Robust performance and adaptation using receding horizon $\mathcal{H}_\infty$ control of time varying systems.

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Abstract

In this paper we construct suboptimal $\mathcal{H}_\infty$ controllers which satisfy a new robust performance condition, using the receding horizon technique. A method is described for the synthesis of $\mathcal{H}_\infty$ controllers online, making use of the exact plant model only on a finite interval extending into the future. Inequalities based on the two Riccati differential equation solution to the finite horizon $\mathcal{H}_\infty$ problem are derived, and the resulting freedom is exploited to construct $\mathcal{H}_\infty$ controllers which have a closed loop induced norm less than a prespecified value for all plants within a set, which is described in terms of the future variation of the plant. Dual results, with a possible adaptive interpretation, are also constructed.

1 Introduction

The construction of $\mathcal{H}_\infty$ controllers for finite dimensional linear systems has been derived in a number of ways, using operator theory, the state space solutions in [4], using game theoretic techniques [1], and in the time domain[11]. The time domain and game theoretic techniques both allow solution of finite horizon problems with non-zero initial conditions, and therefore allow a *receding horizon* control formulation. The receding horizon technique solves, at each time t, an optimization problem based on a cost function evaluated on the interval $[t, t+T]$, for some fixed T. This controller is implemented on some shorter interval, say $[t, t+\delta]$. Then at time $t+\delta$, the optimization is again performed, with the cost function applied to the interval $[t+\delta, t+T+\delta]$.

The use of game theoretic techniques to solve a moving horizon $\mathcal{H}_\infty$ problem was first applied by Tadmor[10], where it was shown that with a terminal state constraint, the receding horizon control law not only gave stability, but also gave an infinite horizon $\mathcal{H}_\infty$ norm bound. Later these results were extended by Lall and Glover[8] to allow a quadratic terminal state weight. The linear quadratic moving horizon problem was studied by Kwon[6, 7], but in this case there is no global suboptimal property satisfied, and the only criterion which was desired was stability.

One interesting property of this controller construction in the game theory problem is that it is not necessary to know exactly the model for the system being controlled for all time. Provided we know the model up to some finite time ahead, we can construct a controller which satisfies some prespecified nominal performance with respect to the induced 2-norm on this finite interval. We then show that a succession of such controllers, under some simple additional conditions, also satisfies the same nominal performance with respect to the induced 2-norm on $[0, \infty)$.

This appears to offer some theoretical advantages. Typically in current robust control design schemes, one starts with a nominal model of the system to be controlled, and an uncertainty description. Then it is assumed that the *real* system lies within some ball around the nominal. However, one possible drawback to this scheme is that the plant may change over time. It may be that if we identify an FDLTI model for our plant now, and repeat the identification in the future, that the models identified will be different. This may be due to parameter variation (for example, ambient temperature) or it may be due to the more subtle considerations of plant ageing.

Typically we cannot predict over a long period of time how the system will age. We may have good knowledge of the plant now, but be much more uncertain about the future. We would like to design for the future plant without having a good model for it.

In this paper we show that the moving horizon control strategy allows this, and we describe an extension of the strategy to allow discrete update of the continuous time controller. Provided that the future plant satisfies a certain condition, it is shown that the controller designed guarantees a $\mathcal{H}_\infty$ norm performance bound over all future time.

The central idea involved in this methodology is that the synthesis of the controller can be an online process. The plant model is assumed to be linear time varying, and known up to a finite time ahead. As new information is gained about the plant model further into the future while the controller is running, we can take this into account and re-synthesise. All that is required is that we know the model for the future plant some finite time in advance. The overall, time varying compensator still gives an $\mathcal{H}_\infty$ performance bound. This bound is satisfied for all plants described by the model over a future interval of time, and which satisfy a particular condition thereafter. Note that we do not give any methods for actually determining the model, for example online identification or parameter estimation.

In some sense dual to the ideas of future variation of the plant given above, is the idea of past error in plant model. Suppose we have designed and implemented an $\mathcal{H}_\infty$ controller using some nominal model, and that this controller has been running for some time. After this time, it is discovered that the nominal model used was incorrect, and we would like to know how much error was allowed. In this case we show how it is possible to construct an estimator which again guarantees an infinite horizon induced norm bound, for all plants which satisfy a condition on the past model.

Throughout this paper, the final time is denoted at t_f. Except where stated, this can be replaced by $t_f = \infty$, and the controllers constructed will be stable, for $C_1'C_1 > \varepsilon I$ and $B_1 B_1' > \varepsilon I$ for some $\varepsilon > 0$.

2 Preliminaries

We consider the system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) \qquad x(t_0) = x_0 \\ z(t) &= C_1(t)x(t) + D_{12}(t)u(t) \qquad\qquad (1) \\ y(t) &= C_2(t)x(t) + D_{21}(t)w(t) \end{aligned}$$

where the coefficient matrices are bounded matrix valued functions of time. We use the standard assumptions, that $D_{12}'C_1 = 0$, $D_{12}'D_{12} = I$, $D_{21}B_1' = 0$ and $D_{21}D_{21}' = I$,

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which can be removed by suitable changes in the loop description. Further, we shall assume there exists $\varepsilon > 0$ such that $B_1(t)B_1(t)' \geq \varepsilon I$ and $C_1(t)'C_1(t) \geq \varepsilon I$ for all $t > 0$ in the infinite horizon case.

Let $\mathcal{M}$ be the set of state feedback strategies possible for player u on a fixed time interval. Similarly for $\mathcal{N}$ as the strategy space for player w. We will consider dynamic game problems of the form

$$\inf_{u\in\mathcal{M}} \sup_{w\in\mathcal{N}} \int_{t_0}^{t_f} |z(t)|^2 - \gamma^2|w(t)|^2\,dt + x(t_f)'Q_f x(t_f).$$

This problem is solved in [1], where it is shown that if there exists a saddle point to the above problem, then the infimising feedback strategy for u gives a control law which satisifies the finite horizon $\mathcal{H}_\infty$ condition

$$\sup_{w\in\mathcal{N}} \frac{\|z\|_2}{\|w\|_2} < \gamma$$

where the 2-norms are defined on the interval $[t_0, t_f]$. Note that the above problem is *not* a static game. In the state feedback case it can be solved by applying dynamic programming:

$$\begin{aligned}&\inf_{u\in\mathcal{M}} \sup_{w\in\mathcal{N}} \left\{ \int_{t_0}^{t_f} |z(t)|^2 - \gamma^2|w(t)|^2\,dt + x(t_f)'Q_f x(t_f) \right\}\\ &\quad= \inf_{u_{[t_0,t_1]}} \sup_{w_{[t_0,t_1]}} \left\{ \int_{t_0}^{t_1} |z(t)|^2 - \gamma^2|w(t)|^2\,dt + \right.\\ &\left.\inf_{u_{[t_1,t_f]}} \sup_{w_{[t_1,t_f]}} \left\{ \int_{t_1}^{t_f} |z(t)|^2 - \gamma^2|w(t)|^2\,dt + x(t_f)'Q_f x(t_f) \right\} \right\}\end{aligned}$$

Writing

$$\begin{aligned}V(t, x(\tau)) = \inf_{u\in\mathcal{M}} \sup_{w\in\mathcal{N}} \int_{\tau}^{t_f} |z(t)|^2 - \gamma^2|w(t)|^2\,dt\\ + x(t_f)'Q_f x(t_f)\end{aligned}$$

this gives a recursive expression for V, which in the continuous limit becomes the *Isaacs* equation

$$\begin{aligned}&-\frac{\partial}{\partial t}V(t,x) =\\ &\qquad \inf_{u(t)} \sup_{w(t)} \left\{ |z(t)|^2 - \gamma^2|w(t)|^2 + \frac{\partial}{\partial x}V(t,x)\dot{x}(t) \right\}\end{aligned}$$

with boundary condition $V(t_f, x(t_f)) = x(t_f)'Q_f x(t_f)$.

For this standard problem, it is known that $V(t) = x(t)'Z(t)x(t)$, where Z satisfies the Riccati differential equation

$$-\dot{Z} = ZA + A'Z + C_1'C_1 - Z(B_2B_2' - \gamma^{-2}B_1B_1')Z$$

with boundary condition $Z(t_f) = Q_f$. We will write the solution to this equation as $Z(t) = \Psi(t, t_f, Q_f)$ for the solution to the above Riccati equation with boundary condition $Z(t_f, t_f, Q_f) = Q_f$. In the receding horizon formulation in [11, 8] the problem formulation was as follows. At each time t, solve

$$\begin{aligned}\inf_{u\in\mathcal{M}} \sup_{w\in\mathcal{N}} \left\{ \int_{t}^{t+T} |z(r)|^2 - \gamma^2|w(r)|^2 dr \right.\\ \left. + x(t+T)'Q_f(t+T)x(t+T) \right\}.\end{aligned}$$

For each time t there is a new optimization problem, and a new controller designed which is implemented only at time t. In [8] it was shown that provided $Q_f(t)$ satisfied certain conditions, then the resulting controller would be stable. Further, the resulting controller satisfies the $\mathcal{H}_\infty$ criterion $\sup_w \frac{\|z\|_2}{\|w\|_2} < \gamma$, where the signal norms are defined over the whole interval $[t_0, \infty)$.

A particularly interesting feature of this controller is that, in order to synthesise a suboptimal $\mathcal{H}_\infty$ controller for a time-varying plant, we have only used the exact model of the plant on the interval $[t, t+T]$. The plant model further in the future has to satisfy a certain condition, but need not be exactly known in order to construct the control law. All plants which are equal up until time T ahead of the current time, subject to this condition, will be controlled with this performance level, and in this sense this gives a new robust performance condition.

In this paper we generalise the results of [10, 8], in particular to the case when the update of the controller is not continuous but discrete. We show how, for arbitary overlapping or touching intervals it is possible to solve finite horizon differential games in each interval and still achieve an infinite (or possibly longer finite) horizon performance level.

Lemma 1. *The solution to the above Riccati equation satisfies, for all* $0 < t_1 < t_2 < t_3$, *(i)* $Q > 0 \implies \Psi(t_1, t_2, Q) > 0$, *(ii)* $Q_1 \geq Q_2 > 0 \implies \Psi(t_1, t_2, Q_1) \geq \Psi(t_1, t_2, Q_2)$, *(iii)* $\Psi(t_1, t_2, \Psi(t_2, t_3, Q)) = \Psi(t_1, t_3, Q)$.

Proof. Statement (i) is a well known result. For statement (ii), the result is given by the fact that if both saddle points exist, then $f(u,w) \leq g(u,w)$ implies $\inf_{u\in\mathcal{M}} \sup_{w\in\mathcal{L}_2} f(u,w) \leq \inf_{u\in\mathcal{M}} \sup_{w\in\mathcal{L}_2} g(u,w)$. Statement (iii) is a direct consequence of the definition. ■

Lemma 2. *Suppose there exists a saddle point under some given information structure for*

$$\begin{aligned}x(t_0)'Z_1x(t_0) = \inf_{u\in\mathcal{M}} \sup_{w\in\mathcal{N}} \int_{t_0}^{t_f} z(t)'z(t) - \gamma^2 w(t)'w(t)\,dt\\ + x(t_f)M_1x(t_f).\end{aligned}$$

Let M_1, M_2 be a symmetric positive definite matrices such that $M_2 < M_1$. With u equal to the saddle point solution of the above differential game, there exists a solution to the following quadratic problem under the given information structure

$$\begin{aligned}x(t_0)'Z_2x(t_0) = \sup_{w\in\mathcal{N}} \int_{t_0}^{t_f} z(t)'z(t) - \gamma^2 w(t)'w(t)\,dt\\ + x(t_f)M_2x(t_f).\end{aligned}$$

and $Z_2 < Z_1$.

Proof. By assumption, we can reverse the order of the inf and sup operations in the first problem. Then the first game simply gives a choice of u, after which the result is shown by comparing the two linear-quadratic maximization problems. ■

3 State feedback

Theorem 3. *Let $t_1 \in (t_0, t_f)$ be a fixed time. Let u_1 be the infimising feedback strategy on $[t_0, t_1]$ for the game*

$$\inf_{u\in\mathcal{M}} \sup_{w\in\mathcal{N}} \int_{t_0}^{t_1} z(t)'z(t) - \gamma^2 w(t)'w(t)\,dt + x(t_f)M_1x(t_f)$$

Also, let u_2 be the infimising feedback strategy on $[t_1, t_f]$ for the game

$$\inf_{u\in\mathcal{M}} \sup_{w\in\mathcal{N}} \int_{t_1}^{t_f} z(t)'z(t) - \gamma^2 w(t)'w(t)\,dt + x(t_f)Q_fx(t_f)$$

Then, using $u = u_1$ on $[t_0, t_1)$ and $u = u_2$ on $[t_1, t_f]$, if $M_1 \geq \Psi(t_1, t_f, Q_f)$, and the corresponding Riccati differential equations on both intervals do not have conjugate points, then

$$\begin{aligned}\sup_{w\in\mathcal{N}} \left\{ \int_{t_0}^{t_f} z(t)'z(t) - \gamma^2 w(t)'w(t)\,dt + x(t_f)Q_fx(t_f) \right\}\\ \leq x(t_0)'\Psi(t_0, t_1, M_1)x(t_1)\end{aligned}$$

Hence, when $x(t_0) = 0$, the controller defined by u above results in $\|z\|_2^2 \leq \|w\|_2^2$ for all $w \in \mathcal{N}$ where the norms are defined on the interval $[t_0, t_f]$.

Proof. Applying dynamic programming, we can see that for any state feedback strategy for u on $[t_0, t_f]$,

$$\sup_{w\in\mathcal{N}} \left\{ \int_{t_0}^{t_f} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + x(t_f)' Q_f x(t_f) \right\}$$
$$= \sup_{w_{[t_0,t_1]}} \left\{ \int_{t_0}^{t_1} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt \right.$$
$$\left. + \sup_{w_{[t_1,t_f]}} \left\{ \int_{t_1}^{t_f} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + x(t_f)' Q_f x(t_f) \right\} \right\} \tag{2}$$

With the choice of u in the lemma,

$$\sup_{w_{[t_1,t_f]}} \int_{t_1}^{t_f} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + x(t_f)' Q_f x(t_f) = x(t_1)' \Psi(t_1, t_f, Q_f) x(t_1)$$

so

$$\begin{aligned}
&\sup_{w\in\mathcal{N}} \int_{t_0}^{t_f} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + x(t_f)' Q_f x(t_f) \\
&= \sup_{w_{[t_0,t_1]}} \int_{t_0}^{t_1} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt \\
&\qquad + x(t_1)' \Psi(t_1, t_f, Q_f) x(t_1) \\
&\le \sup_{w_{[t_0,t_1]}} \int_{t_0}^{t_1} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + x(t_1)' M_1 x(t_1) \\
&= x(t_0)' \Psi(t_0, t_1, M_1) x(t_0)
\end{aligned}$$

■

This result has the following interpretation; suppose we would like to design a control law for the interval $[t_0, t_f]$, for a particular system. Suppose we have a linear time varying model for the system. Then, given γ, we can calculate $\Psi(t_1, t_f, Q_f)$. However, if we expect that the model is very poor on $[t_1, f_f]$, then we can choose M_1 greater than $\Psi(t_1, t_f, Q_f)$, and design using M_1 on $[t_0, t_1]$. Then at time t_1, we might expect to have some experience with which to assume a better nominal model on $[t_1, t_f]$. Provided that with this new model $\Psi(t_1, t_f, Q_1) < M_1$, the resulting control law will still give a bound of γ on the induced 2-norm on the whole interval $[t_0, t_f]$. All plants which agree with the model on $[t_0, t_1]$ and satisfy $\Psi(t_1, t_f, Q_1) < M_1$, can be controlled with performance level γ from time t_0. It is not until time t_1 that we need to know the exact model for the plant on $[t_1, t_f]$.

The above result dealt with the case of two non-overlapping intervals. The next two theorems show that similar results hold for arbitrary numbers of non-overlapping and overlapping intervals.

Theorem 4. *Divide the interval $[t_0, t_K]$ into subintervals as $\{t_0, t_1, \ldots, t_{K-1}, t_K\}$. Let M_i, $0 \le i \le K$, be a sequence of symmetric positive definite matrices, such that $M_i \ge \Psi(t_i, t_{i+1}, M_{i+1})$, and $M_K \ge Q_f$. Construct u on $[t_i, t_{i+1})$ as the solution to*

$$\inf_{u_{[t_i,t_{i+1}]}} \sup_{w\in\mathcal{N}} \int_{t_i}^{t_{i+1}} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + x(t_{i+1})' M_{i+1} x(t_{i+1})$$

If the corresponding Riccati differential equations for the above problems exist and do not have conjugate points on their respective intervals, then with this control law,

$$\sup_{w\in\mathcal{N}} \left\{ \int_{t_0}^{t_K} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + x(t_K)' Q_f x(t_K) \right\} \le x(t_0)' M_0 x(t_0)$$

Hence, when $x(t_0) = 0$, the controller defined by u above results in $\|z\|_2^2 \le \|w\|_2^2$ for all $w \in \mathcal{N}$ where the norms are defined on the interval $[t_0, t_K]$.

Proof. We shall show that, for a particular choice of matrices $\Delta_i \ge 0$, that the controller designed actually is the saddle point solution for the problem:

$$\inf_{u\in\mathcal{M}} \sup_{w\in\mathcal{N}} \left\{ \int_{t_0}^{t_f} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + x(t_f)' Q_f x(t_f) + \sum_{i=1}^{K} x(t_i)' \Delta_i x(t_i) \right\}$$

If we rewrite this as a standard problem with discontinuous C_1 matrix, it is solved by Tadmor[11] using Riccati integral equations to avoid difficulties with the discontinuities. This can be rewritten as a Riccati differential equation with jumps as follows. We split this into subproblems

$$\begin{aligned}
&\inf_{u\in\mathcal{M}} \sup_{w\in\mathcal{N}} \left\{ \int_{t_0}^{t_f} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + x(t_f)' Q_f x(t_f) + \sum_{i=1}^{K} x(t_i)' \Delta_i x(t_i) \right\} \\
&= \inf_{u_{[t_0,t_1]}} \sup_{w_{[t_0,t_1]}} \left\{ \int_{t_0}^{t_1} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + x(t_1)' \Delta_1 x(t_1) \right. \\
&\quad + \cdots \\
&\quad + \inf_{u_{[t_{K-1},t_f]}} \sup_{w_{[t_{K-1},t_f]}} \left\{ \int_{t_{K-1}}^{t_f} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt \right. \\
&\qquad \left.\left. + x(t_f)' (Q_f + \Delta_K) x(t_f) \right\} \right\}
\end{aligned}$$

The value of the last subproblem is $x(t_{K-1})' (\Psi(t_{K-1}, t_f, Q_f) + \Delta_K) x(t_{K-1})$. By hypothesis $M_K > Q_f$ and therefore we can set $\Delta_K = M_K - Q_f$. This subproblem then corresponds to the game we have solved on $[t_{K-1}, t_f]$ to find u. Hence for previous subproblems we can choose

$$\Delta_i = M_i - \Psi(t_i, t_{i+1}, M_{i+1}).$$

For the subproblem on $[t_0, t_1]$, let the value be $x(t_0)' Z x(t_0)$. Then we know

$$Z = \Psi(t_0, t_1, M_1)$$

and

$$x(t_0)' Z x(t_0) = \inf_{u\in\mathcal{M}} \sup_{w\in\mathcal{N}} \left\{ \int_{t_0}^{t_f} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + x(t_f)' Q_f x(t_f) + \sum_{i=1}^{K} x(t_i)' \Delta_i x(t_i) \right\}$$

therefore, with this strategy u,

$$\sup_{w\in\mathcal{N}} \left\{ \int_{t_0}^{t_f} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + x(t_f)' Q_f x(t_f) \right\} \le x(t_0)' Z x(t_0).$$

Also, by definition $M_0 \ge \Psi(t_0, t_1, M_1) = Z$. ■

In order to apply this result, we would like to be able to construct the matrices M_i such that $M_i \ge \Psi(t_i, t_{i+1}, M_{i+1})$. One simple way to do this for time invariant plants is to choose $M_1 = M_2 = \ldots M_K = M$ where M is chosen so that $\Psi(t, t_i, M)$ is an increasing function of t for all t_i. This can be done simply in the LTI case.

Theorem 5. *Consider the sequence of overlapping intervals $[t_i, \tau_i]$, where $0 \le t_0 < t_1 < \ldots t_{K-1} < t_K$ and $0 < \tau_0 < \tau_1 < \ldots \tau_{K-1} = t_K$, where $t_i < \tau_{i-1}$. Also* *let $N_i, 0 \le i \le K$ be a sequence of postive definite matrices such that*

$$N_{K-1} \ge Q_f \quad \text{and}$$

$$N_i \ge \Psi(\tau_i, \tau_{i+1}, N_{i+1}), \quad 0 \le i \le K-2.$$

Construct u on $[t_i, t_{i+1})$ by solving the game

$$\inf_{u_{[t_i,\tau_i]}} \sup_{w \in \mathcal{N}} \int_{t_i}^{\tau_i} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + x(\tau_i)' N_i x(\tau_i) \quad (3)$$

on each interval $[t_i, \tau_i]$, using only the value for u on $[t_i, t_{i+1})$. Then with this control law,

$$\sup_{w \in \mathcal{N}} \left\{ \int_{t_0}^{t_K} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + x(t_K)' Q_f x(t_K) \right\} \le x(t_0)' \Psi(t_0, \tau_0, N_0 x(t_0). \quad (4)$$

Hence, when $x(t_0) = 0$, the controller defined by u above results in $\|z\|_2^2 \le \|w\|_2^2$ for all $w \in \mathbb{N}$ where the norms are defined on the interval $[t_0, t_K]$.

Proof. We can construct the same u as in (3) by solving

$$\inf_{u_{[t_i,t_{i+1}]}} \sup_{w \in \mathcal{N}} \int_{t_i}^{t_{i+1}} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + x(t_{i+1})' \Psi(t_{i+1}, \tau_i, N_i) x(t_{i+1})$$

Using theorem 4, if

$$N_{K-1} \ge Q_f, \quad \text{and}$$

$$\Psi(t_{i+1}, \tau_i, N_i) \ge \Psi(t_{i+1}, t_{i+2}, \Psi(t_{i+2}, \tau_{i+1}, N_{i+1}))$$

for $0 \le i \le K-2$ then equation (4) is satisfied. The result follows by applying part (ii) of lemma 1. ■

4 Estimation

We now wish to consider *output feedback* strategies. Essentially the above results rely on monotonicity and convergence of solutions to the Riccati differential equation associated with the state feedback $\mathcal{H}_\infty$ problem and the corresponding differential game. Since the estimator and its associated Riccati equation can be constructed by dual arguments [4], the question arises as to what the dual estimation problem is in the moving horizon case.

However, since we are not dealing with linear time invariant estimators, and we are not constructing controllers "in advance" of implementing them, the difference between the problems of estimation and control is more subtle. The control problem is solved backwards in time, by integrating a Riccati differential equation from some possibly fictitious future endpoint. The estimation problem is, however, solved by integrating a Riccati differential equation forwards in time from the initial time. Further, the construction of the state estimate is dynamic, that is it must be updated continuously by the output from the plant. If we change our mind while the controller is in operation about how to construct a state estimate, then it is necessary to have information about the history of the plant inputs and outputs up until that time. In order to avoid the practical computational problems of trying to maintain such a history, we restrict attention to cases where the history is not needed when updating the estimator.

We will denote the strategy spaces for u and w by $\hat{\mathcal{M}}$ and $\hat{\mathcal{N}}$ respectively, the spaces of causal functions of y. We state the following certainty equivalence principle which can be obtained from [1, Theorem 5.1].

Theorem 6 (Başar and Bernhard[1]). *Define*

$$G^\tau(x(t_0), u_{[0,\tau]}, w_{[0,\tau]}) = \int_{t_0}^{\tau} z(t)'z(t) - \gamma^2 w(t)'w(t) \, dt + V(\tau, x(\tau))$$

where $V(\tau, x(\tau))$ is the value of the game in the state feedback case as defined above. Suppose that, for every τ and every $y_{[t_0,\tau]}$, there exists a unique $\hat{w}$ which maximizes G^τ subject to the constraint that system behaviour is consistent with the measured past system input output behaviour. Then this worst case *system behaviour corresponds to a worst case state $\hat{x}(\tau)$. This worst case state should be used in the state feedback law at time τ as if it were the actual state. The output feedback controller is then the min-sup controller and the overall game has value $V(t_0, x(t_0))$.*

Note that this states that the value for the overall game in the output feedback case is the same as that in the state feedback case. At time τ, this gives the following expression for the output feedback value of the game:

$$\int_{t_0}^{t_f} z(t)'z(t) - \gamma^2 w(t)'w(t) \, dt + x(t_f)' Q_f x(t_f) \le \sup_{\substack{w_{[t_0,\tau]} \\ \text{consistent}}} \left\{ \int_{t_0}^{\tau} z(t)'z(t) - \gamma^2 w(t)'w(t) \, dt + \inf_{u \in \mathcal{M}} \sup_{w \in \mathcal{N}} \left\{ \int_{\tau}^{t_f} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + x(t_f)' Q_f x(t_f) \right\} \right\} \quad (5)$$

where $u_{[t_0,\tau]}$ is the controller as prescribed above, and w is arbitrary. This therefore gives an upper bound on the output feedback value of the game at time τ, given all measurements and inputs up to that time. Further, when w is the worst case disturbance, then equality is achieved in the above expression.

Suppose there exists a solution to the Riccati differential equation

$$\dot{Y} = AY + YA' + Y(\gamma^{-2} C_1' C_1 - C_2' C_2) Y + B_1 B_1' \quad (6)$$

with boundary condition $Y(0) = 0$. Since G^τ is a quadratic function of w, the existence of the maximum depends solely on the homogeneous part of the quadratic form, and the effect of y is as a constraint on w in this problem to a linear subspace of $\mathcal{L}_2$. Hence we can set $u = 0$ and $y = 0$ when testing to see if the maximum exists. By completion of squares,

$$\sup_{w} G^\tau(0, 0, w) = x(\tau)'(\Psi(\tau, t_f, Q_f) - \gamma^2 Y^{-1}(\tau)) x(\tau) - \int_{t_0}^{\tau} (w - B_1' Y^{-1} x(t))^2 \, dt$$

The maximum of G^τ with respect to w exists if and only if there exists a solution to the Riccati equation for Y, and $\Psi(\tau, t_f, Q_f) - \gamma^2 Y^{-1}(\tau)) x(\tau) < 0$. In this case $G^\tau(0, 0, w) < 0$ with equality for the worst case w. See for example Basar and Bernhard[1], Limebeer et al[9]. We will write $Y(t) = \Phi(t, t_0, Q_0)$ for the solution to the above Riccati equation with boundary condition $\Phi(t_0, t_0, Q_0) = Q_0$. We then have the following results, which are analogues of those in the state feedback case.

Theorem 7. *Let $t_1, t_2 \in [t_0, t_f]$, $t_1 < t_2$. For $\tau \in [t_0, t_2]$ Let*

$$G^\tau = \int_{t_0}^{\tau} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + V(\tau, x(\tau))$$

and for $\tau \in [t_2, t_f]$ let

$$G^\tau = \int_{t_0}^{\tau} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + x(t_1)' \Delta_1 x(t_1) + V(\tau, x(\tau))$$

where $\Delta_1 > 0$. Calculate the state estimator according to the certainty equivalence principle above, maximising G^τ as defined here with respect to past w consistent with known information about u and y, and using the worst case state as if it were the actual state. If the maximum exists, then with the resulting output feedback control law

$$\sup_{w \in \mathcal{N}} \left\{ \int_{t_0}^{t_f} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + x(t_f)' Q_f x(t_f) \right\} \le x(t_0)' \Psi(t_0, t_f, Q_f) x(t_0)$$

Note that the maximum exists if and only if $\Psi(\tau, t_f, Q_f) - \gamma^2\Phi(\tau, t_0, 0)^{-1} < 0$ for all τ.

Proof. We omit the details of this proof due to shortage of space. However, intuitively the result follows directly from the certainty equivalence principle above. We can see this by regarding the addition of Δ as a change in our idea of what the system was in the past. At time t_2, it is as if we have realised that $C_1(t_1)$ was bigger than we thought, by $\Delta^{\frac{1}{2}}$. ∎

Note that in order to implement this change, we need to know, at time t_2, not only $\hat{x}(t_1)$, but also the history of u and y on $[t_1, t_2]$. This is therefore practically unrealisable, when $t_2 > t_1$, but we can apply the theorem to the simpler case when $t_2 = t_1$. In this case, while integrating the Ricatti equation forwards, it is possible to subtract any $\Delta > 0$ from $\Phi(\tau, t_0, 0)^{-1}$.

By direct application of the certainty equivalence principle above, it is clear that we can combine the state feedback and output feedback results above to construct output feedback moving horizon controllers which result in a $\mathcal{H}_\infty$ norm performance bound over all time.

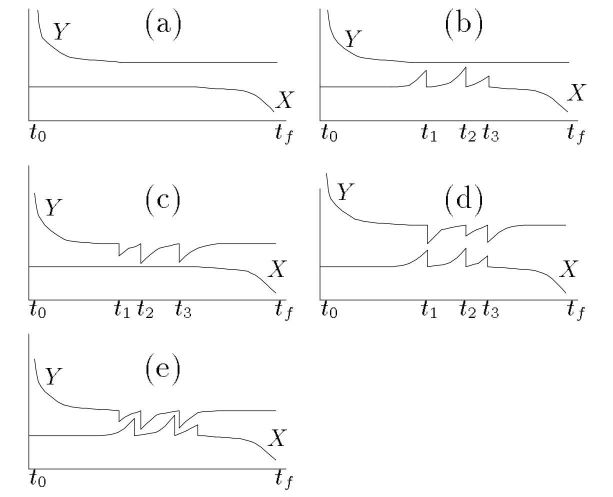

Figure 1: Graphs of $X(t) = \|\Psi(t, t_f, Q_f)\|$ and $Y(t) = \gamma^2\|\Phi(t, t_0, 0)^{-1}\|$ on a finite horizon

5 Conclusions

A pictorial representation of the ideas can be found in figure 1. Here $X(t) = \|\Psi(t, t_f, Q_f)\|$ and $Y(t) = \gamma^2\|\Phi(t, t_0, 0)^{-1}\|$. Graph (a) shows the normal suboptimal situation on a finite horizon. Typically in the suboptimal case, there is a gap between X and Y, since the spectral radius constraint $\Psi(\tau, t_f, Q_f) - \gamma^2\Phi(\tau, t_0, 0)^{-1} < 0$ is not active.

In graph (b) we take advantage of this by choosing, at time t_0, a weighting matrix $M > 0$ to use at time t_1, the first peak of the graph of X, as if it were a final state weight. This matrix must be chosen to satisfy two conditions, firstly that Ψ is increasing on $[t_0, t_1]$, and secondly so that $M - \gamma^2\Phi(\tau, t_0, 0)^{-1} < 0$, the spectral radius condition. Then computation of the controller depends only on the model of the plant from time t_0 to t_1. At time t_1, a new weight is chosen for time t_2, and so on. The situation is similar, but harder to represent pictorially, in the case where the intervals overlap.

In graph (c), the parallel situation for the observer is shown. At time t_1, a new model for the past plant is decided upon, but it is desired to maintain the cost function as an infinity norm constraint on the whole interval $[t_0, t_f]$. In this way we can adapt to the changed idea of the plant model without reinitializing the plant, and this method can be regarded as a method of reducing transients in such changes.

Graphs (d) and (e) show the effect of both of these changes. In graph (d) we see what happens with the simple addition of state weights $\Delta > 0$ to the cost function when output feedback is used. This can also be regarded as a simultaneous forwards/backwards update/adaptation. Then in graph (e) we see that it is possible to achieve a lower gamma value by staggering the times at which update/adaptation has occurred.

Note that it is necessary to choose a value of γ greater than the optimal value achievable by solving the infinite horizon problem directly, since by increasing the weights M we are increasing the cost function for the corresponding game problems and therefore existence of a solution to the corresponding Riccati differential equation is a more stringent condition.

In this paper, we have shown that $\mathcal{H}_\infty$ controllers can be synthesised with only knowledge of the plant model for some finite time in the future, and that the performance level can be maintained provided that the future plant is not, in some sense, "too large", as measured by the solution of a Riccati equation. These results also indicate that, for suboptimal controllers, if the model changes slowly over time, then the performance level need not degrade.

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