Analytic Comparison of Nonlinear H_∞-Norm Bounding Techniques for Low Order Systems with Saturation

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Abstract

In this paper, we will compare differing nonlinear $\mathcal{H}_\infty$ analysis techniques for linear systems with saturation nonlinearities, as applied to low order examples. The examples effectively show the inter-relationship of stability and the behaviour of the induced norm and incremental gain, and the necessity of using bounded input sets. Finally, the state feedback synthesis problem receives preliminary attention, and two broad approaches are illustrated. The first uses differential inequality methods to generate a controller, the second method amounts to being a preliminary attempt at nonlinear $\mathcal{H}_\infty$ anti-windup synthesis.

Keywords: nonlinear $\mathcal{H}_\infty$ control, systems with saturation, finite gain stability.

1 Introduction

This paper consists of a series of examples which illustrate the inter-relationship of nonlinear $\mathcal{H}_\infty$ analysis techniques for feedback loops which contain saturation nonlinearities. As the examples themselves are all low order, it was possible to undertake closed form analysis (in general), and give analytic comparisons, which would not appear to be possible presently with higher order cases.

There are two competing generalisations of the $\mathcal{H}_\infty$ norm available for nonlinear systems, being the incremental gain, and the induced $\mathcal{L}_{2+}$ norm. The incremental gain has many advantages over the induced norm in a theoretical sense, but it appears to be a quite conservative measure. For the class of systems studied, open loop unstable plants cannot be stabilised in an incremental gain sense, which appears to limit the incremental gain's usefulness as an analytical tool.

The methods used to give bounds for the operator norms were developed elsewhere, although some have only appeared in the dissertation [8]. For continuous time systems, the Small Gain Theorem (Circle Criterion) estimate, the bound given in the paper by Liu, Chitour, Sontag [6], the bounds for the induced norm using dissipation functions as calculated by the method outlined in [7], and the bounds for the incremental gain using linear matrix inequalities developed in [8] are compared herein for the general first order plant.

The behaviour of the induced norm as the plant pole crosses from the closed left half plane to the open right half plane is pathological if one takes the input set to be $\mathcal{L}_{2+}$. The introduction of bounded input sets causes the induced norm to behave in a much more sensible fashion. It is believed that the use of such sets is necessary to give non-conservative analysis. This has been noted by other researchers, as in the paper by Lin, Saberi and Teel [4].

Analysis techniques developed in [8] are used to compute the induced ℓ_{2+} norm of a first order discrete time example. Many of the results on continuous time induced norm computation in [7] have been extended to discrete time, but it is not possible to give the results in as clean a form.

Finally, a preliminary examination of the state feedback synthesis problem has been made. Two differing approaches to state feedback synthesis are illustrated for a first order example. The first method is an extension of that which appears in the paper [9] by A.J. van der Schaft, and which relies on the use of differential inequalities. Even the first order example shows some difficulties with a naive extension of the methods in [9] to this class of systems. The second method relies on a parameterised piecewise linear controller, and the analysis undertaken to fix suitable parameter values. The advantage of this approach is that it allows one to cleanly force the linearisation of the controller to be the linear controller derived by the much better understood linear design methodology, and the goal of the nonlinear synthesis is to best preserve the linear operating characteristics. In other words, this is viewed to be a preliminary attempt of using nonlinear $\mathcal{H}_\infty$ methods to synthesize anti-windup schemes.

2 Mathematical Preliminaries

We will now provide the background notation and theory used in this paper. The dissertation [8] contains a fulller development of this problem framework and these surveyed results.

The problems studied herein are the gain computations for a class of nonlinear operators. In order to measure the gain of an operator, we need to be able to measure signal size, which is done herein in terms of $\mathcal{L}_{p+}$ spaces.

The space norm of a vector in $\mathbb{R}^n$ used herein is the Euclidean (or 2-) Norm. It is defined by

$$\|x\| := \sqrt{x'x},$$

and is the only norm used in this paper which is not distinguished by a subscript. The $\mathcal{L}_{p+}$ norm (for $p < \infty$) of a (real) signal y is defined by

$$\|x\|_p := \left(\int_0^\infty \|x(t)\|^p \, dt\right)^{\frac{1}{p}},$$

and in the case $p = \infty$,

$$\|x\|_\infty := \operatorname{ess}\sup_{t\in\mathbb{R}_+} \{\|x(t)\|\}.$$

The space $\mathcal{L}_{p+}$ is the set of all signals with finite $\mathcal{L}_{p+}$ norm. For notational convenience, the dimension of the signal space is supressed, as it should be clear from the context. (We will implicitly use *extended spaces* (see the text by Willems [10] for an exposition using extended spaces), when we refer to test signals which give rise to unbounded outputs. From an operator theoretic point of view, if one does not use extended spaces, unbounded outputs do not formally exist, and the inputs correspond to a defect in the domain of the operator. It is felt that this distinction is not of supreme importance for the present work, and so the notational overhead of introducing extended spaces has been avoided. This distinction should be kept in mind when developing operator theoretic analysis, however.)

In order to analyse systems with saturation nonlinearities, it is necessary to introduce bounded subsets of $\mathcal{L}_{p+}$ spaces, for reasons which will be made apparent within this paper. We will now formally define the bounded sets used.

Definition 2.1: The set $\mathcal{W}_{p,q}(N)$ is the intersection of the sets, $\mathcal{L}_{p+} \bigcap \{w: \|w\|_q \leq N\}$.

Having defined our signal sets, it is possible to define the gains to be bounded by the analysis undertaken in this paper. The *induced $\mathcal{L}_{p+}$ norm* of an operator $\mathbf{\Upsilon}: \mathcal{L}_{p+} \mapsto \mathcal{L}_{p+}$ with domain $\mathcal{L}_{p+}$ (resp. $\mathcal{W}_{p,q}(N)$) is defined (if it exists):

$$\|\mathbf{\Upsilon}\|_{i,p} = \sup_{w\in\mathcal{L}_{p+}(\text{resp. } \mathcal{W}_{p,q}(N)), w\neq 0} \frac{\|\mathbf{\Upsilon} w\|_p}{\|w\|_p}.$$

The $\mathcal{L}_{p+}$ *incremental gain* of an operator $\mathbf{\Upsilon}: \mathcal{L}_{p+} \mapsto \mathcal{L}_{p+}$ with domain $\mathcal{L}_{p+}$ (resp. $\mathcal{W}_{p,q}(N)$) is defined (if it exists):

$$\|\mathbf{\Upsilon}\|_{\Delta,p} = \sup_{w,\tilde{w}\in\mathcal{L}_{p+}(\text{resp. } \mathcal{W}_{p,q}(N)), w\neq \tilde{w}} \frac{\|\mathbf{\Upsilon}\tilde{w} - \mathbf{\Upsilon} w\|_p}{\|\tilde{w} - w\|_p}.$$

The motivation for the study of operator gains in robust control theory follows from the Small Gain Theorem (stated next).

Theorem 2.2: Fix a nonlinear operator $\mathbf{\Upsilon}: \mathcal{L}_{p+} \mapsto \mathcal{L}_{p+}$, with domain $\mathcal{L}_{p+}$. Then if $\|\mathbf{\Upsilon}\|_{\Delta,p} < 1$, $(\mathbf{I}+\mathbf{\Upsilon})^{-1}$ exists, with

$$\left\|(\mathbf{I}+\mathbf{\Upsilon})^{-1}\right\|_{\Delta,p} < \frac{1}{1-\|\mathbf{\Upsilon}\|_{\Delta,p}}.$$

Additionally, if $(\mathbf{I}+\mathbf{\Upsilon})^{-1}$ exists, and $\|\mathbf{\Upsilon}\|_{i,p} < 1$, then,

$$\left\|(\mathbf{I}+\mathbf{\Upsilon})^{-1}\right\|_{i,p} < \frac{1}{1-\|\mathbf{\Upsilon}\|_{i,p}}.$$

Proof: See [10]. □

Remark 2.3: It is an important caveat that the inverse of $(\mathbf{I}+\mathbf{\Upsilon})$ must exist if the induced norm version of the Small Gain Theorem is to be used. If the system is to be generated by a state space realisation, it would be necessary to verify the existence and uniqueness of the solution using standard theory on differential equations.

Analysis of generic nonlinear operators is quite difficult, hence it is usual to restrict ones attention to subclasses of systems. The class to be studied herein are those which arise when memoryless saturation nonlinearities (defined next) are inserted into linear state space equations. The *standard saturation function*, denoted $\sigma_0(\cdot)$, is defined:

$$\sigma_0(x) := \begin{cases} -1 & x < -1, \\ x & |x| \leq 1, \\ 1 & x > 1. \end{cases}$$

The nonlinear operators to be studied are of the form $\mathbf{\Upsilon}: w \mapsto z$, generated by the state space equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) - B\sigma_0(w(t) - K(x(t))), && (1) \\ x(0) &= 0, && (2) \\ z(t) &= Cx(t). && (3) \end{aligned}$$

Restrictions on the terms appearing in (1) - (3) has been imposed, listed in the following Assumption.

Assumption 2.4: We assume that $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, (i.e., single input), $C \in \mathbb{R}^{p\times n}$, $[A, B]$ controllable, $[C, A]$ observable, and that $\left.\frac{dK(x)}{dx}\right|_{x=0}$ exists.

We define $K \in \mathbb{R}^{1\times n}$ as:

$$K := \left.\frac{dK(x)}{dx}\right|_{x=0},$$

and the *local linearisation* of $\mathbf{\Upsilon}$ is the linear operator $\mathbf{\Upsilon}_L$, which has the transfer function

$$\Upsilon_L(s) = \left[\begin{array}{c|c} A-BK & B \\ \hline C & 0 \end{array}\right] := C(sI-(A-BK))^{-1}B. \tag{4}$$

As is well known, $\|\mathbf{\Upsilon}_L\|_{\Delta,2} = \|\Upsilon_L(s)\|_\infty$. The Algebraic Riccati Equation can be used to compute this quantity, as a consequence of the following result.

Theorem 2.5: The linear operator $\mathbf{\Upsilon}_L$ with the transfer function $\Upsilon_L(s)$ defined in (4) is such that $\|\mathbf{\Upsilon}_L\|_{\Delta,2} \leq \gamma$ if and only if there exists a symmetric, positive definite matrix $P \in \mathbb{R}^{n\times n}$ which solves the *Algebraic Riccati equation*:

$$(A-BK)'P + P(A-BK) + \frac{1}{\gamma^2}PBB'P + C'C = 0. \tag{5}$$

Proof: This is standard, see [11]. □

It has been found to be possible to bound above the incremental gain using the solution of a coupled pair of linear matrix inequalities (see the text [1] for an exposition on linear matrix inequalities). This has been developed in [8], and will be stated in an appendix as an aid to readers. The two inequalities amount to being the simultaneous solution of the open loop Lyapunov equation, and the closed loop Riccati equation. There has been further assumptions on the form of systems added, as the present result is in a preliminary form, and some technical issues still appear to be open with regards to the extension to more general configurations.

Theorem 2.6: Fix an operator $\mathbf{\Upsilon}: w \mapsto z$, generated by (1) - (3), with the assumption that $K(x) = Cx$ added to those listed in Assumption 2.4. Then, $\|\mathbf{\Upsilon}\|_{\Delta,2} \leq \gamma$ (over the set $\mathcal{L}_{2+}$) if there exists a (symmetric) positive definite matrix $P \in \mathbb{R}^{n\times n}$ which solves the pair of linear matrix inequalities:

$$(A-BC)'P + P(A-BC) + \frac{1}{\gamma^2}PBB'P + C'C \leq 0, \tag{6}$$

$$A'P + PA + C'C \leq 0. \tag{7}$$

Proof: Given in the appendix. Originally in [8]. □

It is conjectured that the "if" statement in Theorem 2.6 may be strengthened to "if and only if", but this is still an open problem. Examination of the inequalities (6) - (7) indicates that no solution P will exist if either A or $A - BC$ is unstable, and this has been shown to be true for the corresponding discrete time problem. More importantly, bounding the signal sizes allowed is not sufficient to allow less conservative analysis, as the construction uses signals of relatively small ∞ norm, and with arbitrarily small differences between the test pairs. (This is discussed in the appendix.)

Finally, we will give the dissipation theory based results which are used to bound above the induced $\mathcal{L}_{2+}$ norm. We will need the following extra definitions and results.

Definition 2.7: A function $V: \mathbb{R}^n \mapsto \mathbb{R}_+$ is a *candidate dissipation function* if it is uniformly Lipschitz over compact sets, $V(0) = 0$, and $V(\xi) > 0$ if $\xi \neq 0$.

Theorem 2.8 (Rademacher's): A candidate dissipation function V is differentiable almost everywhere.
Proof: See [12, Theorem 2.2.1] for details. □

Definition 2.9: The operator $\mathbf{\Upsilon}: w \mapsto z$ generated by (1) - (3) is $\mathcal{L}_2$*-detectable* if $w, z \in \mathcal{L}_{2+}$ implies that $x \in \mathcal{L}_{2+}$.

Lemma 2.10: If the form of $K: \mathbb{R}^n \mapsto \mathbb{R}$ is restricted to be of the form $K(x) = KCx$, then $\boldsymbol{\Upsilon}$ is $\mathcal{L}_2$-detectable if $[C, A]$ is detectable.
Proof: This preliminary result is taken from [8, Theorem 3.6.11].

Theorem 2.11: Fix the operator $\boldsymbol{\Upsilon}: w \mapsto z$. Assume that $\boldsymbol{\Upsilon}$ is $\mathcal{L}_2$-detectable, and that for $w \in \mathcal{W}_{2,q}(N)$, the reachable set lies within $\Omega_0 \subseteq \mathbb{R}^n$. Let $V(x)$ be a candidate dissipation function defined on Ω_0, and $\Omega_d \subseteq \Omega_0$ be defined

$$\Omega_d = \left\{ \xi: \xi \in \Omega_0, \left. \frac{dV}{dx} \right|_{\xi} \text{exists} \right\}.$$

Then, $\|\boldsymbol{\Upsilon}\|_{i,2} \leq \gamma$ (over $\mathcal{W}_{2,q}(N)$) if

$$\sup_{\xi \in \Omega_d} \{\max \{\Phi_A(\xi), \Phi_B(\xi)\}\} \leq 0,$$

with $\Phi_A(\xi), \Phi_B(\xi)$ defined in (8),(9). $\Phi_{A.1}$ is defined by

$$\Phi_A(\xi) := \begin{cases} \Phi_{A.1}(\xi) & |\theta_a| \leq 1, \\ \Phi_{A.2}(\xi) & |\theta_a| > 1, \end{cases} \tag{8}$$

with $\theta_a, \Phi_{A.1}, \Phi_{A.2}$ defined:

$$\theta_a(\xi) := -K(\xi) + \frac{1}{\gamma^2} B' \left. \frac{d^T V}{dx} \right|_{\xi},$$

$$\begin{aligned} \Phi_{A.1}(\xi) &:= 2 \left. \frac{dV}{dx} \right|_{\xi} (A\xi + B\theta_a(\xi)) - \gamma^2 \|K(\xi) + \theta_a(\xi)\|^2 + \xi' C' C \xi, \\ \Phi_{A.2}(\xi) &:= 2 \left. \frac{dV}{dx} \right|_{\xi} (A\xi + B\mathrm{sign}(\theta_a(\xi))) - \gamma^2 \|K(\xi) + \mathrm{sign}(\theta_a(\xi))\|^2 + \xi' C' C \xi. \end{aligned}$$

The function $\Phi_B(\xi)$ is defined:

$$\Phi_B(\xi) := 2 \left. \frac{dV}{dx} \right|_{\xi} (A\xi - B\sigma_0(K(\xi))) + \xi' C' C \xi. \tag{9}$$

Proof: This is taken from [7]. □

Remark 2.12: There has been a minor re-definition of Φ_B to take advantage of the simpler form of the system given here than that appears in [7], the equivalence of the test condition given here to that in [7] follows from the observations in [7, Remark 3.3].

The problem with characterising the induced norm in terms of dissipation functions is the computation of the functions themselves. There are generic existence theorems available, which rely on viscosity solution theory (see the paper by James [3]). The dissipation functions used herein were computed via the use of a (sub-optimal) heuristic algorithm developed in [7]. Preliminary definitions required will now be stated.

We assume that $K(x)$ is linear, denoted $K(x) = Kx$. Let P be the positive definite (symmetric) solution to the algebraic Ricatti equation

$$(A - BK)'P + P(A - BK) + \frac{1}{\gamma_l^2} PBB'P + C'C + Q = 0,$$

for some arbitrary $Q > 0$. Let $\psi: \mathbb{R}^n \mapsto \mathbb{R}_-$ be defined:

$$\psi(\xi) := \min\{0, 2\xi' P(A\xi - B\sigma_0(K\xi)) + \xi' C'C\xi\}.$$

Let the *induced trajectory from* ξ ($\xi \in \mathbb{R}^n$) (denoted $x(t;\xi)$) be the state space trajectory generated by:

$$\begin{aligned} \dot{x}(t) &= Ax(t) - B\sigma_0(Kx(t)), \\ x(0) &= \xi. \end{aligned}$$

Assumption 2.13: It is assumed that for any fixed disturbance set $\mathcal{W}_{2,q}(N)$, the reachable set (denoted Ω) has compact closure, and for all points $\xi \in \partial\Omega$, $x(t;\xi)$ enters Ω in finite time. (The latter essentially entails that there neither be equilibria nor limit cycles on the boundary of the reachable set.)

Theorem 2.14: Fix the operator $\mathbf{\Upsilon}: w \mapsto z$ generated by (1) - (3), with $K(x) = Kx$. The following statements are then equivalent:

1. for all $w \in \mathcal{W}_{2,q}(N)$, $\mathbf{\Upsilon} w$ is an element of $\mathcal{L}_{2+}$;
2. for all $\xi \in \Omega$, $x(t;\xi)$ (i.e., with $w = 0$) is an element of $\mathcal{L}_{2+}$;
3. the function $V: \bar{\Omega} \mapsto \mathbb{R}_+$ defined:

$$V(\xi) = \frac{1}{2} \int_0^\infty (x(t;\xi)'C'Cx(t;\xi) - \psi(x(t;\xi)))\, dt,$$

is a valid dissipation function for some finite $\gamma \geq \gamma_l$. (Hence $\|\mathbf{\Upsilon}\|_{i,2} \leq \gamma$ over $\mathcal{W}_{2,q}(N)$).

Proof: In [7]. $\square$

3 The First Order Continuous Time Plant

We will now apply the analysis techniques surveyed in Section 2 to the simplest possible non-trivial case, being the general first order plant with proportional feedback. Although it may appear simple, this system appears to display most of the important properties which distinguish feedback loops with saturation nonlinearities from purely linear systems. Additional, as the analysis can be done in closed form, it is easier to compare differing analysis techniques.

The nonlinear system operator $\mathbf{\Upsilon}: w \mapsto z$ has the state space realisation:

$$\begin{aligned} \dot{x} &= -ax(t) + \sigma_0(-kx(t) + w(t)), \\ z(t) &= x(t), \\ x(0) &= 0, \end{aligned}$$

with $a, k \in \mathbb{R}$. The remainder of this section will consist of the application of the analysis techniques to the cases of interest which are determined by the values of a, k.

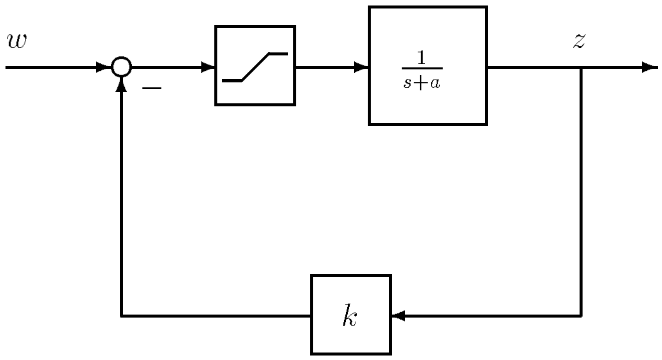

Figure 1: Example System

3.1 Case $a > 0$, $k < -a$

This is a somewhat unusual case, as the choice of feedback destabilises the stable linear plant. We can easily verify that the induced $\mathcal{L}_{p+}$ norm does not exist for all p (take as a test input $w(t) = \epsilon$ for $0 \leq t \leq \epsilon$, $w(t) = 0$ otherwise). However, it is also simple to verify that any trajectory is confined to $\Omega = (-\frac{1}{a}, \frac{1}{a})$, and so the state is confined to a compact set for all choices of input, and could even be considered to be stable under a loose enough definition of state space stability.

The importance (albeit limited) of this example is that it is possibly one of the simplest which shows that global state space properties are not necessarily determined by the behaviour at the origin, as is the case for linear systems. Additionally, it underlines the fact that the behaviour of the (closed loop) local linearisation provides a lower bound for operator gain properties.

3.2 Case $a > 0$, $k = 1$

We will now fix $k = 1$, and then examine the stability properties as the plant pole is shifted along the real axis. Examination of this case will provide a basic motivation as to the importance of the use of bounded input sets. Additionally, we can compare the bounds generated by different analysis techniques.

Once again, as $a > 0$, we have that the state is confined to $\Omega = (-\frac{1}{a}, \frac{1}{a})$. The system is locally stable, and the local linearisation has the transfer function $\Upsilon_L(s) = \frac{1}{s+a+1}$, which is in $\mathcal{RH}_\infty$, with $\|\Upsilon_L(s)\|_\infty = \frac{1}{a+1}$.

We will now examine the different bounds for the operator gain given by the use of the Small Gain Theorem (Circle Criterion), incremental gain analysis, and the result provided by the computational heuristic.

3.2.1 Small Gain Estimate

The first basic approach to bound the operator gain is to use the estimate provided by the small gain theorem, equivalent to the Circle Criterion. The means by which the estimate is made is via the use of *loop shifting*, a tactic long used in this situation (for example, see [10,

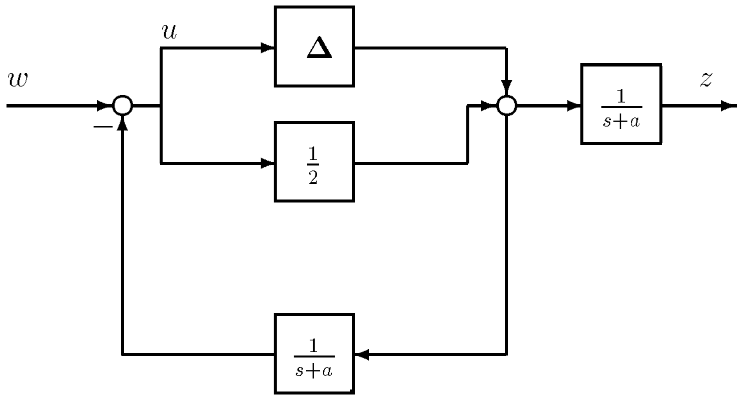

Figure 2: The System after Loopshifting

Section 6.2]). Loop transformations are the usual means of proving the Circle Criterion, and so the details are of enough interest to be given here.

The first step is to realise that we can express the function σ_0 as

$$\sigma_0(x) = \frac{1}{2}x + \Delta(x),$$

with $\Delta(x)$ given by:

$$\Delta(x) := \begin{cases} \frac{1}{2}x & |x| \leq 1, \\ \mathrm{sign}(x) - \frac{1}{2}x & |x| > 1. \end{cases}$$

The importance of this is that $\frac{1}{2}x$ is the centre of the sector $[0,1]$, and Δ has the minimal norm to cover the sector.

Now examine Figure 2. For ease of notation, let the linear operator with transfer function $\frac{1}{s+a}$ be denoted as $\mathbf{G}$, and let $\mathbf{H} = \frac{\mathbf{G}}{\mathbf{I}+\frac{1}{2}\mathbf{G}}$. We then can see that

$$u = -\mathbf{G}\boldsymbol{\Delta}u - \frac{1}{2}\mathbf{G}u + w.$$

Then,

$$u = (\mathbf{I} + \mathbf{H}\boldsymbol{\Delta})^{-1}(\mathbf{I} + \frac{1}{2}\mathbf{G})^{-1}w,$$

and so,

$$z = \mathbf{G}\boldsymbol{\Delta}(\mathbf{I} + \mathbf{H}\boldsymbol{\Delta})^{-1}(\mathbf{I} + \frac{1}{2}\mathbf{G})^{-1}w.$$

The incremental gain can thus be bounded by the use of the Small Gain Theorem,

$$\|\boldsymbol{\Upsilon}\|_{\Delta,p} \leq \|\mathbf{G}\|_{\Delta,p} \|\boldsymbol{\Delta}\|_{\Delta,p} \left(\frac{1}{1 - \|\mathbf{H}\|_{\Delta,p} \|\boldsymbol{\Delta}\|_{\Delta,p}} \right) \left\| (\mathbf{I} + \frac{1}{2}\mathbf{G})^{-1} \right\|_{\Delta,p}. \tag{10}$$

For the signal space $\mathcal{L}_{2+}$, we arrive at

$$\|\boldsymbol{\Upsilon}\|_{\Delta,2} \leq \frac{1}{2a}\left(1 + \frac{1}{2a}\right). \tag{11}$$

(The most difficult part of the transition from from the inequality (10) to (11) is to verify that

$$\left\| \left(1 + \frac{1}{2}G(s)\right)^{-1} \right\|_{\infty} = \left\| \frac{s+a}{s+a+\frac{1}{2}} \right\|_{\infty} = 1,$$

which is true for all $a > -\frac{1}{4}$. This can be verified by the use of the Riccati equation (5), with $\gamma^2 = 1 + \epsilon$, for $\epsilon > 0$.)

This analysis agrees with the expected result from the graphical analysis used in the Circle Criterion that the gain is finite if the system is stable, but becomes unbounded as $a \to 0$.

3.2.2 Incremental Gain Analysis

We will next bound above the incremental gain of the system, by using the result given in Theorem 2.6. The continuous time incremental gain algebraic inequalities are particularly easy to solve as they are given in terms of a scalar variable. The inequalities which result are:

$$\begin{aligned} -2ap + 1 &\leq 0, & (12)\\ -2(a+1)p + \frac{1}{\gamma^2}p^2 + 1 &\leq 0, & (13)\\ p &> 0. & (14) \end{aligned}$$

It is an obvious consequence of linear system analysis that the minimal value of γ for which there exists a solution to the inequalities (13) - (14) is

$$\gamma_l^* = \frac{1}{a+1}.$$

The solution p^* which determines this value of γ is given by:

$$p^* = \frac{1}{a+1}.$$

However, from (12), it is necessary that $p > \frac{1}{2a}$. Hence, if $a \geq 1$ we can use this value of p^* for solution to (12) - (13). This implies that the nonlinear system has the same incremental gain as the local linearisation. If $a < 1$, the minimising choice of p is to take $p = \frac{1}{2a}$. In this case, substitution into (13) yields:

$$\gamma^2 \geq \frac{1}{4a}, \qquad (15)$$

which gives an upper bound of $\frac{1}{2\sqrt{a}}$ for γ^*. Once again, the bound for the incremental gain becomes unbounded as $a \to 0$, but the bound given for $a > 0$ is tighter than that provided by Small Gain analysis.

Remark 3.1: It is interesting that no deterioration from the lower bound provided by the local linearisation for $a > 1$. This may in fact point to a general principle that if one is "stable enough", the introduction of the saturation does not affect the performance (in the present system configuration, at least). How this conjecture could be proven is not readily apparent, however.

3.2.3 Induced $\mathcal{L}_{2+}$ Norm Bounding

We will now apply the heuristic computational method given in Theorem 2.14. The analysis here gives non-surprisingly the least conservative results for bounding the induced $\mathcal{L}_{2+}$ norm. When the analysis is extended to the case where the plant is unstable, the importance of doing such a direct analysis will be underlined.

We will first need the solution for the local linearisation. The algebraic Riccati equation which results from (5) is:

$$2p(-a-1)+\frac{1}{\gamma_l^2}p^2+1=0.$$

We have that $\gamma_l^*=\frac{1}{a+1}$, with a corresponding $p=\frac{1}{a+1}$. The dissipation function V_l associated with the linear problem is given by:

$$V_l(\xi)=\frac{1}{2}\frac{1}{a+1}\xi^2.$$

It is then easy to verify that the state is confined to the set $\Omega=(-\frac{1}{a},\frac{1}{a})$ for all finite time. We now need to see over what subset of Ω the solution $V=V_l$ is a valid candidate function. Referring to Theorem 2.11 for the definitions of Φ_A, etc., it is obvious that V_l satisfies $\Phi_A(\xi)\leq 0$ for ξ; we need only verify the region where $\Phi_B(\xi)\leq 0$.

We have that

$$\begin{aligned}\Phi_B(\xi) &= 2\left.\frac{dV_l}{dx}\right|_{\xi}(-a\xi-\mathrm{sign}(\xi))+\xi^2,\\ &= -\frac{2}{a+1}(a\xi^2+|\xi|)+\xi^2.\end{aligned}$$

And so, $\Phi_B(\xi)\leq 0$ if $(1-a)|\xi|-2\leq 0$, or $|\xi|\leq\frac{2}{1-a}$. We then have verified that $V=V_l$ is sufficient if $\Omega\subseteq(\frac{-2}{1-a},\frac{2}{1-a})$, which occurs if $a\geq\frac{1}{3}$. For such cases then, $\gamma^*=\gamma_l^*=\frac{1}{a+1}$. Considering the result of the incremental gain analysis, it is obvious that the norm would not be affected for $a>1$.

For $a<\frac{1}{3}$, we take $V(\xi)$ to be such that $\Phi_B(\xi)=0$ for $|\xi|>\frac{2}{1-a}$. Then,

$$\left.\frac{dV}{dx}\right|_{\xi}=\frac{\xi^2}{2(a|\xi|+1)}. \tag{16}$$

As an aside, the anti-derivative of (16) is

$$V(\xi)=\frac{1}{2a^3}\left\{\frac{1}{2}(a|\xi|+1)^2-2(a|\xi|+1)+\ln(a|\xi|+1)\right\}+\kappa,$$

(although the actual analytic form of $V(\xi)$ is somewhat unimportant - its derivative is the important quantity, and is of a simple form).

As $\Phi_B(\xi) \leq 0$ is automatically satisfied by the construction of V, the Φ_A expression needs to be examined to determine the value of γ. The functions of interest are given by:

$$\begin{aligned}
\theta_a(\xi) &= -\xi + \frac{1}{\gamma^2} \left.\frac{dV}{dx}\right|_{\xi}, \\
&= -\xi + \frac{1}{\gamma^2} \frac{\xi^2}{2(a|\xi|+1)}, \\
\Phi_{A.1}(\xi) &= 2 \left.\frac{dV}{dx}\right|_{\xi} (-a\xi + \theta_a(\xi)) - \gamma^2 \left\| \frac{1}{\gamma^2} \left.\frac{dV}{dx}\right|_{\xi} \right\|^2 + \xi^2 \\
&= \frac{\xi^2}{a|\xi|+1} - (a+1)|\xi| + \frac{1}{4\gamma^2} \frac{\xi^4}{(a|\xi|+1)^2} + \xi^2, \\
\Phi_{A.2}(\xi) &= 2 \left.\frac{dV}{dx}\right|_{\xi} (-a\xi + \mathrm{sign}(\theta_a(\xi))) - \gamma^2 \left\| \xi + \mathrm{sign}(\theta_a(\xi)) \right\|^2 + \xi^2.
\end{aligned}$$

As $\Phi_{A.1}(\xi) \geq \Phi_{A.2}(\xi)$, we will examine it first. For the expression to be negative (semi)-definite, we need that

$$\gamma \geq \beta(\xi) := \sqrt{\frac{1}{4} \frac{\xi^2}{a\xi^2 + (1-a)|\xi| - 1}}.$$

If we express the bounding expression β as a function of ξ, basic calculus indicates that the minimum choice occurs at $\xi = \frac{1}{1-a}$, and then monotonically increases. When matched at the beginning of the interval of interest (i.e., $\xi = \frac{2}{1-a}$), the non-surprising result is that a choice of $\gamma = \frac{1}{1+a}$ is indicated.

An interesting case is where we solve the equations for all of $\mathbb{R}^1$, we see that $\lim_{\xi \to \infty} \beta(\xi) = \frac{1}{2\sqrt{a}}$. This is precisely the bound determined by the inequality (15)! This is not a mere coincidence; this is the behaviour that should be expected in this case. The $\Phi_{A.1}$ equation corresponds directly to the algebraic Riccati equation, and the Φ_B expression to the algebraic Lyapunov equation. Their simultaneous solution corresponds to the continuous incremental gain inequalities.

Reflection upon this example gives the reasons why the more detailed analysis undertaken in [7] is necessary to give less conservative results. Approaches which rely on quadratic properties will correspond to the $\Phi_{A.1}$ expression, and neglect the fact that we can use the less strict $\Phi_{A.2}$ expression if θ_a is larger than 1, which is extremely likely to occur in the large. The other cause for conservatism which can be eliminated is that the limiting properties are used to determine the norm, which neglects the fact that the state will be confined to a bounded set for any fixed initial condition.

When the full analysis is undertaken, we find that γ is determined by the $\Phi_{A.2}$ expression, and by the fact that $\Omega = (-\frac{1}{a}, \frac{1}{a})$. The full analysis is rather tedious, and in fact will not be given here explicitly. The important point of interest is in fact that $\lim_{a \to 0} \gamma(a) = \sqrt{2}$, which is obviously much better behaved than the prediction given by the small gain or incremental gain analysis, or relying upon the limiting behaviour of $\Phi_{A.1}$. The bound determined as a function of a was numerically computed, and is plotted in Figure 3, along with the bounds given by the small gain and incremental gain analysis.

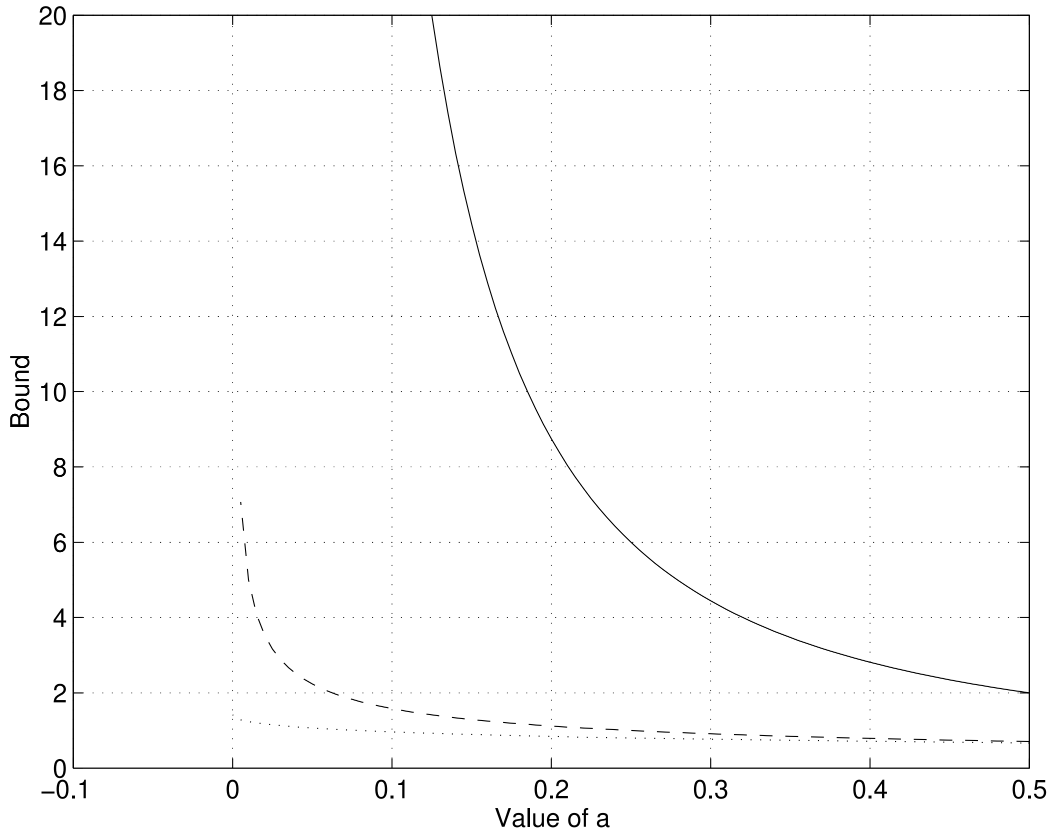

Figure 3: Bounds for γ^* as a function of a determined by (from top): Small Gain, incremental gain, induced $\mathcal{L}_{2+}$ norm analysis.

3.3 Case $a = 0,\ k = 1$

The case where $a = 0$ is of particular interest, being the case of an integrator following the standard saturation with unity negative feedback.

3.3.1 Incremental Gain

In the paper by Chitour, Liu, and Sontag, there is a proof that the incremental gain does not exist for $\mathcal{L}_{2+}$ spaces for this system ([2, Example 5]). However, the construction therein relies on a parameterised pair of signals which become arbitrarily large in infinity norm as the differential gain becomes infinite. It is possible to construct signal pairs which have much smaller infinity norms which display unbounded incremental gain behaviour. The following construction is a continuous time generalisation of the generic construction developed for discrete time systems in [8].

Let $M > 1$ be a free parameter, and define the signals $w(t), \tilde{w}(t)$ for $t \in [0, M]$ as follows:

$$w(t) = \begin{cases} 1 + t - 2T & 2T \leq t < 2T + 1, \\ -.1 - t + 2T & 2T + 1 \leq t < 2T + 2, \end{cases}$$

$$\tilde{w}(t) = \begin{cases} 0 & 0 \leq t < .1, \\ t - .1 & .1 \leq t < 1, \\ w(t) & \text{otherwise}, \end{cases}$$

with $T \in \{0, 1, 2, \ldots\}$, and with $w(t) = \tilde{w}(t) = 0$ for $t > M$. We have that $\|w\|_\infty = \|\tilde{w}\|_\infty = 2$. To prove that $\|\boldsymbol{\Upsilon}\|_{\Delta,2}$ does not exist, assume the converse, i.e., that there exists some finite γ such that $\frac{\|\boldsymbol{\Upsilon}w - \boldsymbol{\Upsilon}\tilde{w}\|_2}{\|w - \tilde{w}\|_2} < \gamma$. It can be verified that $\|w - \tilde{w}\|_2^2 = \frac{358}{3000}$, and that $\|\boldsymbol{\Upsilon}w - \boldsymbol{\Upsilon}\tilde{w}\|_2^2 > \frac{M-1}{100}$. Thus, by choosing M such that

$$M \geq \frac{358}{30}\gamma^2 + 1,$$

we contradict the assumption that incremental gain is bounded by γ.

The importance of verifying that small signals give unbounded incremental gain behaviour is to show that the tactic of using bounded input spaces is not likely to be helpful to give less conservative bounds. Additionally, when one attempts to use Small Gain arguments to bound the operator norm, in the induced norm case one can re-define the standard saturation at large value so as to keep the lower sector bound away from 0. This cannot be done for the incremental gain, as it is the "incremental sector" which is important. This example (at least) shows that this is not a conservative assumption, as the small signal sizes show that re-definition at large values could not help in any event.

3.3.2 Induced $\mathcal{L}_{2+}$ Norm

Despite the fact that the incremental gain is unbounded, the induced $\mathcal{L}_{2+}$ norm is finite. If the input set is taken to be $\mathcal{L}_{2+}$, we see that the reachable set $\Omega = \mathbb{R}^1$, and that $\gamma = \sqrt{2}$. A candidate dissipation function which determines this value is given by:

$$V(\xi) = \begin{cases} \frac{1}{2}\xi^2 & |\xi| \leq 2, \\ \frac{1}{6}\xi^3 + \frac{2}{3} & \text{otherwise}. \end{cases}$$

Once again, the full analysis will be omitted, but the norm bound results by taking the limiting value of $\Phi_{A.2}$.

It is interesting to note that $\gamma = \sqrt{2}$ is a tight bound; this can be shown by examining the limiting properties of the input signal w_T given by

$$w_T(t) = \begin{cases} t+1 & 0 \le t \le T, \\ 0 & t > T, \end{cases}$$

as $T \to \infty$. (The response is given by:

$$x(t) = \begin{cases} t & 0 \le t \le T, \\ 2T - t & T \le t \le 2T-1, \\ e^{-t+2T-1} & t \ge 2T-1.) \end{cases}$$

In this case, we are able to compare this norm bound with that given in the paper by Liu, Chitour and Sontag [6]. The bound given by their analysis is that $\|\boldsymbol{\Upsilon}\|_{i,2} \le 2(1+1)\,\|\boldsymbol{\Upsilon}_L\|_{\infty} = 4$. The analysis in [6] holds of course for general nonlinearities, and so it is not surprising that some deterioration would occur in the bound.

3.4 Case $a < 0$, $k = 1$

When the plant is open loop unstable, all signal bounds become undefined, at least over unbounded sets. Once again, the incremental gain displays explosive behaviour, even within bounded input sets. The test signal w given by:

$$w(t) = \begin{cases} 1 + (a+1)t & 0 \le t \le -\frac{1}{a} + 1, \\ 0 & t > -\frac{1}{a} + 1, \end{cases}$$

lies in all $\mathcal{L}_{p+}$ spaces, yet gives rise to an output which does not lie in any $\mathcal{L}_{p+}$ space. Therefore, the failure of all analysis techniques is not conservative in this case.

3.5 Analysis with Bounded Inputs

The conclusions in Section 3.4 may seem somewhat unusual from point of view of robust control theory. When operator theoretic analysis is applied to feedback loops, one expects results to conform to the so-called Meta-Theorem of robust control (stated next).

Theorem 3.2 (Meta-Theorem of Robust Control): If a (closed loop) control system is stable (in some input/output sense), then the system will remain stable (in the same sense) for all small (in some sense) perturbations to the system model.

For linear systems, the Meta-Theorem underlies $\mathcal{H}_\infty$ control (for example), as there is a characterisation of classes of perturbations. Such reasoning is exemplified in the Gap Metric approaches to robust control. Much of the underlying belief in the validity of the Meta-Theorem comes from the stability implied by the Small Gain Theorem. This belief gave the author the initial justification for the examination of the gain properties of these system operators.

If we confine ourselves to the use of incremental gain, problems do not occur. The incremental gain increases in a continuous fashion, becoming unbounded as a approaches zero. In fact, we can use the stronger version of the Small Gain Theorem, and so we could develop quite a strong theory. The only major source of conservatism introduced may be due to that associated with the Small Gain Theorem.

The restriction of theory to the study of the incremental gain does not seem a completely satisfactory option, as this restriction would *a priori* exclude plants with unstable modes which are to be controlled by bounded inputs. From an engineering standpoint, this might seem to largely confine its usefulness to be an academic exercise. This is not to say that there could be no useful results which could be developed, as in areas such as process control, plants may be more likely to be stable. In such a case, the fact that the computation may be done as the solution of linear matrix inequalities may allow easily understood and implemented algorithms to be developed. Although conservative, incremental gain analysis may still be sufficient for many purposes.

However, the preference is to use the induced norm of the systems, as it will give rise to less conservative bounds. The uniqueness/existence difficulties associated with the induced norm version of the Small Gain theorem are likely to be unimportant in non-pathological classes of nonlinear systems, allowing the development of the corresponding robust control theory.

The fact that the induced norm bound behaves in a sensible fashion for $a > 0$ seems encouraging. But the the model seems to display remarkable lack of robustness with $a = 0$. This could be predicted by the explosion in the incremental gain as a goes to zero, but this seems to be a very unsatisfactory situation. The Meta-Theorem seems to have a a counter-example to it, as it seems unreasonable that replacing $\frac{1}{s}$ by $\frac{1}{s-\epsilon}$ is a dramatic perturbation.

We therefore need to re-examine the situation. It would appear something would have to be eliminated: the Meta-Theorem, the class of systems or perturbations, or the theoretical framework. The first two options are unsatisfying, so a need to re-examine the operator theoretic based definitions of stability used. Examination of the destabilising signals indicates that they may need to be pathologically large in order to destabilise the system. Extremely large excursions in the state space are necessary if the plant is "nearly stable". As it is expected that our models/systems would fail long before such excursions became possible, such test inputs seem suspect in terms of an engineering intuition.

The problem is eliminated if we confine the state to bounded subsets of the state space. In order for this to make sense from an operator theoretic standpoint (and to put the means of restriction into a realisation independent form), $\mathcal{W}_{p,q}(N)$ sets are introduced. And, in fact the pathological behaviour exhibited for the induced norm disappears when we examine the bound given in terms of a.

If we take our input set to be $\mathcal{W}_{2,\infty}(N_\infty)$, it is readily verified that the reachable set $\Omega = (-\frac{N_\infty}{a+1}, \frac{N_\infty}{a+1})$, if $\frac{N_\infty}{a+1} < \frac{1}{|a|}$. The basic existence result given by Theorem 2.14 then indicates a finite norm bound exists. The analysis was made on a digital computer, and the bound dependence vs. a for various values of N_∞ are given in Figure 4. The behaviour of the bound is now sensible, as although there is a value of a at which it becomes unbounded, the failure occurs in a continuous fashion.

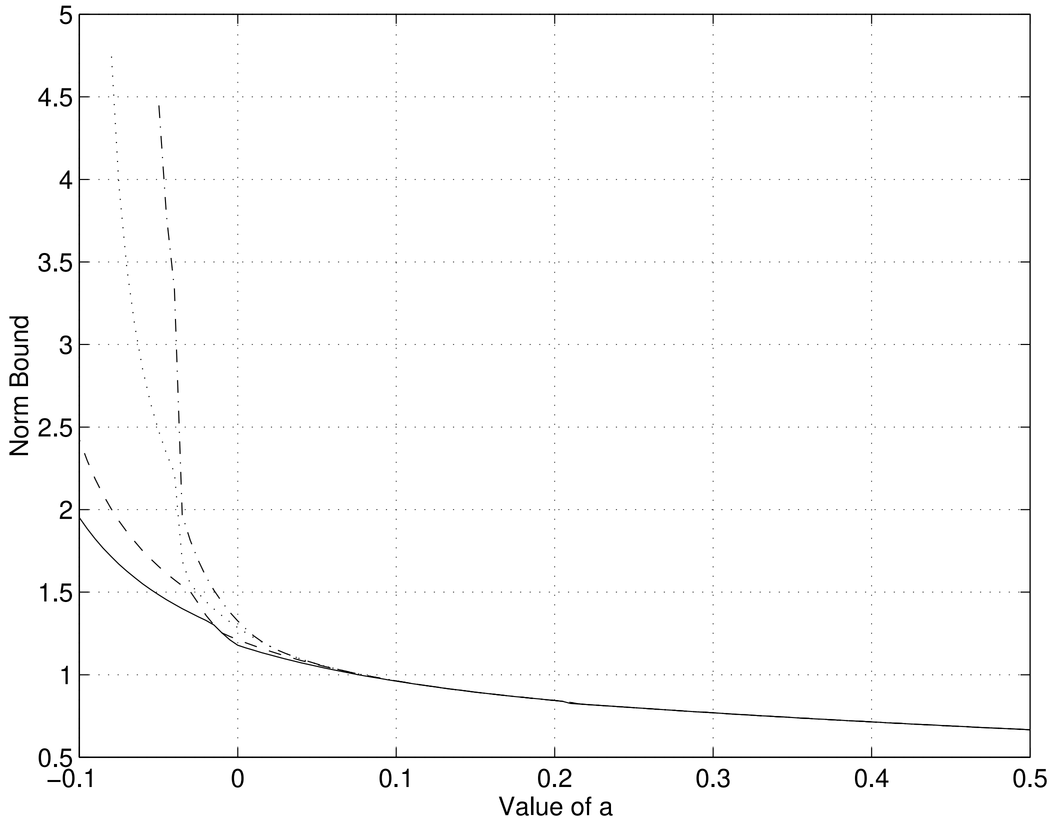

Figure 4: Bound for γ^* as a function of a for values of (from top) $N_\infty = 15, 10, 6, 5$.

4 The Double Integrator

We will now briefly comment on a result given the paper by Liu, Chitour and Sontag [5]. In that paper, they prove the following Proposition.

Proposition 4.1: Consider the system operator generated by:

$$\begin{aligned} \dot{x_1}(t) &= x_2, \\ \dot{x_2}(t) &= -\sigma(ax_1(t) + bx_2(t) + w(t)), \\ x(0) &= 0, \end{aligned}$$

with $a, b > 0$. Then the mapping $w \mapsto x$ is not $\mathcal{L}_{p+}$ stable for all $p \in [1, \infty]$.
Proof: This requires a fairly detailed analysis of test signals, and is done in [2, Section 8]. $\square$

As pointed out in [2], the system in the proposition is globally state space stable. This system provides a very good understanding as to the differences between linear and nonlinear systems when it comes to global properties. By the use of Theorem 2.14, the fact that the system is globally state space stable means that for any compact subset of $\mathbb{R}^2$, trajectories confined to that set will exhibit a finite gain property. The result is that the norm bound will deteriorate as the set size increases in an unbounded fashion. This dependence upon the set size may be of extreme importance for numerical computations.

5 Discrete Time Case

The induced norm computation problem for discrete time systems was examined in [8], but the results were less satisfactory than in the continuous time case. The inability to assume a quadratic form for the dissipation function makes the straight extension of the discrete time completion of squares impossible. The following test theorem on dissipation functions will now be stated without proof.

Theorem 5.1: Take the (discrete time) system operator $\boldsymbol{\Upsilon}\colon w \mapsto z$, which is generated by the difference equations:

$$\begin{aligned} x[k+1] &= Ax[k] + B\sigma_0(-K(x[k]) + w[k]) =: f(x[k], w[k]), \\ z[k] &= Cx[k], \\ x[0] &= 0. \end{aligned}$$

Here, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{p\times n}$, and $K\colon \mathbb{R}^n \mapsto \mathbb{R}$ continuous. Then, let $V\colon \mathbb{R}^n \mapsto \mathbb{R}_+$ be continuous, with $V(0) = 0$, $V(\xi) > 0$ for all $\xi \neq 0$. If for a fixed choice of γ, we have that for all $\xi \in \mathbb{R}^n$,

$$\max_{\omega \in \mathcal{S}(\xi)} \left\{ H(\xi, \omega) = V(f(\xi, \omega)) - V(\xi) - \gamma^2 \|\omega\|^2 + \|h(\xi, \omega)\|^2 \right\} \leq 0, \tag{17}$$

where

$$\mathcal{S}(\xi) = [K(\xi) - 1, K(\xi) + 1] \bigcup \{0\},$$

then $\|\mathbf{\Upsilon}\|_{i,2} \leq \gamma$.

Proof: This is a simplification of [8, Theorem 6.4.1]. $\square$

We will now give an example of the use of Theorem 5.1 for a simple discrete time system. The analysis is complicated by the fact that it is not sufficient to evaluate the derivative, and so it is likely that pen and paper calculations will be of more limited value.

The system is a "summer" following the standard saturation; the difference equations generating the system operator are given by:

$$\begin{aligned} x[k+1] &= x[k] + \sigma_0(w[k] - x[k]) = f(x[k], w[k]), \\ x[0] &= 0, \end{aligned}$$

with $x[k] \in \mathbb{R}$.

The induced $\mathcal{L}_{2+}$ norm for the equivalent linear system is 1. When the saturation is included, the norm deteriorates to $\sqrt{2}$, as will be shown below. The fact that these match the values for the continuous time equivalent is somewhat suggestive; this may point to a more general principle. The nature of the saturation would discourage the use of high amplitude high frequency test signals, and so there may be some form of sampling property which could be exploited.

The dissipation function to be used is:

$$V(\xi) = \begin{cases} \xi^2 & |\xi| \leq 1, \\ \frac{|\xi|(|\xi|+1)(2|\xi|+1)}{6} & |\xi| > 1 \end{cases}$$

(this was generated by the dicrete time version of Theorem 2.14 [8, Theorem 6.5.6]). The function ξ^2 is the solution to the corresponding (discrete) algebraic Riccati equation, and so for test values of the Hamiltonian which correspond to differences between two points lying in $[-1, 1]$, the test holds with a test value of 1.

If we examine some ξ such that $|\xi| \leq 1$, but $|f(\xi, \omega)| > 1$, we first note that for all values of ω in the test interval, $f(\xi, \omega) = \omega$. Then,

$$\begin{aligned} H(\xi, \omega) &= V(f(\xi, \omega)) - V(\xi) - \gamma^2\omega^2 + \xi^2 \\ &= V(\omega) - \gamma^2\omega^2 \\ &= \frac{1}{3}|\omega|^3 - \frac{3}{2}\omega^2 + \frac{1}{6}|\omega|. \end{aligned}$$

As $|\omega| \leq 2$ over the test interval, we find that $H(\xi, \omega) \leq 0$.

For $|\xi| > 1$, it is easily tested that $H(\xi, 0) = 0$. Otherwise, we need to test $H(\xi, \omega)$ for $\xi - 1 \leq \omega \leq \xi + 1$. Without loss of generality, assume that $\xi > 1$. If we write $\omega = \xi + \Delta$, we have that:

$$\begin{aligned} H(\xi, \omega) &= \left[\frac{\xi + \Delta(\xi + \Delta + 1)(2\xi + \Delta + 1)}{6} - \frac{\xi(\xi + 1)(2\xi + 1)}{6}\right] \\ &\quad -2(\xi + \Delta)^2 + \xi^2, \\ &= \frac{1}{6}\Big((6\Delta - 6)\xi^2 + (6\Delta^2 + 6\Delta - 24)\xi + 2\Delta^3 - 9\Delta^2 + \Delta\Big). \end{aligned}$$

The maximising choice of Δ over the interval $[-1, 1]$ is $\Delta = 1$, and then we see that

$$H(\xi, \omega) = -2|\xi| - 1.$$

If one verifies the algebra without setting $\gamma = \sqrt{2}$, one will see that the choice of $\gamma = \sqrt{2}$ is the minimal one such that the leading coefficient in the $H(\xi, \omega)$ expression is non-positive.

6 A First Order State Feedback Synthesis

We will now examine a simple application which illustrates the potential power of the analysis technique developed in [7]. The first order integrator with input saturation will once again be examined, but we will now synthesize a general nonlinear (state) feedback, using two different approaches. The first technique uses partial differential inequalities, following the sort of technique developed by A.J. van der Schaft in [9]. The second technique is similar in principle to the most basic anti-windup schemes, as the controller is fixed to be piecewise linear, and then algebraic conditions solved to fix the value of free parameters.

We take the plant to be controlled as:

$$\begin{aligned}\dot{x}(t) &= \sigma_0(-K(x(t)) + w(t)),\\ x(0) &= 0.\end{aligned}$$

6.1 Controller Synthesis Using Partial Differential Inequalities

The analysis here follows closely that given in [9], and is of a preliminary nature. It may be possible to solve the state feedback problem using dynamic programming methods, however the preference is once again is to see whether more class specific algorithms may be developed. The somewhat unusual nature of the test condition in Theorem 2.11 seems to indicate that the rather naive extension attempted here will not be sufficient.

The controller is a state feedback, and is to be generated by the equation

$$K(\xi) = -\left.\frac{dV}{dx}\right|_{\xi}, \tag{18}$$

with $V(\xi)$ a smooth function mapping $\mathbb{R} \mapsto \mathbb{R}$. We would like that the induced $\mathcal{L}_{2+}$ norm for the mapping $w \mapsto z$ to be less than or equal one.

As is typical in this situation, the function V in (18) is to be a valid dissipation function for $\gamma = 1$. The first unusual point to note is that the "optimal" input function $\theta_a(\xi)$ is identically zero, as

$$\theta_a(\xi) = -K(\xi) + \frac{1}{\gamma^2}\left.\frac{dV}{dx}\right|_{\xi}.$$

Therefore, V needs to satisfy $\Phi_{A.1}(\xi), \Phi_B(\xi) \leq 0$ for all ξ. The inequalities of interest become:

$$\begin{aligned} -\left\|\left.\frac{dV}{dx}\right|_{\xi}\right\|^2 + \xi^2 &\leq 0,\\ 2\left.\frac{dV}{dx}\right|_{\xi}\left(-\sigma_0\left(\left.\frac{dV}{dx}\right|_{\xi}\right)\right) + \xi^2 &\leq 0.\end{aligned}$$

The constraining equations then become:

$$
\begin{aligned}
x^2 &\leq \left\| \left. \frac{dV}{dx} \right|_\xi \right\|^2, \\
\frac{1}{2}x^2 &\leq \begin{cases} \left(\left. \frac{dV}{dx} \right|_\xi \right)^2 & \text{if } \left| \left. \frac{dV}{dx} \right|_\xi \right| < 1, \\ \left| \left. \frac{dV}{dx} \right|_\xi \right| & \text{otherwise.} \end{cases}
\end{aligned}
$$

One possible smooth solution is to take $\left. \frac{dV}{dx} \right|_\xi = \xi + \frac{\xi^3}{2}$.

The intuition behind the solution is that the controller has unit gain near the origin, giving the correct local properties. The gain is then increased as the output get larger, and reduces the "system sensitivity". The controller is thus able to preserve the local norm bound.

An interesting issue arises as a consequence of the behaviour of θ_a with $\gamma = 1$. The synthesis does not take advantage of the fact that the $\Phi_{A.2}(\xi) \leq \Phi_{A.1}(\xi)$, and hence is a weaker condition. It that if we are extend the methods in [9], this issue will have to be seriously examined if the controller performance is to be satisfactory.

6.2 Synthesis of a Piecewise Linear Controller

The control synthesised by the straight use of partial differential inequalities may be unsatisfactory, however. The additional nonlinear terms may complicate linear system analysis, and lead to too complex controllers. One of the important inplication of the analysis in [7] is that linear controllers will preserve their norm properties over a non-negligible region around the origin. As the nonlinear analytic techniques and design intuition are nowhere near as developed as that for linear control theory, the preference is that the nonlinear techniques do not affect linear design processes unnecessarily.

As such, an alternative state feedback will be given which preserves the induced norm, but is in fact piecewise linear. The control is fixed to be of the form

$$K(\xi) = k_1\xi + k_2 M\sigma_0\left(\frac{\xi}{M}\right). \tag{19}$$

There are three free parameters (k_1, k_2, M), but as we assume that the linearisation of the controller is fixed by *a priori* linear analysis, there are in fact only two.

In order to motivate setting the problem in this framework, it may be noted that the most basic anti-windup schemes are also piecewise linear, although in that case the controller has dynamics. If this method is generalised, it would then amount to providing a synthesis method for anti-windup schemes using nonlinear $\mathcal{H}_\infty$ methods.

We assume that the linearisation of the controller is fixed to be $K_l = k_1 + k_1 = 1$, as a result of linear analysis (for example, this controller gives the optimal (unweighted) robustness measure with respect to the Gap Metric). The resulting (closed loop) local linearisation has an infinity norm of $\gamma_l^* = 1$. We now wish to find a controller $K(\xi)$ of the form determined by (19), with $k_1 + k_2 = 1$, and which ensures that $\|\boldsymbol{\Upsilon}\|_{i,2} = 1$.

We choose $M = \frac{1}{2}$, by reflection upon the fact that $V_l(\xi) = \frac{\xi^2}{2}$ is a valid dissipation function for the unity feedback controller for $|\xi| \leq 2$. We then see that the controller has one free parameter, and the control form may be expressed as:

$$K(\xi) = \begin{cases} \xi & |\xi| \leq 2, \\ k\xi + 2(1-k) & |\xi| > 2. \end{cases}$$

It is readily apparent that in order to preserve the operator norm of 1, $k \geq 1$. Take a candidate dissipation function V to be:

$$V(\xi) = \begin{cases} \frac{1}{2}\xi^2 & |\xi| \leq 2, \\ \frac{1}{6}\xi^3 + \frac{2}{3} & |\xi| > 2. \end{cases}$$

Examination of the asymptotic value of $\Phi_{A.2}(\xi)$ (with $\gamma = 1$) gives:

$$\Phi_{A.2}(\xi) = 2\xi^2 - (k\xi + 3 - 2k)^2,$$

for large enough ξ ($\theta_a(\xi) \gg 1$ for $\xi \gg 2$). Choosing $k = \sqrt{2}$ is sufficient for the candidate function to be valid for $\gamma = 1$.

In order to generalise this synthesis technique, many computational issues would need to be explored. However, by the tactic of fixing the form of the controller to be piecewise linear with free parameters, it is conjectured that the resulting computations should be algorithmically tractable.

7 Conclusions

The examples in this paper are felt to provide a very good test bed of illustrations of the properties of feedback loops with saturation nonlinearities. Most of the observed theoretical peculiarities of these systems, and how they differ from linear systems, are given concrete manifestation.

The induced norm analysis for the plant as the plant pole shifts from the left half plane to the right half plane gives a very good motivation for the use of bounded input sets if one is to undertake less conservative analysis. If one is to confine one's attention to purely stable systems, it may be preferable to use the incremental gain, which is far more easily computed, and has nice properties for analysis.

Finally, the preliminary state feedback synthesis example illustrates some of the problems with a straightforward extension of differential inequality based methods to controller synthesis, and how it may be possible to synthesize anti-windup schemes based on nonlinear $\mathcal{H}_\infty$ methods in the near future.

Appendix

We will now state the proof to Theorem 2.6.

Theorem 2.6: Fix an operator $\mathbf{\Upsilon}: w \mapsto z$, generated by (1) - (3), with the assumption that $K(x) = Cx$ added to those listed in Assumption 2.4. Then, $\|\mathbf{\Upsilon}\|_{\Delta,2} \leq \gamma$ (over the set

$\mathcal{L}_{2+}$) if there exists a (symmetric) positive definite matrix $P \in \mathbb{R}^{n\times n}$ which solves the pair of linear matrix inequalities:

$$(A-BC)'P + P(A-BC) + \frac{1}{\gamma^2}PBB'P + C'C \leq 0, \tag{6}$$

$$A'P + PA + C'C \leq 0. \tag{7}$$

Proof: Take any two signals $w, \tilde{w} \in \mathcal{L}_{2+}$, with $\|w-\tilde{w}\|_2 \neq 0$. Let $x, \tilde{x}$ be the state trajectories associated with $w, \tilde{w}$, respectively. Let $\eta = x - \tilde{x}$. Fix $V(\xi) = \frac{1}{2}\xi' P\xi$. It is then a sufficient condition to prove that $\|\boldsymbol{\Upsilon}\|_{\Delta,2} \leq \gamma$ if

$$\sup_{t\in\mathbb{R}_+}\left\{H(t) := 2\frac{dV}{dx}_{\eta}(t)\dot{\eta}(t) - \gamma^2 \|(w-\tilde{w})(t)\|^2 + \|C\eta(t)\|^2\right\} \leq 0,$$

by the use of standard dissipation arguments.

Note that

$$\dot{\eta}(t) = A\eta(t) + B(\sigma_0(-Cx(t)+w(t)) - \sigma_0(-C\tilde{x}(t)+\tilde{w}(t))).$$

Then, by using the fact that we can write $\sigma_0(a)-\sigma_0(b) = \lambda(a-b)$, for some λ with $0 \leq \lambda \leq 1$, we can re-write $H(t)$ as (suppressing t dependence):

$$\begin{aligned} H &\leq \lambda\left(2\frac{dV}{dx}(A-BC)\eta - \gamma^2\|w-\tilde{w}\|^2 + \eta' C' C\eta\right) \\ &\quad +(1-\lambda)\left(2\frac{dV}{dx}A\eta + \eta' C' C\eta\right) \\ &=: \lambda\zeta_1 + (1-\lambda)\zeta_2. \end{aligned}$$

It is a standard consequence of linear dissipation theory that $\zeta_1 \leq 0$ is implied by P being the solution to (6), and that $\zeta_2 \leq 0$ is implied by P also being the solution to (7). Hence, $H(t) \leq 0$, and so Theorem 2.6 has been verified. □

Remark 7.1: Although similar linear matrix inequalities have been used to verify upper bounds for induced norms, the author is unaware of a reference which explicitly points to the fact that the linear matrix inequality is being used to compute the incremental gain. Additionally, in [8], discrete time constructions have been made to show that if either the open or closed loop dynamics matrices are unstable, the incremental gain does not exist. These constructions use signals of small infinity norm, so bounding signal sizes appears to be unhelpful, unlike the induced norm problem. The result will be now stated without proof. (There is a further test condition for the non-existence of the incremental gain in [8], but the result will not be stated due to its added notational complexity, and the fact that the result is somewhat less satisfactory in that there is no *a priori* characterisation of the size of the "destabilising" (incremental gain sense) test signal pairs.) See also the paper by Chitour, Liu and Sontag [2] for further characterisations of the incremental gain properties of systems with saturation nonlinearities.

Theorem 7.2: Fix a discrete time operator $\mathbf{\Upsilon}: w \mapsto z$ generated by the difference equations:

$$\begin{aligned} x[k+1] &= Ax[k] + B\sigma_0(w[k] - Cx[k]), \\ x[0] &= 0, \\ z[k] &= Cx[k], \end{aligned}$$

with $[A, B]$ controllable, $[C, A]$ observable, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$,$C \in \mathbb{R}^{1\times n}$. Then $\|\mathbf{\Upsilon}\|_{\Delta,p}$ does not exist if $\rho(A) \geq 1$ (if $p < \infty$), or if $\rho(A) > 1$ (if $p = \infty$). Additionally, the test signals constructed to show that the incremental gain is unbounded have an $\ell_{\infty+}$ norm of $1 + \epsilon$, for any $\epsilon > 0$.
Proof: See [8]. □

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