On the Computation of the Induced $\mathcal{L}_2$ Norm of Single Input Linear Systems with Saturation

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Abstract

In this paper, a means of determining an upper bound of the induced $\mathcal{L}_2$ norm for a class of single input linear systems with saturation is given in terms of the existence of a candidate function which satisfies three differential inequalities. This amounts to a generalisation of finding the solution of a single algebraic Riccati equation in the linear problem. A technique to calculate such a function for systems with linear controllers is also developed. A worked first order example is then presented. The operator norm is tightly bounded by the use of a piecewise smooth candidate function which is quadratic within a certain region and cubic outside.

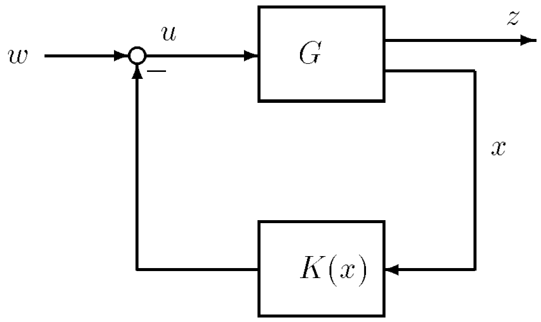

Figure 1: System Configuration

1 Introduction

In the study of input/output based approaches to robust control, the ability to compute induced operator norms on signal spaces of interest is a useful technical tool. For shift-invariant linear systems on $\mathcal{L}_{2+}$ spaces, the induced norm is the $\mathcal{H}_\infty$ norm of the transfer function matrix, and can be bounded by the solution of a single algebraic Riccati equation. Extending these results to nonlinear systems has been a topic of recent interest, such as in the papers [3],[7],[8],[10].

In this paper, we are studying the case of single input linear systems with input saturation. By applying the concept of dissipativity introduced by Willems in [11], we are able to reduce the problem to finding a (piecewise) smooth candidate function whose derivatives satisfy three differential inequalities.

For a subclass of closed loop systems, a technique is presented to calculate a valid candidate function. This technique is generalisable to more general cases, but restricting assumptions were made to make the presentation more readable. The result is presented as an existence theorem, which formalises the connection between state space and input-output stability, and gives the requisite test function in a closed form.

A fuller survey of background papers and recent, related, results will not be presented herein due to space limitations; these are available in [9]. However, it should be noted that the same problem framework as this paper has been addressed in the recent work by Chitour, Liu, and Sontag in [8]. Their results hold for a subclass of linear plants (specifically, it is assumed that the plant is neutrally stable and there is no direct feedthrough term, and a specific feedback controller is fixed), but a more general class of nonlinearities, multiple inputs, and all $\mathcal{L}_\mathcal{P}$ norms are dealt with. However, for the most basic system where a comparison can be made (the plant being an integrator following the standard saturation, with unity negative feedback), the bound for the induced $\mathcal{L}_{2+}$ norm given in [8] is conservative, whereas the technique given in this paper determines a tight bound.

2 Mathematical Preliminaries

Definition 2.1: The operators whose norm we are measuring are the closed loop mappings Υ (resp. Υ_x), which are the mappings from the external disturbance w to z (resp. x), and generated by the following differential equations:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 u(t) + B_2 \sigma_0(u(t)), \\
&= f(x(t), u(t)), \\
u(t) &= w(t) - K(x(t)), \\
x(0) &= 0, \\
z(t) &= h(x(t)) + D_1 u(t) + D_2 \sigma_0(u(t)), \\
&= g(x(t), u(t)).
\end{aligned}$$

In order to simplify expressions, the following notations are used:

$$B = B_1 + B_2, D = D_1 + D_2, d(u) = D_1 u + D_2 \sigma_0(u).$$

Additionally, we impose the following condtions on G and K:

1. The system is n-dimensional, single input with p outputs; i.e. $A \in \mathbb{R}^{n\times n}$, $B_1, B_2 \in \mathbb{R}^{n\times 1}$, $h(x)\colon \mathbb{R}^n \mapsto \mathbb{R}^p$, $D_1, D_2 \in \mathbb{R}^{p\times 1}$.
2. The state feedback K, and the state readout function $h(x)$ are assumed to be *uniformly Lipschitz* over any arbitrary compact set, with $K(0) = 0$, $h(0) = 0$.
3. The derivatives of $K(x), h(x)$ are assumed to exist at the origin, and the linearisation about the origin for the closed loop system has only open left half plane poles.

The function $\sigma_0(\cdot)$ appearing in the description is the standard saturation function;

$$\sigma_0(x) = \operatorname{sign}(x) \min\{|x|, 1\},$$

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{otherwise.} \end{cases}$$

The space norm used throughout is the Euclidean norm, denoted by $\|\cdot\|$. We are interested in the $\mathcal{L}_{2+}$ norm of signals, defined as:

$$\|y\|_2^2 = \int_0^\infty \|y(t)\|^2 \,\mathrm{dt}.$$

Additionally, the infinity norm of signals is used, where

$$\|y\|_\infty = \operatorname{ess}\sup_{t\in\mathbb{R}_+} \{\|y(t)\|\}.$$

The results in this paper are given in *semi-global* form, and in particular, the results apply to systems where the state is confined to a bounded subset of $\mathbb{R}^n$ in response to bounded disturbances. The proofs can (in general) be extended to the global case, in fact the example in Section 4.1 relies on taking the limit as the state goes to infinity. The reason for this restriction is not merely a technical one; rather it allows the study of a more interesting sets of problems (such as open loop unstable plants), as has been noted in recent literature (see the discussion in[7] and justification given in [9]).

Assumption 2.2: Let $\mathcal{W}_2(N_\infty) \subset \mathcal{L}_{2+} \bigcap \mathcal{L}_\infty$ be the set of $\mathcal{L}_{2+}$ disturbance signals with a bound on their infinity norm:

$$\mathcal{W}_2(N_\infty) = \{w : w \in \mathcal{L}_{2+}, \|w\|_\infty < N_\infty\}.$$

It is then assumed that there exists a set $\Lambda \subset \mathbb{R}^n$ with compact closure such that for all $w \in \mathcal{W}_2$, $\boldsymbol{\Upsilon}_x w(t) \in \Lambda$ for all $t \in \mathbb{R}_+$.

The following two definitions are standard system properties, expressed using operator terminology. A fuller discussion of these properties is found in [9].

Definition 2.3: The system is *Λ-reachable over $\mathcal{W}_2$* if for all $\xi \in \Lambda$, there exists a $w \in \mathcal{W}_2$ such that for some $t \in \mathbb{R}_+$, $(\boldsymbol{\Upsilon}_x w)(t) = \xi$.

Definition 2.4: The system meets the *observability condition* if for all $w \in \mathcal{W}_2$ such that $\boldsymbol{\Upsilon} w \in \mathcal{L}_{2+}$, then $\boldsymbol{\Upsilon}_x w \in \mathcal{L}_{2+}$.

2.1 Candidate Functions and Norm Computation

The basic background theory for the norm computation will now be given. The results rely on the use of Lyapunov-like functions, termed *candidate functions* in this paper. The functions are tested against a dissipation condition, which is derived in this paper.

Definition 2.5: The operator $\boldsymbol{\Upsilon}$ has an *induced $\mathcal{L}_{2+}$ norm* of γ^* over $\mathcal{W}_2$ if for all $w \in \mathcal{W}_2$, $z = \boldsymbol{\Upsilon} w$ lies in $\mathcal{L}_{2+}$, and

$$\gamma^* = \inf_{\gamma \in (0,\infty)} \{\gamma : \gamma^2 \|w\|_2^2 \geq \|z\|_2^2, \forall w \in \mathcal{W}_2\}.$$

Definition 2.6: A function $V : \mathbb{R}^n \mapsto \mathbb{R}_+$ is a *candidate function* if V is Lipschitz, and $V(0) = 0$.

Theorem 2.7 (Rademacher's): A candidate function V is differentiable almost everywhere.
Proof: See [13, Theorem 2.2.1] for details. □

Theorem 2.8: Given the system in Definition 2.1, which is reachable and meets the observability condition over the sets $\mathcal{W}_2, \Lambda$. Fix a candidate function V. Let $\Lambda_d \subseteq \Lambda$ be the set where $\frac{dV}{dx}$ exists. Then if γ is fixed, the Hamiltonian H^* is defined on Λ_d:

$$\begin{aligned} H^*(\xi) \quad = \quad \sup_{\omega \in \mathbb{R}} \Bigg\{ & H(\xi,\omega) = 2 \left.\frac{dV}{dx}\right|_\xi f(\xi, \omega - K(\xi)) \\ & -\gamma^2 \|\omega\|^2 + \|g(\xi, \omega - K(\xi))\|^2 \Bigg\}. \end{aligned}$$

Then $\|\boldsymbol{\Upsilon} w\|_2^2 \leq \gamma^2 \|w\|_2^2$ for all $w \in \mathcal{W}_2$ is implied by $H^*(\xi) \leq 0$ for all $\xi \in \Lambda_d$.
Proof: This theorem is originally due to Willems ([11], [12]), however later work (as in James [5]) has studied the consequences of relaxing the smoothness conditions. □

Lemma 2.9: For a system of the form given in Definition 2.1, the induced norm γ^* over any nontrivial set $\mathcal{W}_2$ is greater than or equal to the $\mathcal{H}_\infty$ norm of the linearised system at the origin.
Proof: This can be shown by scaling linear signals which (in the limit) achieve the closed loop norm. □

Assumption 2.10: In order for the algebraic manipulations used in this paper to hold, it is assumed that $\gamma \geq \max\{\|D\|, \|D_1\|\}$. (The assumption that $\gamma > \|D\|$ is not a restriction, as this is a consequence of Lemma 2.9.)

3 A Test Condition to Bound the Norm

In order to apply the dissipation results, it is preferable to eliminate the dependence upon the external disturbance in the point-wise test condition. This is done by the technique of the completion of squares.

Theorem 3.1: Consider a state space system with the same state and input/output dimensions as the system in Definition 2.1, described by the following equations:

$$\begin{aligned} \dot{x} &= f(x) + B(\omega - K(x)), && (1)\\ z &= h(x) + D(\omega - K(x)). && (2) \end{aligned}$$

Given a candidate function V which is differentiable at ξ, and a fixed choice for the quantity γ. We can then define the real-valued functions θ,Φ,and H:

$$\begin{aligned} \theta(\xi) &= -K(\xi) + \frac{1}{\gamma^2 - \|D\|^2}\left(B^T \left.\frac{d^T V}{dx}\right|_\xi \right. \\ &\quad \left. + D^T h(\xi) - \|D\|^2 K(\xi)\right), \\ \Phi(\xi) &= 2\left.\frac{dV}{dx}\right|_\xi (f(\xi) + B\theta(\xi)) \\ &\quad -\gamma^2 \|K(\xi) + \theta(\xi)\|^2 + \|h(\xi) + D\theta(\xi)\|^2, \\ H(\xi, W) &= 2\left.\frac{dV}{dx}\right|_\xi f(\xi) - \gamma^2 \|W\|^2 \\ &\quad + \|h(\xi) + D(W - K(\xi))\|^2. \end{aligned}$$

Then,

$$H(\xi, \omega) = -(\gamma^2 - \|D\|^2)\|\omega - K(\xi) - \theta(\xi)\|^2 + \Phi(\xi),$$

and hence $H(\xi, \omega) \leq 0$ is implied by $\Phi(\xi) \leq 0$.
Proof: See [10], and references therein. □

The motivation for result and notation is that for systems of the class described in Theorem 3.1, $\theta(\xi)$ represents a maximising $u(t)$ at the point ξ. To apply the result to linear systems with saturation, the completion of squares is done along all three piecewise linear segments of $\sigma_0(u)$, giving rise to three test conditions.

Theorem 3.2: Take the system given in Definition 2.1, which is observable and reachable over the sets $\mathcal{W}_2, \Lambda$. The closed loop mapping $\boldsymbol{\Upsilon}: w \mapsto z$ will have an upper bound γ to its induced $\mathcal{L}_{2+}$ norm over $\mathcal{W}_2$, with $\gamma > \max\{\|D\|, \|D_1\|\}$, if there exists a candidate function V, whose derivative exists on the set Λ_d (with $\Lambda \setminus \Lambda_d$ having zero measure), and

$$\sup_{\xi \in \Lambda_d} \left\{ \max \Phi_A(\xi), \Phi_B^+(\xi), \Phi_B^-(\xi) \right\} \leq 0. \tag{3}$$

The functions Φ_A, Φ_B^+, and Φ_B^- will now be defined upon the set Λ_d. Let

$$\Phi_A(\xi) = \begin{cases} \Phi_{A.1}(\xi) & \text{if } |\theta_a(\xi)| \leq 1, \\ \Phi_{A.2}(\xi) & \text{if } |\theta_a(\xi)| > 1, \end{cases}$$

where $\theta_a, \Phi_{A.1}, \Phi_{A.2}$ are defined as:

$$\begin{aligned} \theta_a(\xi) &= -K(\xi) + \frac{1}{\gamma^2 - \|D\|^2} \left(B^T \left. \frac{d^T V}{dx} \right|_\xi \right. \\ &\qquad \left. + D^T h(\xi) - \|D\|^2 K(\xi) \right), \\ \Phi_{A.1}(\xi) &= 2 \left. \frac{dV}{dx} \right|_\xi (A\xi + B\theta_a(\xi)) \\ &\qquad -\gamma^2 \|K(\xi) + \theta_a(\xi)\|^2 + \|g(\xi, \theta_a(\xi))\|^2, \\ \Phi_{A.2}(\xi) &= 2 \left. \frac{dV}{dx} \right|_\xi (A\xi + B\operatorname{sign}(\theta_a(\xi))) \\ &\qquad -\gamma^2 \|K(\xi) + \operatorname{sign}(\theta_a(\xi))\|^2 \\ &\qquad + \|g(\xi, \operatorname{sign}(\theta_a(\xi)))\|^2. \end{aligned}$$

The functions Φ_B^+, Φ_B^- are defined as:

$$\begin{aligned} \Phi_B^+(\xi) &= \begin{cases} \Phi_{B.1}^+(\xi) & \text{if } \theta_b^+(\xi) \geq 1, \\ \Phi_{B.2}^+(\xi) & \text{if } \theta_b^+(\xi) < 1, \end{cases} \\ \Phi_B^-(\xi) &= \begin{cases} \Phi_{B.1}^-(\xi) & \text{if } \theta_b^-(\xi) \leq -1, \\ \Phi_{B.2}^-(\xi) & \text{if } \theta_b^-(\xi) > -1. \end{cases} \end{aligned}$$

The functions $\theta_b^+, \theta_b^-, \Phi_{B.1}^+, \Phi_{B.2}^-, \Phi_{B.2}^+, \Phi_{B.1}^-$ are defined:

$$\begin{aligned} \theta_b^\pm(\xi) &= -K(\xi) + \frac{1}{\gamma^2 - \|D_1\|^2} \left(B_1^T \left. \frac{d^T V}{dx} \right|_\xi \right. \\ &\qquad \left. + D_1^T (h(\xi) \pm D_2) - \|D_1\|^2 K(\xi) \right), \\ \Phi_{B.1}^\pm(\xi) &= 2 \left. \frac{dV}{dx} \right|_\xi \left(A\xi + B_1 \theta_b^\pm(\xi) \pm B_2 \right) \\ &\quad -\gamma^2 \left\| K(\xi) + \theta_b^\pm(\xi) \right\|^2 + \left\| h(\xi) + D_1 \theta_b^\pm(\xi) \pm D_2 \right\|^2, \\ \Phi_{B.2}^\pm(\xi) &= 2 \left. \frac{dV}{dx} \right|_\xi (A\xi \pm B) - \gamma^2 \|K(\xi) \pm 1\|^2 \\ &\qquad + \|h(\xi) \pm D\|^2. \end{aligned}$$

Proof:

It will be shown that $H(\xi,\omega) \leq 0$ is implied by (3) holding. Then by Theorem 2.8, the norm condition follows.

Fix any point $\xi \in \Lambda_d$, and a choice of $\omega \in \mathbb{R}$. One of three cases is true: either (1) $\omega - K(\xi) > 1$,(2) $-1 \leq \omega - K(\xi) \leq 1$, or (3) $\omega - K(\xi) < -1$.

Examine case (1), and note that case (3) is symmetric. Then, there are two possibilities; either $\theta_b^+(\xi) \geq 1$, or $\theta_b^+(\xi) < 1$.

In either possible case, we have

$$\begin{aligned} H(\xi,\omega) = 2 \left.\frac{dV}{dx}\right|_\xi & (A\xi + B_1(\omega - K(\xi)) + B_2) \\ -\gamma^2 \|\omega\|^2 & + \|h(\xi) + D_1(\omega - K(\xi)) + D_2\|^2 . \end{aligned}$$

If $\theta_b^+(\xi) \geq 1$, by the condition that $\Phi_B^+(\xi) \leq 0$, we have that $\Phi_{B.1}^+(\xi) \leq 0$. It then is a straightforward application of Theorem 3.1 to see that this implies $H(\xi,\omega) \leq 0$. If $\theta_b^+(\xi) < 1$, we have by assumption that $\Phi_{B.2}^+(\xi) \leq 0$. Then,

$$\begin{aligned} H(\xi,\omega) &\leq H(\xi,\omega) - \Phi_{B.2}^+(\xi), \\ &= (H(\xi,\omega) - \Phi_{B.1}^+(\xi)) - (\Phi_{B.2}^+(\xi) - \Phi_{B.1}^+(\xi)), \\ &\leq -\Big\{(\gamma^2 - \|D_1\|^2) \\ &\quad \left(\left\|\omega - K(\xi) - \theta_b^+(\xi)\right\| - \left\|1 - \theta_b^+(\xi)\right\|\right) \\ &\quad \left(\left\|\omega - K(\xi) - \theta_b^+(\xi)\right\| + \left\|1 - \theta_b^+(\xi)\right\|\right)\Big\}, \\ &\leq 0, \end{aligned}$$

as we have that $\omega - K(\xi) > 1$, and $\theta_b^+(\xi) < 1$.

For case (2) (i.e. $|\omega - K(\xi)| \leq 1$), there are (once again) two possibilities: either $|\theta_a(\xi)| \leq 1$ or $|\theta_a(\xi)| > 1$. If $|\theta_a(\xi)| \leq 1$, it follows from Theorem 3.1 that $H(\xi,\omega) \leq 0 \impliedby \Phi_A(\xi) = \Phi_{A.1}(\xi) \leq 0$.

If $|\theta_a(\xi)| > 1$, by the stated condition, $\Phi_{A.2}(\xi)$ is less than or equal 0. Then,

$$\begin{aligned} H(\xi,\omega) &\leq H(\xi,\omega) - \Phi_{A.2}(\xi), \\ &= (H(\xi,\omega) - \Phi_{A.1}(\xi)) - (\Phi_{A.2}(\xi) - \Phi_{A.1}(\xi)). \\ &\leq -\Big\{(\gamma^2 - \|D\|^2)(\|\omega - K(\xi) - \theta_a(\xi)\| \\ &\quad - \|\theta_a(\xi)\| + 1)(\|\omega - K(\xi) - \theta_a(\xi)\| \\ &\quad + \|\theta_a(\xi)\| - 1)\Big\}, \\ &\leq 0, \end{aligned}$$

by the Triangle Inequality. We have thus shown that $H(\xi,\omega) \leq 0$, and hence the norm condition follows. □

Remark 3.3: Some of the test conditions given in Theorem 3.2 are possibly redundant, and the tests collapse down to a lesser number, in some cases one.

1. Note that $\Phi_{A.1}(\xi) \leq 0 \implies \Phi_A(\xi), \Phi_{A.2}(\xi) \leq 0$. This follows as $\Phi_{A.2}(\xi) - \Phi_{A.1}(\xi) = -(\gamma^2 - \|D\|^2)\|\mathrm{sign}(\theta_a(\xi)) - \theta_a(\xi)\|^2$.

2. $\Phi^{\pm}_{B.2} \leq \Phi_{A.2}$, hence $\Phi_A(\xi) \leq 0 \implies \Phi^{\pm}_{B.2}(\xi) \leq 0$.

3. If $|\frac{D_1^T D_2}{\gamma^2 - \|D_1\|^2}| \leq 2$, it is not possible for $\theta_b^+(\xi) > 1$ and $\theta_b^-(\xi) < -1$ simultaneously, and one (or both) of the conditions that $\Phi^{\pm}_B(\xi) \leq 0$ will be implied by $\Phi_A(\xi) \leq 0$.

4. If $D_1^T D_2 = 0$, $\theta_b^+ = \theta_b^-$. It is then possible to define $\Phi_B(\xi), \theta_b(\xi)$ as follows:

$$\begin{aligned}\theta_b(\xi) &= -K(\xi) + \frac{1}{\gamma^2 - \|D_1\|^2}\left(B_1^T \left.\frac{d^T V}{dx}\right|_\xi \right.\\ &\quad \left. + D_1^T h(\xi) - \|D_1\|^2 K(\xi)\right),\end{aligned}$$

$$\Phi_B(\xi) = \begin{cases} \Phi_{B.1}(\xi) & \text{if } |\theta_b(\xi)| \geq 1, \\ \Phi_{B.2}(\xi) & \text{otherwise,} \end{cases}$$

with

$$\begin{aligned}\Phi_{B.1}(\xi) &= 2\left.\frac{dV}{dx}\right|_\xi (A\xi + B_1\theta_b(\xi) \\ &+ B_2 \mathrm{sign}(\theta_b(\xi))) - \gamma^2 \|K(\xi) + \theta_b(\xi)\|^2 \\ &+ \|g(\xi, \theta_b(\xi))\|^2, \\ \Phi_{B.2}(\xi) &= 2\left.\frac{dV}{dx}\right|_\xi (A\xi + B\mathrm{sign}(\theta_b(\xi))) \\ &- \gamma^2 \|K(\xi) + \mathrm{sign}(\theta_b(\xi))\|^2 \\ &+ \|g(\xi, \mathrm{sign}(\theta_b(\xi)))\|^2.\end{aligned}$$

The norm condition then amounts to requiring that $\Phi_A(\xi), \Phi_B(\xi) \leq 0$ holds for all ξ in Λ_d.

4 A Computational Heuristic

Having set forth the test condition on candidate functions, some means of computing these functions is needed. There are basic existence theorems, which are expressed in terms of the solution of dynamic programming problems. The paper by van der Schaft [10] (for example) makes use of such results. The paper by James [6], discusses a generic algorithm to make such a computation, which is in fact applicable to this class of systems.

In this section, a heuristic approach to generating candidate functions explicitly is given. The computation is reminiscent of the inverse Lyapunov theorems developed in the papers by Willems [12], and by Hill and Moylan [4]. In fact, the present treatment captures the Lyapunov results by letting γ tend to infinity. The result is presented as an existence result, which in fact shows the relationship between state space stability (null input) and stability in the input/output sense.

It is not claimed that this technique will determine the optimal bound for the norm, as it is not presented as an iterative technique. It may be possible to iteratively improve the results, as there are free parameters which may be tuned; there are no guarantees on such a procedure, however. The most attractive possibility is to solve the constraint conditions algebraically in a piecewise fashion over the state space; it is unclear how this could be implemented at present.

In order to cleanly present the results, an added set of assumptions on the class of systems is given. It is believed that all of these could be relaxed, but at the cost of making the presentation quite difficult.

Assumption 4.1: The systems for which results will be derived form a subset of the systems given in Definition 2.1; the following restrictions being added:

1. The matrix B_1 is assumed to be the zero vector; hence $B = B_2$.
2. The controller is linear, or $K(x) = Kx$.
3. The output readout function is linear, or $h(x) = Cx$.
4. In order to simplify the algebra, it is assumed that $C^T D = 0$, and that $D_1^T D_2 = 0$.

Definition 4.2: Let $V_l(\xi) = \xi^T P \xi$, with $P > 0$ being the solution to the algebraic Riccati equation (4):

$$\begin{aligned} &(A + B\Theta_L)'P + P(A + B\Theta_L) \\ &\quad -\gamma_l^2 (K + \Theta_L)'(K + \Theta_L) \\ &+(C + D\Theta_L)'(C + D\Theta_L) + Q = \quad 0, \end{aligned} \tag{4}$$

with $\Theta_L = -K + \frac{1}{\gamma_l^2 - \|D\|^2}(B'P - \|D\|^2 K)$, Q an arbitrary positive definite matrix, γ_l being a suboptimal choice for which a solution exists. It is assumed that

$$\begin{aligned} &\tfrac{1}{\gamma_l^2 - \|D\|^2}(B^T P - \|D\|^2 K)^T (B^T P - \|D\|^2 K) \\ &\qquad +Q > \tfrac{\|D_1\|^4}{\gamma_l^2 - \|D_1\|^2} K^T K. \end{aligned} \tag{5}$$

Definition 4.3: Let $\psi(\xi)\colon \mathbb{R}^n \mapsto \mathbb{R}$ be defined:

$$\begin{aligned} \psi(\xi) = \min \Bigg\{ &\frac{-1}{\gamma_l^2 - \|D_1\|^2} \left\| \|D_1\|^2 K\xi \right\|^2, \\ &2\xi' P(A\xi - B\sigma_0(K\xi)) + \|C\xi - d(K\xi)\|^2 \Bigg\}. \end{aligned}$$

Remark 4.4: The function ψ has some properties which should be noted. Firstly, $\psi(\xi)$ is uniformly Lipschitz over a compact set, as it is formed by the sums and products of functions which are piecewise uniformly Lipschitz over compact sets. Secondly, by applying the completion of squares to the linear system, and using the assumed relation in equation (5), we have that $\psi(\xi) = -\frac{1}{\gamma_l^2 - \|D\|^2} \left\| B^T P\xi - \|D\|^2 K\xi \right\|^2 - \xi^T Q \xi$ if $|K\xi| \leq 1$.

Definition 4.5: Let the *induced trajectory* from the point $\xi \in \bar{\Lambda}$ (written as $x(t;\xi)$), be the trajectory defined by:

$$\begin{aligned} \dot{x}(t) &= Ax(t) - B\sigma_0(Kx(t)), \\ x(0) &= \xi. \end{aligned}$$

In order to eliminate any technical difficulties arising due to behaviour along the boundary, the following assumption is made.

Assumption 4.6: It is assumed that for all points $\xi \in \partial\Lambda$, there exists a finite $t(\xi) > 0$ such that $x(t;\xi) \in \Lambda$.

Definition 4.7: Let $P_\infty = P_\infty^T > 0$ be the solution to the Lyapunov equation of the linear dynamic system, i. e. :

$$(A - BK)^T P_\infty + P_\infty(A - BK) = -I_n.$$

Definition 4.8: Let $\Lambda_l \subset \mathbb{R}^n$ be the set

$$\Lambda_l = \{\xi : \xi^T P_\infty \xi \leq b\},$$

with $b = \min_{\xi:|K\xi|=1} \xi^T P_\infty \xi$.

Lemma 4.9: For all $\xi \in \Lambda_l$, $x(t;\xi) \in \Lambda_l$, and $|Kx(t;\xi)| \leq 1$ for all $t \in \mathbb{R}_+$.
Proof: This follows from the linear system properties. □

Lemma 4.10: If $x(\xi) \in \mathcal{L}_{2+}$ for all $x \in \bar{\Lambda}$, there then exists a finite constant T_Λ such that $T_\Lambda = \max_{\xi\in\bar{\Lambda}} T(\xi)$, with $T : \bar{\Lambda} \mapsto \mathbb{R}_+$ defined as

$$T(\xi) = \min_{t\in R_+}\{t : x(t;\xi) \in \Lambda_l\}.$$

Proof: The details of this proof consist in showing that T is a continuous function, and are omitted. □

Lemma 4.11: Let $U : \bar{\Lambda} \mapsto \mathbb{R}$ be defined:

$$U(\xi) = \int_0^\infty \alpha(x(t;\xi))dt,$$

with $\alpha : \bar{\Lambda} \mapsto \mathbb{R}$ having a (uniform) Lipshitz constant k, and with $\alpha(0) = 0$. Then if $x(t;\xi) \in \mathcal{L}_{2+}$ for all ξ in $\bar{\Lambda}$, U is uniformly Lipschitz.
Proof: This proof is omitted; it follows from Lemma 4.10 and the Bellman-Gronwall Inequality. □

Theorem 4.12: Fix a closed loop system in the class of systems specified by Assumption 4.1, which is also reachable and observable over the sets $\mathcal{W}_2, \Lambda$, and meets the condition in Assumption 4.6. The following statements are then equivalent:

1. For all $w \in \mathcal{W}_2$, $\boldsymbol{\Upsilon} w$ is an element of $\mathcal{L}_{2+}$.

2. For all $\xi \in \Lambda$, $x(t;\xi)$ is an element of $\mathcal{L}_{2+}$.

3. The function $V\colon \bar{\Lambda} \mapsto \mathbb{R}_+$ defined:

$$\begin{aligned} V(\xi) &= \frac{1}{2}\int_0^\infty \|g(x(t;\xi) - Kx(t;\xi))\|^2 \\ &\quad -\psi(x(t;\xi))dt, \end{aligned} \tag{6}$$

is uniformly Lipschitz on $\bar{\Lambda}$, and satisfies the conditions of Theorem 3.2 over the set Λ for some finite γ.

Proof: (1) $\implies$ (2): This follows from the assumptions of reachability and stability.

(2) $\implies$ (3): By the assumption of continuity and the given implication, we have that $x(\xi) \in \mathcal{L}_{2+}$ for all $\xi \in \bar{\Lambda}$. By applying Lemma 4.11, we see that V defined in (6) is uniformly Lipschitz. It only remains to be shown that there exists a finite choice of $\gamma \geq \gamma_l$ such that the conditions of Theorem 3.2 hold.

The claim will be shown by verifying that for any fixed point in Λ_d there exists a choice of γ which forces Φ_A, Φ_B (see Remark 3.3) to be less than or equal to zero, expressed in terms of functions which remain bounded over $\bar{\Lambda}$.

To prove the above claim, fix any point $\xi \in \Lambda_d$. It first should be noted that by its construction, V satisfies the partial differential equation:

$$\left.\frac{dV}{dx}\right|_\xi (A\xi - B\sigma_0(K\xi)) + \|g(\xi, -K\xi)\|^2 = \psi(\xi). \tag{7}$$

There then are three possible cases to examine, which are considered in turn.

1. The first possibility is that ξ lies in the set $\Lambda_l \bigcap \{\xi\colon (1+\frac{\|D\|^2}{\gamma_l^2 - \|D\|^2})|K\xi| < 1\}$. Over the set Λ_l, $V(\xi)$ is forced to equal $V_l(\xi)$ by the construction of $\psi(x)$. It is easy to verify that if γ is chosen to be γ_l,$\Phi_A(\xi) = -\xi^T Q\xi$. And over this subset, we have that $|\theta_b(\xi)| < 1$ for all choices of $\gamma \geq \gamma_l$, and so $\Phi_B(\xi) = \Phi_{B.2}(\xi)$. By noting Remark 3.3, this implies that $\Phi_B(\xi) \leq 0$ for all γ greater than or equal γ_l.

2. The second case is where $|K\xi| \leq 1$, but $\xi \notin \Lambda_l \bigcap \{\xi\colon (1 + \frac{\|D\|^2}{\gamma_l^2 - \|D\|^2})|K\xi| < 1\}$. Examine $\Phi_{A.1}$ (if $|\theta_a(\xi)| > 1$, the condition will be weaker than that indicated here): by noting the constitutive equation for V (equation (7)), the relation on ψ given in Remark 4.4, and performing the manipulations analogous to the completion of squares, it results that

$$\begin{aligned} \Phi_{A.1}(\xi) &= \frac{1}{\gamma^2 - \|D\|^2}\left\| \left.\frac{dV}{dx}\right|_\xi B - \|D\|^2 K\xi \right\|^2 \\ &\quad -\frac{1}{\gamma_l^2 - \|D\|^2}\left\|\xi^T PB - \|D\|^2 K\xi\right\|^2 \\ &\quad -\xi^T Q\xi. \end{aligned}$$

Therefore, a finite choice of γ exists which makes $\Phi_A(\xi) \leq 0$. When $\Phi_B(\xi)$ is examined, there are two possibilities, depending whether $\left(1 + \frac{\|D_1\|^2}{\gamma^2 - \|D_1\|^2}\right)|K\xi| \leq 1$. If so, $\Phi_B(\xi) =$

$\Phi_{B.2}(\xi)$, and then $\Phi_B(\xi) \leq 0 \iff \Phi_A(\xi) \leq 0$. Otherwise, as $\left|\left(1+\frac{\|D_1\|^2}{\gamma^2-\|D_1\|^2}\right) K\xi\right| > 1$, $1 \geq |K\xi| > 1 - \frac{\|D_1\|^2}{\gamma^2}$. If the set Z is defined $Z = \bar{\Lambda} \bigcap \{\xi : 1 - \frac{\|D_1\|^2}{\gamma_l^2} \leq |K\xi| \leq 1\}$, then a conservative choice of γ to force $\Phi_B(\xi) < 0$ is

$$\gamma^2 > \frac{\max_{\xi \in Z} 2\|D_1\|^2 \left\{\left\| \left.\frac{dV}{dx}\right|_\xi B\right\| + \|D_2\|^2\right\}}{\min_{\xi \in Z}\left\{\frac{\gamma_l^2}{\gamma_l^2-\|D_1\|^2}\, \|\|D_1\|\, K\xi\|^2 - \psi(\xi)\right\}}.$$

This expression is bounded, as $\left\|\frac{dV}{dx}\right\|$ is bounded.

3. The final possibility is that $|K\xi| > 1$. Here, $|\theta_b(\xi)| > 1$. Then, examining Φ_B, we have that

$$\begin{aligned}\Phi_B(\xi) &= -\gamma^2 \|K\xi + \theta_b(\xi)\|^2 - \psi(\xi) \\ &\quad - \|C\xi + d(K\xi)\|^2 + \|C\xi + d(\theta_b(\xi))\|^2 .\end{aligned}$$

Since $\psi(\xi) \leq \frac{1}{\gamma_l^2-\|D_1\|^2}\left\|\|D_1\|^2 K\xi\right\|^2$,

$$\begin{aligned}\Phi_B(\xi) &\leq \frac{1}{\gamma^2 - \|D_1\|^2}\left\|\|D_1\|^2 K\xi\right\|^2 \\ &\quad - \frac{1}{\gamma_l^2-\|D_1\|^2}\left\|\|D_1\|^2 K\xi\right\|^2 \leq 0.\end{aligned}$$

By using arguments parallel to that in the previous case, $\Phi_A(\xi)$ can be bounded above by zero over some set $\Lambda_d \bigcap \{\xi : 1 < |K\xi| \leq 1 + \Delta\}$, for some $\Delta > 0$. Then, over the remainder of the set, Φ_A can be conservatively bounded as

$$\begin{aligned}\Phi_A(\xi) &\leq 2 \left.\frac{dV}{dx}\right|_\xi B\left(\mathrm{sign}(K\xi) - \sigma_0(\theta_a(\xi))\right) \\ &\quad -\gamma^2 \|\Delta\|^2 .\end{aligned}$$

Hence, the choice of γ required is bounded over Λ_d.

(3) $\implies$ (1): This is a consequence of Theorem 3.2. □

4.1 Example

We will now examine the case of the calculation of the norm of the parallel projection operator for the case of an integrator with a saturation element, and with unity negative feedback. The paper by Doyle, Georgiou, and Smith ([1]) defines and discusses the importance of such operators in robust control theory. The parallel projection operator $\mathbf{\Pi_1}$ is defined as the mapping

$$\mathbf{\Pi_1}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ y_1 \end{bmatrix},$$

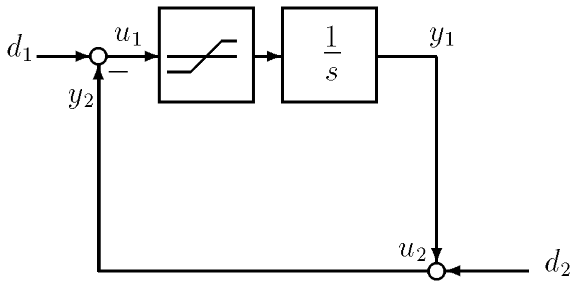

Figure 2: Example System

for the signals as defined in Figure 2.

Although the operator has two inputs, we define $w = d_1 - d_2$, and the readily checked claim is that if γ^* is the norm for the thus defined single input system, the parallel projection operator itself has a norm of $\sqrt{2}\gamma^*$.

The system has the realisation

$$\begin{aligned} \dot{x}(t) &= \sigma_0(w(t) - x(t)), \\ x(0) &= 0, \\ z &= \begin{bmatrix} w - x \\ x \end{bmatrix}, \end{aligned}$$

with $Kx = x$. In the absence of saturation, the linear system has norm $\gamma_l^* = 1$. With the saturation present, the induced norm rises to $\gamma^* = \sqrt{2 + \sqrt{2}}$. This will be shown by providing a candidate $V(x)$ which meets the conditions of Theorem 3.2 with this upper bound, and then generate a sequence of test inputs whose limiting output gain tends to the same limit.

We take as our candidate dissipation function (with $r = \sqrt{\gamma^{*2} - 1}$):

$$V(\xi) = \begin{cases} \xi^2 & \text{if } |\xi| \leq \frac{4}{r}, \\ \frac{r}{6}|\xi^3| + \frac{16}{3r^2} & |\xi| > \frac{4}{r}. \end{cases}$$

Verification of the fact that the above defined function implies the claimed upper bound is left to the reader to save space.

Now, we would like to verify that we can find a sequence of signals in $\mathcal{L}_{2+}$ such that their output gain tends to γ^*. The fact that this class of signals tended to the optimal value, as well as the original motivation for the study of this example, was provided in [2]. Take as the test input $w_{T,N}$ as

$$w_{T,N}(t) = \begin{cases} t + 1 & t \leq T, \\ T + 2 & t \in (T + 2n, T + 2n + 1], \\ -\frac{1}{r}T & t \in (T + 2n + 1, T + 2n + 2], \\ 0 & t > T + 2N, \end{cases}$$

for $n = 0, 1, \ldots, N - 1$.

We then have that if $T > 2, N > 0$,

$$\|\mathbf{\Upsilon} w_{T,N}\|_2^2 = T^3 + T + \frac{1}{3} + N(4T^2 + (3 + \frac{1}{r})T + \frac{10}{3}),$$

$$\|w_{T,N}\|_2^2 = \frac{(T^3+1)-1}{3} + N((1+\frac{1}{r^2})T^2 + 4T + 4).$$

In taking the limit, with $T \gg 1$,$N \gg T$, we see that this lower limit for γ^{*2} tends to $2+\sqrt{2}$.

5 Concluding Remarks

The contribution of this paper is to set forth the mathematical framework of the norm computation problem for linear systems with saturation, and then by using the dissipation results in the literature, convert the problem of bounding the operator norm to that of finding a candidate function whose derivative meets three inequality constraints. A heuristic technique is presented to compute a suboptimal function with a closed form computation.

There are many possible extensions to this work, such as state feedback controller synthesis, and allowing multiple inputs. The author has successfully extended this technique to discrete time, using an alternative method of proof. This, and other related results, are documented in [9].

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