RICCATI DIFFERENTIAL INEQUALITIES: SUBOPTIMAL $\mathcal{H}_\infty$ CONTROLLERS FOR FINITE HORIZON TIME VARYING SYSTEMS

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Abstract

In this paper we give a direct derivation of the solution to the continuous time finite horizon $\mathcal{H}_\infty$ synthesis problem. An explicit separation of the time domain cost function into past and future pieces is used. Perturbations are made to these pieces which lead to Riccati differential inequalities in the form of the solution, and demonstrate the correspondence between the inequalities and the cost function halves.

1. Introduction

A complete solution to the continuous time $\mathcal{H}_\infty$ problem was first given in terms of two algebraic Riccati equations and a coupling condition [2]. In this paper we give an explicit game theoretic derivation of the solution to the suboptimal $\mathcal{H}_\infty$ synthesis problem on a finite horizon, and express this solution in terms of Riccati differential inequalities. In particular, the method of derivation illustrates the correspondence between the solutions to these inequalities and variations in the past and future cost functions.

2. Preliminaries

We will consider the system described by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) \\ z(t) &= C_1(t)x(t) + D_{12}(t)u(t) \\ y(t) &= C_2(t)x(t) + D_{21}(t)w(t) \end{aligned}$$

on a finite time interval $[t_0, t_f]$, with initial state $x(t_0) = x_0$. We will make the standard assumptions that $D_{21}(t)D_{21}(t)' = I$, $D_{12}(t)'D_{12}(t) = I$, $D_{21}(t)B_1(t)' = 0$, and $D_{12}(t)'C_1(t) = 0$ for all $t \in [t_0, t_f]$.

The problem is a dynamic game, where the minimising player's input u is required to be a causal function of the measured output y. Define the cost by

$$J(x_0, \mu, w) = -\gamma^2 x_0' Q_0 x_0 + \int_{t_0}^{t_f} |z(t)|^2 - \gamma^2 |w(t)|^2 \, dt + x(t_f)' Q_f x(t_f)$$

where $Q_0 > 0$ and $Q_f > 0$. We would then like to solve the problem

$$\inf_{\mu \in \mathcal{U}_{of}} \sup_{w \in \mathcal{L}_2, x_0 \in \mathbb{R}^n} J(x_0, \mu, w).$$

The solution to the above problem is well known, and a suboptimal controller exists if and only if there exists bounded solutions to the two Riccati differential equations

$$\begin{aligned} \dot{Y}_\infty &= AY_\infty + Y_\infty A' - Y_\infty (C_2'C_2 - \gamma^{-2} C_1'C_1) Y_\infty + B_1 B_1' \\ -\dot{X}_\infty &= A'X_\infty + X_\infty A - X_\infty (B_2 B_2' - \gamma^{-2} B_1 B_1') X_\infty + C_1'C_1 \end{aligned}$$

with boundary conditions $X_\infty(t_f) = Q_f$ and $Y_\infty(t_0) = Q_0^{-1}$, satisfying the coupling condition $X_\infty(t) - \gamma^2 Y_\infty^{-1}(t) < 0$ for all $t \in [t_0, t_f]$.

In this paper we will show that subject to a further coupling condition, these equations can be generalised to differential inequalities. This result can be derived also using similar techniques to those used in the LTI case to derive solutions in terms of Linear Matrix Inequalities; however the primary purpose here is to illustrate the relationship between this form of solution and the related dynamic game problem.

3. State feedback

Let X satisfy the following differential inequality on $[t_0, t_f]$.

$$-\dot{X} \le A'X + XA - X(B_2B_2' - \gamma^{-2} B_1 B_1')X + C_1'C_1$$

with boundary condition $X(t_f) = Q_f$. We can rewrite this as

$$-\dot{X} = A'X + XA - X(B_2B_2' - \gamma^{-2} B_1 B_1')X + C_1'C_1 + \Delta_1(t)$$

for some time varying matrix $\Delta(t) \ge 0$. A simple completion of squares argument gives the following result.

Theorem 1. *Suppose that the above Riccati differential inequality has a solution on* $[0, t_f]$*. Then*

$$J(x(0), \mu, w) \le x(t_0)'X(t_0)x(t_0) + \|u - u^*\|_2^2 - \gamma^2 \|w - w^*\|_2^2$$

for any controller μ *and any input* w*. Here* $u^*(t) := -B_2'(t)X(t)x(t)$ *and* $w^*(t) := \gamma^{-2} B_1'(t)X(t)x(t)$*. If* $x(t_0) = 0$*, then with the controller* $u(t) = \mu_0(t, x(t))$ *in place, the closed loop norm satisfies* $\|z\|_2 \le \gamma \|w\|_2$*.*

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4. Output feedback

We will split the cost function into past and future pieces, and give an explicit algebraic derivation for this special case of the more general separation theory developed by Whittle [3] and Başar and Bernhard [1]. It is shown in [3] that, given information up to time τ, if the cost is maximised with respect to all disturbances (w, x_0) consistent with measurements of u and y on $[t_0, \tau]$ assuming that future behaviour will be generated by an optimal state feedback controller, then the worst case state trajectory can be used in the state feedback controller as if it were the actual state, and the resulting controller will be optimal for the dynamic game with output feedback.

However, we will not use the separation principle as derived there, since in this case we wish to use separate Δ_1 and Δ_2 for the control and estimation halves of the problem, and thus the forward cost and the backward cost come from *different* cost functions. Instead we derive directly the required monotonicity property for this problem, in the spirit of [3] but avoiding reliance on complex dynamic programming arguments.

Let $V(t, x(t))$ be the *value* of the game, that is define $V(t, x(t)) = x(t)'X(t)x(t)$. Also define $\Omega(\tau) = \{(x_0, w^\tau)$ consistent with y^τ, $u^\tau\}$ and

$$G(\tau, x_0, u^\tau, w^\tau) = x_0' Q_0 x_0 + \int_{t_0}^{\tau} z'z + x'\Delta_2 x - \gamma^2 w'w\, dt + V(\tau, x(\tau)).$$

The following lemma is easily shown.

Theorem 2. *Suppose that, for every y^τ and every $\tau \in [t_0, t_f]$, the maximisation problem*

$$W(\tau, u^\tau, y^\tau) = \sup_{(x_0, w^\tau) \in \Omega_0[t_0,\tau]} G(\tau, x_0, u^\tau, w^\tau)$$

has a unique maximum $(\hat{x}_0^\tau, \hat{w}^\tau)$ generating a state trajectory $\hat{x}^\tau$. Suppose also that $\hat{\mu}$ is such that, for all $t_1, t_2 \in [t_0, t_f]$ with $t_2 \geq t_1$, and for all y, $W(t_2, u^{t_2}, y^{t_2}) \leq W(t_1, u^{t_1}, y^{t_1})$. That is, W is a decreasing function of τ when the signal u is defined by the feedback law $\hat{\mu}$. Then

$$\sup_{\substack{x_0 \in \mathbb{R}^n \\ w \in \mathcal{L}_2}} J(x_0, \hat{\mu}, w) \leq \max_{x_0} \{V(t_0, x_0) + \alpha(x_0)\}.$$

and hence the feedback law μ results in a closed loop induced norm bound on the finite horizon of γ.

Proof. Clearly $J(x_0, \hat{\mu}, w) \leq W(t_f, u^{t_f}, y^{t_f})$. Since W is decreasing,

$$\begin{aligned} J(x_0, \hat{\mu}, w) &\leq W(t_0, u^{t_0}, y^{t_0}) \\ &= \max_{x_0 \in \mathbb{R}^n} \{V(t_0, x_0) + \alpha(x_0)\} \qquad \forall w \in \mathcal{L}_2 \end{aligned}$$

and the result follows. ∎

Now suppose that S satisfies the Riccati differential inequality

$$-\dot{S} \geq SA + A'S - C_2'C_2 + \gamma^{-2} C_1'C_1 + SB_1B_1'S$$

with boundary condition $S(0) = Q^0$. This can be written as the Riccati differential equation

$$\dot{Y}_\infty = AY + YA' - Y(C_2'C_2 - \gamma^{-2}(C_1'C_1 + \Delta_2))Y + B_1B_1'$$

where $Y = S^{-1}$, for some time varying positive definite matrix Δ_2. The following theorem is the key result of this paper. It shows that, if we choose $\Delta_1 > \Delta_2$, then the above form of the separation principle is still applicable.

Theorem 3. *If $X(\tau) - \gamma^2 Y^{-1}(\tau) < 0$, then the unique maximum $W(\tau, u^\tau, y^\tau)$ for the above problem satisfies*

$$\frac{d}{d\tau} W(\tau, u^\tau, y^\tau) = -\gamma^2 (y(\tau) - C_2(\tau)\hat{x}^\tau(\tau))^2 + (u(\tau) + B_2(\tau)'X(\tau)\hat{x}^\tau(\tau))^2 - \hat{x}^\tau(\tau)'(\Delta_1 - \Delta_2)\hat{x}^\tau(\tau)$$

Hence, if $\Delta_1 - \Delta_2 > 0$, then if $u(\tau) = -B_2(\tau)'X(\tau)\hat{x}^\tau(\tau)$ for all $\tau \in [t_0, t_f]$, then

$$\frac{d}{d\tau} W(\tau, u^\tau, y^\tau) \leq 0$$

for all y^τ, and hence this is a suboptimal controller for the original problem.

5. Conclusions

We have presented a solution for the finite horizon time varying problem where the standard Riccati differential equations have been replaced by Riccati differential inequalities. By assuming that there is a distinction between the past and future cost function, this corresponds at each time τ to the revision of the estimate of the previous worst case w.

The computational implications are that it is not necessary to solve the RDEs exactly. It is enough to solve the RDEs approximately, provide we can guarantee that the solution satisfies the required inequality. The resulting controller is time varying and of the same order as the plant, as expected.

References

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