t)$. Then, compute the corresponding *tracking error* $e(t) = \bar{r}(t) - y(t)$. Even though plant $P(z)$ is not known, we know that when $P(z)$ is fed by $u(t)$ (the actually measured input signal), it generates $y(t)$ as an output. Therefore, a good controller is one that generates $u(t)$ when fed by $e(t)$. The idea is then to search for such a controller. Since both signals $u(t)$ and $e(t)$ are known, this task reduces to the *identification problem* of describing the dynamical relationship between $e(t)$ and $u(t)$. □

The above idea can be implemented by the following 3-step algorithm (where a filtering of data through a user-chosen filter $L(z)$ is also considered). It represents the bulk of the VRFT method.

Given a set of measured I/O data $\{u(t), y(t), t = 1, \ldots, N\}$, do the following:

1. calculate:
   - a virtual reference $\bar{r}(t)$ such that $y(t) = M(z)\bar{r}(t)$, and
   - the corresponding tracking error $e(t) = \bar{r}(t) - y(t)$ (we assume $M(z) \neq 1$, otherwise $e(t) = 0$);
2. filter the signals $e(t)$ and $u(t)$ with a suitable filter $L(z)$:
$$e_L(t) = L(z)e(t), \qquad u_L(t) = L(z)u(t);$$
3. select the controller parameter vector, say $\hat{\theta}_N$, that minimizes the following criterion
$$J_{VR}^N(\theta) = \frac{1}{N}\sum_{t=1}^{N}\left(u_L(t) - C(z;\theta)e_L(t)\right)^2. \qquad (2)$$
   Note that when $C(z;\theta) = \beta^T(z)\theta$, the criterion (2) can be given the form
$$J_{VR}^N(\theta) = \frac{1}{N}\sum_{t=1}^{N}(u_L(t) - \varphi_L^T(t)\theta)^2, \qquad \varphi_L(t) = \beta(z)e_L(t), \qquad (3)$$
   and the parameter vector $\hat{\theta}_N$ is given by
$$\hat{\theta}_N = \left[\sum_{t=1}^{N}\varphi_L(t)\varphi_L(t)^T\right]^{-1}\sum_{t=1}^{N}\varphi_L(t)u_L(t).$$

In the next section we show that, by a suitable selection of the pre-filter $L(z)$, the controller $C(z;\hat{\theta}_N)$ is nearly-optimal for the cost criterion (1) and it is in fact optimal provided that the selected controller class contains a controller that gives perfect matching between the closed-loop transfer fucntion and $M(z)$.

3 Shaping the filter

Consider the performance index $J_{MR}(\theta)$ of the model reference control problem (eqn.(1)) and the criterion of the Virtual Reference approach (eqn.(2)): they look different. In this section it will be shown that their minimum arguments can in fact be made close to each other by a suitable selection of the filter $L(z)$. In this way, the Virtual Reference approach can be used to solve the model reference control problem stated in the introduction.

It is important to note that in the derivations below we do not make the assumption that a controller exists in the controller class that leads to perfect matching. This would be unrealistic. As for the presence of noise, in this section we assume that $u(t)$ and $y(t)$ are noise-free. This is for ease of explanation and the presence of noise will be treated in the next section.

The choice of the filter

To start with, note that, using the definition of 2-norm of a discrete-time linear transfer function, $J_{MR}(\theta)$ can be written as:

$$J_{MR}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{P(e^{j\omega})C(e^{j\omega};\theta)}{1+P(e^{j\omega})C(e^{j\omega};\theta)} - M(e^{j\omega})\right|^2 |W(e^{j\omega})|^2 d\omega,$$

or, more compactly, by dropping the argument $e^{j\omega}$:

$$J_{MR}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{PC(\theta)}{1+PC(\theta)} - M\right|^2 |W|^2 d\omega. \tag{4}$$

Introduce now the rational function $C_0(z)$ which exactly solves the model-matching problem, namely $C_0(z)$ is such that ($C_0(z)$ exists because $M(z) \neq 1$)

$$\frac{P(z)C_0(z)}{1+P(z)C_0(z)} = M(z). \tag{5}$$

Note that, in general, $C_0(z)$ does not belong to the family of parameterized controllers $\{C(z;\theta)\}$ and, even more so, it need not be a proper rational function. Such a $C_0(z)$ is only used for analysis purposes.

Using $C_0(z)$, after some manipulations, the performance index (4) can be rewritten as:

$$J_{MR}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|P|^2|W|^2}{|1+PC(\theta)|^2}\frac{|C(\theta)-C_0|^2}{|1+PC_0|^2}d\omega. \tag{6}$$

Consider now the criterion $J_{VR}^N(\theta)$. It is well-known that, if the measured signals $u(t)$ and $y(t)$ can be considered realizations of stationary and ergodic stochastic processes, when the number of available data grows ($N \to \infty$), the following holds:

$$J_{VR}^N(\theta) \longrightarrow J_{VR}(\theta) = E[(u_L(t) - C(z;\theta)e_L(t))^2]. \tag{7}$$

$J_{VR}(\theta)$ is the *asymptotic* counterpart of $J_{VR}^N(\theta)$. Accordingly, as $N \to \infty$, the minimum $\hat{\theta}_N$ of $J_{VR}^N(\theta)$ will converge to the minimum of $J_{VR}(\theta)$, say $\hat{\theta}$. In the rest of the paper, for analysis purposes, $J_{VR}(\theta)$ will be used extensively in place of $J_{VR}^N(\theta)$.

Using the definitions of $u_L(t)$ and $e_L(t)$ given in the previous section, the definition of $C_0(z)$ in (5), and the Parseval theorem (see e.g.[14]), the asymptotic criterion (7) can be given the following frequency domain representation:

$$J_{VR}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}|P|^2|C(\theta)-C_0|^2|1-M|^2\frac{|L|^2}{|M|^2}\Phi_u d\omega, \tag{8}$$

where Φ_u is the spectral density of $u(t)$.

Comparing $J_{MR}(\theta)$ and $J_{VR}(\theta)$ (eqn.(6) and eqn.(8), respectively), the following observation can be made:

- If $C_0(z) \in \{C(z;\theta)\}$ and $J_{VR}(\theta)$ has a unique minimum, minimizing $J_{VR}(\theta)$ gives $C_0(z)$, no matter what $L(z)$ is.

- Suppose instead that $C_0(z) \notin \{C(z;\theta)\}$. If the following identity holds

$$|L|^2 = \frac{|M|^2|W|^2}{|1+PC(\theta)|^2}\frac{1}{\Phi_u}, \qquad \forall\omega \in [-\pi;\pi], \tag{9}$$

  then $J_{VR}(\theta) = J_{MR}(\theta)$. As a consequence, minimizing $J_{VR}(\theta)$ is the same as minimizing $J_{MR}(\theta)$.

Clearly, the choice of the filter $L(z)$ suggested by equation (9) is not feasible since $P(z)$ is not known and it also depends on θ. Here, the following choice of $L(z)$ is instead proposed:

- Select $L(z)$ such that

$$|L|^2 = |1-M|^2|M|^2|W|^2\frac{1}{\Phi_u}, \qquad \forall\omega \in [-\pi;\pi]. \tag{10}$$

First notice that all quantities in the right-hand-side of equation (10) are known and therefore $L(z)$ can be actually computed (in fact Φ_u can be considered known only when the input signal has been selected by the designer. In other situations, Φ_u needs to be estimated). Moreover, it is readily seen that expression (10) is equivalent to

$$|L|^2 = \frac{|M|^2|W|^2}{|1+PC_0|^2}\frac{1}{\Phi_u}, \qquad \forall\omega \in [-\pi;\pi].$$

Hence, choice (10) corresponds to substituting $|1+PC(\theta)|^2$ with $|1+PC_0|^2$ in equation (9), which appears to be a sensible selection since we expect that $|1+PC(\theta)|^2 \approx |1+PC_0|^2$ for $\theta = \bar{\theta}$, where $\bar{\theta}$ is the minimum of $J_{MR}(\theta)$. In Proposition 1 below, we show that choice (10) is in fact optimal in a sense precisely stated in the Proposition.
Remark. One should note that the analysis is based on asymptotic results. Should the signals be poorly exciting over certain frequency ranges of interest, the asymptotic results would start to hold for a very large amount of data points.

Analysis of the proposed filter
Set $\Delta C(z) = C_0(z) - \beta^T(z)\bar{\theta}$, where $\bar{\theta}$ is the parameter vector which minimizes $J_{MR}(\theta)$. Note that $\Delta C(z)$ is the part of $C_0(z)$ which cannot be explained by the chosen family of controllers. Obviously, if $C_0(z) \in \{C(z;\theta)\}$, then $\Delta C(z) = 0$.
Introduce now the following extended vector of transfer functions

$$\beta^+(z) = [\beta_1(z) \quad \beta_2(z) \quad \ldots \quad \beta_n(z) \quad \Delta C(z)]^T,$$

and the following extended parameter vector

$$\theta^+ = [\vartheta_1 \quad \vartheta_2 \quad \ldots \quad \vartheta_n \quad \vartheta_{n+1}]^T.$$

Then, define an extended family of controllers $C^+(z;\theta^+) = \beta^+(z)^T\theta^+$. Clearly, $C_0(z) \in \{C^+(z;\theta^+)\}$ with $\bar{\theta}^+ = [\bar{\theta}^T \quad 1]^T$. Finally, consider the extended performance index

$$J^+_{MR}(\theta^+) = \left\| \left(\frac{P(z)C^+(z;\theta^+)}{1 + P(z)C^+(z;\theta^+)} - M(z) \right) W(z) \right\|_2^2 .$$

Note that the difference between $J_{MR}(\theta)$ and $J^+_{MR}(\theta^+)$ is that the latter is parameterized by the family of extended controllers $\{C^+(z;\theta^+)\}$. The second order Taylor expansion around its global minimizer $\bar{\theta}^+$ is denoted by $\bar{J}^+_{MR}(\theta^+)$, namely:

$$J^+_{MR}(\theta^+) = \bar{J}^+_{MR}(\theta^+) + o(\|\theta^+ - \bar{\theta}^+\|_2^2).$$

We have now the following result.

Proposition 1. The parameter vector $\bar{\theta}$ which minimizes the performance index $J_{MR}(\theta)$, and the parameter vector $\hat{\theta}$ which minimizes $J_{VR}(\theta)$ when $L(z)$ is selected according to (10) are such that:

$$\bar{\theta} = \arg\min_{\theta} J^+_{MR}([\theta^T \quad 0]^T). \tag{11}$$

$$\hat{\theta} = \arg\min_{\theta} \bar{J}^+_{MR}([\theta^T \quad 0]^T). \tag{12}$$

proof: *see the Appendix.* □

The above result is interesting since it provides a formal relationship between the parameter vector $\hat{\theta}$ obtained using the Virtual Reference approach and the "optimal" parameter vector $\bar{\theta}$, which minimizes the original performance index $J_{MR}(\theta)$. In particular, Proposition 1 states that $\bar{\theta}$ minimizes the restriction - over the n-dimensional vector space of the first n elements of θ^+ - of $J^+_{MR}(\theta^+)$, whereas $\hat{\theta}$ minimizes the restriction over the same vector space of the 2-nd order expansion (around $\bar{\theta}^+$) of the extended performance index $J^+_{MR}(\theta^+)$. The reason why this is so is that, by the choice (10), $J_{VR}(\theta)$ is in fact equal to $\bar{J}^+_{MR}([\theta^T \quad 0]^T)$. Based on this result, we conclude that if the transfer function $\Delta C(z)$ plays a marginal role in determining $C_0(z)$, namely the family of controllers $\{C(z;\theta)\}$ is only slightly under-parameterized, then $C(z;\hat{\theta})$ is a good approximation to $C(z;\bar{\theta})$ since $J^+_{MR}(\theta^+)$ is well approximated in a neighborhood of its minimum by its second order expansion $\bar{J}^+_{MR}(\theta^+)$.

Example 1. Consider a plant, a reference model, and a family of controllers characterized by the following transfer functions:

$$P(z) = \frac{z^{-1}}{1 - 0.6z^{-1}}, \qquad M(z) = \frac{0.6z^{-1}}{1 - 0.4z^{-1}}, \qquad C(z;\theta) = \frac{\theta}{1 - z^{-1}}.$$

Notice that $C(z;\theta)$ can be rewritten as $C(z;\theta) = \beta(z)^T\theta$, where $\beta(z) = 1/(1 - z^{-1})$. The "ideal" controller which exactly solves the model-matching problem is given by

$$C_0(z) = \frac{0.6 - 0.36z^{-1}}{1 - z^{-1}}$$

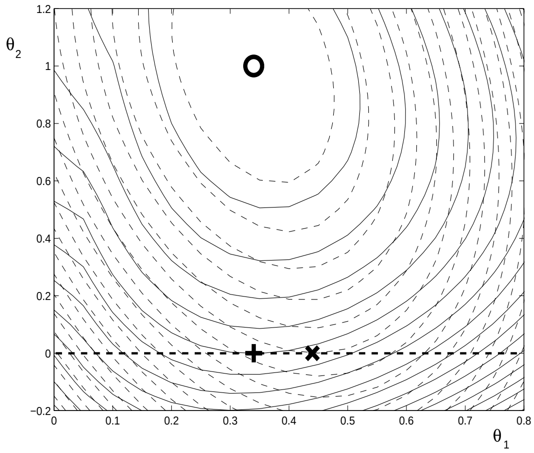


Figure 2: Contour plots of $J^+_{MR}(\theta^+)$ (continuous line) and $\bar{J}^+_{MR}(\theta^+)$ (dashed line); (**O**)=$\bar{\theta}^+$; (**X**)=$[\hat{\theta} \quad 0]$; (+)=$[\bar{\theta} \quad 0]$.

(note that $C_0(z) \notin \{C(z;\theta)\}$). The global minimum of the model-reference criterion $J_{MR}(\theta)$ is achieved by $\bar{\theta} = 0.34$, whereas the global minimum of $J_{VR}(\theta)$ (when the filter is chosen as in (10)) is achieved by $\hat{\theta} = 0.44$. Correspondingly, $J_{MR}(\bar{\theta}) = 1.35$, and $J_{MR}(\hat{\theta}) = 1.45$. Using the definitions introduced above, the extended controller class is given by $C^+(z;\theta^+) = \beta^+(z)^T\theta^+$, where

$$\beta^+(z) = \left[\frac{1}{1-z^{-1}} \quad \frac{0.26 - 0.36z^{-1}}{1-z^{-1}} \right]^T, \qquad \theta^+ = [\vartheta_1 \quad \vartheta_2]^T,$$

and $C_0(z) = \beta^+(z)^T\bar{\theta}^+$, with $\bar{\theta}^+ = [\bar{\theta}^T \quad 1]^T = [0.34 \quad 1]$. A graphical interpretation of the results is given in Fig.(2), where the contour plots of $J^+_{MR}(\theta^+)$ and $\bar{J}^+_{MR}(\theta^+)$ are displayed. As stated in Proposition 1, it is apparent that $\bar{\theta}$ is the minimum of the extended performance index restricted to $\vartheta_2 = 0$, whereas $\hat{\theta}$ is the minimum of the quadratic approximation around $\bar{\theta}^+$ of the extended performance index restricted to $\vartheta_2 = 0$. □

4 The use of noisy data

In this section we discuss the behavior of the VRFT method when the plant output $y(t)$ is affected by additive noise $d(t)$, namely

$$\tilde{y}(t) = P(z)u(t) + d(t).$$

We make the assumption that the processes $u(\cdot)$ and $\xi(\cdot)$ are *uncorrelated*, namely that the data are collected when the plant is working in open-loop configuration. Closed-loop data collection is not dealt with in detail in this paper for space limitations; however, extending the presented ideas to a closed-loop setting is straightforward and briefly discussed at the end of the present section. If the Virtual Reference algorithm of Section 2 is applied to the data-set $\{u(t), \tilde{y}(t)\}_{t=1,..,N}$, one obtains a biased parameter vector and this results in a significant deterioration of the performance.

This can be easily understood by inspecting the frequency-domain expression of the asymptotic criterion $J_{VR}(\theta)$, when using noise-free and noisy data:

- Asymptotic criterion using noise-free data $\{u(t), y(t)\}_{t=1,..,N}$:

$$J_{VR}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |P|^2 |C(\theta) - C_0|^2 |1 - M|^2 \frac{|L|^2}{|M|^2}\Phi_u d\omega. \tag{13}$$

- Asymptotic criterion using noisy data $\{u(t), \tilde{y}(t)\}_{t=1,..,N}$:

$$J_{VR}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[|P|^2 |C(\theta) - C_0|^2 |1 - M|^2 \frac{|L|^2}{|M|^2}\Phi_u + \frac{|C(\theta)|^2}{|P|^2|C_0|^2}|L|^2\Phi_d \right] d\omega \tag{14}$$

  (Φ_d is the spectral density of the noise).

Apparently, the minima of (13) and (14) are different because the term due to $d(\cdot)$ in (14) depends on θ.

In the following, we propose the use of an Instrumental Variable method to counteract the effect of noise ([14]).

Introduce the symbol

$$\tilde{\varphi}_L(t) = \beta(z)L(z)(M(z)^{-1} - 1)\tilde{y}(t),$$

which denotes the regressors when the system is affected by noise (compare with (3)). Letting $\zeta(t)$ be the instrumental variable, the parameter is estimated according to equation

$$\hat{\theta}_N^{IV} = \left[\sum_{t=1}^{N} \zeta(t)\tilde{\varphi}_L(t)^T\right]^{-1} \left[\sum_{t=1}^{N} \zeta(t)u_L(t)\right]. \tag{15}$$

Choice of the instrumental variables

We propose two different choices for the instrumental variables. The first one guarantees that asymptotically $\hat{\theta}^{IV} = \hat{\theta}$. However an additional experiment on the plant is required. The second one does not guarantee that $\hat{\theta}^{IV} = \hat{\theta}$ rigorously, but the residual error is expected to be small. It does not require an additional experiment on the plant.

The proposed choices are as follows:

- **Repeated experiment**

  Perform a second experiment on the plant using the same input $\{u(t)\}_{t=1,..,N}$ and collect the corresponding output sequence $\{\tilde{y}(t)'\}_{t=1,..,N}$. Then construct the instrumental variables as:

$$\zeta(t) = \beta(z)L(z)(M(z)^{-1} - 1)\tilde{y}'(t). \tag{16}$$

  Notice that $\{\tilde{y}(t)'\}_{t=1,..,N}$ will be different from $\{\tilde{y}(t)\}_{t=1,..,N}$ since the two sequences are affected by two different realizations of the noise in the two experiments. If we assume, as it is reasonable, that the noise signals in the two experiments are uncorrelated, then, asymptotically, (15) gives $\hat{\theta}$, the same result as in the noiseless case.

- **Identification of the plant**

  Identify a model $\hat{P}(z)$ of the plant from the set of data $\{u(t), \tilde{y}(t)\}_{t=1,..,N}$ and generate the simulated output $\hat{y}(t) = \hat{P}(z)u(t)$. Then construct the instrumental variables as:

$$\zeta(t) = \beta(z)L(z)(M(z)^{-1} - 1)\hat{y}(t). \tag{17}$$

  The identification of the plant is a standard open-loop identification problem. The model $\hat{P}(z)$ can be estimated using different techniques among which a high-order ARX model ([14]), or a high-order state space model ([22]).
  Due to the possible inaccuracy of the estimated $\hat{P}(z)$, this second method does not guarantee that the estimate asymptotically tends precisely to $\hat{\theta}$.

The following remarks are in order.
- Using (16) is possible only if two independent experiments characterized by the same input signal can be made on the plant.
- When (17) is used, strictly speaking, we can no longer claim that the method is fully direct since $\hat{P}(z)$ has to be estimated. However, it is important to note that the estimated plant is used with the only objective of generating an instrumental variable signal and its actual expression is not directly used to design the controller. This in particular implies that a high order model can be used in the identification of $P(z)$ without affecting the controller complexity.

We conclude this section by summarizing the complete VRFT algorithm in the case when choice (17) is made.

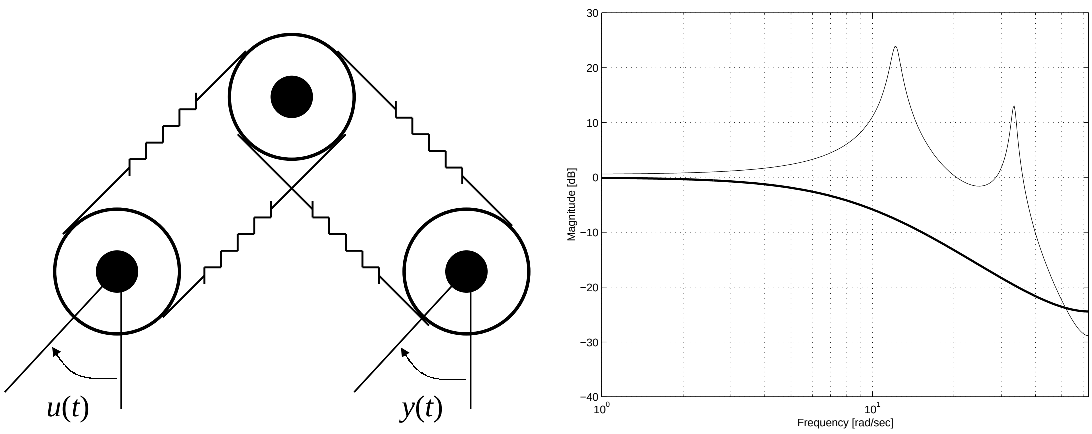
VRFT ALGORITHM

- Set $L(z) = M(z)(1 - M(z))U(z)^{-1}W(z)$, where $U(z)$ is such that $|U(e^{j\omega})|^2 = \Phi_u(\omega)$.
- Compute $u_L(t)$ as: $u_L(t) = L(z)u(t)$.
- Compute $\tilde{\varphi}_L(t)$ as: $\tilde{\varphi}_L(t) = \beta(z)L(z)(M(z)^{-1} - 1)\tilde{y}(t)$.
- Identify a high-order model $\hat{P}(z)$ from $\{u(t)\}_{t=1,..,N}$ to $\{\tilde{y}(t)\}_{t=1,..,N}$.
- Compute $\zeta(t)$ as: $\zeta(t) = \beta(z)L(z)(M(z)^{-1} - 1)\hat{P}(z)u(t)$.
- Compute the parameter vector of the controller as:

$$\hat{\theta}_N^{IV} = \left[\sum_{t=1}^{N} \zeta(t)\tilde{\varphi}_L(t)^T\right]^{-1} \sum_{t=1}^{N} \zeta(t)u_L(t).$$

Closed-loop noisy data

The VRFT method can be successfully applied to data collected in closed-loop as well. An extended discussion on the use of closed-loop data goes beyond the scope of this paper. Here, it suffices to say that the above procedure can still be used by replacing the identification step of $\hat{P}(z)$ with the



(a) (b)

Figure 3: **(a)** Layout of the flexible transmission; **(b)** Bode magnitude plots: the plant (thin line) and the reference-model (bold line).

identification of the complementary sensitivity of the closed-loop system. The reader is referred to [13] for details and comments.

5 A simulation example

In order to better illustrate the main features of the VRFT technique, a simulation example is now presented.

The plant we consider is the flexible transmission system proposed in [12] as a benchmark for digital control design. Here the unloaded case is considered. The flexible transmission consists of three horizontal pulleys connected by two elastic belts (see Fig.(3.a)). The system input is the angular position of the first pulley; the system output is the angular position of the third pulley. The control objective is to make the angular position of the third pulley as close as possible, over a suitable bandwidth, to the angular position of the first one.

The input-output dynamic behavior of the plant can be described by the following discrete-time linear transfer function $P(z)$:

$$\begin{aligned} P(z) &= z^{-3}B(z)/A(z) \\ A(z) &= 1 - 1.41833z^{-1} + 1.58939z^{-2} - 1.31608z^{-3} + .88642z^{-4} \\ B(z) &= .28261 + .50666z^{-1}, \end{aligned}$$

which is the discrete-time model (using a sampling time $T_s = 0.05s$) of the system described in [12]. The control objective is expressed by:

$$M(z) = \frac{z^{-3}(1-\alpha)^2}{(1-\alpha z^{-1})^2}, \qquad \alpha = e^{-T_s\bar{\omega}}, \qquad \bar{\omega} = 10,$$

where $\bar{\omega}$ is the desired bandwidth. The magnitude Bode plots of $P(z)$ and $M(z)$ are shown in Fig.(3.b). The weighting factor is $W(z) = 1$ and the class of controllers is:

$$C(z;\theta) = \frac{\vartheta_0 + \vartheta_1 z^{-1} + \vartheta_2 z^{-2} + \vartheta_3 z^{-3} + \vartheta_4 z^{-4} + \vartheta_5 z^{-5}}{1 - z^{-1}}.$$

Note that, being $P(z)$ nonminimum-phase, a perfect model matching would lead to an unstable closed-loop. In order to compute $\hat{\theta}_N$ via the VRFT method, a set of data have been obtained by feeding $P(z)$ in open loop with $N = 512$ samples of a zero-mean Gaussian white noise ($\Phi_u(\omega) = 0.01$). In the following, we will present three different VRFT design cases. The first two cases aim to illustrate the effect of a bad/good shaping of the filter $L(z)$. Specifically, in Case 1 the trivial filter $L(z) = 1$ is used, whereas in Case 2 $L(z)$ is designed as proposed in Section 2. The third design case is instead characterized by the presence of noise.

Case 1: $L(z) = 1$ - no noise.

The estimated parameter vector is $\hat{\theta}_N^1$ =[0.14724 -0.25016 0.29166 -0.25678 0.18587 -0.03717]. We obtain $J_{MR}(\hat{\theta}_N^1) = 0.232$. The magnitude Bode plot and step response of the corresponding closed-loop transfer function are shown in Fig.(4). Apparently, the control system has a behavior which remarkably differs from that of $M(z)$.

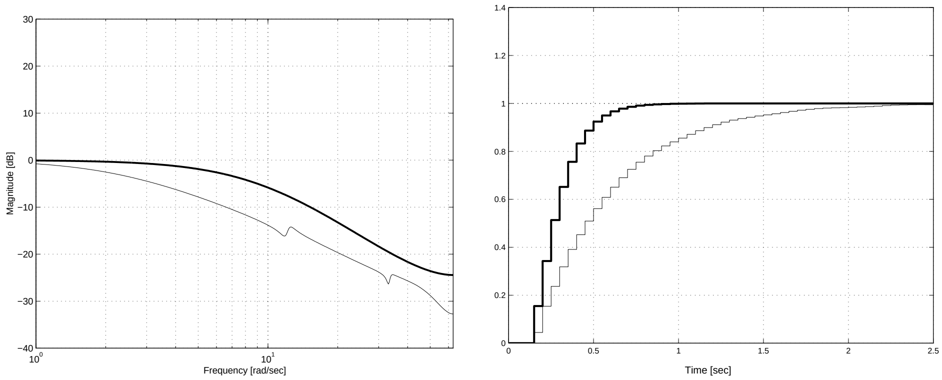
Case 2: $L(z) = M(z)(1 - M(z))$ - no noise.

The estimated parameter vector is $\hat{\theta}_N^2$ =[0.32905 -0.59771 0.70728 -0.64010 0.46499 -0.11763]. In this case we obtain $J_{MR}(\hat{\theta}_N^2) = 0.0343$. The magnitude Bode plot and step response of the achieved closed-loop transfer function are shown in Fig.(5). Note that the so-obtained control system tracks almost perfectly the behaviour of the reference model. It is interesting to compare the performance of the controller $C(z;\hat{\theta}_N^2)$ obtained via VRFT with the optimal controller $C(z;\bar{\theta})$, for which $\bar{\theta}$ =[0.33324 -0.60964 0.72401 -0.66020 0.48204 -0.12508], $J_{MR}(\bar{\theta}) = 0.0340$. Since $J_{MR}(\hat{\theta}_N^2) = 0.0343$, the sub-optimality of $C(z;\hat{\theta}_N^2)$ is negligible.

Case 3: $L(z) = M(z)(1 - M(z))$ - noisy data.

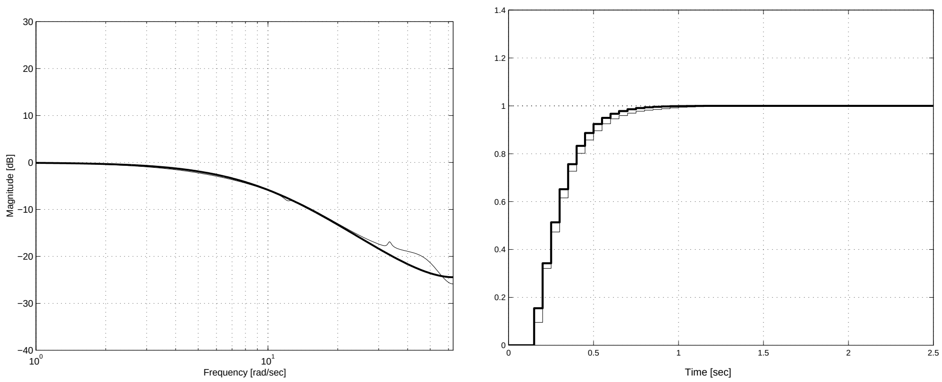
In this case the output signal has been corrupted by a zero mean white disturbance such that the signal to noise ratio is $SNR = 10$. (SNR is the ratio between the variance of $y(t) = P(z)u(t)$ and the variance of the noise signal). The filter $L(z)$ has been chosen as in Case 2. The controller parameter vector $\hat{\theta}_N^3$ = [0.07069 -0.03865 -0.00191 0.00767 0.02166 -0.0077] is estimated without paying attention to the presence of noise. The step response of the achieved control system is shown in Fig.(6.a). Notice the degradation of the performance with respect to the noise-free case (Case 2). The influence of noise can be counteracted via the use of instrumental variables as explained in Section 4. The parameter vector obtained through (15) with the choice (17) (where $\hat{P}(z)$ is a 4th order ARX model) is $\hat{\theta}_N^4$=[0.28381 -0.43055 0.39188 -0.28544 0.23083 -0.04599]. The step response of the so-designed control system is shown in Fig.(6.b).

Note that in the example the unstable zero of the plant gives no problems since it is located in the high frequency region, well beyond the closed-loop bandwidth, and therefore does not tend to be canceled by the controller. Should this be not the case, stability problems could have arisen.



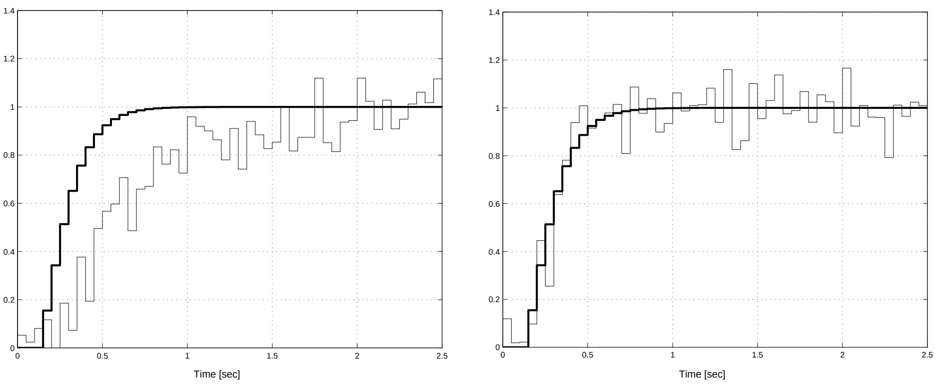
(a) (b)

Figure 4: **(a)** Magnitude Bode plots: the control system with $\hat{\theta}_N^1$ (thin line) and the reference-model (bold line); **(b)** Step responses: the control system with $\hat{\theta}_N^1$ (thin line) and the reference-model (bold line);



(a) (b)

Figure 5: **(a)** Magnitude Bode plots: the control system with $\hat{\theta}_N^2$ (thin line) and the reference-model (bold line); **(b)** Step responses: the control system with $\hat{\theta}_N^2$ (thin line) and the reference-model (bold line);



(a) (b)

Figure 6: Step responses: **(a)** the control system with $\hat{\theta}_N^3$ (thin line) and the reference-model (bold line); **(b)** the control system with $\hat{\theta}_N^4$ (thin line) and the reference-model (bold line).

6 Conclusions

In this work a design technique called Virtual Reference Feedback Tuning has been presented. VRFT has many attractive features, which can be summarized as follows:

- it allows the direct *global* minimization of standard model-reference performance indices using I/O measurements;
- it can be used to tune controllers with a prescribed structure (e.g. PID);
- it does not require a parameter initialization;
- it does not require iterations.

The main application realm of VRFT is represented by applications where a quick (low-cost) controller design has to be performed. However, it can be also regarded as a powerful and effective initialization tool for gradient-based iterative algorithms (like IFT) designed for the "fine-tuning" of controllers.

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Appendix

Proof of Proposition 1. Since $J_{MR}(\theta) = J^+_{MR}([\theta^T \quad 0]^T)$ result (11) is trivially true. As for (12) we start by calculating the Taylor expansion of $J^+_{MR}(\theta^+)$ around $\bar{\theta}^+$. After some cumbersome calculations we obtain:

$$\begin{aligned}
\left[J^+_{MR}(\theta^+)\right]_{\theta^+=\bar{\theta}^+} &= 0, \\
\left[\frac{d}{d\theta^+} J^+_{MR}(\theta^+)\right]_{\theta^+=\bar{\theta}^+} &= 0, \\
\left[\frac{d^2}{d\theta^{+2}} J^+_{MR}(\theta^+)\right]_{\theta^+=\bar{\theta}^+} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2|P|^2|W|^2}{|1+PC_0|^4}\overline{\beta^+}\beta^{+T} d\omega.
\end{aligned}$$

The second order expansion is then given by:

$$\begin{aligned}
\bar{J}^+_{MR}(\theta^+) &= (\theta^+ - \bar{\theta}^+)^T \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|P|^2|W|^2}{|1+PC_0|^4}\overline{\beta^+}\beta^{+T} d\omega\right](\theta^+ - \bar{\theta}^+) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |P|^2|C^+(\theta^+) - C_0|^2 \frac{|W|^2}{|1+PC_0|^4}. \qquad (18)
\end{aligned}$$

Consider now the cost function $J_{VR}(\theta)$.
If the filter $L(z)$ is chosen as in (10) then $J_{VR}(\theta)$ is given by:

$$J_{VR}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |P|^2|C(\theta) - C_0|^2 \frac{|W|^2}{|1+PC_0|^4} d\omega. \qquad (19)$$

Notice, by comparing (18) and (19), that:

if the filter $L(z)$ is chosen as in (10) then $\bar{J}^+_{MR}([\theta^T \quad 0]^T) = J_{VR}(\theta)$,

from which we obtain the result (12). □