Optimal control of constrained, piecewise affine systems with bounded disturbances[1]

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Abstract

The solution to the problem of optimal control of piecewise affine systems with a bounded disturbance is characterised. Results that allow one to compute the value function, its domain (robustly controllable set) and the optimal control law are presented. The tools that are employed include dynamic programming, polytopic set algebra and parametric programming. When the cost is time (robust time-optimal control problem) or the stage cost is piecewise affine (robust optimal and robust receding horizon control problems), the value function and the optimal control law are both piecewise affine and each robustly controllable set is the union of a finite set of polytopes. Conditions on the cost and constraints are also proposed in order to ensure that the optimal control laws are robustly stabilising.

Key words: piecewise linear, constraints, optimal control, receding horizon control, model predictive control, robust control, dynamic programming, parametric programming

1 Introduction

In recent years there has been an increase in the amount of research on the control of piecewise affine systems (sometimes also called piecewise linear systems). The rise in interest in this class of systems is due to the fact that many nonlinear systems can be approximated arbitrarily closely using piecewise affine models [18] and because of the equivalence that has been shown to exist between piecewise affine systems and a large class of hybrid systems [10].

Though many papers address the analysis and optimal control of piecewise affine systems (see [1, 17] and the references therein), the literature on the robust control of this class of systems is relatively sparse. Some of the contributions include reachability-based approaches for the control of uncertain, piecewise linear hybrid systems [3, 13], and LMI-based approaches for H_2 and H_∞ control of piecewise affine systems [9].

This paper considers the problems of robust time-optimal control [4, 16], robust optimal control [5] and robust receding horizon control [2, 14, 15] of a piecewise affine system to a given target set. The results in this paper are an extension of results on the robust optimal control of linear systems to the class of piecewise affine systems. The extension of the results for linear systems presented in [2, 16], to which this paper is most closely related, is unfortunately not straight-forward. The system under consideration in this paper is nonlinear and the resulting domains of attraction non-convex; some of the linearity and convexity arguments exploited in [2, 16] do not hold and extra care has to be taken when computing the control laws.

One of the key results which allow one to compute robust optimal controllers for piecewise affine systems, is the fact that one can compute the robustly controllable sets of the system as the union of a finite set of polytopes, using standard computational geometry tools [11, 12]. The other key observation is that, provided the stage cost is piecewise affine (which includes the case when the cost is ℓ_∞ or ℓ_1), one can set up a sequence of parametric programming problems [7, 8] and compute the explicit solutions to the robust optimal and robust receding horizon control problems, in a similar fashion to [1, 2].

Sections 2 and 3 introduce some notation and set up the optimal feedback control problems that will be considered. Section 4 provides a dynamic programming solution to the set of control problems and Section 5 provides some stability results for the resulting control laws. Geometric results that allow one to compute the control laws are given in Section 6 and some conclusions are drawn in Section 7.

2 Definitions and Notation

A polyhedron in $\mathbb{R}^n$ is a (convex) set given by the intersection of a finite number of open and/or closed half-spaces in $\mathbb{R}^n$. A polytope is a closed and bounded (i.e. compact) polyhedron. A function is said to be piecewise affine if its domain can be partitioned into a finite number of mutually disjoint polyhedra and the function is affine on each polyhedron.

If $Y \subset \mathbb{R}^n$ and $Z \subset \mathbb{R}^n$, then 2^Y is the power set (set of all subsets) of Y, the complement of Y is $Y^c := \{y \in \mathbb{R}^n \,|\, y \notin Y\}$, the set difference $Y \setminus Z := \{y \in Y \,|\, y \notin Z\}$, the Minkowski sum $Y \oplus Z := \{y + z \,|\, y \in Y, z \in Z\}$ and

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the Pontryagin (or Minkowski) difference $Y \sim Z := \{y \in \mathbb{R}^n \,|\, y+z \in Y, \forall z \in Z\}$.

3 Problem Setup

The problem considered in this paper is the robust optimal control of continuous, discrete-time, piecewise affine systems of the form

$$x^+ = f(x,u,w) := A_q x + B_q u + c_q + w,\ \forall (x,u) \in P_q \quad (1)$$

where x, u and w denote, respectively, the state, input and disturbance at a given time instant and x^+ denotes the state at the next time instant. Each $(A_q, B_q, c_q) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^n$, $q \in Q$ (Q a finite set) and $\{P_q \mid q \in Q\}$ is a finite set of polytopes with mutually disjoint interiors. For each $q \in Q$, let $f_q(x,u) := A_q x + B_q u + c_q$ and the continuous function $f_0(\cdot)$ be defined by $f_0(x,u) := f_q(x,u)$ for all $(x,u) \in P_q$, $q \in Q$ so that $f(x,u,w) = f_0(x,u) + w$.

It is assumed that the bounded disturbance w is persistent and satisfies $w \in W$, and that the control and state are required to satisfy the hard constraints $u \in U$ and $x \in X$; X, U and W are all polytopes, with $0 \in W$ and $0 \in \mathrm{int}(X)$. The state x is assumed to be accessible. State, control and disturbance sequences of the system being controlled are denoted by $\{x(i)\}$, $\{u(i)\}$ and $\{w(i)\}$.

To determine a suitable control law an optimal control problem $\mathbf{P}_N$ (defined below) with horizon N is solved. Let $\mathbf{w} := \{w(0), w(1), \ldots, w(N-1)\}$ denote the disturbance sequence over the interval 0 to $N-1$. Effective control in the presence of the disturbance w requires state feedback, so that the decision variable in the optimal control problem (for a given initial state) is a control policy π defined by

$$\pi := \{u(0), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}, \quad (2)$$

where $u(0) \in U$ and $\mu_i : X \to U$, $i \in \{1,\ldots,N-1\}$; $u(0)$ is a control *action* (since the initial state is known) and each $\mu_i(\cdot)$ is a state feedback control *law*. Let $\phi(i;x,\pi,\mathbf{w})$ denote the solution to (1) when the initial state is x at time 0, the control is determined by policy π ($u = \mu_i(x)$ at *event* (x,i), i.e. state x, time i) and the disturbance sequence is $\mathbf{w}$; similarly, $\phi(i;x,\kappa,\mathbf{w})$ denotes the solution to (1) when the initial state is x at time 0, the disturbance sequence is $\mathbf{w}$ and a time-invariant control law $\kappa : X \to U$ is employed ($u = \kappa(x)$ at state x).

Given a target set $T \subset X$, for each initial state $x \in X$, let $\Pi_N(x)$ denote the set of *admissible* policies π:

$$\begin{aligned}\Pi_N(x) := \{\pi \mid u(0) \in U,\ \mu_i(\phi(i;x,\pi,\mathbf{w})) \in U,\\ \phi(i;x,\pi,\mathbf{w}) \in X,\ \phi(N;x,\pi,\mathbf{w}) \in T,\\ \forall i \in \{1,\ldots,N-1\}, \forall \mathbf{w} \in W^N\}. \quad (3)\end{aligned}$$

Conditions on the target set (also called terminal constraint set) T, together with the stage cost $\ell(\cdot)$ and terminal cost $F(\cdot)$, will be given in the subsequent sections in order to ensure that the solution to the finite horizon optimal control problems are stabilising.

The cost due to a policy π, initial state x and an individual realization $\mathbf{w}$ of the disturbance process is

$$J_N(x,\pi,\mathbf{w}) := \sum_{i=0}^{N-1} \ell(x_i,u_i) + F(x_N), \quad (4)$$

where $x_i := \phi(i;x,\pi,\mathbf{w})$ if $i \in \{0,1,\ldots,N\}$, $u_i := \mu_i(\phi(i;x,\pi,\mathbf{w}))$ if $i \in \{1,\ldots,N-1\}$ and $u_0 := u(0)$. In order to define the optimal control problem, a cost $V_N(\cdot)$ that is independent of $\mathbf{w}$ is defined; the conventional choice is

$$V_N(x,\pi) := \max_{\mathbf{w}} \{J_N(x,\psi,\mathbf{w}) \,|\, \mathbf{w} \in W^N\}. \quad (5)$$

The robust optimal control problem $\mathbf{P}_N$ can now be defined as

$$\mathbf{P}_N(x):\ V_N^0(x) := \inf_{\pi} \{V_N(x,\pi) \,|\, \pi \in \Pi_N(x)\}. \quad (6)$$

Let $\pi_N^0(x)$ denote the solution to $\mathbf{P}_N(x)$:

$$\begin{aligned}\pi_N^0(x) &:= \{u_0^0(x), \mu_1^0(\cdot;x), \ldots, \mu_{N-1}^0(\cdot;x)\} \quad (7)\\ &:= \arg\inf_{\pi} \{V_N(x,\pi) \,|\, \pi \in \Pi_N(x)\}. \quad (8)\end{aligned}$$

The robust time-optimal control problem is defined as

$$\mathbf{P}(x):\ N^0(x) := \min_{\pi,N} \{N \,|\, (\pi,N) \in \Pi_N(x) \times \mathfrak{N}\}, \quad (9)$$

where $\mathfrak{N} := \{0,1,\ldots,N_{\max}\}$ and $N_{\max}$ is an upper bound on the horizon. The solution to $\mathbf{P}(x)$ is

$$\left(\pi^0(x), N^0(x)\right) := \arg\min_{\pi,N} \{N \,|\, (\pi,N) \in \Pi_N(x) \times \mathfrak{N}\}. \quad (10)$$

4 Dynamic Programming Solution

Dynamic programming provides a recursive procedure for computing sequentially the partial return functions $V_i^0(\cdot)$ (defined in (6) with $N = i$), the associated set-valued control laws $\kappa_i(\cdot)$ as well as their domains (here i denotes 'time-to-go' so that $\kappa_i(\cdot) = \mu_{N-i}^0(\cdot)$ if $i \in \{1,\ldots,N-1\}$ and $\kappa_N(\cdot) = u_0^0(\cdot)$). The domain of $V_i^0(\cdot)$ and $\kappa_i(\cdot)$ is X_i, the set of states that can be robustly steered (steered for all $\mathbf{w} \in W^N$) to the target set T in i steps or less. Standard optimal control implements the time-varying policy $\pi_N^0(x) = \{\kappa_N(x), \kappa_{N-1}(\cdot), \ldots, \kappa_1(\cdot)\}$ ($u \in \kappa_{N-i}(x)$ at event (x,i), i.e. at state x, time i), whereas receding horizon control uses the time-invariant control law $\kappa_N(\cdot)$ ($u \in \kappa_N(x)$ at state x).

4.1 Robust time-optimal problem

For the robust time-optimal control problem $\mathbf{P}$, the value function $N^0(x)$ takes the discrete values $i \in \{0,1,2,\ldots,N_{\max}\}$. For each i, the robustly controllable set

$X_i := \{x \mid N^0(x) \le i\}$ is the set of initial states that can be *robustly* steered (steered for all $\mathbf{w} \in W^i$) to the target set T, in i steps or less while satisfying all state and control constraints. Thus $N^0(x) = i$ for all $x \in X_i \setminus X_{i-1}$. The robustly controllable set X_i and the associated robust time-optimal control law $\kappa_i : X_i \to 2^U$ are yielded by the following recursion:

$$X_i := \{x \in X \mid \exists u \in U : f(x,u,W) \subset X_{i-1}\} \quad (11)$$

$$\kappa_i(x) := \{u \in U \mid f(x,u,W) \subset X_{i-1}\},\ \forall x \in X_i \quad (12)$$

for $i \in \{1, 2, \dots, N_{\max}\}$, with boundary condition $X_0 = T$. The control law $\kappa_0 : T \to 2^U$ is defined by

$$\kappa_0(x) := \{u \in U \mid f(x,u,W) \subset T\},\ \forall x \in T. \quad (13)$$

A condition on T that ensures $\kappa_0(x) \neq \emptyset$ for all $x \in T$ is given in §5.1. Note that the control law $\kappa_i(\cdot)$ is set-valued; at event (x,i) (i.e. at state x, time i) *any* control in the set $\kappa_{N-i}(x)$ may be employed. The time-invariant control law $\kappa^0 : X_{N_{\max}} \to 2^U$ defined by

$$\kappa^0(x) := \begin{cases} \kappa_i(x), & \forall x \in X_i \setminus X_{i-1},\ \forall i \in \{1,\dots,N_{\max}\} \\ \kappa_0(x), & \forall x \in X_0 \end{cases} \quad (14)$$

robustly steers any $x \in X_N$ to X_0 in N steps or less, while satisfying state and control constraints, and thereafter maintains the state in X_0.

Finally, it is useful to note that $X_i = \mathrm{Proj}_X(S_i)$, where $\mathrm{Proj}_X(S_i)$ is the orthogonal projection onto x-space of the set S_i defined by

$$S_i = \{(x,u) \in X \times U \,|\, f(x,u,W) \subset X_{i-1}\}. \quad (15)$$

4.2 Robust optimal control problem

Problem $\mathbf{P}_N$ is embedded in the sequence of problems $\{\mathbf{P}_i\}$ (defined by (6) with $N = i$), yielding the recursions:

$$\begin{aligned} V_i^0(x) &:= \min_{u\in U} \max_{w\in W} \{\ell(x,u) + V_{i-1}^0(f(x,u,w)) \mid \\ &\qquad f(x,u,W) \subset X_{i-1}\},\ \forall x \in X_i \\ &= \min_u \{\ell(x,u) + \max_{w\in W} V_{i-1}^0(f(x,u,w)) \mid \\ &\qquad (x,u) \in S_i\},\ \forall x \in X_i \end{aligned} \quad (16)$$

$$\begin{aligned} \kappa_i(x) &:= \arg\min_{u\in U} \max_{w\in W} \{\ell(x,u) + V_{i-1}^0(f(x,u,w)) \mid \\ &\qquad f(x,u,W) \subset X_{i-1}\},\ \forall x \in X_i \end{aligned} \quad (17)$$

$$X_i := \{x \in X \,|\, \exists u \in U : f(x,u,W) \subset X_{i-1}\} \quad (18)$$

for $i \in \{1,\dots,N\}$, with boundary conditions

$$X_0 = T \text{ and } V_0^0(x) = F(x),\ \forall x \in T. \quad (19)$$

The control law $\kappa_0 : T \to 2^U$ is defined by

$$\begin{aligned} \kappa_0(x) &:= \arg\min_{u\in U} \max_{w\in W} \{\ell(x,u) + F(f(x,u,w)) \\ &\qquad \mid f(x,u,W) \subset T\},\ \forall x \in T. \end{aligned} \quad (20)$$

Note again that all the $\kappa_i : X_i \to 2^U$ are set-valued.

5 Stability

Because of the persistent, additive disturbance w, convergence of the state of the controlled system to the origin is not possible; convergence to a set T containing the origin can be proven instead. The corresponding notions of stability and attractivity are as follows. If $d(z,Z) := \inf_{y\in Z} |z-y|$ for any set $Z \subset \mathbb{R}^n$, then the set T is robustly stable iff, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(x(0),T) \le \delta$ implies $d(x(i),T) \le \varepsilon$, for all $i \ge 0$ and for all admissible disturbance sequences. The set T is robustly asymptotically (finite-time) attractive with domain of attraction $\mathfrak{X}$ iff, for all $x(0) \in \mathfrak{X}$, $d(x(i),T) \to 0$ as $i \to \infty$ (there exists a time I such that $x(i) \in T$ for all $i \ge I$) for all admissible disturbance sequences. The set T is robustly asymptotically (finite-time) stable with domain of attraction $\mathfrak{X}$ iff it is robustly stable and robustly asymptotically (finite-time) attractive with domain of attraction $\mathfrak{X}$.

A set $Z \subset \mathbb{R}^n$ is said to be *robustly controlled invariant* [6] (sometimes called controlled disturbance invariant) for (1) iff, for all $x \in Z$, there exists a $u \in U$ such that $f(x,u,W) \subset Z$. A set Z is *disturbance invariant* (sometimes called robustly positively invariant) for the system $x^+ \in f(x,W)$ iff $f(x,W) \subset Z$ for all $x \in Z$. A set Z is robustly controlled invariant iff there exists a control law $\kappa : Z \to U$ that maintains the state in Z if the initial state is in Z ($\phi(i;x,\kappa,\mathbf{w}) \in Z$ for all $x \in Z$, all $i \ge 0$ and all admissible $\mathbf{w}$). In this case Z is disturbance invariant for the closed-loop system $x^+ \in f_\kappa(x,W) := f(x,\kappa(x),W)$.

5.1 Robust time-optimal control

The following assumption is made:

A1: The set T is compact, robustly controlled invariant and contains the origin in its interior.

Proposition 1 *The sets $\{X_i\}$ computed using the recursion* (11) *with boundary condition $X_0 = T$ satisfy $X_0 \subset X_1 \subset \dots X_{N_{\max}} \subset X$. Moreover, for each $i > 1$, X_i is compact, is robustly controlled invariant, and contains the origin in its interior.*

Using the discontinuous value function $N^0(\cdot)$ as a Lyapunov function, robust stability cannot be established. However, analogous results (including robust attractivity) can be obtained. The stabilising properties of the set-valued control law $\kappa^0(\cdot)$ defined in (14) are stated next.

Theorem 1 *The target set T is robustly finite-time attractive for the closed-loop system $x^+ \in f(x,\kappa^0(x),W)$ with a region of attraction $X_{N_{\max}}$. Any state x in $X_i \subset X_{N_{\max}}$ ($i \le N_{\max}$) is robustly steered by the controller $\kappa^0(\cdot)$ to T in i steps or less and, thereafter, remains in T, while satisfying all state and control constraints.*

5.2 Robust optimal and receding horizon control

Robust finite-time attraction of T for the closed-loop system with the time-varying, optimal control policy $\pi_N^0(x) = \{\kappa_N(x), \kappa_{N-1}(\cdot), \ldots, \kappa_1(\cdot), \kappa_0(\cdot), \kappa_0(\cdot), \ldots\}$, where each $\kappa_i(\cdot)$, $i \in \{1, \ldots, N\}$ is defined in (17) and $\kappa_0(\cdot)$ is defined in (20), follows similar arguments as for the robust time-optimal control problem. The rest of this section will therefore only consider the stabilising properties of the time-invariant receding horizon control law $\kappa_N : X_N \to 2^U$ defined in (17) with $i = N$.

A2: The set T is compact, robustly controlled invariant, and contains the origin in its interior. The terminal cost $F(x) := 0$ for all $x \in T$. The path cost $\ell(\cdot)$ is piecewise affine, is zero in $T \times U$, continuous in $(X \setminus T) \times U$, and satisfies $\ell(x,u) \geq c|x|_\infty$ for all $(x,u) \in (X \setminus T) \times U$, for some $c > 0$.

The above assumption satisfies axioms A3a and A4a in [14] for problem $\mathbf{P}_N$; T is robustly controlled invariant, $T \subset X$ and $\min_{u \in U} \max_{w \in W} \{[\overset{*}{F} + \ell](x,u,w)\} \leq 0$, $\forall x \in T$, where $\overset{*}{\alpha}(x,u,w) := \alpha(f(x,u,w)) - \alpha(x)$ for any function $\alpha(\cdot)$. With these assumptions it follows [15] that $[\overset{*}{V_N^0} + \ell](x,u,w) \leq 0$ for all $x \in X_N \setminus T$, all $u \in \kappa_N(x)$ and all $w \in W$. It follows from A2 that $V_N^0(x) = 0$, $\forall x \in T$ and that T is disturbance invariant for the closed-loop system $x^+ \in f(x, \kappa_0(x), W)$ (so that $\kappa_0(\cdot)$, which is set-valued, keeps the state in T irrespective of the disturbance). Since $F(x)$ and $\ell(x,u)$ are zero if $(x,u) \in T \times U$, it follows that $\kappa_N(x) = \kappa_0(x)$ for all $x \in T$.

Theorem 2 *The set T is robustly finite-time stable for the closed-loop system $x^+ \in f(x, \kappa_N(x), W)$ with a region of attraction X_N.*

6 Geometric Solution

Necessary results are given in §6.1 and the sets X_i and S_i are characterised in §6.2; solutions to the robust time-optimal and robust optimal control problems are given in §6.3 and §6.4 respectively. In the subsequent sections it will be assumed that T is a polytope or is the union of a finite set of polytopes.

6.1 Preliminary results

Proposition 2 justifies the set operations employed in §6.2. Propositions 3 and 4 and Theorem 3 justify the result given in §6.4.

The next key result establishes a relation between the Pontryagin difference and Minkowski sum. As discussed in [11, §4.5] and [12, §3.2], Proposition 2 allows one to develop an algorithm for computing the Pontryagin difference between the union of a finite set of polytopes and a polytope.

Proposition 2 *If $Y \sim Z \neq \emptyset$, where $Y \subset \mathbb{R}^n$ and $Z \subset \mathbb{R}^n$, then*

$$Y \sim Z = [Y^c \oplus (-Z)]^c. \tag{21}$$

Furthermore, if Y is the union of a finite set of closed (open) polyhedra and Z is a closed polyhedron, then $Y \sim Z$ is the union of a finite set of closed (open) polyhedra.

Proposition 3 characterises the solution to a multi-parametric linear program (mp-LP), where the cost is an affine function of the decision variables and a set of parameters and the constraints on the decision variables and parameters are given by a polytope. The reader is referred to [7] for a geometric algorithm for computing the solution to an mp-LP.

Proposition 3 *If*

$$V^0(z) := \min_y \{l'z + m'y + n \mid (z,y) \in C\} \tag{22}$$

$$y^0(z) := \arg\min_y \{l'z + m'y + n \mid (z,y) \in C\} \tag{23}$$

where $(l,m,n) \in \mathbb{R}^{n_z} \times \mathbb{R}^{n_y} \times \mathbb{R}$, C is a polytope and

$$Z := \{z \mid \exists y : (z,y) \in C\}, \tag{24}$$

then $V^0 : Z \to \mathbb{R}$ is a convex, piecewise affine function. Furthermore, there exists a continuous, piecewise affine function[1] $\upsilon : Z \to \mathbb{R}^{n_y}$ such that $\upsilon(z) \in y^0(z)$ for all $z \in Z$.

Proposition 4 will be used in Theorems 3 and 6 to characterise the solution to the robust optimal control problem.

Proposition 4 *If $\{f_1(\cdot), \ldots, f_p(\cdot)\}$ is a finite set of (continuous) piecewise affine functions, where each $f_i : X_i \to \mathbb{R}^n$, then $x \mapsto \min\{f_1(x), \ldots, f_p(x)\}$, $x \mapsto \max\{f_1(x), \ldots, f_p(x)\}$ and $x \mapsto f_1(x) + \ldots + f_p(x)$ are (continuous) piecewise affine functions on $\bigcap_{i=1}^p X_i$. If $g : X \to Y$ and $h : Y \to Z$ are (continuous) piecewise affine functions, then the composite $h \circ g : X \to Z$ is a (continuous) piecewise affine function.*

Theorem 3 characterises the solution to a multi-parametric piecewise affine program (mp-PAP) where the cost is a piecewise affine function of the decision variables and parameters and the constraints on the decision variables and parameters are given by the union of a set of (possibly overlapping) polytopes. This result will be used in §6.4 to characterise the optimal cost and control law.

Theorem 3 *Let $V : D \to \mathbb{R}$ be a piecewise affine function with*

$$V(z,y) := l_s'z + m_s'y + n_s,\ \forall (z,y) \in D_s, \tag{25}$$

where $\{D_s \mid s \in S\}$ is a finite set of polytopes such that $D := \cup_{s \in S} D_s$ and each $(l_s, m_s, n_s) \in \mathbb{R}^{n_z} \times \mathbb{R}^{n_y} \times \mathbb{R}$.

If $\{C_r \mid r \in R\}$ is a finite set of polytopes and $C := \cup_{r \in R} C_r$,

$$V^0(z) := \min_y \{V(z,y) \mid (z,y) \in C\} \tag{26}$$

$$y^0(z) := \arg\min_y \{V(z,y) \mid (z,y) \in C\} \tag{27}$$

[1] Recall that, in general, $y^0(z)$ is set-valued for all $z \in Z$.

and

$$Z := \{z \,|\, \exists y : (z,y) \in C \cap D\}, \tag{28}$$

then $V^0 : Z \to \mathbb{R}$ is a piecewise affine function. Furthermore, there exists a piecewise affine function $\upsilon : Z \to \mathbb{R}^{n_y}$ such that $\upsilon(z) \in y^0(z)$ for all $z \in Z$.

Proof: For each $(r,s) \in R \times S$, let $Z_{r,s}$ be the orthogonal projection of the polytope $C_r \cap D_s$ onto the z-space, i.e.

$$Z_{r,s} := \{z \,|\, \exists y : (z,y) \in C_r \cap D_s\} \tag{29}$$

and let

$$V^0_{r,s}(z) := \min_y \{l_s' z + m_s' y + n_s \mid (z,y) \in C_r \cap D_s\}. \tag{30}$$

Since the non-empty $Z_{r,s}$ are polytopes, by Proposition 3 it follows that each $V^0_{r,s} : Z_{r,s} \to \mathbb{R}$ is a convex (hence continuous) piecewise affine function. Finally, note that $Z = \cup_{(r,s)\in R\times S} Z_{r,s}$ and that for all $z \in Z$,

$$\begin{aligned} V^0(z) &= \min_{y,r} \{V(z,y) \,|\, (z,y) \in C_r \cap D, r \in R\} \\ &= \min_{y,r,s} \{l_s' z + m_s' y + n_s \mid \\ &\qquad (z,y) \in C_r \cap D_s, (r,s) \in R \times S\} \\ &= \min_{r,s} \{V^0_{r,s}(z) \,|\, (r,s) \in R \times S\}. \end{aligned}$$

The statement regarding $V^0(\cdot)$ follows from Proposition 4. That there exists a piecewise affine function $\upsilon : Z \to \mathbb{R}^{n_y}$, such that $\upsilon(z) \in y^0(z)$ for all $z \in Z$, follows from Proposition 3 by using similar arguments as above. ■

Remark 1 *Note that, unlike the proof of [2, Lem. 1], the proof of Theorem 3 does not require the introduction of integer variables and obtaining the solution to a multi-parametric mixed-integer linear program (mp-MILP) [8]; it is sufficient to compare the solutions to the finite number of mp-LPs defined by* (30)*. An algorithm for comparing the solutions to different mp-LPs is described in [8, App. A].*

6.2 Characterisation of the sets X_i and S_i

The recursions (11) and (18) are employed by noting that

$$S_i = \{(x,u) \in X \times U \,|\, f_0(x,u) \in X_{i-1} \sim W\} \tag{31}$$

and recalling that $X_i = \mathrm{Proj}_X(S_i)$. Because $f_0(\cdot)$ is nonlinear, the sets X_i and S_i are not necessarily convex even if X_{i-1} is. However, as noted in [11, Chap. 4] and [12, §4],

Theorem 4 *If T is the union of a finite set of polytopes then, for all $i \geq 1$, S_i and X_i are each the union of a finite set of polytopes.*

Proof: Suppose $X_{i-1} := \bigcup_{j\in L_{i-1}} \Omega_j^{i-1}$, where each Ω_j^{i-1} is a closed polyhedron; X_{i-1}^c is then the union of a finite set of open polyhedra. By Proposition 2, $X_{i-1} \sim W = \left[X_{i-1}^c \oplus (-W)\right]^c$, hence, $X_{i-1}^c \oplus (-W)$ is the union of a finite set of open polyhedra and $X_{i-1} \sim W := \bigcup_{j\in M_i} \Phi_j^i$, where each Φ_j^i is a closed polyhedron. From (31), $S_i = \bigcup_{j\in M_i} \left\{(x,u) \in X \times U \,\middle|\, f_0(x,u) \in \Phi_j^i\right\}$. Since $f_0(\cdot) = f_q(\cdot)$ on each polytope P_q, $S_i = \bigcup_{(q,j)\in Q\times M_i} \left(P_q \cap \Psi_{q,j}^i\right) = \bigcup_{j\in L_i} \Sigma_j^i$, where each $\Psi_{q,j}^i := \left\{(x,u) \in X \times U \,\middle|\, f_q(x,u) \in \Phi_j^i\right\}$ and Σ_j^i are closed polyhedra. Finally, $\Omega_j^i := \mathrm{Proj}_X\left(\Sigma_j^i\right)$ and $X_i = \bigcup_{j\in L_i} \Omega_j^i$. ■

6.3 Solution to the robust time-optimal control problem

Recalling the discussion in §4.1 and §5.1, one can now characterise the optimal cost and control law and derive an algorithm for computing the solution to the robust time-optimal control problem.

Theorem 5 *$X_{N_{\max}}$ is the union of a finite set of polytopes and, for each $x \in X_{N_{\max}}$, the value $\kappa^0(x)$ of the set-valued control law $\kappa^0(\cdot)$ is the union of a finite set of polytopes.*

Proof: That $X_{N_{\max}}$ is the union of a finite set of polytopes follows from Theorem 4. It also follows from the proof of Theorem 4 that if $x \in X_i \setminus X_{i-1}$ for an arbitrary $i \in \{1,2,\ldots N_{\max}\}$ then

$$\begin{aligned} \kappa_i(x) &= \{u \in U \,|\, f_0(x,u) \in X_{i-1} \sim W\} = \{u \,|\, (x,u) \in S_i\} \\ &= \left\{u \,\middle|\, (x,u) \in \cup_{j\in L_i} \Sigma_j^i\right\} = \left\{u \,\middle|\, \exists j \in L_i : (x,u) \in \Sigma_j^i\right\} \\ &= \cup_{j\in L_i} \left\{u \,\middle|\, (x,u) \in \Sigma_j^i\right\}, \end{aligned}$$

where each Σ_j^i is a polytope and hence each $\left\{u \,\middle|\, (x,u) \in \Sigma_j^i\right\}$ is a polytope. The result follows by recalling (14). ■

In order to compute $\kappa^0(x)$ for a given x, one first needs to compute $N^0(x) = \min_i \{i \,|\, x \in X_i, i \in \{1,2,\ldots N_{\max}\}\}$. Since each X_i is the union of a finite set of polytopes, computing $N^0(x)$ amounts to checking a finite number of linear inequalities. From the proof of Theorem 4, $S_{N^0(x)} := \bigcup_{j\in L_{N^0(x)}} \Sigma_j^{N^0(x)}$ and from the proof of Theorem 5 it follows that $\kappa^0(x) = \bigcup_{j\in L_{N^0(x)}} \left\{u \,\middle|\, (x,u) \in \Sigma_j^{N^0(x)}\right\}$. Hence, for a given x, if the control input is selected from any of the polytopes $\left\{u \,\middle|\, (x,u) \in \Sigma_j^{N^0(x)}\right\}$, $j \in L_{N^0(x)}$ then the state of the system will be steered from $X_{N^0(x)}$ to $X_{N^0(x)-1}$ for all admissible disturbances.

6.4 Solution to the robust optimal control problem

Recalling the discussion in §4.2 and §5.2, one can now characterise the optimal cost and control law and derive an algorithm for computing the solutions to the robust optimal and robust receding horizon control problems.

Theorem 6 *For each $i \geq 0$, X_i is the union of a finite set of polytopes and the value function $V_i^0 : X_i \to \mathrm{I\!R}$ is a piecewise affine function. Furthermore, there exists a piecewise affine control law $\upsilon_i : X_i \to U$ such that $\upsilon_i(x) \in \kappa_i(x)$ for all $x \in X_i$.*

Proof: That X_i is the union of a finite set of polytopes follows from Theorem 4. From (16) it follows that if $V_i^*(x,u) := \max_{w \in W} V_{i-1}^0(f(x,u,w))$ for all $(x,u) \in S_i$, then $V_i^0(x) = \min_u \{\ell(x,u) + V_i^*(x,u) \,|\, (x,u) \in S_i\}$ for all $x \in X_i$. Since $f(\cdot)$ is piecewise affine, if $V_{i-1}^0(\cdot)$ is piecewise affine over X_{i-1}, then by Proposition 4 and Theorem 3 it follows that $V_i^*(\cdot)$ is piecewise affine over S_i. Since $\ell(\cdot)$ is piecewise affine over S_i, by Proposition 4 and Theorem 3 it follows that $V_i^0(\cdot)$ is piecewise affine over X_i. The statement regarding $V_i^0(\cdot)$ is completed by recalling that $V_0^0(x) = 0$ for all $x \in X_0$. That there exists a piecewise affine function $\upsilon_i : X_i \to U$ such that $\upsilon_i(x) \in \kappa_i(x)$ for all $x \in X_i$ follows similar arguments by noting that, for all $x \in X_i$, $\kappa_i(x) = \arg\min_u \{\ell(x,u) + V_i^*(x,u) \,|\, (x,u) \in S_i\}$. ■

Remark 2 *Note that it is sufficient to compute $V_i^0(x)$ and $\kappa_i(x)$ only for all $x \in X_i \setminus T$, since $\ell(x,u) = 0$ and $F(x) = 0$ for all $(x,u) \in T \times U$ and hence $V_i^0(x) = 0$ and $\kappa_i(x) = \kappa_0(x)$ for all $x \in T$.*

Remark 3 *A typical choice for a piecewise affine stage cost that satisfies A2 is to use $\ell(x,u) := |Qx|_1 + |Ru|_1$ or $\ell(x,u) := |Qx|_\infty + |Ru|_\infty$, where Q is a positive definite matrix and R is a positive, semi-definite matrix. Since it is assumed that $\ell(x,u) = 0$ for all $(x,u) \in T \times U$, continuity of the value function and control law over X_i cannot be guaranteed. However, the optimal cost is still piecewise affine on X_i and it is still possible to compute an optimal control law that is piecewise affine on X_i.*

7 Conclusions

The solutions to the problems of robust time-optimal, robust optimal and robust receding horizon control of a piecewise affine system with a persistent, but bounded, disturbance have been characterised. For the robust time-optimal control problem, the robustly controllable sets were shown to be finite unions of polytopes that can be computed using the results given above; the time-optimal control is determined by simple optimization over these sets. For the robust optimal and robust receding horizon control problems, the optimal value functions and control laws were shown to be piecewise affine (provided the stage cost is piecewise affine) and their domains the union of a finite set of polytopes; the optimal control is determined by simply checking a finite number of inequalities at each time step. Finally, it is worth mentioning that algorithms based directly on the results presented here might be too inefficient to be realisable for large or complex systems. As such, current research is aimed at finding more efficient algorithms for the computation and implementation of the control laws discussed in this paper.

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