Continuity Properties of LQG Optimal Controllers

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Abstract

It is shown that the LQG optimal controller is a continuous function of the plant. The result is proved for a class of plants which contains the class of strictly proper finite-dimensional plants. The topology employed is the one generated by convergence of the closed loop transfer functions in an induced $\mathcal{L}_\infty$ sense. This topology is slightly stronger than the usual ($\mathcal{H}_2$) gap metric convergence on transfer functions.

Notation

$\mathbb{R}$ and $\mathbb{C}$ denote the real and complex fields. $\mathcal{H}_2$, $\mathcal{H}_\infty$ denote the standard Hardy spaces in the right half of the complex plane. $\mathcal{L}_p$ denotes the standard Lebesgue space and $\|\cdot\|_p$ its norm. If $G(s)$ is a matrix function of s, $G^*(s) := \overline{G(-\overline{s})}'$ is the conjugate of $G(s)$. If $G(s)$ is real rational, $G^*(s) = G(-s)'$. $\mathcal{H}_p^n$, $\mathcal{L}_p^n$ etc. denote corresponding spaces of vector-valued functions. We denote by $\mathbf{\Pi}_+$ the orthogonal projection from $\mathcal{L}_2^n(-j\infty, j\infty)$ onto $\mathcal{H}_2^n$. Dimensions will be suppressed if they can be inferred from the context.

1. Introduction

This paper examines the question of the continuity of LQG optimal controllers as a function of the plant. Continuity is proved for gap-type perturbations, where the convergence of the closed loop transfer functions is demanded in an induced $\mathcal{L}_\infty$ (rather than the usual $\mathcal{L}_2$) sense. As usual with gap topologies, small perturbations may change the order of the plant (or even make it infinite dimensional), and possibly allow changes in the number of right half plane poles etc. We formulate the LQG problem as an $\mathcal{H}_2$-optimal control problem over a suitably general space of (infinite-dimensional) transfer functions defined in terms of the Wiener algebra. The continuity result exploits properties of the Wiener algebra and in particular the possibility of continuous spectral factorisation for functions in this class.

2. The Wiener algebra

The Wiener algebra $\mathcal{W}$ is defined to be the set of functions of the form $g = \hat{f} + c$ where $\hat{f}$ is the Fourier transform of a function $f \in \mathcal{L}_1(-\infty, \infty)$ and c is a constant. The norm of an element $g = \hat{f} + c$ is defined by

$$\|g\|_{\mathcal{W}} = |c| + \int_{-\infty}^{\infty} |f(t)| dt.$$

It is well-known (see [2]) that $\mathcal{W}$ is a Banach algebra in the continuous functions on $j\mathbb{R} \cup \{\infty\}$ and that it contains all rational functions with poles off $j\mathbb{R} \cup \{\infty\}$.

For any $f \in \mathcal{L}_1(-\infty, \infty)$ we can write $f = f_+ + f_-$ where $f_+ \in \mathcal{L}_1[0, \infty)$ and $f \in \mathcal{L}_1(-\infty, 0]$. We define the natural projections $\mathbf{P}(\hat{f} + c) = \hat{f}_+ + c$ and $\mathbf{Q}(\hat{f} + c) = (\mathbf{I} - \mathbf{P})(\hat{f} + c) = \hat{f}_-$ and denote $\mathcal{W}^+ = \mathbf{P}\mathcal{W}$ and $\mathcal{W}_0^- = \mathbf{Q}\mathcal{W}$. This gives a direct sum decomposition of $\mathcal{W}$ into the two closed sub-algebras: $\mathcal{W} = \mathcal{W}^+ \oplus \mathcal{W}_0^-$. Note that the elements of $\mathcal{W}^+$ (resp. $\mathcal{W}_0^-$) have holomorphic continuations into the right (resp. left) half plane. It is also true that $\mathcal{W}^+ \subset \mathcal{H}_\infty$ and $\|g\|_\infty \le \|g\|_{\mathcal{W}}$ for any $g \in \mathcal{W}^+$.

The corresponding spaces of matrix functions are denoted by $\mathcal{W}_{r\times c}$, $(\mathcal{W}^+)_{r\times c}$ and $(\mathcal{W}_0^-)_{r\times c}$ with the norm given by

$$\|g\|_{\mathcal{W}} = \max_i \sum_{j=1}^{c} \|g_{ij}\|_{\mathcal{W}}.$$

We remark that $\|\cdot\|_{\mathcal{W}}$ is an induced norm on $(\mathcal{W}^+)_{r\times c}$ in the following sense: if $g = \hat{f} + c \in (\mathcal{W}^+)_{r\times c}$ and $u \in \mathcal{L}_\infty^c[0, \infty)$ then

$$\|g\|_{\mathcal{W}} = \sup_{u \ne 0} \frac{\|cu + f * u\|_\infty}{\|u\|_\infty}$$

where $*$ denotes convolution and $\|u\|_\infty = \max_j \|u_j(t)\|_\infty$. We will use the same notation $\mathbf{P}$ and $\mathbf{Q}$ for the natural projections on the matrix spaces.

3. $\mathcal{H}_2$ control problem

In the feedback configuration of Fig. 1, let the transfer function from $w = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ to $z = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ be denoted by $T(s)$. If the closed-loop is stable and G, K are strictly proper rational transfer functions the 2-norm of $T(s)$:

$$\|T(s)\|_2 = \left(\frac{1}{2\pi}\int_{\infty}^{\infty} \text{trace}(T(j\omega)^* T(j\omega) d\omega\right)^{1/2}$$

is well-defined. We will consider the problem of minimising $\|T(s)\|_2$ over all stabilizing controllers K. To this end let $G(s) = C(sI - A)^{-1}B$ be a minimal realisation, $u = y_2$, $y = y_1 + d_2$ and consider the "standard plant" P, defined by the equations $z = P_{11}w + P_{12}u$, $y = P_{21}w + P_{22}u$, which is given by:

$$P = \begin{bmatrix} \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} G \\ I \end{bmatrix} \\ \begin{bmatrix} G & I \end{bmatrix} & G \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A & \begin{bmatrix} B & 0 \end{bmatrix} & B \\ \hline \begin{bmatrix} C \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ C & \begin{bmatrix} 0 & I \end{bmatrix} & 0 \end{array}\right]. \tag{1}$$

The unique optimal controller which minimises the 2-norm of $T_{zw} = T$ is given (see, for example, [3, Theorem 2], [6]) by $K_{\text{opt}}(s) = -F(sI - \hat{A})^{-1}H$ where $\hat{A} = A - BF - HC$, $F = B'X$, $H = YC'$ and X, Y are the unique positive definite solutions of the algebraic Riccati equations:

$$\begin{aligned} A'X + XA - XBB'X + C'C &= 0, & (2) \\ AY + YA' - YC'CY + BB' &= 0. & (3) \end{aligned}$$

It can be seen that $K_{\text{opt}}(s)$ is a standard LQG optimal controller for $G(s)$ (see [1]).

The above minimisation problem can be posed and solved, at least abstractly, for very general (not necessarily finite-dimensional) plants $G(s)$. We will now discuss such a procedure. First of all, any linear shift-invariant plant which is $\mathcal{L}_2$-stabilizable has normalised coprime factorisations $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ and corresponding solutions of the Bezout identity

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix}\begin{pmatrix} M & U \\ N & V \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{4}$$

where all matrices have elements in $\mathcal{H}_\infty$. Moreover, any stabilising controller has transfer function

$$\begin{aligned} K &= (U + MQ)(V + NQ)^{-1} & (5) \\ &= (\tilde{V} + Q\tilde{N})^{-1}(\tilde{U} + Q\tilde{M}) & (6) \end{aligned}$$

for some Q over $\mathcal{H}_\infty$. From (1) we obtain

$$
\begin{aligned}
T_{zw} &= \begin{pmatrix} I \\ K \end{pmatrix} (I - GK)^{-1} \begin{pmatrix} G, & I \end{pmatrix} - \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} V + NQ \\ U + MQ \end{pmatrix} \begin{pmatrix} \tilde{N} & \tilde{M} \end{pmatrix} - \begin{pmatrix} I \\ 0 \end{pmatrix} \begin{pmatrix} 0 & I \end{pmatrix}. \qquad (7)
\end{aligned}
$$

If $G(s)$ has sufficiently fast "roll-off" at infinity in the sense that the 2-norms of N and $\tilde{N}$ exist, then solutions of (4) can be found for which the 2-norms of U and $\tilde{U}$ exist. In this case, $\|T_{zw}\|_2$ exists if $\|Q\|_2$ does. We now assume that these conditions are satisfied. Using the fact that the 2-norm is unitarily invariant we can pre- and post-multiply (7) by

$$
\begin{pmatrix} N^* & M^* \\ -\tilde{M} & \tilde{N} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \tilde{N}^* & M \\ -\tilde{M}^* & -N \end{pmatrix}
$$

to find that

$$
\|T_{zw}\|_2 = \left\| \begin{pmatrix} N^*V + M^*U - N^*\tilde{M}^* + Q & N^*N \\ -I + \tilde{M}\tilde{M}^* & -\tilde{M}N \end{pmatrix} \right\|_2 .
$$

It therefore follows that the optimal controller is given by (5) or (6) with

$$
\begin{aligned}
Q = Q_{\mathrm{opt}} &= \mathbf{\Pi}_+(-N^*V - M^*U + N^*\tilde{M}^*) \\
&= -\mathbf{\Pi}_+(N^*V + M^*U).
\end{aligned}
$$

4. Controller convergence

We now restrict attention to the class of plants $\mathcal{G}_{p\times m}$ which possess normalised coprime factorisations and corresponding solutions of (4) over $\mathcal{W}^+$. This class strictly contains the class of finite-dimensional plants but is a subset of those plants for which the solutions of (4) belong to $\mathcal{H}_\infty$. We now denote the transfer function T_{zw} in (7) by $T_{[G,K]}$ when it exists. Given G, $G_i \in \mathcal{G}_{p\times m}$ we say that $G_i \overset{\mathcal{W}}{\to} G$ if, given any $K = (U + MQ)(V + NQ)^{-1}$ ($Q \in (\mathcal{W}^+)_{m\times p}$), K stabilises G_i for sufficiently large i and $\|T_{[G,K]} - T_{[G_i,K]}\|_\mathcal{W} \to 0$ as $i \to \infty$. Since $\|\cdot\|_\infty \le \|\cdot\|_\mathcal{W}$, this implies convergence in the usual $\mathcal{H}_2$-gap sense. One consequence of the following theorem is that the above definition is independent of the choice of K.

Theorem. $G_i \overset{\mathcal{W}}{\to} G$ if and only if there exist *normalised* coprime factorisations of G, G_i and corresponding solutions of (4) which converge in norm, i.e. $\|M - M_i\|_\mathcal{W} \to 0$, $\|N - N_i\|_\mathcal{W} \to 0$, $\|U - U_i\|_\mathcal{W} \to 0$, etc.

Proof. The weaker statement of the theorem which does not require the coprime factorisations of G_i to be normalised is easily proved as follows (cf. [8]). We will use the alternative expression

$$T_{zw} = \begin{pmatrix} G \\ I \end{pmatrix} (I - KG)^{-1} \begin{pmatrix} I, & K \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$$

and for notational convenience we take $Q = 0$. Then

$$\begin{aligned} & \|T_{[G,K]} - T_{[G_i,K]}\|_{\mathcal{W}} \to 0 \\ \Leftrightarrow \quad & \left\| \begin{pmatrix} N \\ M \end{pmatrix} \begin{pmatrix} \tilde{U}, & \tilde{V} \end{pmatrix} - \begin{pmatrix} N_i \\ M_i \end{pmatrix} (\tilde{V} M_i - \tilde{U} N_i)^{-1} \begin{pmatrix} \tilde{U}, & \tilde{V} \end{pmatrix} \right\|_{\mathcal{W}} \to 0 \\ \Leftrightarrow \quad & \left\| \begin{pmatrix} N \\ M \end{pmatrix} - \begin{pmatrix} N_i \\ M_i \end{pmatrix} (\tilde{V} M_i - \tilde{U} N_i)^{-1} \right\|_{\mathcal{W}} \to 0. \end{aligned} \tag{8}$$

The last step follows since $\| \begin{pmatrix} \tilde{U}, & \tilde{V} \end{pmatrix} \|_{\mathcal{W}}$ is bounded and $\begin{pmatrix} N \\ -M \end{pmatrix}$ is a right inverse of $\begin{pmatrix} \tilde{U}, & \tilde{V} \end{pmatrix}$ of bounded $\mathcal{W}$-norm.

The key part of the proof is to show that the factorisations can be normalised while preserving convergence. Denote

$$\begin{pmatrix} \overline{N}_i \\ \overline{M}_i \end{pmatrix} = \begin{pmatrix} N_i \\ M_i \end{pmatrix} (\tilde{V} M_i - \tilde{U} N_i)^{-1}$$

and note that

$$\|I - (\overline{N}_i^* \overline{N}_i + \overline{M}_i^* \overline{M}_i)\|_{\mathcal{W}} \to 0$$

as $i \to \infty$. We now seek a spectral factorisation

$$\Phi_i = \overline{N}_i^* \overline{N}_i + \overline{M}_i^* \overline{M}_i = R_i^* R_i$$

using the procedure of [2]. Accordingly we seek a canonical factorisation $\Phi_i = \Phi_{i-}\Phi_{i+}$ where $\Phi_{i+}, \Phi_{i+}^{-1} \in (\mathcal{W}^+)_{m\times m}$ and $\Phi_{i-} - I, \Phi_{i-}^{-1} - I \in (\mathcal{W}_0^-)_{m\times m}$. For i sufficiently large the operators

$$\begin{aligned} \mathbf{S}_{\Phi_i}(X) &= \mathbf{P}(\Phi_i X) + \mathbf{Q}(X), \\ \mathbf{R}_{\Phi_i}(X) &= \mathbf{P}(X) + \mathbf{Q}(X\Phi_i) \end{aligned}$$

satisfy

$$\begin{aligned} \|\mathbf{I} - \mathbf{S}_{\Phi_i}\| &\leq \|\mathbf{P}\| \; \|I - \Phi_i\|_{\mathcal{W}} < 1, \\ \|\mathbf{I} - \mathbf{R}_{\Phi_i}\| &\leq \|\mathbf{Q}\| \; \|I - \Phi_i\|_{\mathcal{W}} < 1 \end{aligned}$$

and hence have bounded inverses. Thus, $\Phi_{i+} = X_i^{-1}$, $\Phi_{i-} = Y_i^{-1}$ where X_i and Y_i are the unique solutions of the equations

$$\begin{aligned} \mathbf{P}(\Phi_i X_i) + \mathbf{Q}(X_i) &= I, \\ \mathbf{P}(Y_i) + \mathbf{Q}(Y_i \Phi_i) &= I. \end{aligned}$$

(See [2, Corollary 1.2, p. 39]). Moreover, the solutions must converge to the corresponding factorisations of I, namely

$$\begin{aligned} \|\Phi_{i+} - I\|_{\mathcal{W}} &\to 0, \\ \|\Phi_{i-} - I\|_{\mathcal{W}} &\to 0. \end{aligned}$$

Since $\Phi_i^* = \Phi_i$ and canonical factorisations are unique then we must have $R_i = \Phi_{i+} = \Phi_{i-}^*$. We can now conclude that

$$\begin{pmatrix} \hat{N}_i \\ \hat{M}_i \end{pmatrix} = \begin{pmatrix} \overline{N}_i \\ \overline{M}_i \end{pmatrix} R_i^{-1}$$

is normalised (i.e. $\hat{M}_i^* \hat{M}_i + \hat{N}_i^* \hat{N}_i + I$) and

$$\left\| \begin{pmatrix} N \\ M \end{pmatrix} - \begin{pmatrix} \hat{N}_i \\ \hat{M}_i \end{pmatrix} \right\|_{\mathcal{W}} \to 0$$

as $i \to \infty$.

Resuming the simplified notation of the theorem statement we have shown that $G_i \stackrel{\mathcal{W}}{\to} G$ implies the existence of normalised factorisations so that $\left\| \begin{pmatrix} N_i \\ M_i \end{pmatrix} - \begin{pmatrix} N \\ M \end{pmatrix} \right\|_{\mathcal{W}} \to 0$ and (similarly) $\| \begin{pmatrix} \tilde{M}_i, & \tilde{N}_i \end{pmatrix} - \begin{pmatrix} \tilde{M}, & \tilde{N} \end{pmatrix} \|_{\mathcal{W}} \to 0$. It now follows that $\tilde{V} M_i - \tilde{U}_i N_i = I + \tilde{V}(M_i - M) + \tilde{U}(N - N_i)$ is invertible over $\mathcal{W}$ for large enough i, and the same is also true of $-\tilde{N}_i U + \tilde{M}_i V$. We can therefore define

$$\begin{aligned} \begin{pmatrix} U_i \\ V_i \end{pmatrix} &= \begin{pmatrix} U \\ V \end{pmatrix} (-\tilde{N}_i U + \tilde{M}_i V)^{-1}, \\ \begin{pmatrix} \tilde{V}_i, & -\tilde{U}_i \end{pmatrix} &= (\tilde{V} M_i - \tilde{U}_i N_i)^{-1} \begin{pmatrix} \tilde{V}, & -\tilde{U} \end{pmatrix} \end{aligned}$$

to achieve the required convergence properties of U_i, V_i, $\tilde{U}_i$, $\tilde{V}_i$.

The converse direction of the proof is immediate since

$$\left\| \begin{pmatrix} N_i \\ M_i \end{pmatrix} - \begin{pmatrix} N \\ M \end{pmatrix} \right\|_{\mathcal{W}} \to 0$$

implies (8). $\square$

Corollary. Let the plants G, $G_i \in \mathcal{G}_{p\times m}$ satisfy $G_i \xrightarrow{\mathcal{W}} G$. Let $Q_{\rm opt} = -\mathbf{P}(N^*V + M^*V)$ and $Q_{{\rm opt},i} = -\mathbf{P}(N_i^*V_i + M_i^*V_i)$, where N, N_i etc. are given as in Theorem 1. Define

$$\begin{aligned} K &= (U + MQ_{\rm opt})(V + NQ_{\rm opt})^{-1}, \\ K_i &= (U_i + M_iQ_{{\rm opt},i})(V_i + N_iQ_{{\rm opt},i})^{-1}. \end{aligned}$$

Then $K_i \xrightarrow{\mathcal{W}} K$.

Proof. Since $\mathbf{P}$ is a bounded operator then $\|Q_{\rm opt} - Q_{{\rm opt},i}\|_{\mathcal{W}} \to 0$ as $i \to \infty$. This implies

$$\left\| \begin{pmatrix} V + NQ_{\rm opt} \\ U + NQ_{\rm opt} \end{pmatrix} - \begin{pmatrix} V_i + N_iQ_{{\rm opt},i} \\ U_i + N_iQ_{{\rm opt},i} \end{pmatrix} \right\|_{\mathcal{W}} \to 0.$$

Therefore, from (7), we have $\|T_{[G,K]} - T_{[G,K_i]}\|_{\mathcal{W}} \to 0$, which proves the result. $\square$

Remark. The above Corollary establishes the promised continuity result for LQG controllers. This is because the natural projection $\mathbf{P}$ and $\mathbf{\Pi}_+$ coincide on $\mathcal{W} \cap \mathcal{L}_2$, so that K and K_i are optimal LQG controllers if N (and consequently $\tilde{N}$) has elements in $\mathcal{H}_2$ (and U, $\tilde{U}$ are chosen similarly).

5. Example

It is interesting to compare the continuity properties of LQG controllers with gap-optimal controllers. The controller which is optimally robust to perturbations in the gap metric can be calculated by performing a Nehari extension on $-(N^*V + M^*U)$ instead of an orthogonal projection onto $\mathcal{H}_2$ [4]. It is known that optimal interpolants for the Nehari problem can have discontinuities in an $\mathcal{L}_\infty$ sense [7] when the associated Hankel operator has a repeated top singular value. A plant at which gap discontinuity actually occurs has been given in [5]: $G(s) = s/(s^2+1)$. On the other hand a Lipschitz continuity property for suboptimal gap-controllers does hold, which has been shown in [9].

We will now work out the optimal LQG controller for $G(s) = (s+\epsilon)/(s^2+1)$ explicitly. For the state space realisation:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} \epsilon & 1 \end{bmatrix}$$

we can directly compute that the positive definite solution $X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}$ of (2) is given by: $x_2 = -1 + \sqrt{1+\epsilon^2}$, $x_3 = \sqrt{1+2x_2}$, $x_1 = x_3(1+x_2) - \epsilon$. To find the solution $Y = \begin{bmatrix} y_1 & y_2 \\ y_2 & y_3 \end{bmatrix}$ of (3) we need to solve the equations:

$$2y_2 - (\epsilon y_1 + y_2)^2 = 0,$$

$$\begin{aligned} y_3 - y_1 - (\epsilon y_1 + y_2)(\epsilon y_2 + y_3) &= 0, \\ -2y_2 - (\epsilon y_2 + y_3)^2 + 1 &= 0. \end{aligned}$$

At $\epsilon = 0$ the positive definite solution is: $y_1 = y_3 = 1$ and $y_2 = 0$. We can then directly calculate the successive terms in a Taylor expansion to find:

$$\begin{aligned} y_1 &= 1 - \epsilon + \epsilon^3 + O(\epsilon^4), \\ y_2 &= \epsilon^2/2 - \epsilon^3/2 + O(\epsilon^4), \\ y_3 &= 1 - \epsilon^2/2 + O(\epsilon^4). \end{aligned}$$

After substitution and some manipulation we obtain the following expression for the optimal controller:

$$K_{\mathrm{lqg}}(s) = \frac{-(1 + \epsilon^3 + O(\epsilon^4))s + (\epsilon - \epsilon^2 - \epsilon^3/2 + O(\epsilon^4))}{s^2 + (2 + \epsilon^2 + O(\epsilon^4))s + (1 + 2\epsilon^2 - \epsilon^3 + O(\epsilon^4))},$$

which illustrates the above-derived continuity property. In contrast, the optimal gap controller, which was derived in [5], is given by:

$$K_{\mathrm{gopt}}(s) = \begin{cases} \dfrac{(-1 + O(\epsilon))s + \epsilon/2 + O(\epsilon^2)}{s + \epsilon/2 + O(\epsilon^2)} & (\epsilon > 0) \\ -1 & (\epsilon = 0) \\ \dfrac{(1 + |\epsilon| + O(\epsilon^2))s - 2|\epsilon|^{-1} + O(1)}{s + 2|\epsilon|^{-1} + O(1)} & (\epsilon < 0) \end{cases}$$

with the discontinuity being clearly evident.

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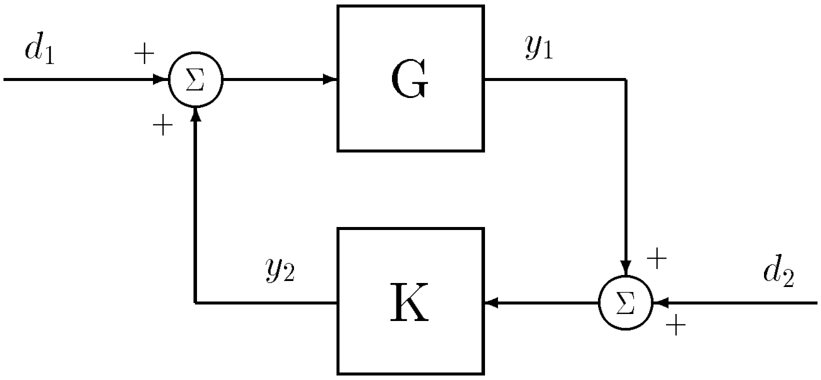

Figure 1: Standard feedback configuration