A Polytope Based Algorithm to Compute Regions of Attraction: Planar Case

B.G. Romanchuk

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B.G. Romanchuk
University of Cambridge
Department of Engineering
Trumpington Street
Cambridge CB2 1PZ
email: bgr@eng.cam.ac.uk

Abstract

An algorithm which estimates the region of attraction of the origin using a convex polytope is developed for piecewise linear dynamical systems. This is equivalent to the problem of computing a Lyapunov function. The $\mathbb{R}^2$ case is studied herein for simplicity.

Keywords: nonlinear dynamical systems, Lyapunov stability, invariant polytopes.

1 Introduction

The determination of the stability of the origin for a nonlinear dynamical system is an old problem in mathematics. The classical means of achieving non-local results is the use of Lyapunov functions. Although powerful, the results are limited by the fact that one has to produce the correct Lyapunov function. In many cases, one almost has to have the problem solved in order to produce such a function; the function only serves to formalise the result.

It is preferable to develop general purpose algorithms for analysis. This has meant solving partial differential inequalities (PDI's) by the use of finite difference methods over $\mathbb{R}^n$. The study of PDI's has seen a resurgence of interest, mainly in the form of nonlinear $\mathcal{H}_\infty$ control.

In this paper, an alternative means of determining the region of stability of systems with piecewise linear dynamics is presented. The method uses a convex polytope of a regular form to approximate sets which are invariant under the system dynamics. It is believed that this method will prove to have algorithmic advantages over finite difference methods.

For ease of presentation, the results are given for the planar case ($\mathbb{R}^2$). This allows geometric explanations to be given for proofs, and a good deal of the imposing mathematics associated with the higher order cases is eliminated. The higher order cases are to be developed elsewhere, but understanding this case is very useful in order to develop an intuition with regards to the mathematics.

Some related work is that which appears in a series of papers [1],[2],[6],[10],[11] (amongst others, see the further references therein). The results in the listed papers are concerned with constrained input systems, which form an important subset of the piecewise linear systems studied herein. The previous work has been concerned with determining the polyhedral sets which are invariant and respect input constraints - i.e., lie within the the linear operating region. The present work is aimed at determining the largest invariant sets within the full range of operating conditions; the reduction of conservatism is at the cost of a more involved set of computations.

The form of the paper will now be summarised. A nonlinear transformation on a polytope is developed which preserves the geometric connection information - an isomorphism. This allows a standard polytope to be distorted in a controlled fashion, allowing it to be used as an approximant. This general isomorphism is then applied to the problem of approximating the region of attraction of the origin, by the tactic of ensuring that the polytope remains invariant under the flow at all times. This generic algorithm on piecewise linear systems is applied to a the case of a linear system with a saturating feedback, and graphical results shown. There then follow a series of comments on issues raised by its development, and brief concluding remarks are given. Finally, the proofs to statements in the body of the paper are deferred to an appendix.

2 An Isomorphism on Standard Polytopes

In order to determine the region of attraction of the origin, it is necessary to develop a means of expressing the geometry of a non-ellipsoidal subset of $\mathbb{R}^2$ in an efficient fashion. One means would be to use the level curves of non-quadratic functions (e.g., Lyapunov functions), the means used herein is to utilise convex polytopes.

Ordinarily, computations involving polytopes are quite involved, such as the problem of determining the vertices of an arbitrary polytope, or finding the convex hull of an arbitrary set of points. These are long studied problems, for example, see [4, Chapter 18] (amongst others). In this case, by restricting our attention to a special class of "standard" polytopes, it is possible to develop efficient computational schemes. The important advantage is that the geometric connection information is known *a priori*, and it is ensured that this connection information is preserved.

We will now review some basic definitions on polytopes; the notation follows that of the text by A. Brøndsted ([3]).

The hyperplane (line) $H(u,\alpha)$ is the set $H(u,\alpha) := \{x: \langle u, x\rangle = \alpha\}$. The half-plane $K(u,\alpha)$ is the set $K(u,\alpha) := \{x: \langle u, x\rangle \geq \alpha\}$. A *polyhedral set* (in $\mathbb{R}^n$) is the intersection of a finite number of closed half planes in $\mathbb{R}^n$, or $\mathbb{R}^n$ itself. A *convex polytope* is the convex hull of of a finite number of points in $\mathbb{R}^n$. We have the following relation between polytopes and polyhedral sets (which is non-trivial, belying its obvious appearance):

Theorem 2.1 ([3, Corr. 8.7]): If Ω is a non-empty bounded polyhedral set in $\mathbb{R}^n$, Ω is a polytope.

We will now clarify what we refer to as "standard" polytopes. Henceforth, let the constant $N > 4$ be fixed. Then the set $\mathcal{U}$ is the set of vectors

$$\mathcal{U} := \left\{ u^i \colon u^i = \begin{bmatrix} \cos(\frac{2\pi i}{N}) \\ \sin(\frac{2\pi i}{N}) \end{bmatrix}, \quad i \in 0, \ldots, N-1 \right\}.$$

The vector $a := [a_0, \ldots, a_{N-1}]'$ is an element of $\mathbb{R}^N_+$. We introduce the partial orderings $\prec, \preceq, \precneqq$ on vectors in $\mathbb{R}^N$ as follows: $x \prec y$ (resp. $x \preceq y$) if for all $i \in 0, \ldots, N-1$, $x_i < y_i$ (resp. $x_i \leq y_i$), and $x \precneqq y$ if $x \preceq y$ and $x \neq y$. We can then formally define the polytopes used in the algorithm, which are closely tied to the issue of computational representation.

Definition 2.2: An *ext-representation* determined by $a \in \mathbb{R}^N_+, 0 \prec a$, denoted $\Omega^{\mathcal{U}}_a$, is the polyhedral set with the representation

$$\Omega^{\mathcal{U}}_a = \bigcap_{i=0}^{N-1} K(u^i, a_i).$$

Remark 2.3: We will hereafter bend our notation, and treat all indices of the elements of a and $\mathcal{U}$ as being modulo N, hence if $i = N-1$, $u^{i+1} = u^0$.

Definition 2.4: An ext-representation $\Omega^{\mathcal{U}}_a$ is *proper* if:

1. it is irreducible, i.e.,

$$\Omega^{\mathcal{U}}_a \subset \bigcap_{i=0, i\neq j}^{N-1} K(u^i, a_i), \qquad j \in 0, \ldots, N-1;$$

2. it is bounded (i.e., a polytope);

3. the set $\Omega^{\mathcal{U}}_a \bigcap H(u^i, a_i) \bigcap H(u^{i+1}, a_{i+1})$ is non-empty for all $i \in 0, \ldots, N-1$ (remembering Remark 2.3).

Definition 2.5: The facets of a proper ext-representation $\Omega^{\mathcal{U}}_a$ are denoted F_i, and are the sets

$$F_i = H(u^i, a_i) \bigcap \Omega^{\mathcal{U}}_a.$$

Remark 2.6: It is an important property of planar geometry that every 2-polytope is simple and simplicial ([3, p. 83]), i.e., every vertex lies in the intersection of only 2 facets, and every facet contains 2 vertices.

Lemma 2.7: The vertices of a proper ext-representation are the points $F_i \bigcap F_{i+1}$ (remembering Remark 2.3).

We now need to define the geometry matrices which are used to generate the vertices of $\Omega_a^{\mathcal{U}}$. Define T^i $(i \in 0, \ldots, N-1)$ as:

$$T^i := \frac{1}{sin\left(\frac{2\pi}{N}\right)} \begin{bmatrix} sin\left(\frac{2\pi(i+1)}{N}\right) & -sin\left(\frac{2\pi i}{N}\right) \\ -cos\left(\frac{2\pi(i+1)}{N}\right) & cos\left(\frac{2\pi i}{N}\right) \end{bmatrix}.$$

Theorem 2.8: The vertices of a proper ext-representation $\Omega_a^{\mathcal{U}}$ is the set

$$\mathcal{X} := \left\{ x^i : x^i = T^i \begin{bmatrix} a_i \\ a_{i+1} \end{bmatrix}, \quad i \in 0, \ldots, N-1 \right\}. \tag{1}$$

We will now turn to the algorithm development. The importance of the definition of ext-representations is made apparent by Theorem 2.8, as it is possible to quickly generate the geometry of such a polytope. In general, it would be difficult to determine which facet intersections give rise to vertices; that is easily done here. The logic of the algorithm is to start with a "standard" polytope, which is guaranteed to be a proper ext-representation, and then distort its geometry in a fashion which guarantees that the representation remains proper.

Theorem 2.9: Let $a = \alpha[1, \ldots, 1]'$, with $\alpha > 0$. Then, $\Omega_a^{\mathcal{U}}$ is a proper ext-representation.

The function maxstep will now be defined; it is the furthest "step" outwards a facet can be moved whilst remaining a proper ext-representation.

Definition 2.10: For a vector $a \in \mathbb{R}_+^N$, and $i \in 0, \ldots, N-1$, define $maxstep$: $\mathbb{R}_+^N \times \mathbb{Z} \mapsto \mathbb{R}^N$ as:

$$(\text{maxstep}(a, i))_j := \begin{cases} a_j & j \neq i, \\ \frac{\sin\left(\frac{2\pi}{N}\right)}{\sin\left(\frac{4\pi}{N}\right)} [1 \quad 1] \begin{bmatrix} a_{i-1} \\ a_{i+1} \end{bmatrix} & j = i. \end{cases}$$

Theorem 2.11: Let $\Omega_a^{\mathcal{U}}$ be a proper ext-representation $(0 \prec a)$. Then for all $i \in 1, \ldots, N$, $a^* = \text{maxstep}(a, i)$ is such that $a \precneqq a^*$, and for all a^+ with $a \preceq a^+ \precneqq a^*$, $\Omega_a^{\mathcal{U}}$ is a proper ext-representation.

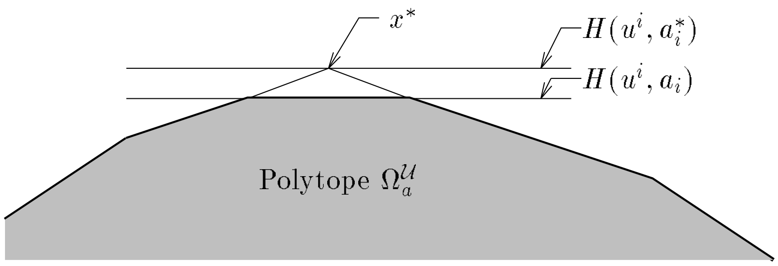

Figure 1: Maximum step determination.

The motivation for the means of proof is seen by examination of Figure 1. The point x^* is the intersection of the $i-1$-th and $i+1$-th facets, independent of the value of a_i. The value of a_*, is the value at which $H(u^i, a_i^*)$ passes through x^*. In order for the representation to remain irreducible, we can increase a_i to strictly less than this value only.

Remark 2.12: It is possible to examine a step in backwards direction - i.e., how much can we decrease a_i and still remain a valid ext-representation? This will not be examined herein, but is quite similar in how it is to be determined.

Theorem 2.11 can represent the basis for a variety of algorithms. By adding external constraints upon growth, it is possible to arrive at an approximating polytope for a variety of problems. It should also be possible to adapt the theory to algorithms which move all the facets simultaneously, which may be necessary for practical implementations.

3 Stability Theorems on Piecewise Linear Systems

We will now develop the stability theorems which are to be used in conjunction with the polytope based algorithm. The previous set of results detailed how one can expand the polytope in a nonlinear fashion but still preserving the connection information; what is needed is some means of constraining the growth so that the polytopes are invariant sets under the dynamical flow.

These theorems will be developed in terms of general piecewise linear dynamical systems, but is not complete for an actual implementation. Additionally, one needs to verify behaviour where the expansion of the polytope crosses the boundaries of regions over which the function is piecewise linear.

We will now formally define what is meant by a general piecewise linear system with a regional representation. It is the dynamical system generated by:

$$\dot{x}(t) = f(x(t)), \quad x(0) = \xi_0, \tag{2}$$

with $f(\xi)$ defined by:

$$f(\xi) = A^{(i)}\xi + B^{(i)}, \text{if } \xi \in \Gamma_i, \tag{3}$$

with $A^{(i)} \in \mathbb{R}^{n\times n}$, $B^{(i)} \in \mathbb{R}^n$. The sets Γ_i and function f in (3) are such that:

1. f is continuous;
2. each Γ_i is a convex polyhedral set with a non-empty interior;
3. $\bigcup_{i=1}^{n_\sigma} \Gamma_i = \mathbb{R}^2$;
4. $\Gamma_i \bigcap \Gamma_j$ has an empty interior for all $i \neq j$;
5. it is assumed that a non-zero open ball surrounding the origin lies in Γ_1, and we fix $B^{(1)}$ to be zero.

(The rationale for the final requirement is that the linearisation at the origin is well defined. This presumably could be relaxed at some extra cost.) We will denote the trajectory generated by (2) as $x(t;\xi_0)$.

Remark 3.1: As we insist that there be a finite number of regions, it is likely that the requirement that the regions Γ_i be polyhedral is redundant if the system matrices are distinct. It may be possible to alternative represent such functions as cascaded memoryless simple piecewise linear functions.

Remark 3.2: As we will not be taking the powers of matrices, we will henceforth suppress the parentheses in $A^{(i)}$,$B^{(i)}$.

We are now able to formally define the notion of stability used.

Definition 3.3: The system (2) is *uniformly asymptotically stable* over a (simply connected) compact set Ω if:

1. for all $\xi_0 \in \Omega$, $x(t;\xi_0) \in \Omega$ for all $t \in \mathbb{R}_+$;
2. for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $\xi_0 \in \Omega$ with $\|\xi_0\| < \epsilon$, then $\|x(t;\xi_0)\| < \delta$ for all $t \in \mathbb{R}_+$ (the norm used is any space norm on $\mathbb{R}^2$);
3. $\lim_{t\to\infty} x(t;\cdot) = 0$.

We will now formally define what is meant by a polytope being invariant under a dynamical flow.

Definition 3.4: An ext-representation $\Omega_a^{\mathcal{U}}$ is *strictly invariant* under the flow $f(x)$ in (3) if for all $\xi \in \partial\Omega_a^{\mathcal{U}}$,

$$\langle u^i, f(\xi)\rangle < 0,$$

for all i such that $\langle u^i, \xi\rangle = a_i$.

It is fairly straightforward to see that this is in fact equivalent to the typical notion of the invariance of the set - i.e., that trajectories all lie within the interior from any initial condition within the polytope for all $t > 0$. The following is an important convexity result upon which the algorithm development is built.

Lemma 3.5: Let the set $\hat{X}$ be the set of vertices of a proper ext-representation $\Omega_a^{\mathcal{U}}$ in union with the set

$$\partial\Omega_a^{\mathcal{U}} \bigcap \{x \colon x \in \partial\Gamma_k, \text{for any } k \in 1,\ldots,n_\sigma\}.$$

Then, $\Omega_a^{\mathcal{U}}$ is strictly invariant under the flow $f(x)$ if for all $\xi \in \hat{X}$,

$$\langle u^i, f(\xi)\rangle < 0,$$

for all i such that $\langle u^i, \xi\rangle = a_i$.

We will for reasons of simplicity not treat the more obvious extensions of the problem formulation, which include the notion of the invariance of set including ∞ (an anti-stability result), or invariance under piecewise linear mappings, which corresponds to discrete time systems (the discrete time appears more complicated than the continuous time case, but will also reduce to a problem on matrices).

Theorem 3.6: Fix a proper ext-representation $\Omega_a^{\mathcal{U}}$, and a^*, $a \precneqq a^* \precneqq \mathrm{maxstep}(a,i)$. We assume that the system (2) is uniformly asymptotically stable over $\Omega_a^{\mathcal{U}}$. Then, if for all α_+ with $a \precneqq a^+ \preceq a^*$, $\Omega_{a^+}^{\mathcal{U}}$ is strictly invariant under the flow $f(x)$, the system (2) is uniformly asymptotically stable over $\Omega_{a^*}^{\mathcal{U}}$.

We are now able to use polytopes to determine stable sets. It is possible to exploit the fact that $f(x)$ is piecewise linear to determine the furthest possible step forward which preserves strict invariance as a simple test. The simplest case is where the changes all occur within the same region of definition, which will be detailed next in its full generality.

We now need to define matrices and functions which are used in the computation. Define $P^{k,i}$ as:

$$P^{k,i} := A^k T^i, \quad k \in 1,\ldots,n_\sigma, i \in 0,\ldots,N-1.$$

Define the step(a,k,i) function as:

$$\text{step}(a,k,i) := \min_{j \in 1,\ldots,4} \left\{ \text{step}_j(a,k,i) \right\},$$

with the step$_j$ functions defined below. Note that x^{i-1}, x^i are the vertices connected to F_i. Then,

$$\text{step}_1(a,k,i) := \begin{cases} \dfrac{\left\langle u^i, \begin{bmatrix} P_{11}^{k,i} \\ P_{12}^{k,i} \end{bmatrix} \right\rangle a_i - \left\langle u^i, A^k x^i + B^k \right\rangle}{\left\langle u^i, \begin{bmatrix} P_{11}^{k,i} \\ P_{12}^{k,i} \end{bmatrix} \right\rangle} & \text{if } \left\langle u^i, \begin{bmatrix} P_{11}^{k,i} \\ P_{12}^{k,i} \end{bmatrix} \right\rangle > 0, \\ \infty & \text{otherwise.} \end{cases}$$

$$\text{step}_2(a,k,i) := \begin{cases} \dfrac{\left\langle u^{i+1}, \begin{bmatrix} P_{11}^{k,i} \\ P_{12}^{k,i} \end{bmatrix} \right\rangle a_i - \left\langle u^{i+1}, A^k x^i + B^k \right\rangle}{\left\langle u^{i+1}, \begin{bmatrix} P_{11}^{k,i} \\ P_{12}^{k,i} \end{bmatrix} \right\rangle} & \text{if } \left\langle u^{i+1}, \begin{bmatrix} P_{11}^{k,i} \\ P_{12}^{k,i} \end{bmatrix} \right\rangle > 0, \\ \infty & \text{otherwise.} \end{cases}$$

$$\text{step}_3(a,k,i) := \begin{cases} \dfrac{\left\langle u^i, \begin{bmatrix} P_{21}^{k,i-1} \\ P_{22}^{k,i-1} \end{bmatrix} \right\rangle a_i - \left\langle u^i, A^k x^{i-1} + B^k \right\rangle}{\left\langle u^i, \begin{bmatrix} P_{21}^{k,i-1} \\ P_{22}^{k,i-1} \end{bmatrix} \right\rangle} & \text{if } \left\langle u^i, \begin{bmatrix} P_{21}^{k,i-1} \\ P_{22}^{k,i-1} \end{bmatrix} \right\rangle > 0, \\ \infty & \text{otherwise.} \end{cases}$$

$$\text{step}_4(a,k,i) := \begin{cases} \dfrac{\left\langle u^{i-1}, \begin{bmatrix} P_{21}^{k,i-1} \\ P_{22}^{k,i-1} \end{bmatrix} \right\rangle a_i - \left\langle u^{i-1}, A^k x^{i-1} + B^k \right\rangle}{\left\langle u^{i-1}, \begin{bmatrix} P_{21}^{k,i-1} \\ P_{22}^{k,i-1} \end{bmatrix} \right\rangle} & \text{if } \left\langle u^{i-1}, \begin{bmatrix} P_{21}^{k,i-1} \\ P_{22}^{k,i-1} \end{bmatrix} \right\rangle > 0, \\ \infty & \text{otherwise.} \end{cases}$$

Remark 3.7: The motivation behind the form of the step$_j$ functions is that the dot products appearing therein are the derivative of the dot product of the flow with the hyperplanes forming an edge (evaluated at the two vertices connected at the edge), with respect to the variation in a_i (which causes a linear movement of the vertex).

Theorem 3.8: Fix a proper ext-representation $\Omega_a^{\mathcal{U}}$ which is strictly invariant under $f(x)$, and (2) is uniformly asymptotically stable over $\Omega_a^{\mathcal{U}}$. Assume that there exists some a^* with $a \precneqq a^* \precneqq$ maxstep(a,i) such that:

1. $\Omega_{a^*}^{\mathcal{U}} \setminus \Omega_a^{\mathcal{U}} \subset \Gamma_k$;
2. a_i^* is less than step(a,k,i).

Then, $\Omega_{a^*}^{\mathcal{U}}$ is also proper, strictly invariant under $f(x)$, and the system (2) is uniformly asymptotically stable over $\Omega_{a^*}^{\mathcal{U}}$.

In order to motivate the form of the algorithm, note Figure 2. As the hyperplane $H(u^i,a_i)$ is moved out, the vertex x^* moves in a linear fashion, proportional to the change. As long as the vertex remains within Γ_k, the value of $\langle f(x^*), u^i \rangle$ will change linearly.

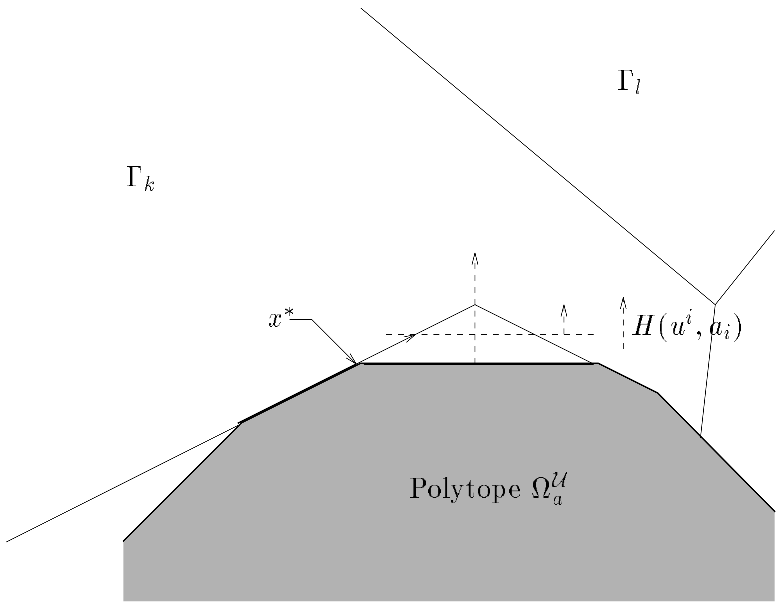

Figure 2: Constrained step determination.

4 Implementation and Examples

The result in Theorem 3.8 is the main result which is used during the algorithm development, but there remain practical issues which need to be dealt with before an implementation is possible. The first point to note is that only the preservation of uniform asymptotic stability is guaranteed; it is first necessary to ensure that the algorithm starts in a valid condition. The second problem is that in order for the algorithm to be non-trivial, the polytope will have to cross boundaries of the regions of definition of $f(x)$. We will examine these issues for a specific system - a second order linear system with saturating feedback. The generalisation is straightforward in terms of analysis, but the algorithmic details are presently unclear for more general systems.

We are interested in determining the region of asymptotic attraction of the origin for the (autonomous) system:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) := Ax(t) - B\sigma_0(Kx(t)), && (4)\\ x(0) &= \xi_0, && (5) \end{aligned}$$

with $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 1}$, $K \in \mathbb{R}^{1\times 2}$. In order for the problem to be non-trivial, we assume that $BK \neq 0$. The function σ_0 appearing in (4) is the standard saturation function; $\sigma_0(x) := \mathrm{sign}(x)(\min\{|x|, 1\})$.

It should be noted that there are two simple conditions which would be checked first, which result from algebraic tests on the matrices. If $A - BK$ is not Hurwitz, the origin is not stable, and there is no need to continue. Additionally, by the use of linear matrix inequalities, it may be possible to find a global quadratic Lyapunov function if the system is stable (both open and closed loop). It is therefore without loss of generality that we can assume that the local linearisation of the origin is Hurwitz. Typically, we are interested in the case where the linearisation is stable, but the open loop system is unstable, which ensures that the system is not globally stable (a long known result; see [8] for a recent treatment of the combined continuous and discrete time results).

Assumption 4.1: It is assumed that $A - BK$ is Hurwitz. Additionally, we assume that the system has been normalised so that there exists a positive definite matrix P^{-1} such that:

$$(A - BK)' + (A - BK) + P^{-1} = 0. \tag{6}$$

The assumption of the normalisation implied by (6) is not restrictive; one only needs to compute a Lyapunov function for the original system, and then transform the state space so that (6) holds. This is a rudimentary form of conditioning, and is sufficient for present development. As with many other topics associated with this algorithm, the initialisation step is likely to require further study.

It is straightforward that the system defined by (4) - (5) has three regions of definition as a piecewise linear function: $\Gamma_1 = \{\xi\colon |K\xi| \leq 1\}$, $\Gamma_2 = \{\xi\colon K\xi > 1\}$, $\Gamma_3 = \{\xi\colon K\xi < -1\}$.

Theorem 4.2: Let $0 < \alpha < \frac{1}{\|K\|}$. There exists an N such that if $a = \alpha[1, \ldots, 1]'$, $\Omega_a^{\mathcal{U}}$ is a proper ext-representation which is strictly invariant under the flow $f(x)$ defined by (4) and the system defined by (4) - (5) is uniformly asymptotically stable over $\Omega_a^{\mathcal{U}}$.

The remaining implementation issue is the problem of the region boundaries. It appears that the most efficient method would be to keep track of the dot product of the flow at vertex, and keep track of how it is affected by each algorithmic step. When a step causes an edge or vertex to cross a region boundary, it would then be necessary to verify that invariance is preserved.

The algorithm has been applied to the system generated by:

$$\dot{x}(t) = \left[\begin{array}{cc} -.333 & -.86 \\ 1.42 & .53 \end{array}\right] x(t) - \left[\begin{array}{c} -.41 \\ 2.27 \end{array}\right] \sigma_0\left([.3564.284]x(t)\right).$$

In Figure 3, we see the growth of the polytope as the iteration proceeds at selected intervals. The outer (dotted) line is the limit cycle for the system, which forms the boundary of the region of stability. The resemblance to the level curves of a Lyapunov function is quite obvious. Figure 4 shows the dependence of the limit to growth upon the number of facets; more facets allows a better approximation of the region of attraction. The use of an algorithm which moves all of the facets simultaneously may allow a better approximation with a lower number of facets, which is beyond the scope of the current paper.

5 Generalisation to Higher Orders

The present algorithm development has been tied to the $\mathbb{R}^2$ case quite heavily. This has been advantageous in that it is possible to give geometrical interpretations of proofs, and the development is intuitively easy to follow. The mathematical development for the higher order cases is quite involved, and uses many fundamental properties of polytopes. It seems necessary to understand the second order case first, or else the development will be too imposing. Additionally, it is quite difficult to present graphical examples for higher order cases.

The polytopes which are used in the algorithm are simple polytopes, which are polytopes in which the vertices arise at the intersection of n facets (in $\mathbb{R}^n$). These polytopes have important combinatorial properties which are exploited in the algorithm development.

The higher order version of Theorem 2.11 is quite similar to the second order case. For higher order polytopes, the facets contain a variable number of vertices and edges (1-faces). The maximum allowable step is determined by examining at which point moving a facet outwards causes an edge to disappear - i.e., the two vertices which are at its endpoints are identified. Once again, this is a problem which can be solved via linear algebra.

Presently, the main difficulty is the generation of the initial polytope. Although an algorithm has been developed in principle, its efficiency in practice has yet to be determined. There is much more emphasis on the combinatorial connection properties in the higher order cases, which do not appear here. Basic examination of the combinatorial problems indicates that computational

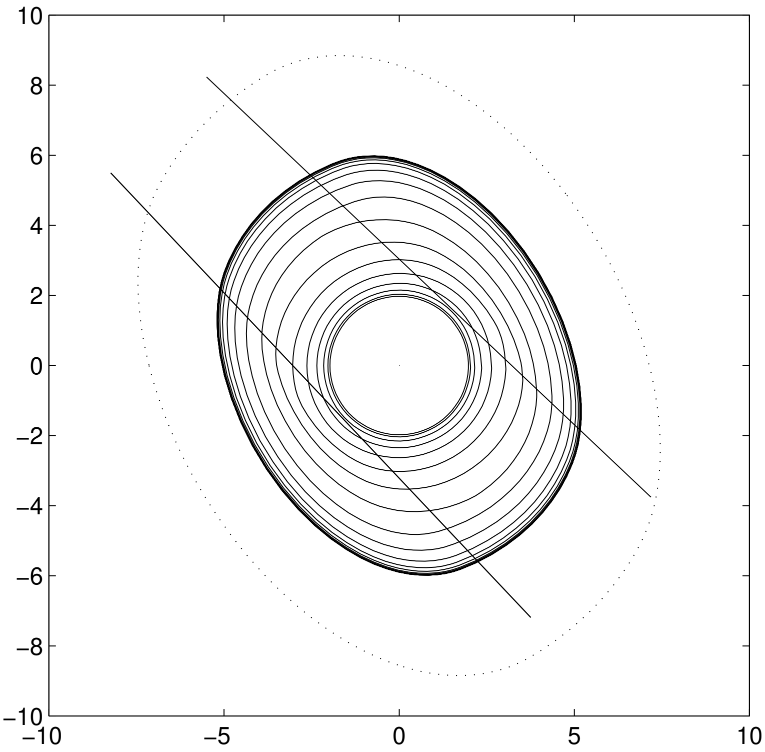

Figure 3: Growth of the polytope, with 75 facets.

order is still explosive as a function of the dimension of the state space, but should represent a reduction when compared to finite difference methods. And it should be noted that the difficult combinatorial problems associated with the initialisation step can in fact be done off line, which is a useful property of the algorithm.

6 Comparison to PDI based methods

The usual means of treating the problem of the determination of the domain of attraction of the origin is to use a Lyapunov function. If one is able to develop and analyse such a function by hand, algorithmic difficulties are not a problem. In general, this is difficult, as it almost amounts to one having the solution to the problem in advance. There are general existence results (as in the paper by J.C. Willems [12]), but these are not practical in terms of a computational algorithm.

In order to develop a general purpose algorithm, one would typically attempt to solve the partial differential inequality:

$$\left.\frac{dV}{dx}\right|_{\xi} f(x) < 0 \quad (x \neq 0), \tag{7}$$

with $V\colon \mathbb{R}^n \mapsto \mathbb{R}_+$ such that $V(0) = 0$, $V(\xi) > 0$ for $\xi \neq 0$. It is likely that the solution is non-smooth; (7) should be interpreted in a weak sense (see the paper by M.R. James [7] for details on the use of weak sense (non-smooth) interpretations). Viscosity solution theory can be used to examine such non-smooth partial differential inequalities; see the text [5] by Fleming and Soner for an exposition on viscosity solutions). Typically, solutions are computed by finite difference

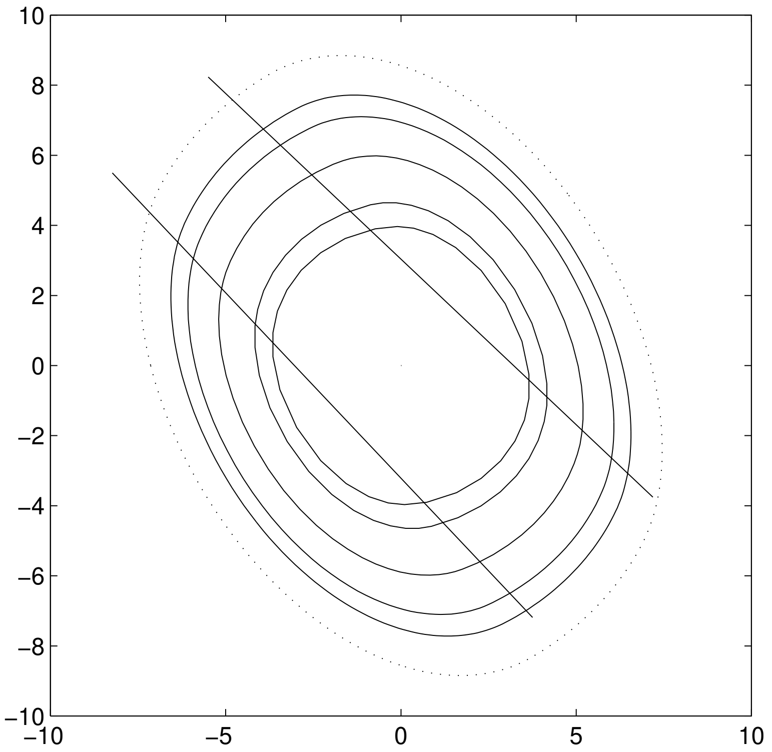

Figure 4: Limits to growth with 30,45,75,150,250 facets.

methods: a grid is imposed on the state space, and a solution is iteratively computed based on values at the grid points (for example, see the discussion in [5, Chapter IX]).

The algorithm developed herein can be interpreted as being an attempt to solve the PDI (7) in a novel fashion. The boundary of the polytope can be seen as a level curve of a solution, although it is more difficult to determine the form of the global solution. It is expected that this algorithm can be generalised to solve other useful PDI's in control - such as the Hamilton-Jacobi-Bellman equations which are used to compute the nonlinear $\mathcal{H}_\infty$ norm.

There are believed to be advantages to the presented algorithm over finite difference methods.

1. Imposing a grid over the state space runs into the problem that the number of grid points increases geometrically at the power of the state space dimension. That occurs here as well, of course. The added disadvantage grid based methods face is that in order for the method to work, the grid spacings need to be small in order to approximate the nonlinear geometries. It is hoped that extremely tight griddings are not needed for polytope based algorithms. Also, the spacings determined by the growth steps will be as large as possible to keep the solution valid. Such variable spacings are difficult to implement for a finite difference scheme.

2. When solving a PDI, the region is determined *a priori*. If one is within a finite region of attraction, repetitive runs will be needed to determine its extent. The polytope "grows" until stopped by constraints or the user, and so this problem disappears.

3. The solution is guaranteed to be valid at all times once the initial conditions are met; one cannot be sure in an iterative scheme, especially near the limit of convergence/divergence. Additionally, one cannot be sure if a grid density is sufficient; guarantees are typically given only in the limit as the spacing goes to zero.

4. The only "boundary condition" which appears corresponds to the required $V(0) = 0$.
5. The data structures are quite compact, and only basic multiplications and additions are needed. The algorithm can be quite naturally be developed in a parallel fashion.
6. The rather involved mathematics surrounding viscosity solution theory is not needed for the algorithm development.

7 Non-Convexity and Conditioning

One important issue which arises is the implicit assumption that the invariant sets are convex. One can easily write down nonlinear systems with power series representations which have non-convex invariant sets, but are stable. It is conjectured that this can be viewed as a conditioning problem - that the closer to convex the invariant sets are, the better conditioned the system representation. All practical algorithms require some form of numeric conditioning; how that is to be attempted here is not yet clear. It presumably would take the form of a nonlinear transformation on the state space.

One point to be noted is that if one writes down a power series representation of a nonlinear system, one obscures how complex a piecewise linear system needed to approximate it. It is believed that the algorithmic complexity is inherently tied in some fashion to the number of regions of definition, as well as the state dimension. The extent of the regions might need to be made more uniform, which would partially provide the nonlinear transformation required.

It may be possible to identify problems by examining whether there are abrupt changes in the dynamics between regions. Again, problems of this sort may be identifiable algebraically, and hopefully fixed by some conditioning algorithm.

It should be noted that this convexity issue is also likely to be a problem for finite difference based methods. This arises as one typically solves the PDI over a convex set such as a hypercube. Only some subset of the region examined would provide a convergent solution, as there can be no guarantees over points whose induced trajectories leave the region of computation.

8 Concluding Remarks

We have detailed herein a novel algorithm for determining the region of attraction of the origin for piecewise linear dynamical systems. It is hoped that this method can provide a useful alternative to finite difference methods.

There are many open problems surrounding this algorithm, as is to be expected. The generalisation to higher orders is to be detailed elsewhere. There are considerable algorithmic details still open for the higher order cases. Conditioning issues are still unclear. Most importantly, whether it is possible to extend this technique to solving other PDI based problems is yet open - e.g., the induced norm computation problem, and controller synthesis.

Appendix: Proofs

Proof of Lemma 2.7: By definition, the intersection $F_i \bigcap F_{i+1}$ is non-empty; and they are distinct facets due to the fact that we assume it is an irreducible representation. The intersection of the two 1-faces gives rise to a 0-face (a vertex). Then, remembering Remark 2.6, that every vertex lies in the intersection of only 2 facets, and every facet contains 2 vertices. Hence, the points $F_i \bigcap F_{i+1}$ enumerates all possible vertices. ■

Proof of Theorem 2.8: Noting Lemma 2.7, the proof consists of showing that the intersection of the i-th and $i+1$-th facet is given by (1) (remembering the comment in Remark 2.3).

It is a matter of basic trigonometric identities to arrive at the expression (1), once it is noted that at the intersection of $H(u^i, a_i)$,$H(u^{i+1}, a_{i+1})$ (denoted x^i) is characterised by:

$$\left\langle x^i, u^i \right\rangle = a_i, \qquad \left\langle x^i, u^{i+1} \right\rangle = a_{i+1},$$

hence,

$$\begin{bmatrix} (u^i)' \\ (u^{i+1})' \end{bmatrix} x^i = \begin{bmatrix} a_i \\ a_{i+1} \end{bmatrix}.$$

■

Proof of Theorem 2.9: The difficult part of the proof is to verify that the representation is irreducible. Fix some $i \in 0, \ldots, N-1$. Take the point $x = \alpha u^i$. Then, we see that

$$\left\langle x, u^j \right\rangle = \alpha \left\langle u^i, u^j \right\rangle = \cos\left(\frac{2\pi(i-j)}{N}\right) \leq \alpha, \tag{8}$$

hence $x \in \Omega_a^{\mathcal{U}}$. Then note that $\left\langle x, u^j \right\rangle < \alpha(1-\epsilon)$ for some $\epsilon > 0$, for all $j \neq i$. Then, we readily see that $\alpha\frac{1}{1-\epsilon}x \notin \Omega_a^{\mathcal{U}}$, but,

$$\alpha\frac{1}{1-\epsilon}x \in \bigcap_{j=0,j\neq i}^{N-1} K(u^j, a_j).$$

Hence $\Omega_a^{\mathcal{U}} \subset \bigcap_{j=0,j\neq i}^{N-1} K(u^j, a_j)$ (inclusion is obvious). We then need to show that the vertices of the polytope are given by the intersections of F_i,F_{i+1}.

One means of verifying this will be sketched. If the polytope generated by $\bigcap_{j=0,j\neq i}^{N-1} K(u^j, a_j)$ is examined, we see that the point x^* which is the intersection of F_{i-1} and F_{i+1} is given by $\alpha^* u^i$ $(\alpha^* > \alpha)$. Examination of the relation in (8) can be applied to show that x^* is a vertex of the new polytope. Then, as x^* does not lie in the original polytope, the relation between the number of vertices and facets ensures that the intersection between F_i and F_{i-1} (resp. F_{i+1}) is the second vertex connected to F_{i-1} (resp. F_{i+1}). The fact that $\Omega_a^{\mathcal{U}}$ is bounded follows from the fact that its vertices are finite. ■

Proof of Theorem 2.11: (Please note that the full details have been omitted, and some details are only sketched herein. The missing steps are felt to be straightforward from examination of the geometry.) We will need to show that $\Omega_{a+}^{\mathcal{U}}$ is irreducible if $a_i < a_i^+ < a_i^*$, but is not for $a_i^+ \geq a_i^*$. Let the intersection of $H(u^{i-1}, a_{i-1})$, $H(u^{i+1}, a_{i+1})$ be denoted x^*. Then, we see that:

$$x^* = \hat{T}\begin{bmatrix} a_{i-1} \\ a_{i+1} \end{bmatrix} := \begin{bmatrix} \cos(\frac{2\pi(i-1)}{N}) & \sin(\frac{2\pi(i-1)}{N}) \\ \cos(\frac{2\pi(i+1)}{N}) & \sin(\frac{2\pi(i+1)}{N}) \end{bmatrix}^{-1} \begin{bmatrix} a_{i-1} \\ a_{i+1} \end{bmatrix}.$$

We then can explicitly express $\hat{T}$ as (using a geometric identity):

$$\hat{T} = \frac{1}{\sin(\frac{4\pi}{N})}\begin{bmatrix} \sin(\frac{2\pi(i+1)}{N}) & -\sin(\frac{2\pi(i-1)}{N}) \\ -\cos(\frac{2\pi(i+1)}{N}) & \cos(\frac{2\pi(i-1)}{N}) \end{bmatrix}.$$

Then, by writing $\alpha_i^* = \left\langle u^i, x^* \right\rangle$, with $u^i = \begin{bmatrix} \cos(\frac{2\pi i}{N}) \\ \sin(\frac{2\pi i}{N}) \end{bmatrix}$, then geometric identities give us that:

$$a_i^* = \begin{bmatrix} \cos(\frac{2\pi i}{N}) & \sin(\frac{2\pi i}{N}) \end{bmatrix} \hat{T} \begin{bmatrix} a_{i-1} \\ a_{i+1} \end{bmatrix} = \frac{\sin\left(\frac{2\pi}{N}\right)}{\sin\left(\frac{4\pi}{N}\right)}[1 \quad 1]\begin{bmatrix} a_{i-1} \\ a_{i+1} \end{bmatrix},$$

which is $(\text{maxstep}(a,i))_i$. By definition, $x^* \notin \Omega_a^{\mathcal{U}}$ (otherwise, it would be a vertex), hence $a_i^* > a_i$. If we examine the polytope

$$\Omega_- = \bigcap_{j=0, j\neq i}^{N-1} K(u^j, a_j),$$

the results on 2-polytopes found in Remark 2.6 readily indicate that x^* is a vertex of Ω_-.(To sketch the proof, note that all previous vertices of $\Omega_a^{\mathcal{U}}$ remain vertices of Ω_-, except for the two which were elements of F_i. This follows from noting that $\Omega_{a+}^{\mathcal{U}}$ consists of the union of $\Omega_a^{\mathcal{U}}$ and some new polytope which lies between $H(u^i, a_i)$ and $H(u^i, a_i^+)$. As F_{i+1}, F_{i-1} have then only one vertex from the old set, their intersection is the new vertex, which is x^*.)

Then, we note that if $a_i^+ \geq a_i^*$, then we can readily see that $\Omega_{a+}^{\mathcal{U}} = \Omega_-$, hence, the polytope is not irreducible. But if $a_i^+ < a_i^*$, then $x^* \notin \Omega_a^{\mathcal{U}}$. Hence, $\Omega_{a+}^{\mathcal{U}}$ is irreducible. Once again, examination of the properties results quoted in Remark 2.6 ensure that the intersection of F_{i-1} and F_i, as well as the intersection of F_i and F_{i+1} are vertices of $\Omega_{a+}^{\mathcal{U}}$, which is another requirement for $\Omega_{a+}^{\mathcal{U}}$ to be proper. The final condition, the boundedness of $\Omega_{a+}^{\mathcal{U}}$, follows from the boundedness of all its vertices. ■

Proof of Lemma 3.5: This lemma is a straightforward application of convexity. Fix facet F_i for examination. If it lies completely within a single region Γ_k, the convexity of the function $A^i x + B^i$ ensures that if $\langle u^i, f(x^i)\rangle < 0$, $\langle u^i, f(x^{i-1})\rangle < 0$ implies that $\langle u^i, f(\lambda x^i + (1-\lambda)x^{i-1})\rangle < 0$ for all $0 < \lambda < 1$, i.e., all of the set F_i. Otherwise, the convexity argument is carried out between the endpoints and all other members of $\hat{X}$. Note that if a region boundary coincides with some subset of the facet, it is only necessary to verify the invariance at the endpoints of coincidence. ■

Proof of Theorem 3.6: (The details of this proof are only sketched, as they follow standard control theory methods.) Examine the set $\Delta_\Omega := \text{closure}\{\Omega_{a*}^{\mathcal{U}} \setminus \Omega_a^{\mathcal{U}}\}$. The theorem statement is implied if all trajectories originating in Δ_Ω enter $\Omega_a^{\mathcal{U}}$ in a finite time.

Due to the continuity of trajectories, the Intermediate Value Theorem indicates that the invariance of the set $\Omega_{a*}^{\mathcal{U}}$ makes it impossible that any trajectory originating in Δ_Ω can leave that set except through the facet F_i in $\Omega_a^{\mathcal{U}}$. It remains to be verified that any such trajectory will do so in a finite time.

Define a Lyapunov-like function $V(x)$ as:

$$V(x) := \langle u^i, x\rangle - a_i.$$

Then by definition, $V(x) = 0$ on the facet F_i of $\Omega_a^{\mathcal{U}}$, and $V(x) > 0$ for all other points in Δ_Ω. The invariance condition specified in the theorem statement, along with the fact that for all $x \in \Delta_\Omega$, there exists some a_i^+ with $a_i \leq a_i^+ \leq a_i^*$ such that $x \in \partial\Omega_{a+}^{\mathcal{U}}$, implies that $\frac{dV}{dx}\big|_x < 0$. Then, as Δ_Ω is compact, there exists a maximal value of $\frac{dV}{dx}\big|_x$, and standard Lyapunov results indicate that $V(x)$ goes to zero in a finite time (i.e., enters $\Omega_a^{\mathcal{U}}$). (See [9, Section 4.6] for an exposition on Lyapunov function theory (amongst others).) ■

Proof of Theorem 3.8: Examination of Lemma 3.5 shows that if the dot product of the flow with the tangent vectors remains negative definite evaluated at the connected vertices as the value of a_i varies, along with Theorem 3.6 is sufficient for asymptotic stability to be preserved over the larger polytope.

Noting then that if we let $g(a_i) = \langle u^i, f(x^i(a_i))\rangle$, then

$$\frac{dg}{da_i}\bigg|_{a_i} = \left\langle u^i, A^k \frac{dx^i}{d_{a_i}}\bigg|_{a_i}\right\rangle = \left\langle u^i, \begin{bmatrix} P_{11}^{k,i} \\ P_{12}^{k,i} \end{bmatrix}\right\rangle.$$

As we are given that $g(a_i) = \langle u^i, A^k x^i + B^k \rangle < 0$ initially, if $\frac{dg}{da_i} \leq 0$, the invariance is preserved for all $a_i^* > a_i$. Otherwise, if $a_i^* < a_i - \frac{g(a_i)}{\frac{dg}{da_i}}$, then $g(a_i^*) < 0$. This corresponds to the step$_i$ conditions. The remaining step$_j$ functions correspond to the other 3 conditions which arise when the i-th facet is shifted, and result in analogous calculations. ■

Proof of Theorem 4.2: The polytope in the theorem statement is a regular polygon. The value of α will be chosen so that polytope lies within the circle $\|x\| < \frac{1}{\|K\|}$, i.e., within the linear operating region.

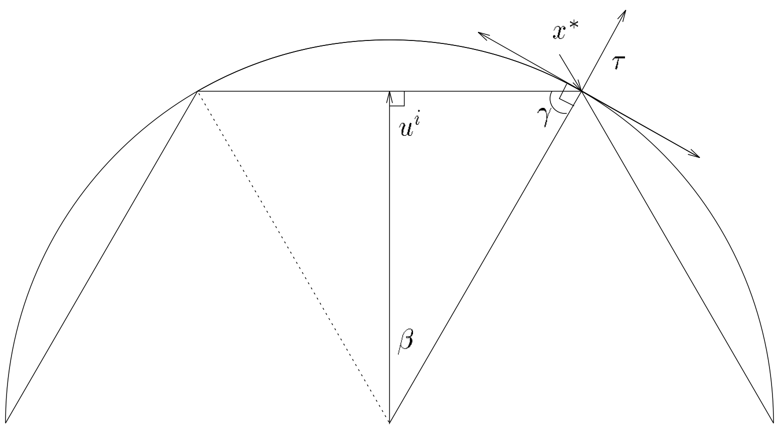

Figure 5: Geometry of the initial polytope.

Examine Figure 5, with τ a unit vector normal to the enscribed circle at the vertex x^*. Examine a scaled system so that $\|x^*\| = 1$. Then,

$$\begin{aligned} \langle (A - BK)x^*, u^i \rangle &= \langle (A - BK)x^*, \tau \rangle + \langle (A - BK)x^*, u^i - \tau \rangle \\ &= -x^{*\prime} P^{-1} x^* + \langle (A - BK)x^*, u^i - \tau \rangle . \end{aligned}$$

Examination of the geometry in Figure 5 shows that the angle between u^i and τ is $\beta = \frac{\pi}{N}$, hence $\|u^i - \tau\|^2 = \|u^i\|^2 - 2\langle u^i, \tau \rangle + \|\tau\|^2 = 2 - 2\cos(\frac{\pi}{N})$. This implies that

$$\langle (A - BK)x^*, u^i \rangle \leq -\underline{\sigma}(P^{-1}) + 2\|(A - BK)\| \left(1 - \cos\left(\frac{\pi}{N}\right)\right),$$

hence a choice of N exists to ensure that the polytope is invariant under the flow in the linear operating region. Once N is fixed, α is chosen so that $\alpha < 1/\|K\| \left\|T^1[11]'\right\|$ (symmetry ensures that all the vertices lie within the linear operating region).

The asymptotic stability condition follows from noting that the trajectory cannot exit the initial polytope (hence the linear operating region) due to its invariance, and then application of standard Lyapunov theory for linear systems (for example, see [9, Lemma 4.6.4]). Alternatively, one could directly prove uniform asymptotic stability by noting that the trajectory must enter the polytope generated by $a = (1 - \epsilon)\alpha[1, \ldots, 1]'$ in a finite time; hence multiple applications of this property via scaling indicates that the trajectory will enter an arbitrarily small non-zero set surrounding the origin in finite time. ■

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