Computation of Voronoi Diagrams and Delaunay Triangulation via Parametric Linear Programming

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Abstract

This note illustrates how Voronoi diagrams and Delaunay triangulations of point sets can be computed by applying parametric linear programming techniques. We specify parametric linear programming problems that yield the Delaunay triangulation or the Voronoi Diagram of an arbitrary set of points S in $\mathbb{R}^n$.

1 Introduction

It is the purpose of this note to establish a link between the methodology of *parametric* linear programming (PLP) [3, 10, 14], Voronoi diagrams and Delaunay triangulations. Voronoi diagrams, Dirichlet tesselations and Delaunay tesselations are the concepts introduced by the mathematicians: Dirichlet [7], Voronoi [15, 16] and Delaunay [6].

It is shown in this note that the solution of an appropriately specified PLP yields the Voronoi diagram or the Delaunay triangulation, respectively. The following introduction to Voronoi diagrams and Delaunay triangulation is derived from [9].

Given a set S of d distinct points p in $\mathbb{R}^n$, the Voronoi diagram is a partition of $\mathbb{R}^n$ into d polyhedral regions. The region associated with the point p is called the Voronoi cell of p and is defined as the set of points in $\mathbb{R}^n$ that are closer to p than to any other point in S. Voronoi diagrams are well known in computational geometry and have been studied by many authors [4, 5, 8, 9, 13, 17]. Voronoi Diagrams are a fundamental tool in many fields due to their wide ranging applications. For example the breeding areas of fish, the optimal

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placement of cellular base stations in a city and searches of large databases can all be described by Voronoi Diagrams [1].

The Delaunay complex of a set of points S is a partition of the convex hull of S into polytopical regions whose vertices are the points in S. The convex hull of the nearest neighbour set of a Voronoi vertex v is called the Delaunay cell of v. The Delaunay complex of S is a partition of the convex hull of S into the Delaunay cells of Voronoi vertices together with their faces. The Delaunay complex is not in general a triangulation (we use the term triangulation, throughout this note, in the sense that it represents the generalized triangulation, i.e., a division of polytope in $\mathbb{R}^n$ into n- dimensional simplicies) but becomes a triangulation when the input points are in general position or non-degenerate (i.e. no points are cospherical or equivalently there is no point $c \in \mathbb{R}^n$ whose nearest neighbour set has more than $n + 1$ elements and the convex hull of the set of points has non–empty interior). The Delaunay complex is dual to the Voronoi diagram in the sense that there is a natural bijection between the two complexes which reverses the face inclusions. The Delaunay triangulation has been used in several publications in the control literature (e.g., [2, 12]), in order to compute explicit minimum time state-feedback controllers.

Both the Voronoi diagram and the Delaunay triangulation of a random set of points are illustrated in Figure 1.
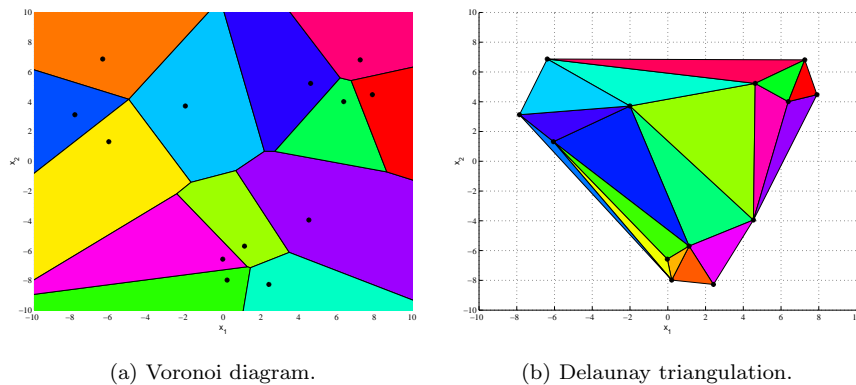

(a) Voronoi diagram.　　　　　(b) Delaunay triangulation.

Figure 1: Illustration of a Voronoi diagram and Delaunay triangulation of a random set of points.

2   Preliminaries

Before proceeding, the following definitions and preliminary results are needed.

Definition 1 *A convex* polyhedron *is the intersection of a finite number of closed half-spaces. A convex* polytope *is a closed and bounded* polyhedron.

Let $\mathbb{N} \triangleq \{1, 2, \ldots, \}$ denote the set of positive integers and for a positive integer q let $\mathbb{N}_q \triangleq \{1, 2, \ldots, q\}$ denote the first q positive integers. Let $d(x, y) \triangleq ((x - y)'(x - y))^{1/2}$ denote the Euclidan distance between two points x and y in $\mathbb{R}^n$.

Definition 2 *Given a set of $S \triangleq \{p_i \in \mathbb{R}^n \mid i \in \mathbb{N}_q\}$, the Voronoi cell associated with point p_i is the set $V(p_i) \triangleq \{x \mid d(x, p_i) \leq d(x, p_j), \ \forall j \neq i, \ i, j \in \mathbb{N}_q\}$ and the Voronoi diagram of the set S is given by the union of all of the Voronoi cells: $\mathcal{V}(P) \triangleq \bigcup_{i \in \mathbb{N}_q} V(p_i)$.*

Definition 3 *The convex hull of a set of points $S \triangleq \{p_i \mid i \in \mathbb{N}_q\}$ is defined as*

$$\mathrm{convh}(S) = \left\{ x \,\middle|\, \exists \lambda, \ x = \sum_{i=1}^{q} p_i \lambda_i, \ \lambda_i \geq 0, \ \sum_{i=1}^{q} \lambda_i = 1 \right\}$$

Remark 1 *Note that the convex hull of a finite set of points is always a convex polytope.*

Definition 4 *A triangulation of a point set S is the partition of the convex hull of S into a set of simplices such that each point in S is a vertex of a simplex. A simplex is a polytope defined as the convex hull of $n + 1$ vertices.*

Definition 5 *For a set of points S in $\mathbb{R}^n$, the Delaunay triangulation is the unique triangulation $DT(S)$ of S such that no point in S is inside the circumcircle of any triangle in $DT(S)$.*

Definition 6 *The map $\mathcal{L}(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is called the* lifting map *and is defined as $\mathcal{L}(x) \triangleq \begin{bmatrix} x' & x'x \end{bmatrix}'$.*

Definition 7 *Given the surface $f(z) = 0$, where $f(\cdot) : \mathbb{R}^p \to \mathbb{R}$ the tangent plane $H_f(z^*)$ to the surface $f(z) = 0$ at the point $z = z^*$ is*

$$H_f(z^*) \triangleq \{z \mid (\nabla f(z^*))'(z - z^*) = 0\} \tag{2.1}$$

where $\nabla f(z^*)$ is gradient of $f(z)$ evaluated at $z = z^*$.

Let $x \in \mathbb{R}^n$ and $\theta \in \mathbb{R}$ and let $g(\cdot) : \mathbb{R}^{n+1} \to \mathbb{R}$ be defined as $g(x, \theta) \triangleq x'x - \theta$, then the gradient of $g(x, \theta)$ is $\nabla g(x, \theta) = \begin{bmatrix} 2x' & -1 \end{bmatrix}'$. The tangent hyperplane at a point $r_i = \mathcal{L}(p_i) = \begin{bmatrix} p_i' & p_i'p_i \end{bmatrix}'$ for any $i \in \mathbb{N}_q$ is then given by:

$$\begin{aligned} H_g(r_i) &= \{(x, \theta) \mid 2p_i'(x - p_i) - (\theta - p_i'p_i) = 0\} \\ &= \{(x, \theta) \mid h_i(x, \theta) = 0\} \end{aligned}$$

where

$$h_i(x, \theta) = 2p_i'(x - p_i) - (\theta - p_i'p_i) \tag{2.2}$$

Definition 8 *Let $S \triangleq \{p_i \in \mathbb{R}^n \mid i \in \mathbb{N}_q\}$ be a set of points and $U \triangleq \{r_i = \mathcal{L}(p_i) \mid i \in \mathbb{N}_q\}$ be the lifting of S. The upper envelope of the set of points R is defined by $\mathcal{U}_E(R) \triangleq \{(x, \theta) \mid h_i(x, \theta) \leq 0, \forall i \in \mathbb{N}_q\}$ where $h_i(x, \theta)$ is defined in (2.2).*

Definition 9 *Let $S \triangleq \{p_i \in \mathbb{R}^n \mid i \in \mathbb{N}_q\}$ be a set of points, $U \triangleq \{r_i = \mathcal{L}(p_i) \in \mathbb{R}^{n+1} \mid i \in \mathbb{N}_q\}$ be the lifting of S and let $\mathrm{convh}(U) \subset \mathbb{R}^{n+1}$ be the convex hull of U. A facet of $\mathrm{convh}(R)$ is called a* lower facet *if the halfspace defining the facet is given by $\{(x, \theta) \in \mathbb{R}^n \times \mathbb{R} \mid \alpha'x + \beta\theta \leq \gamma\}$ and β is less than zero. The surface formed by all the lower facets of $\mathrm{convh}(U)$ is called the lower convex hull of U and is denoted by $\mathrm{lconvh}(R)$.*

In the following the properties of parametric linear programs are restated [3, 10]:

Theorem 1 *Consider the parametric linear program*

$$\mathbb{P}(x): \quad J^\circ(x) = \min_\theta \{\langle c, \begin{bmatrix} x' & \theta' \end{bmatrix}' \rangle \mid Mx + N\theta \leq p\},$$

where $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^m$ and the objective $c \in \mathbb{R}^{n+m}$. Then the set of feasible parameters $\mathcal{X}_f$ given by $\{x \mid \exists \theta : Mx + N\theta \leq p\}$ is convex and there exists an optimiser $\theta^\circ : \mathcal{X}_f \to \mathbb{R}^m$ that is continuous and piecewise affine (PWA):

$$\begin{array}{rcl} J^\circ(x) & = & \langle c, \begin{bmatrix} x' & \theta^\circ(x)' \end{bmatrix}' \rangle \\ \theta^\circ(x) & = & T_i x + t_i \end{array} , \qquad if\ x \in R_i \qquad (2.3)$$

where R_i form a polyhedral partition of $\mathcal{X}_f$.

3 Computation of Voronoi Diagrams

In this section it will be shown how to compute the Voronoi diagram via a PLP for a given *finite* set of points $S \triangleq \{p_i \in \mathbb{R}^n \mid i \in \mathbb{N}_q\}$.

3.1 Introduction to Voronoi Diagrams

We will here show how the equality (2.2) relates to the Euclidian distance between two points, which is used in the Voronoi diagram Definition 2.

Let $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) \triangleq x'x$ and let x and y be two points in $\mathbb{R}^n$, then:

$$d^2(x, y) = (x - y)'(x - y) = f(x) - \hat{\theta}(x, y) \qquad (3.1)$$

where $\hat{\theta}(x, y) \triangleq 2y'x - y'y$. Let

$$\theta_i(x) \triangleq \hat{\theta}(x, p_i) = 2p_i'x - p_i'p_i, \qquad\qquad i \in \mathbb{N}_q \qquad (3.2)$$

Obviously, the square of the Euclidian distance of any point $x \in \mathbb{R}^n$ from any point $p_i \in S$ is given by $d^2(x, p_i) = f(x) - \theta_i(x)$. Furthermore, given any $x \in \mathbb{R}^n$, $\theta_i(x)$ is just a solution of Equation (2.2):

$$h_i(x, \theta) = 0 \Leftrightarrow 2p_i'(x - p_i) - (\theta - p_i'p_i) = 0. \qquad (3.3)$$

Let $\bar{\theta}(x)$ be defined by:

$$\bar{\theta}(x) \triangleq \max_{i \in \mathbb{N}_q^+} \theta_i(x) \qquad (3.4)$$
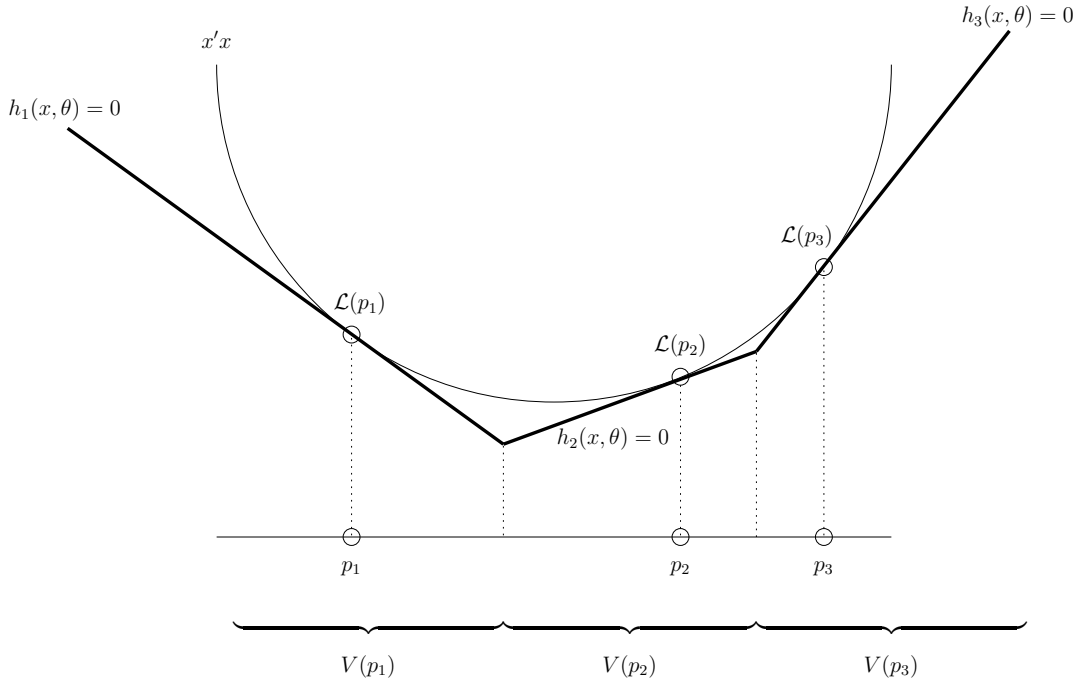
Figure 2: Voronoi Lifting

It is clear that,

$$h_i(x, \theta) \leq 0, \ \forall i \in \mathbb{N}_q \quad \Leftrightarrow \quad \theta \geq \bar{\theta}(x) \tag{3.5}$$

The lifting of the set S and the resulting calculation of the Voronoi cells is shown in Figure 2.

Lemma 1 *Let x be in $\mathbb{R}^n$ and $S = \{p_i \in \mathbb{R}^n \mid i \in \mathbb{N}_q\}$, then the Voronoi cell associated with the point p_i is:*

$$V(p_i) = \{x \mid \bar{\theta}(x) = \theta_i(x)\} \tag{3.6}$$

where $\bar{\theta}(x)$ and $\theta_i(x)$ are defined in (3.4) and (3.2), respectively.

Proof: The proof uses (3.1) and the fact that $d(x, y) \geq 0$, so that:

$$
\begin{aligned}
V(p_i) &\triangleq \{x \mid d(x, p_i) \leq d(x, p_j), \ \forall j \in \mathbb{N}_q\} \\
&= \{x \mid d^2(x, p_i) \leq d^2(x, p_j), \ \forall j \in \mathbb{N}_q\} \\
&= \{x \mid f(x) - \theta_i(x) \leq f(x) - \theta_j(x), \ \forall j \in \mathbb{N}_q\} \\
&= \{x \mid -\theta_i(x) \leq -\theta_j(x), \ \forall j \in \mathbb{N}_q\} \\
&= \{x \mid \theta_i(x) \geq \theta_j(x), \ \forall j \in \mathbb{N}_q\} \\
&= \{x \mid \bar{\theta}(x) = \theta_i(x)\}
\end{aligned}
$$

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3.2 Parametric linear programming formulation of Voronoi Diagrams

In this section we will assume that the parameter $\theta(x)$ is no longer a function of x but is instead a free variable, henceforth denoted by θ. It will be shown how a parametric optimization problem can be posed for the variables θ and x, such that the solution to the PLP is a Voronoi diagram.

Let the set $\Psi \subseteq \mathbb{R}^{n+1}$ be defined by:

$$\Psi \triangleq \{(x,\theta) \mid h_i(x,\theta) \leq 0, \forall i \in \mathbb{N}_q\} \tag{3.7a}$$
$$= \{(x,\theta) \mid \theta \geq \bar{\theta}(x)\} \tag{3.7b}$$

¿From (3.2), (3.4) and (3.7) we have

$$\Psi = \{(x,\theta) \mid Mx + N\theta \leq p\} \tag{3.8}$$

where M, N and p are given by:

$$M = \begin{bmatrix} 2p_1' \\ 2p_2' \\ \vdots \\ 2p_q' \end{bmatrix}, \quad N = \begin{bmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}, \quad p = \begin{bmatrix} p_1'p_1 \\ p_2'p_2 \\ \vdots \\ p_q'p_q \end{bmatrix} \tag{3.9}$$

Consider the cost function

$$g(x,\theta) = \mathbf{0}x + 1\theta \tag{3.10}$$

and the following parametric program $\mathbb{P}_V(x)$:

$$\mathbb{P}_V(x): \quad g^o(x) = \min_{\theta}\{g(x,\theta) \mid (x,\theta) \in \Psi\} \tag{3.11}$$

$$= \min_{\theta}\{\mathbf{0}x + 1\theta \mid Mx + N\theta \leq p\}. \tag{3.12}$$

The parametric form of $\mathbb{P}_V(x)$ is a standard form encountered in the literature on parametric linear programming [3, 10]. It is obvious from (3.7) and (3.11) that the optimiser $\theta^\circ(x)$ of $\mathbb{P}_V(x)$ is equal to $\bar{\theta}(x)$.

Theorem 2 *Let* $S \triangleq \{p_i \in \mathbb{R}^n \mid i \in \mathbb{N}_q\}$. *The* explicit *solution of the parametric problem* $\mathbb{P}_V(x)$ *defined in* (3.11) *yields the Voronoi Diagram of the set of points* S.

Proof:

The optimiser $\theta^\circ(x)$ for problem $\mathbb{P}_V(x)$ is a piecewise affine function of x [3,10] and it satisfies, for all $x \in \mathbb{R}^n = \bigcup_{i \in \mathbb{N}_q} R_i$:

$$\theta^o(x) = T_i x + t_i = \bar{\theta}(x), \qquad\qquad \forall x \in R_i \qquad\qquad (3.13)$$

By Lemma 1, R_i is the Voronoi cell associated with the point p_i. Hence, computing the solution of $\mathbb{P}_V(x)$ via PLP yields the Voronoi Diagram of S. ∎

4 Computation of the Delaunay Triangulation

In this section it will be shown how to compute the Delaunay triangulation via a PLP for a given *finite* set of points $S \triangleq \{p_i \in \mathbb{R}^n,\ i \in \mathbb{N}_q\}$.

4.1 Introduction to Delaunay Triangulation

The Delaunay triangulation of the set $S \triangleq \{p_i \in \mathbb{R}^n \mid i \in \mathbb{N}_q\}$ of vertices is a projection on $\mathbb{R}^n$ of the lower convex hull of the set of lifted points $U \triangleq \{\mathcal{L}(p_i) \mid p_i \in S\} \subset \mathbb{R}^{n+1}$. It is well known [9] that the Delaunay triangulation of the set S can be computed in two steps. First, the lower convex hull of the lifted point set S is computed: $\mathcal{U} \triangleq \mathrm{lconvh}(\{\mathcal{L}(p_i) \mid p_i \in S\})$. Second, each facet $\mathcal{F}_i$ of $\mathcal{U}$, is projected to $\mathbb{R}^n$: $\mathcal{T}_i \triangleq \mathrm{Proj}_{\mathbb{R}^n} \mathcal{F}_i$. If $\mathcal{U}$ has $\mathbb{N}_t$ facets, then the Delaunay triangulation of S is given by:

$$DT(S) \triangleq \bigcup_{i=1}^{\mathbb{N}_t} \mathcal{T}_i$$

This is illustrated in Figure 3.

4.2 Parametric linear programming formulation of Delaunay triangulation

This section shows how the Delaunay triangulation can be computed via an appropriately formulated parametric linear program. Let $S = \{p_i \in \mathbb{R}^n \mid i \in \mathbb{N}_q\}$ be a finite point set and $U = \{\mathcal{L}(p_i) \mid p_i \in S\}$ be the lifted point set. From Definition 9, the lower convex hull of U can be written as:

$$\mathrm{lconvh}(U) = \{(x, \gamma^\circ) \in \mathrm{convh}(U) \mid \gamma^\circ = \mathrm{argmin}\{\gamma \mid (x, \gamma) \in \mathrm{convh}(U)\}\}. \tag{4.1}$$

Equation 4.1 is equivalent to the following parametric linear program:

$$\mathbb{P}_D(x): \quad \gamma^o(x) = \min_{\gamma}\{\gamma \mid (x, \gamma) \in \mathrm{convh}(U)\} \qquad (4.2)$$

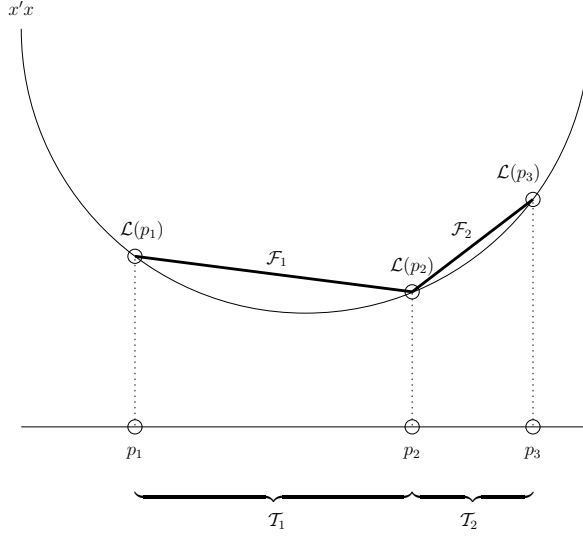**A 4.1** *Throughout this section we assume that:*

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Figure 3: Calculation of a Delaunay Triangulation

(i) *Convex hull of S has non–empty interior,* $\text{interior}(\text{convh}(S)) \neq \emptyset$ *and*

(ii) *There does not exist $n + 2$ points that lie of the surface of the same n-dimensional ball.*

Remark 2 *Assumption 4.1 ensures that the Delaunay triangulation exists, is unique and is in fact a triangulation. [9, 13, 17].*

We will now show how (4.7) can be formulated in the standard form for parametric linear program solvers. The convex hull of the lifted point set U can be written as:

$$\text{convh}(U) \triangleq \left\{ (x, \gamma) \,\middle|\, \exists \lambda, \begin{bmatrix} x \\ \gamma \end{bmatrix} = \sum_{i=1}^{q} \lambda_i \begin{bmatrix} p_i \\ p_i' p_i \end{bmatrix}, \sum_{i=1}^{q} \lambda_i = 1, \lambda_i \geq 0 \right\} \tag{4.3}$$

$$= \left\{ (x, \gamma) \,\middle|\, \exists \lambda, \begin{array}{l} M_I x \ + \ N_I \gamma \ + \ L_I \lambda \ \leq \ b_I \\ M_E x \ + \ N_E \gamma \ + \ L_I \lambda \ = \ b_E \end{array} \right\} \tag{4.4}$$

where M_I, N_I, L_I and b_I are given by:

$$M_I = \mathbf{0}, \quad N_I = \mathbf{0}, \quad L_I = -I, \quad b_I = \mathbf{0} \tag{4.5}$$

and M_E, N_E, L_E and b_e are given by:

$$M_E = \begin{bmatrix} I \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \quad N_E = \begin{bmatrix} \mathbf{0} \\ 1 \\ \mathbf{0} \end{bmatrix}, \quad L_E = \begin{bmatrix} -\mathcal{S} \\ -\mathcal{Y} \\ \mathbf{1} \end{bmatrix}, \quad b_E = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ 1 \end{bmatrix} \tag{4.6}$$

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where $\mathcal{S} \triangleq [p_1 \ p_2 \ \dots p_q]$ and $\mathcal{Y} \triangleq \begin{bmatrix} p'_1 p_1 & p'_2 p_2 & \dots & p'_q p_q \end{bmatrix}$.

Problem $\mathbb{P}_D(x)$ can now be written in standard form as:

$$\mathbb{P}_D(x): \quad \gamma^o(x) = \min_{\gamma} \left\{ \mathbf{0}x + \mathbf{0}\lambda + 1\gamma \ \middle| \ \begin{matrix} M_I x & + & N_I \gamma & + & L_I \lambda & \leq & b_I \\ M_E x & + & N_E \gamma & + & L_I \lambda & = & b_E \end{matrix} \right\} \quad (4.7)$$

The explicit solution of the parametric problem $\mathbb{P}_D(x)$ is a piecewise affine function:

$$\gamma^o(x) = G_i x + g_i, \qquad\qquad x \in R_i, \ i \in \mathbb{N}_t \qquad (4.8)$$

Theorem 3 *Let $S \triangleq \{p_i \in \mathbb{R}^n \mid i \in \mathbb{N}_q\}$ be a given point set. Then the explicit solution of parametric form of $\mathbb{P}_D(x)$ defined in (4.7) yields the Delaunay triangulation.*

Proof: From Assumption 4.1 it follows that the Delaunay triangulation exists and is unique [9, 13, 17], i.e., the facets of the lower convex hull of the lifted point set U are n dimensional simplices. It follows from the construction of lconvh(U) that the optimiser for problem $\mathbb{P}_D(x)$ is a piecewise affine function of x and that $(x, \gamma^o(x))$ is in lconvh(U), for all $x \in$ convh(S). Furthermore, the optimiser $\gamma^o(x)$ in each region R_i obtained by solving $\mathbb{P}_D(x)$ (4.7) as a parametric program is affine, i.e., $\gamma^o(x) = T_i x + t_i$ if $x \in R_i$. Thus, each region R_i is equal to the projection of a facet $\mathcal{F}_i$, i.e. $R_i = \mathrm{Proj}_{\mathbb{R}^n} \mathcal{F}_i$, $i \in \mathbb{N}_t$. Hence the PLP $\mathbb{P}_D(x)$ defined in (4.7) solves the Delaunay triangulation problem. ∎

5 Numerical Examples

In order to illustrate the proposed PLP Voronoi and Delaunay algorithms a random set of points S in $\mathbb{R}^2$ was generated and the corresponding Voronoi diagram and Delaunay triangulation are shown in Figures 4. Figure 5 shows the Voronoi partition and Dalaunay triangulation for a unit-cube.

6 Conclusion

This note demonstrated that Voronoi diagrams, Delaunay triangulations and parametric linear programming are connected. It was shown how to formulate appropriate parametric linear programming problems in order to obtain the Voronoi diagram, or the Delaunay triangulation of a finite set of points S. Numerical examples were provided to illustrate the proposed algorithms. These algorithm are not necessarily the most efficient algorithms for performing computation of Voronoi diagrams and Delanuay triangulations but are easily generalized to arbitrary dimensions. Moreover, the link which is established between recent parametric programming techniques and Voronoi diagrams and
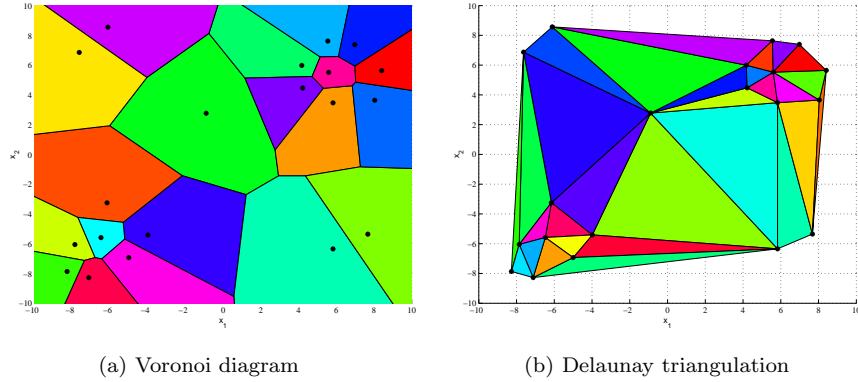
(a) Voronoi diagram

(b) Delaunay triangulation

Figure 4: Illustration of a Voronoi diagram and Delaunay triangulation for a given set of points S.

(a) Voronoi diagram
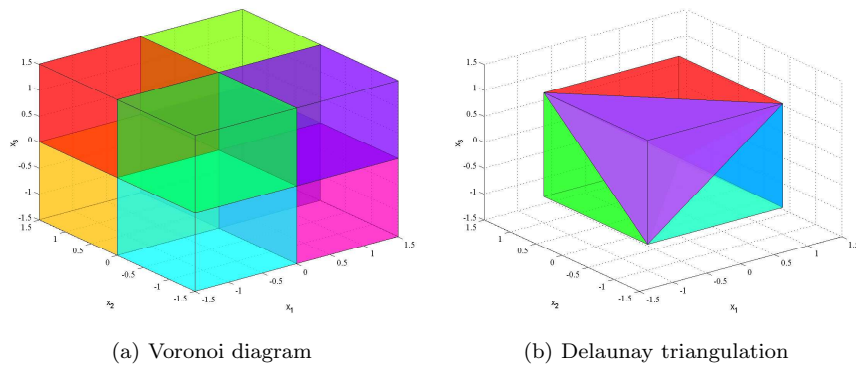
(b) Delaunay triangulation

Figure 5: Illustration of the Voronoi diagram and Delaunay triangulation of a unit-cube $\mathcal{P}$ in $\mathbb{R}^3$.

Delaunay triangulation motivates further research on combining the known results of computational geometry with some of the results in control theory, in which parametric programming has been applied.

The presented algorithms are contained in the MPT toolbox [11] and can be downloaded from http://control.ee.ethz.ch/~mpt.

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