Model reduction in H2 using matrix solutions of polynomial equations *

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Abstract

A method is given for solving an optimal H_2 approximation problem for SISO linear time-invariant stable systems. The method guarantees that the global optimum is found. It is based on constructive algebra, but compared with earlier results, the method has much smaller time and memory requirements, and can therefore be applied to systems of significantly higher McMillan degree. The use of Buchberger's algorithm is avoided by writing the first-order optimality conditions in a special form, from which a Gröbner basis is immediately available. The problem is converted into linear algebra by exhibiting a finite-dimensional basis for a certain space, and can then be solved by eigenvalue calculations. This approach has potential for wider application to the solution of polynomial equations. Two examples are included.

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1 Introduction

In this paper we consider the problem of approximating a stable linear dynamic system by one of lower McMillan degree. We take the L_2 norm as the measure of approximation, namely we solve the problem

$$\min_{\hat{h}\in\mathcal{M}(n)} \int_0^\infty |h(t) - \hat{h}(t)|^2 dt \tag{1}$$

where $h \in \mathcal{M}(N)$ is the impulse response of the original system, $\hat{h}$ is the impulse response of the approximating system, and $\mathcal{M}(N)$ denotes the set of impulse responses of minimal stable systems of McMillan degree N. This problem is equivalent to the problem of finding an approximation which minimizes the H_2 norm of the error in the frequency response:

$$\min_{\hat{H}\in\mathcal{H}(n)} \int_{-\infty}^\infty |H(\omega) - \hat{H}(\omega)|^2 d\omega \tag{2}$$

where H and $\hat{H}$ are the frequency responses of the original and the approximating systems, respectively, and $\mathcal{H}(N)$ denotes the set of Fourier transforms of elements of $\mathcal{M}(N)$. Throughout this paper we consider SISO systems only, with the 'true' system having distinct poles, and we solve the H_2 problem for $n = N-1$. (In section 5 we introduce the set ΣS_N of rational transfer functions, whose impulse responses are elements of $\mathcal{M}(N)$ and frequency responses are elements of $\mathcal{H}(N)$.)

The H_2 problem has many applications and connections to other problems in systems and control theory, including model simplification, system identification, and approximate model matching. Many publications treat this problem, such as [20, 1] and the references cited therein.

We investigate the H_2 approximation problem by means of constructive algebra, in particular by exploiting the theory of polynomial ideals. In [12] two of the present authors already applied constructive algebra to this problem, taking an approach based on state-space realizations of the linear systems involved. By contrast, the approach here is based on a form of the first-order necessary conditions for optimality which arises from transfer function descriptions of both the original and the approximating systems. The solution method which we develop here is quite different from that developed in [12]. Computationally it is much more efficient, as regards both memory and time requirements. This allows us to tackle problems with significantly larger values of N, as can be seen from the examples. As in [12], the use of constructive algebra leads to an algorithm with the important attribute that the solution found is guaranteed to be the global optimum. But the method presented here is much more practical than the one given in [12], so we can say that it is the first practical method which carries with it a guarantee of global optimality for a rather general class of H_2 problems. Since, as will be seen, it relies on

eigenvalue calculations for a set of matrices which can be constructed in a rather straightforward manner, it is in some ways comparable with Glover's method for solving the Hankel-norm approximation problem [10].

In the next section we obtain a special representation of the first-order necessary conditions for optimality. This representation is in the form of a set of quadratic equations, which take a special form which we call *diagonal quadratic*.

The following section investigates such diagonal quadratic equations. It is shown that the polynomials which define these equations form a Gröbner basis for the ideal generated by themselves. It is further shown that these equations have a finite set of solutions, and that in consequence a certain space is finite-dimensional. Furthermore a basis for this space is identified, which allows a solution method based on linear algebra.

We then present such a method of solving a system of polynomial equations. This method relies on obtaining a Gröbner basis, but in the application to the H_2 problem, such a basis is immediately available. This method of solving polynomial equations appears to be of general use, but we confine ourselves here to its application to the H_2 problem. Its wider application will be discussed in another paper. The solution is obtained in the form of a set of matrices which commute pairwise with each other. In order to extract the solution from this set, either numerical methods can be used (requiring the solution of eigenvalue problems), or symbolic methods (for locating roots, such as Sturm sequence methods).

A short section then applies this method to the solution of the H_2 problem, for the case $n = N - 1$. This is followed by two examples.

2 A special representation of the first order conditions.

In this section the first order conditions for a class of H_2 model order reduction problems will be considered. The continuous-time case is treated here, but the discrete-time case is in fact the same up to isometry (see e.g. [14], Theorem 5.4-3; [15], Theorem 3.2-22).

Studying the outcomes of a computer algebra calculation in which a set of symbolic first order conditions for the H_2 model order reduction problem was brought into a recursive form, it was observed that the occurrence of multiple poles in the original system gave rise to a certain singularity in the first order equations. This was the motivation to investigate the class of systems with distinct poles separately from the class of systems with multiple poles (which we hope to treat elsewhere).

Let us consider a continuous-time stable SISO linear system. Without loss of generality we can assume the system to be strictly proper, because if it is not then the direct feedthrough term of the optimal H_2 approximant will be equal to the direct

feedthrough term of the original system, and the strictly proper part of the optimal approximant will not be influenced at all (nor will the strictly proper part of any of the critical points) by the value of the direct feedthrough term. Let the transfer function of the original system (i.e. the system that is to be reduced in order) be given by $e(s)/d(s)$, where e is some polynomial with real coefficients of degree at most $N-1$, and d is a monic polynomial with real coefficients of degree N with all its zeroes (i.e. poles of the transfer function) $\delta_1, \delta_2, \ldots, \delta_N$, within the open left half plane in the complex plane. Let $b(s)/a(s)$, where b is a polynomial with real coefficients of degree at most $n-1$ and a a monic polynomial with real coefficients of degree n, which has all its zeros within the open half plane, denote an arbitrary transfer function of a stable linear system of order n. A well-known first order necessary condition for optimality of an $n-$th order transfer function $b(s)/a(s)$ with real coefficients, as an approximant in H_2 is the following. First let us present a geometric formulation.

If $\frac{b(s)}{a(s)}$ is an optimal approximant within the class of transfer functions of order n in H_2, of the transfer function $\frac{e(s)}{d(s)}$ in H_2, with respect to the H_2-norm, then the difference $\frac{e(s)}{d(s)} - \frac{b(s)}{a(s)}$ is perpendicular to the tangent plane at the manifold of transfer functions of order n at the point $\frac{b(s)}{a(s)}$.
It is well-known (and not hard to show) that the tangent space consists of all strictly proper rational functions of the form $\frac{p(s)}{a(s)^2}$, where p is a polynomial of degree at most $2n-1$. From the theory of Hardy spaces it follows that the orthogonal complement in H_2 of this vector space is given by $a(-s)^2 H_2$, i.e. all H_2-functions which can be written as the product of the function $a(-s)^2$ and an arbitrary H_2 function. Combining this with the first order conditions given above, it follows that the numerator of the difference $\frac{e(s)}{d(s)} - \frac{b(s)}{a(s)}$ has to be divisible by $a(-s)^2$. (Cf. [20], see also [1], [2]). Algebraically this can be written down as follows:

Let $n < N$. If $\frac{b(s)}{a(s)}$ is an optimal approximant within the class of transfer functions of order n in H_2, of the transfer function $\frac{e(s)}{d(s)}$ in H_2, with respect to the H_2-norm, then there exists a polynomial $q(s)$ of degree at most $N-(n+1)$ such that

$$e(s)a(s) - b(s)d(s) = a(-s)^2 q(s). \tag{3}$$

Let us now specialise to the case $n = N-1$. Then the polynomial $q(s)$ has degree zero, so in fact it reduces to a constant $q(s) = q_0$. The unknowns in the polynomial equation are the polynomials $b(s), a(s)$ and the number q_0. Although q_0 is only an auxiliary variable we will not eliminate it. Note that once the polynomial a and the number q_0 are known, the polynomial b follows from the formula

$$b(s) = \frac{e(s)a(s) - q_0 a(-s)^2}{d(s)}. \tag{4}$$

Let us now specialise to the case in which the original system has distinct poles

(i.e. the multiplicity of each of the $N = n + 1$ poles $\delta_1, \ldots, \delta_N$ is one). Substituting $s = \delta_i, i = 1, \ldots, N$ in the polynomial equation one obtains:

$$e(\delta_i)a(\delta_i) = a(-\delta_i)^2 q_0, i = 1, \ldots, N. \tag{5}$$

Note that the polynomials appearing here do not depend on the polynomial b, due to the fact that $d(\delta_i) = 0$ for each $i = 1, \ldots, N$. Further note that the possibility $q_0 = 0$ can be excluded on the grounds that if $q_0 = 0$ then either $e(\delta_i) = 0$ for some value of $i \in \{1, \ldots, N\}$, which implies that there is pole-zero cancellation in the original transfer function and the order of the transfer function will be smaller than N, which can be ruled out without loss of generality, or otherwise it would follow that $a(s) = 0$ in N different points, namely at $s = \delta_i, i = 1, \ldots, N$, which together with the fact that a has degree $n = N - 1$ would imply that $a = 0$, which is in contradiction with the assumption that a is monic. It follows that $q_0 \neq 0$ for each value of q_0 that corresponds to a solution of the first order equations. Therefore multiplying both sides of the polynomial equation with q_0 the first order conditions can be rewritten as

$$e(\delta_i)a(\delta_i)q_0 = \left(a(-\delta_i)q_0\right)^2, i = 1, \ldots, N, q_0 \neq 0. \tag{6}$$

The polynomial a is monic, so q_0 is the leading coefficient of the non-zero polynomial $\tilde{a} := q_0 a$. Using this notation the first order equations can be rewritten as

$$e(\delta_i)\tilde{a}(\delta_i) = \tilde{a}(-\delta_i)^2, i = 1, \ldots, N, \ \tilde{a} \neq 0. \tag{7}$$

The idea is now to consider this as an equation in the unknowns $\tilde{a}(-\delta_i), i = 1, \ldots, N$. In order to do this explicitly we need to express the sequence of numbers $\tilde{a}(\delta_i), i = 1, \ldots, N$ in terms of the sequence of numbers $\tilde{a}(-\delta_i), i = 1, \ldots, N$. This can be done by relating both sequences to the coefficients $\tilde{a}_j, j = 0, \ldots, N - 1$, of the polynomial $\tilde{a}(s) = \tilde{a}_{N-1}s^{N-1} + \tilde{a}_{N-2}s^{N-2} + \ldots + \tilde{a}_0 s^0$. Let $V(\delta_1, \ldots, \delta_N)$ denote the Vandermonde matrix

$$V(\delta_1, \ldots, \delta_N) \ := \ \begin{pmatrix} 1 & \delta_1 & \delta_1^2 & \ldots & \delta_1^{N-1} \\ 1 & \delta_2 & \delta_2^2 & \ldots & \delta_2^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \delta_N & \delta_N^2 & \ldots & \delta_N^{N-1} \end{pmatrix}. \tag{8}$$

Using matrix-vector notation the following linear relations are obtained:

$$\begin{bmatrix} \tilde{a}(\delta_1) \\ \vdots \\ \tilde{a}(\delta_N) \end{bmatrix} \ = \ V(\delta_1, \ldots, \delta_N) \begin{bmatrix} \tilde{a}_0 \\ \vdots \\ \tilde{a}_{N-1} \end{bmatrix} \tag{9}$$

and

$$\begin{bmatrix} \tilde{a}(-\delta_1) \\ \vdots \\ \tilde{a}(-\delta_N) \end{bmatrix} = V(-\delta_1, \ldots, -\delta_N) \begin{bmatrix} \tilde{a}_0 \\ \vdots \\ \tilde{a}_{N-1} \end{bmatrix}. \tag{10}$$

It follows that

$$\begin{bmatrix} \tilde{a}(\delta_1) \\ \vdots \\ \tilde{a}(\delta_N) \end{bmatrix} = V(\delta_1, \ldots, \delta_N)V(-\delta_1, \ldots, -\delta_N)^{-1} \begin{bmatrix} \tilde{a}(-\delta_1) \\ \vdots \\ \tilde{a}(-\delta_N) \end{bmatrix}. \tag{11}$$

Note that $V(-\delta_1, \ldots, -\delta_N)$ is an invertible matrix because, by assumption, for all $i = 1, \ldots, N, j = 1, \ldots, N$, if $i \neq j$ then $\delta_i \neq \delta_j$ and therefore we have $\det(V(-\delta_1, \ldots, -\delta_N)) = \Pi_{1 \leq i < j \leq N}(\delta_i - \delta_j) \neq 0$ (cf. e.g. [19], p.35).

The first order equations can now be rewritten as

$$\begin{bmatrix} \tilde{a}(-\delta_1)^2 \\ \vdots \\ \tilde{a}(-\delta_N)^2 \end{bmatrix} = \operatorname{diag}(e(\delta_1), \ldots, e(\delta_N))V(\delta_1, \ldots, \delta_N)V(-\delta_1, \ldots, -\delta_N)^{-1} \begin{bmatrix} \tilde{a}(-\delta_1) \\ \vdots \\ \tilde{a}(-\delta_N) \end{bmatrix}$$

$$[\tilde{a}(-\delta_1), \ldots, \tilde{a}(-\delta_N)] \neq 0 \tag{12}$$

where $\operatorname{diag}(e(\delta_1), \ldots, e(\delta_N))$ denotes the diagonal matrix with $e(\delta_i)$ in the $(i,i)-$entry, $i = 1, \ldots, N$.
This means that these first order equations can be written as

$$\begin{bmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_N^2 \end{bmatrix} = M \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \; x \neq 0 \tag{13}$$

where $x_i = \tilde{a}(-\delta_i), i = 1, \ldots, N, \; x = (x_1, \ldots, x_N)'$ and

$$M = \operatorname{diag}(e(\delta_1), \ldots, e(\delta_N))V(\delta_1, \ldots, \delta_N)V(-\delta_1, \ldots, -\delta_N)^{-1}. \tag{14}$$

In the next section the solution of equations of the form found here will be treated in general.

3 Diagonal-quadratic systems of equations

In this section we will present results about an arbitrary system of polynomial equations of the form

$$\begin{bmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_N^2 \end{bmatrix} = M \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} + \mu, \tag{15}$$

where $\mu \in \mathbf{R}^N$ is a constant $N-$vector. This will be called a *diagonal-quadratic* system of equations.

Remark. A quadratic equation in x can be written as $x^T Ax + cx + d$ for some symmetric matrix A, a row vector c and a scalar d. If $A = e_i e_i^T$, for some $i \in \{1, \ldots, N\}$, then the equation is one of the form described above. If there are N quadratic equations and the corresponding $A-$matrices are all diagonal, and these diagonal matrices form a basis of the linear vector space of all diagonal $N \times N$ matrices then such a system can (obviously) be rewritten in the form above. That is the motivation for the terminology 'diagonal-quadratic'.

In this paper use will be made of Gröbner basis theory and constructive algebra. For an exposition of this theory one can refer to e.g. [7]. In Gröbner basis theory an important role is played by the so-called *monomial orderings*. Let $\alpha = (\alpha_1, \ldots, \alpha_N)$ denote an arbitrary vector of nonnegative integers, which will be called a multi-index in the sequel, then x^α will denote the monomial $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_N^{\alpha_N}$. The multi-index α is called the *multi-degree* of the monomial x^α. The corresponding total degree is defined as $|\alpha| := \alpha_1 + \alpha_2 + \ldots + \alpha_N$. For a general definition of monomial ordering we refer to [7], p.54, Definition 1. In this paper the so-called total degree ordering will be used. The total degree ordering on monomials is defined by $x^\alpha \succ x^\beta$ if and only if either $|\alpha| > |\beta|$, or $|\alpha| = |\beta|$ and $\alpha_i > \beta_i$ for the smallest integer $i \in \{1, \ldots, N\}$ for which $\alpha_i \neq \beta_i$. This ordering goes under various names; in [7] it is called 'the graded lexicographic order', in [9], p. 432 it is noted that the ordering is known as 'total degree ordering'. In fact more than one total degree ordering can be found in the literature. For our purposes any total degree ordering would do. For the sake of definiteness we choose to work with the total degree ordering as just defined. The total degree of a polynomial is defined as follows. Each polynomial is a unique linear combination of monomials with nonzero coefficients. The maximal total degree of these monomials is called the *total degree of the polynomial*. If we denote the $i-$th row of the matrix M by m_i and the $i-$th entry of the vector μ by μ_i for $i \in \{1, \ldots, N\}$, then the equations can be rewritten as

$$x_i^2 - m_i x - \mu_i = 0, i = 1, \ldots, N.$$

Let $g_i(x_1, \ldots, x_N) := x_i^2 - m_i x - \mu_i, i = 1, \ldots, N$, then we are looking for the zeros

of the ideal I spanned by $G := \{g_1, g_2, \ldots, g_N\}$.

Theorem 3.1 *The set G is a Gröbner basis with respect to total degree ordering.*

Proof. We will show that the set of leading terms of the polynomials in the ideal generated by G is a subset of the (monomial) ideal that is generated by the leading terms of $g_1, \ldots, g_N$, namely by $x_1^2, \ldots, x_N^2$. It then follows that the ideal generated by the leading terms of the polynomials in the ideal generated by G is equal to the ideal generated by the leading terms of the elements of G. By definition (cf. [7], Ch.2, Def. 5) it then follows that G is a Gröbner basis.

Let $h_i(x) = h_i(x_1, \ldots, x_N) := m_i x + \mu_i, i = 1, \ldots, N$, denote the linear polynomial with the property that $g_i(x) = x_i^2 - h_i(x),\ i = 1, \ldots, N$. Let $f \in < g_1, \ldots, g_N >$, i.e. f is an element of the ideal generated by G. Then there exist polynomials $c_1(x), c_2(x), \ldots, c_N(x)$ such that

$$f = \sum_{i=1}^{N} c_i(x) g_i(x).$$

It is important to notice that if $N \geq 2$ the polynomials $c_i(x),\ i = 1, \ldots, N$ are by no means unique. For example if $f = 0$, then one could of course take $c_i := 0,\ i = 1, \ldots, N$, but one could alternatively take $c_1 := g_2,\ c_2 := -g_1$ and $c_i = 0$ for all $i > 2$.

Let R denote the set of multi-indices $R := \{0, 1\}^N$. In other words, R is the set of all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_N)$ with the property that for each $i = 1, \ldots, N$ one has either $\alpha_i = 0$ or $\alpha_i = 1$. Let Q denote the set of all multi-indices outside R. For each polynomial $p = p(x)$ there exists a unique additive decomposition $p = p^R + p^Q$, where the polynomial p^R is a linear combination of monomials with multi-degree in R and p^Q is a linear combination of monomials with multidegree in Q. Using this notation f can be represented as

$$f(x) = \sum_{i=1}^{N} c_i^Q(x) g_i(x) + \sum_{i=1}^{N} c_i^P(x) g_i(x).$$

Now let q_i denote the total degree of $c_i^Q(x)$ for each $i = 1, \ldots, N$ and let $q := \max\{q_1, \ldots, q_N\}$. It should be clear that q depends on the specific choice of the polynomials $c_1, \ldots, c_N$. To stress this write $q = q(c_1, \ldots, c_N)$. For each i a polynomial c_i can of course be considered as a unique linear combination of monomials. For each i let r_i denote the number of different monomials with total degree q which occur with nonzero coefficient in the expression of c_i as a linear combination of monomials. Clearly $q_i = q$ if and only if $r_i > 0$. Let $r = \sum_{i=1}^{N} r_i$. Let us denote $r = r(c_1, \ldots, c_N)$

to stress that r is a function of the $N-$tuple of polynomials $(c_1, \ldots, c_N)$. Let q_f denote the minimal value of q over all possible $N-$tuples of polynomials $(c_1, \ldots, c_N)$ with $f = \sum_1^N c_i g_i$. Let r_f denote the minimal value of r over all possible $N-$tuples of polynomials $(c_1, \ldots, c_N)$ with $f = \sum_1^N c_i g_i$ and $q(c_1, \ldots, c_N) = q_f$.

It is clear by construction that there exist $N-$tuples of polynomials $(c_1, \ldots, c_N)$ with $f = \sum_1^N c_i g_i$, $q(c_1, \ldots, c_N) = q_f$ and $r(c_1, \ldots, c_N) = r_f$. We now assert that in such a case the leading monomial of f is one of the leading monomials of $c_i g_i,\ i \in \{1, \ldots, N\}$. This will be shown shortly. Note that for any $i \in \{1, \ldots, N\}$ the leading monomial of $c_i g_i$ is equal to the leading monomial of $c_i(x)x_i^2$, because we work with a total degree monomial ordering and $g_i(x) = x_i^2 - h_i(x)$, with h_i linear. Therefore the implication of the assertion is that the leading monomial of f is an element of the ideal generated by the leading monomials, viz. $x_1^2, \ldots, x_N^2$, of $g_1, \ldots, g_N$, which is what we want to prove. So it only remains to prove the assertion. In fact we will prove that if the leading monomial of f is *not* equal to any of the leading monomials of $c_i g_i,\ i \in \{1, \ldots, N\}$ then the pair (q, r) is not equal to (q_f, r_f). To be more precise we will show that in such a case either q can be reduced, or r can be reduced without increasing q.

Suppose that indeed the leading monomial of f is not equal to any of the leading monomials of $c_i g_i,\ i \in \{1, \ldots, N\}$. Then there must be at least one pair of nonnegative integers i, j, with $i \neq j$ and $i, j \in \{1, \ldots, N\}$ such that the leading monomial of $c_i g_i$ is equal to the leading monomial of $c_j g_j$. Let this leading monomial be denoted by x^α. Without loss of generality we can assume that the pair i, j is such that $|\alpha|$ is maximal among all such pairs. Because the leading monomial of $c_i g_i$ is equal to the leading monomial of $c_i x_i^2$ it follows that x^α is divisible by x_i^2. Similarly it follows that x^α is divisible by x_j^2. Therefore $\alpha \notin R$. It follows that x^α cannot be the leading monomial of $c_i^R g_i$ and it cannot be the leading monomial of $c_j^R g_j$. Therefore it is in fact the leading monomial of $c_i^Q g_i$ and of $c_j^Q g_j$. It follows that $|\alpha| = q + 2$ and $r_i > 0,\ r_j > 0$. Consider the monomial $x^\beta := \frac{x^\alpha}{x_i^2 x_j^2}$. Let the leading term of c_i^Q by denoted by $\gamma x^\beta x_j^2$. Consider the equality

$$\gamma x^\beta (x_j^2 - h_j) = \gamma x^\beta g_j.$$

Using this we can rewrite $c_i^Q g_i + c_j^Q g_j$ as

$$c_i^Q g_i + c_j^Q g_j = (c_i^Q - \gamma x^\beta x_j^2 + \gamma x^\beta h_j) g_i + (c_j^Q + \gamma x^\beta x_i^2 - \gamma x^\beta h_i) g_j.$$

It follows that using $(c_i^Q - \gamma x^\beta x_j^2 + \gamma x^\beta h_j)$ instead of c_i^Q and $(c_j^Q + \gamma x^\beta x_i^2 - \gamma x^\beta h_i)$ instead of c_j^Q a representation of f is obtained which has a smaller value for q or otherwise an equal value for q and a smaller value for r. □

This result is very important because to apply the results of Gröbner basis theory one needs a Gröbner basis. Usually one needs to apply an algorithm like Buchberger's

algorithm to bring a set of polynomials that generates the ideal in which one is interested in Gröbner basis form. In fact in many cases this is the most difficult part of the calculations. In the case at hand however the set of polynomials of which we want to find the zeros forms itself a Gröbner basis.

But that is not all. We can say more. We know that $G = \{g_1, \ldots, g_N\}$ forms a Gröbner basis and that the leading monomial of g_i is x_i^2 for each $i = 1, \ldots, N$. Applying [7], Chapter 5, Theorem 6, we can conclude the following.

Corollary 3.1 *Let I denote the ideal generated by G.*

(i) The set $V = V(I)$ of zeros in $\mathbf{C}^N$ of the system of polynomial equations $g_i(x) = 0,\ i = 1, \ldots, N$, is finite.

(ii) The $\mathbf{C}-$vector space $S = Span(x^\alpha : x^\alpha \notin < LT(I) >)$ is finite-dimensional.

(iii) The $\mathbf{C}-$vector space $\mathbf{C}[x_1, \ldots, x_N]/I$ is finite-dimensional.

In fact more can be said.

Lemma 3.1 *(i) The set of monomials $\{x^\alpha : \alpha \in R\}$ forms a basis for the vector space S defined in the previous lemma.*

(ii) The dimension of the vector space S is 2^N.

(iii) The dimension of the vector space $\mathbf{C}[x_1, \ldots, x_N]/I$ is 2^N.

Proof. ad (i). Because G is a Gröbner basis the ideal $< \mathrm{LT}(I) >$ is equal to the ideal generated by the leading terms of the elements of G, i.e. the ideal $< x_1^2, \ldots, x_N^2 >$. The monomials in this ideal are precisely those which have multi-degree in the set Q. Therefore the monomials in S are the all the monomials with multidegree in R.
ad (ii). From (i) it follows that the dimension of S is equal to the cardinality of R, which is $card(R) = 2^N$.
ad (iii). According to Proposition 4 of Chapter 5 of [7] the vector space $\mathbf{C}[x_1, \ldots, x_N]/I$ is isomorphic to S and therefore has the same dimension as S. □

From [7], Chapter 5, Proposition 1 it follows that every polynomial in $\mathbf{C}[x_1, \ldots, x_N]$ can be written in a unique way as the sum of an element of S and an element of I. In other words, each equivalence class $f + I$, where f is an arbitrary polynomial in $\mathbf{C}[x_1, \ldots, x_N]$, has a unique representative in S. Let this representative be denoted by $\pi(f) \in S$. Given f, the polynomial $\pi(f)$ can be obtained by a general method from Gröbner basis theory, namely the so-called division algorithm with respect to the Gröbner basis G as described in e.g. [7]. However, for diagonal quadratic equations, the division algorithm simplifies considerably and one can describe in direct terms how one can obtain $\pi(f)$ from f. The 'reduction procedure' can be

described as follows. Using the same notation as above, one can write $f = f^Q + f^R$, where $f^R \in S$ and the monomials of f^R all have multi-degree in Q. This additive decomposition is obviously unique. If $f^Q = 0$ then $f = f^R \in S$ in which case $\pi(f) = f$ and we are done. If $f^Q \neq 0$ then consider any monomial of f^Q with total degree equal to the total degree of f^Q. By construction each such monomial is divisible by at least one of the monomials $x_1^2, x_2^2, \ldots, x_N^2$. If it is divisible by x_i^2 then replacing it by the polynomial that is obtained by multiplying the monomial by $\frac{h_i(x)}{x_i^2}$ the result is a polynomial $\tilde{f}$ that is in the equivalence class $f + I$ and which has the following property. Either the total degree of $\tilde{f}^Q$ is smaller than the total degree of f^Q, or otherwise the total degree of $\tilde{f}^Q$ is equal to the total degree of f^Q but the number of monomials in $\tilde{f}^Q$ with total degree equal to the total degree of f^Q is one less than the number of monomials in f^Q with total degree equal to the total degree of f^Q. Such a replacement of f by $\tilde{f}$ will be called a 'reduction step'. It follows that after a finite number of such reduction steps one arrives at a polynomial in the equivalence class $f + I$ with the property that it lies in S. This is then the unique polynomial $\pi(f)$ that was sought for.

The importance of this reduction procedure in our application will become clear in the examples section.

4 Commutative matrix solutions of polynomial equations

In this section a method to obtain the solutions of a system of polynomial equations in several variables will be presented. We will consider the situation in which the system of polynomial equations will have a finite number of solutions over the field of complex numbers $\mathbf{C}$. However the method appears to be quite general and further generalizations seem to be possible. In the modern constructive algebra approach to the problem of finding the roots of a system of polynomial equations the theory of Gröbner bases plays an important role. For this theory we refer, as before, to [7]. A fundamental theorem of the theory of Gröbner bases is that for any polynomial ideal given by a finite number of polynomials which generate it, a Gröbner basis can be calculated with respect to any admissible monomial ordering (like the lexicographical ordering or the total degree ordering) in a *finite* number of steps. It can for example be obtained by Buchberger's algorithm. However the number of steps required by such an algorithm can be huge. In the literature it is suggested that in order to obtain the roots of a system of polynomial equations, one can construct a Gröbner basis with respect to a lexicographical ordering. Cf e.g. [7],p.233,[9], pp. 459-462. Also in the paper [12] this approach was followed to show that under two hypotheses described in that paper, the H_2 model order reduction problem can be solved in a finite number of steps. However only examples of reduction of third order models were presented in that paper. The bottle-neck in the calculations was the construction of a Gröbner basis with respect to lexicographical ordering. In discussing the use of Buchberger's

algorithm and variations Geddes et al.([9]) note that in general the computation of a lexicographic Gröbner basis is *much* more difficult than the corresponding total degree computation ([9], p. 462; their conclusion is based on several complexity studies mentioned there).

In the approach presented here a Gröbner basis with respect to an *arbitrary* monomial ordering is all that is required to do the calculations. In the previous section it was shown that for the problem of reduction of the model order by one with respect to the H_2 norm, in case of an original model with distinct poles, the first order equations found already are in the form of a total degree Gröbner basis, so *no Gröbner basis construction at all* is required in the application at hand.

The idea is first to construct a *commutative matrix solution* for a system of polynomial equations in Gröbner basis form.

Definition 4.1 *Let N be a positive integer. Let $f \in \mathbf{C}[x_1, \ldots, x_N]$ be a polynomial in the variables $x_1, \ldots, x_N$. Let M be a positive integer and consider an $N-$tuple $(A_1, A_2, \ldots, A_N)$ of square $M \times M$ matrices that commute with each other, i.e. $A_i A_j = A_j A_i$ for each pair $(i, j), i = 1, \ldots, N,\ j = 1, \ldots, N$. Then $(A_1, A_2, \ldots, A_N)$ will be called a commutative matrix solution of the polynomial equation $f = 0$ if $f(A_1, \ldots, A_N) = 0_M$, where the symbol 0_M denotes the $M \times M$ zero matrix.*

In the following, an $M \times M$ zero matrix will often be denoted by the symbol 0, as is usual, instead of the symbol 0_M. The size of the zero matrix should then be clear from the context. An $N-$tuple of $M \times M$ matrices $(A_1, \ldots, A_N)$ will be called a commutative matrix solution of a system of polynomial equations in N unknowns $x_1, \ldots, x_N$, if it is a commutative matrix solution for each of the polynomials in the system.

From a commutative matrix solution a scalar solution can be obtained by considering any common eigenvector of the matrices. The corresponding eigenvalues form an $N-$tuple which is in fact a scalar solution of the system of polynomial equations involved. The commutative matrix solution that will be constructed here for the case of ideals with zero dimensional variety, has the property that ALL (scalar) solutions can be obtained in this way.

It will first be explained how such a commutative matrix solution can be constructed. Then it will be shown how the (scalar) solutions of the system of polynomial equations can be derived from the matrix solution by eigenvalue-eigenvector calculations. If $\mathcal{F}$ is a field containing all the coefficients of the polynomial equations then all the entries of the matrix solution that will be constructed will be contained in $\mathcal{F}$; in other words, only additions, subtractions, multiplications and divisions are required to obtain a matrix solution.

We start with two results which hold for an arbitrary polynomial ideal. For these results to hold, the ideal does *not* have to have the property that the number of

common zeros of the polynomials in the ideal is finite. Let $\mathbf{C}[x_1, \ldots, x_N]$ denote the ring of polynomials with complex coefficients. The two results consist of a number of observations concerning the operation 'multiplication by x_i modulo the ideal', for $i \in \{1, \ldots, N\}$. Composition of a pair of mappings $X,\ Y$ will be denoted (as usual) by $X \circ Y$.

Theorem 4.1 *Let N be a positive integer. Let $I \subset \mathbf{C}[x_1, \ldots, x_N]$ be an ideal and let $\mathcal{V} := \mathbf{C}[x_1, \ldots, x_N]/I$ denote the corresponding residue class ring. Let $i \in \{1, \ldots, N\}$ be fixed. Let $f_1, f_2 \in \mathbf{C}[x_1, \ldots, x_N]$. If f_1 and f_2 are equal modulo I, then $x_i f_1$ and $x_i f_2$ are equal modulo I. The mapping $X_i : \mathcal{V} \to \mathcal{V}, f + I \mapsto x_i f + I$, is a linear endomorphism. For $i, j \in \{1, \ldots, N\}$ arbitrary, $X_i \circ X_j = X_j \circ X_i$ i.e. the linear mappings X_i and X_j commute. The mapping $X_i \circ X_j$ is the mapping given by $f + I \mapsto x_i x_j f + I$.*

Proof. Let $f_1, f_2 \in \mathbf{C}[x_1, \ldots, x_N]$. If f_1 and f_2 are equal modulo I, then $f_1 - f_2 \in I$ and therefore, because I is an ideal, $x_i(f_1 - f_2) \in I$ for any $i \in \{1, \ldots, N\}$. So $x_i f_1$ is equal to $x_i f_2$ modulo I. This implies that the mapping X_i is well-defined.

In order to show that the mapping X_i is a linear endomorphism, we have to verify that X_i is *homogeneous (of order one)* and *additive*. The homogeneity can be seen as follows: Let λ be an arbitrary nonzero complex number and let $f \in \mathbf{C}[x_1, \ldots, x_N]$. Then $X_i(\lambda.(f + I)) = X_i(\lambda.f + I) = x_i.\lambda.f + I = \lambda.x_i.f + \lambda.I = \lambda(x_i.f + I) = \lambda X_i(f + I)$. The additivity can be seen as follows: Let $f_1, f_2 \in \mathbf{C}[x_1, \ldots, x_N]$. Then $X_i((f_1+I)+(f_2+I)) = X_i(f_1+f_2+I) = x_i.(f_1+f_2)+I = (x_i.f_1+I)+(x_i.f_2+I) = X_i(f_1 + I) + X_i(f_2 + I)$.

For arbitrary i and j, both positive integers less than or equal to N, the composition $X_i \circ X_j$ of endomorphisms X_i and X_j maps an element $f+I$ of $\mathcal{V}$ to $X_i(X_j(f+I)) = X_i(x_j f+I) = x_i x_j f+I$. The commutativity of X_i and X_j follows immediately from this.

□

For any pair of linear endomorphisms $X,\ Y$ let us interpret XY as the composition $X \circ Y$, let us interpret X^0 as the identity and for each positive integer k, let us interpret the power X^k as the $k-$fold composition $X \circ X \circ \ldots \circ X$. Using this interpretation for any $N-$tuple of *commutative* linear endomorphisms $X_1, \ldots, X_N$ and any polynomial $f \in \mathbf{C}[x_1, \ldots, x_N]$, the expression $f(X_1, \ldots, X_N)$ denotes a well-defined linear endomorphism.

Theorem 4.2 *Let $N, I, \mathcal{V}$ and $X_i,\ i = 1, \ldots, N$ be as given in the previous theorem. For any polynomial $f \in \mathbf{C}[x_1, \ldots, x_N]$ the linear mapping $f(X_1, X_2, \ldots, X_N) : \mathcal{V} \mapsto \mathcal{V}$ is well-defined.*

The following two statements are equivalent,

(i) $f \in I$,

(ii) $f(X_1, \ldots, X_N)$ *is equal to the zero mapping* $\mathcal{V} \to \mathcal{V}, f + I \mapsto 0 + I$.

Proof. Because the linear mappings $X_i,\ X_j$ commute for each pair $i, j \in \{1, \ldots, N\}$, the linear mapping $f(X_1, \ldots, X_N)$ is well-defined and equal to the mapping $\mathcal{V} \to \mathcal{V}, g + I \mapsto f.g + I$.

If $f \in I$ then for any polynomial $g \in \mathbf{C}[x_1, \ldots, x_N]$ the product $f.g$ is an element of the ideal I, and therefore $f.g + I = I = 0 + I$. It follows that $f(X_1, \ldots, X_N)$ is equal to the linear mapping $\mathcal{V} \to \mathcal{V}, g + I \mapsto 0 + I$, which is in fact the zero mapping.

On the other hand, if $f(X_1, \ldots, X_N)$ is the zero mapping then $f.g \in I$ for all $g \in \mathbf{C}[x_1, \ldots, x_N]$. By choosing g equal to the constant 1 it follows that $f \in I$.

□

Now we will specialize to systems of polynomial equations with finitely many common solutions. We will make extensive use of the results from section 3 of Chapter 5 of [7], pp. 228-235, especially Propositions 1 and 4 and Theorem 6 of that section.

Let $g_1(x_1, \ldots, x_N) = 0, \ldots, g_{N'}(x_1, \ldots, x_N) = 0$ denote a system of N' polynomial equations with complex coefficients in the N variables $x_1, \ldots, x_N$. The complex vector $(\xi_1, \ldots, \xi_N) \in \mathbf{C}^N$ is a root of the system of polynomial equations if for each $j = 1, \ldots, N'$,

$$g_j(\xi_1, \ldots, \xi_N) = 0.$$

Let $I =< g_1, \ldots, g_{N'} > \subset \mathbf{C}[x_1, \ldots, x_N]$ denote the ideal generated by the polynomials $g_1(x_1, \ldots, x_N), \ldots, g_{N'}(x_1, \ldots, x_N)$.

Suppose that $G = \{g_1, \ldots, g_{N'}\}$ is in fact a Gröbner basis for I, with respect to some fixed monomial ordering. Similarly to what was noted in the previous section for the special case of diagonal-quadratic systems of polynomial equations, the following can be said for this more general case. Each polynomial $f \in \mathbf{C}[x_1, \ldots, x_N]$ is congruent modulo I to a polynomial r with leading term (with respect to the monomial ordering) that cannot be reduced by any of the leading terms of the polynomials in the Gröbner basis; for each f the associated polynomial r is unique; it will be denoted by $\overline{f}^G$. The set V of all polynomials r obtained in this way forms a finite dimensional vector space if and only if the number of roots of the system of polynomial equations is finite. If this set is indeed a finite dimensional vector space, then it has a basis consisting of monomials, namely all monomials that cannot be reduced by any of the leading terms of the polynomials in the Gröbner basis. Given the monomial ordering it is a straightforward task to list these monomials ([7]). Let this basis be denoted by B. The mapping $V \to \mathcal{V}, r \mapsto r + I$, is a linear bijection of vector spaces. In case $\mathcal{V}$ is finite dimensional, let $\mathcal{B}$ denote the basis of $\mathcal{V}$ obtained as the image of B under this mapping. Let D denote the dimension of $\mathcal{V}$.

For each $i \in \{1, \ldots, N\}$ let A_{X_i} denote the $D \times D-$matrix of the endomorphism X_i with respect to the basis $\mathcal{B}$.

Using this set-up it we easily obtain the following fundamental result.

Theorem 4.3 *Let a monomial ordering be fixed and let G be a Gröbner basis of the ideal I. Let the associated linear space $\mathcal{V}$ be finite dimensional with dimension D. Let $f \in \mathbf{C}[x_1, \ldots, x_N]$ be given. Let the mappings $X_i,\ i = 1, \ldots, N$ and $f(X_1, X_2, \ldots, X_N)$ be as given in the previous theorems.*

The matrix of the linear mapping $f(X_1, X_2, \ldots, X_N) : \mathcal{V} \to \mathcal{V}$ with respect to the basis of monomials $\mathcal{B}$ of $\mathcal{V}$ is equal to $f(A_{X_1}, A_{X_2}, \ldots, A_{X_N})$.

The following two statements are equivalent,

(i) $f \in I$,

(ii) $f(A_{X_1}, A_{X_2}, \ldots, A_{X_N}) = 0$, *i.e. this matrix is the $D \times D$ zero matrix.*

Proof. If $X,\ Y$ are arbitrary linear endomorphisms of $\mathcal{V}$ with corresponding matrices $A_X,\ A_Y$ with respect to the basis $\mathcal{B}$ then, of course, $A_X A_Y$ is the matrix of the composition $X \circ Y$, the matrix A_X^k corresponds to the $k-$fold composition $X \circ X \circ \ldots \circ X$, and $A_X + A_Y$ is the matrix corresponding to the mapping $X + Y$. Using this repeatedly it is obvious that the matrix $f(A_{X_1}, \ldots, A_{X_N})$ is the matrix with respect to the basis $\mathcal{B}$ that corresponds to the linear endomorphism $f(X_1, \ldots, X_N)$.

It is a basic fact from linear algebra that a linear mapping of a vector space into itself (i.e. an endomorphism) is the zero mapping, if and only if the matrix with respect to a basis of the vector space is zero. Therefore the equivlence of (i) and (ii) in the statement of the theorem follows directly from the previous theorem.

□

This theorem tells us that the $N-$tuple of matrices $(A_{X_1}, \ldots, A_{X_N})$ is in fact a commutative matrix solution of any system of polynomial equations that generates I.

The entries of the $k-$th column of the matrix A_{X_i} are in fact obtained as follows. Let the $k-$th element of the basis B of monomials be denoted by b_k. The monomial $x_i b_k$ is either itself in the basis B or otherwise $\overline{x_i b_k}^G \neq x_i b_k$. In both cases $\overline{x_i b_k}^G$ can be written as a unique linear combination of the elements of B. The coefficients of the linear combination are the entries of the $k-$th column of the matrix A_{X_i}. If $x_i b_k$ is itself in the basis B then the $k-$th column of the matrix A_{X_i} is in fact a standard basis vector.

In the case $N = 1$ then there exists a unique monic polynomial g such that I is generated by g. In that case the matrix A_{X_1} is in fact a so-called a *companion*

matrix of g (cf. e.g. [19],p. 68). (Also, and in fact most often, the transpose of this matrix is called the companion matrix of g).

Now suppose that the vector v is a *common* eigenvector of the matrices $A_{X_1}, \ldots, A_{X_N}$ with corresponding eigenvalues $\xi_1, \xi_2, \ldots, \xi_N$, respectively, i.e. for each $i \in \{1, \ldots, N\}$ the equality $A_{X_i} v = \xi_i v$ holds and $v \neq 0$. Then for each $f \in I$ one has

$$0 = f(A_{X_1}, \ldots, A_{X_N})v = f(\xi_1, \ldots, \xi_N)v$$

and therefore $f(\xi_1, \ldots, \xi_N) = 0$. It follows that $(x_1, \ldots, x_N) = (\xi_1, \ldots, \xi_N)$ is a root of any system of polynomial equations that generates the ideal I.

The following fundamental result states that in fact *each* of the finite number of roots is obtained in this way.

Theorem 4.4 *Let N be a positive integer and let I be an ideal in the ring $\mathbf{C}[x_1, \ldots, x_N]$ such that the corresponding set $Z \subset \mathbf{C}^N$ of common zeros of all the polynomials in I is finite. Let X_i, $i = 1, \ldots, N$ be as defined above. Then for each vector $\xi = (\xi_1, \ldots, \xi_N)' \in Z$ there exists a polynomial $w \in \mathbf{C}[x_1, \ldots, x_N]$, $w \notin I$, with the property that for each $i = 1, \ldots, N$, the following equality holds:*

$$X_i(w + I) = \xi_i . w + I,$$

i.e. w is a common eigenvector of the mappings $X_1, X_2, \ldots, X_N$, with corresponding eigenvalues $\xi_1, \ldots, \xi_N$, respectively.

Proof If Z is empty then there is nothing to prove. Therefore let us assume that Z is non-empty. Let the positive integer $\bar{H}$ denote the number of different roots of the system of polynomial equations and let $\xi(h) := (\xi_1(h), \ldots, \xi_N(h)), h = 1, \ldots, \bar{H}$ denote these $\bar{H}$ different roots. Let $H := \{1, \ldots, \bar{H}\}$ denote the set of all positive integers less than or equal to $\bar{H}$. Furthermore for each $i = 1, \ldots, N$ let $\xi_i(H) := \{\xi_i(h) : \ h \in H\}$.

Let $h_0 \in H$ be arbitrary and consider the root $\xi(h_0) = (\xi_1(h_0), \ldots, \xi_N(h_0))$. Let $u \in \mathbf{C}[x_1, \ldots, x_N]$ be a polynomial which is zero in each of the points of the set $Z \setminus \{\xi(h_0)\}$ and nonzero in the point $\xi(h_0)$, i.e. $u(\xi_1(h_0), \ldots, \xi_N(h_0)) \neq 0$. Such a polynomial u exists and can in fact be constructed as follows. For each $i = 1, \ldots, N$ let u_i be the univariate polynomial given by

$$u_i(x_i) := \prod_{a \in \xi_i(H) \setminus \{\xi_i(h_0)\}} (x_i - a).$$

Then clearly $u_i(\xi_i(h_0)) \neq 0$. Form the product $u(x_1, \ldots, x_N) = \prod_{i=1}^{N} u_i(x_i)$. Then clearly $u(\xi(h_0)) \neq 0$. But for each $h \in H \setminus \{h_0\}$ there will be at least one value of

$i \in \{1, \ldots, N\}$ such that $\xi_i(h) \neq \xi_i(h_0)$ and so $u_i(\xi_i(h)) = 0$ which in turn implies that $u(\xi(h)) = 0$. Therefore this polynomial u indeed has the required properties.

Now note that for each $i \in \{1, \ldots, N\}$ the polynomial $(x_i - \xi_i(h_0))u(x_1, x_2, \ldots, x_N)$ is zero in each of the points of Z and therefore according to Hilbert's Nullstellensatz (see e.g [23], p.164 or [7], p.172) there exists a positive integer q_i such that

$$(x_i - \xi_i(h_0))^{q_i} u(x_1, x_2, \ldots, x_N)^{q_i} \in I.$$

Take q to be an integer upper bound for some fixed choice of such positive integers $q_1, q_2, \ldots, q_N$. Then for each $i = 1, \ldots, N$ one has

$$(x_i - \xi_i(h_0))^{q} u(x_1, x_2, \ldots, x_N)^{q} \in I.$$

Let us now recursively define a multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in \{0, 1, \ldots, q-1\}^N$ as follows.

Note that $u(\xi(h_0)) \neq 0$, so $u(\xi(h_0))^q \neq 0$, and therefore $u^q \notin I$. Therefore there must be a maximal nonnegative integer $\alpha_1 \leq q - 1$ such that $(x_1 - \xi_1(h_0))^{\alpha_1} u^q \notin I$. This defines α_1 uniquely.

If $N = 1$ then α is well-defined. Now let $N > 1$, let $i \in \{2, \ldots, N\}$ and suppose α_k has been defined for all positive integers less than i. Note that $\left(\prod_{k=1}^{i-1}(x_k - \xi_k(h_0))^{\alpha_k}\right) u^q \notin I$, while $(x_i - \xi_i(h_0))^q \left(\prod_{k=1}^{i-1}(x_k - \xi_k(h_0))^{\alpha_k}\right) u^q \in I$. Therefore there exists a maximal positive integer value $\alpha_i \leq q-1$ such that $(x_i - \xi_i(h_0))^{\alpha_i} \left(\prod_{k=1}^{i-1}(x_k - \xi_k(h_0))^{\alpha_k}\right) u^q \notin I$. This defines α completely.

Now consider the polynomial

$$w(x_1, \ldots, x_N) := \left(\prod_{k=1}^{N}(x_k - \xi_k(h_0))^{\alpha_k}\right) u^q \notin I.$$

By construction for each $i \in \{1, \ldots, N\}$ one has

$$(x_i - \xi_i(h_0))w(x_1, \ldots, x_N) \in I.$$

It follows that $w + I \neq I$ and that for each $i \in \{1, \ldots, N\}$, the equality $x_i w(x_1, \ldots, x_N) + I = \xi_i(h_0)w(x_1, \ldots, x_N) + I$ holds and therefore $X_i(w + I) = x_i w(x_1, \ldots, x_N) + I = \xi_i(h_0)w(x_1, \ldots, x_N) + I$. Therefore for each $i \in \{1, \ldots, N\}$, the residue class $w + I$ is an eigenvector of the linear endomorphism X_i with corresponding eigenvalue $\xi_i(h_0)$. □

From this theorem we have the following important corollary.

Corollary 4.1 *Let N, I and Z be as given in the previous theorem. Let X_i, $i = 1, \ldots, N$ be as defined above. Let a monomial ordering be given and let G be a Gröbner basis of I with respect to this monomial ordering. Let B denote the basis of all monomials in $\mathbf{C}[x_1, \ldots, x_N]$ that are not included in the ideal $< LT(G) >$ generated by the leading terms of the elements of G and let $\mathcal{B}$ denote the corresponding basis of the residue class ring $\mathbf{C}[x_1, \ldots, x_N]/I$, as before. Let $A_{X_1}, \ldots, A_{X_N}$ denote the matrices of the linear endomorphisms $X_1, \ldots, X_N$, respectively, with respect to the basis $\mathcal{B}$. Then the following two statements are equivalent.*

(i) $\xi = (\xi_1, \ldots, \xi_N)' \in Z$.

(ii) There exists a common eigenvector $v \in \mathbf{C}^N \backslash \{0\}$ of the (commutative) matrices $A_{X_1}, \ldots, A_{X_N}$ with corresponding eigenvalues $\xi_1, \ldots, \xi_N$ respectively, i.e. there exists a nonzero vector v with the property

$$A_{X_i} v = \xi_i v, \ i = 1, \ldots, N.$$

Proof. $(ii) \Rightarrow (i)$. As we have seen before the $N-$tuple of matrices $A_{X_1}, \ldots, A_{X_N}$ is a commutative matrix solution of each polynomial equation $p = 0$, $p \in I$ and if $v \neq 0$ is any common eigenvector of the matrices $A_{X_1}, \ldots, A_{X_N}$ with corresponding eigenvalues $\xi_1, \ldots, \xi_N$, respectively then $0 = p(A_{X_1}, \ldots, A_{X_N})v = p(\xi_1, \ldots, \xi_N)v$, which implies $p(\xi_1, \ldots, \xi_N) = 0$. Therefore the vector $(\xi_1, \ldots, \xi_N)' \in Z$.

$(i) \Rightarrow (ii)$. For each $i \in \{1, \ldots, N\}$ the linear endomorphism X_i has matrix A_{X_i} with respect to the basis $\mathcal{B}$. Therefore the previous theorem implies that if $\xi = (\xi_1, \ldots, \xi_N)' \in Z$, then there exists a common eigenvector $v \neq 0$ of $A_{X_1}, \ldots, A_{X_N}$ with the property $A_{X_i} v = \xi_i v$, $i = 1, \ldots, N$. □

Various alternatives arise as to how to exploit the theory presented here to solve a system of polynomial equations, starting with a Gröbner basis. The commutative matrix solution presented can be calculated in symbolic form if the original system of equations is in symbolic form and it can be calculated exactly in numerical form if the coefficients of the original system of polynomials are given numerically. From the commutative matrix solution the roots of the system of polynomial equations can be obtained either by exact algebraic methods or by numerical methods that involve round-off errors.

The exact algebraic approach will not be worked out here. We just want to note that methods of isolation of complex zeros in the complex plane can be useful in such an approach. It is possible to isolate the complex zeros of the characteristic polynomial of a matrix, e.g. by the algebraic methods used to determine the inertia of a matrix. (cf. e.g. [13],[21], [22]). Alternative methods are described in [6]; in any case most of the methods for complex root isolation seem to be based somehow on the principle of the argument. To isolate real zeros one can use the Sturm chain method. With these techniques one can in principle isolate the eigenvalues of a matrix with arbitrary precision. We hope to return to this topic in future research.

A (nonexact) numerical approach can be based on numerical calculation of the eigenvalues and eigenvectors of the matrices involved. Note that the eigenvectors can be used to find out which of the eigenvalues of the various matrices should be matched. Therefore in the eigenvector calculation one can possibly allow a slightly lower level of precision generally speaking, in distinction to the eigenvalues which need to be calculated with high precision in order to get a precise answer. As it is well-known that eigenvector calculation is numerically a difficult problem, it can be important to make this distinction.

In the examples section this approach will be applied to the H_2-model order reduction problem.

In fact one can avoid the eigenvector calculation altogether by calculating the eigenvalues not only of $A_{X_1}, A_{X_2}, \ldots, A_{X_N}$ but also of the sums $A_{X_1} + \epsilon A_{X_2}, A_{X_2} + \epsilon A_{X_3}, \ldots, A_{X_{N-1}} + \epsilon A_{X_N}$. If $(\xi_1, \xi_2, \ldots, \xi_N)$ is a solution of the original system of polynomial equations, then $\xi_1, \ldots, \xi_N$ are eigenvalues of $A_{X_1}, A_{X_2}, \ldots, A_{X_N}$ and $\xi_1 + \epsilon\xi_2, \xi_2 + \epsilon\xi_3, \ldots, \xi_{N-1} + \epsilon\xi_N$ are eigenvalues of $A_{X_1} + \epsilon A_{X_2}, A_{X_2} + \epsilon A_{X_3}, \ldots, A_{X_{N-1}} + \epsilon A_{X_N}$, respectively. For sufficiently small positive value of ϵ this allows us to find out which eigenvalues of the matrices correspond to the same eigenvector (without explicitly calculating the eigenvector.) We hope to return to this in a separate study.

The possibility of using a mixture of exact and symbolic calculations with numerical calculations is very promising for obtaining practically useful results. The matrices involved will tend to become huge (in terms of numbers of entries) if the number of variables involved grows; however eigenvalue calculation can be done numerically for quite big matrices. In section 6 matrices with several hundreds of rows and columns are used. One can expect that usage of more refined numerical techniques will make it possible to push the limits quite a bit further.

Remarks.

(i) Note that in order to use this method all that is required is a Gröbner basis with respect to *some* monomial ordering. Therefore it is important to do more research into the possibilities of choosing monomial orderings. Especially if a given set of polynomials is already a Gröbner basis with respect to some monomial ordering it would be very important to find out about this in a systematic way. These kind of questions appear to be related to the results of [8] which allow one to find a Gröbner basis with respect to one monomial ordering from a Gröbner basis with respect to another monomial ordering in case of an ideal with a finite set of common zeros. Further investigations along these lines could possibly lead to useful results.

(ii) One potential advantage of the usage of commutative matrix solutions is the following. Let $f \in \mathbf{C}[x_1, \ldots, x_N]$ and let F be the corresponding linear endomorphism of $\mathbf{C}[x_1, \ldots, x_N]/I$ defined by $g + I \mapsto f.g + I$. If the number of common zeros of the polynomials in I is finite, and we have a basis $\mathcal{B}$ of $\mathbf{C}[x_1, \ldots, x_N]/I$ as before, then we can represent F with respect

to this basis by a matrix A_F. It is now straightforward to see that $A_F = f(A_{X_1}, A_{X_2}, \ldots, A_{X_N})$. More generally if $f = \frac{f_n}{f_d},\ f_n, f_d \in \mathbf{C}[x_1, \ldots, x_N]$ and $f_d(\xi) \neq 0$ for each common zero ξ of the polynomials in I, then F and A_F are again well-defined and $A_F = f_n(A_{X_1}, \ldots, A_{X_N}).(f_d(A_{X_1}, \ldots, A_{X_N}))^{-1}$. For example in optimization problems in which the criterion function f, say, is a rational function this can be used to obtain the matrix A_F which has as its eigenvalues the values of f on the set of common zeros of the first order equations, if these are polynomial and the set of (real and complex) common zeros is finite. As is well-known the values that a function takes on its set of critical points are called the critical values. The matrix A_F could be called a *critical value matrix* and its characteristic polynomial a *critical value polynomial*. This is related to Theorem 9 and the subsequent Remark 10 in [12] concerning the existence and usage of a univariate polynomial which has the critical values of the criterion function as its zeros.

5 Model order reduction by one in H_2

In this section we combine and apply the results of the previous sections to the problem of model order reduction by one in H_2 for real stable SISO systems with distinct poles. First let us set up the problem. In fact there are several equivalent formulations. One formulation which is closest to the form of the first order conditions that we use in this paper is as follows.(For other formulations refer to the literature, e.g. [12])

Let N be an integer larger than one, let $e(s)$ be a real polynomial of degree $N-1$ and $d(s)$ a real polynomial of degree N with N distinct zeros, all lying in the open left half plane in $\mathbf{C}$ and suppose e and d are coprime, i.e. e and d have no common zeros. Consider the rational function $\frac{e(s)}{d(s)}$. It is an element of the Hardy space H_2 of square summable functions on the imaginary axis which are analytic on the open right halfplane and satisfy a certain continuity requirement on the imaginary axis(cf. [18]). In this paper we work with the subspace of real rational functions in H_2. This subspace consists of all strictly proper real rational functions which have the property that all the poles lie in the open left half plane. The space H_2 is in fact a Hilbert space with corresponding norm $\|.\|_2$ of a function $t \in H_2$ given by

$$\|t\|_2^2 = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} |t(i\omega)|^2 d\omega$$

Consider the differentiable manifold ΣS_{N-1} of all real rational functions $\frac{b(s)}{a(s)}$ in H_2 such that $b(s)$ and $a(s)$ are coprime, the coefficients of $a(s)$ and $b(s)$ are real and $a(s)$ is a Hurwitz polynomial of degree $N-1$. For more information about the structure of this differentiable manifold see for example [5] and [16] and the references given there. The model order reduction problem that we treat in this paper can now be

formulated as the following optimization problem:

$$\min_{\frac{b(s)}{a(s)} \in \Sigma S_{N-1}} \left\| \frac{e(s)}{d(s)} - \frac{b(s)}{a(s)} \right\|_2 .$$

Remark. It is well-known that the distance squared $\left\| \frac{e(s)}{d(s)} - \frac{b(s)}{a(s)} \right\|_2^2$ is in fact a rational function of the coefficients of the numerator and denominator polynomials (see the literature, e.g. [14]; in order to obtain explicit rational function formulas one could use the methods proposed in [17])

In order to facilitate the statement of the following theorem let us define the the set $\Xi_{\frac{e}{d}}$ as follows. Let $\frac{e}{d} \in \Sigma S_N$ have N distinct poles $\delta_1, \ldots, \delta_N \in \mathbf{C}$. Let the matrix M be as given in equation (14) and let Ξ denote the set of solutions in $\mathbf{C}^N \setminus \{0\}$ of the equation (13). The diagonal quadratic system of equations (13) is shown to form a total degree Gröbner basis in Theorem 3.1 and in Lemma 3.1 a basis of 2^N monomials of the corresponding vector space S, of monomials outside the ideal generated by the leading terms of all polynomials in the ideal corresponding to the diagonal quadratic equations, is presented. Let this basis be denoted by B. Then Corollary 4.1 can be applied to (13) using the basis of monomials B. The implication is that in this case the set Ξ just defined is equal to the set Z of that Corollary except for the zero vector:

$$\Xi = Z \setminus \{0\}$$

It follows that Ξ contains at most $2^N - 1$ elements, each of which is a vector of N entries that can be found as the eigenvalues corresponding to any common eigenvector of the matrices $A_{X_1}, \ldots, A_{X_N}$ from Corollary 4.1. We therefore have the following theorem

Theorem 5.1 *Let $\frac{e}{d} \in \Sigma S_N$ have N distinct poles $\delta_1, \ldots, \delta_N \in \mathbf{C}$.*

(i) The number of critical points of the criterion function $f : \Sigma S_{N-1} \to [0, \infty)$, $\frac{b}{a} \mapsto \left\| \frac{e}{d} - \frac{b}{a} \right\|_2^2$ is finite and not greater than $2^N - 1$.

(ii) If the rational function $\frac{b}{a} \in \Sigma S_{N-1}$ is a critical point of f then there exists a number q_0 and a vector $\xi \in \Xi \subset \mathbf{C}^N \setminus \{0\}$ such that $q_0 a(-\delta_i) = \xi_i$, $i = 1, \ldots, N$. For given q_0 and ξ the polynomial a is uniquely determined by this linear system of equations and b is uniquely determined by equation (4).

Of course the solutions that will be found for the first order equations will in general not all correspond to rational functions $\frac{b}{a} \in \Sigma S_{N-1}$: it is certainly possible that

some will not correspond to real systems; some may correspond to real but unstable systems. Note that if a and b are *not* coprime and satisfy the first order equation $e(s)a(s) - b(s)d(s) = a(-s)^2 q_0$ then $a(s)$ and $b(s)$ must have a factor in common with $a(-s)$ and therefore $a(s)$ cannot have all its zeros in the open left half plane. This implies that if $a(s)$ is a Hurwitz polynomial and a and b form a solution of the first order equations, then $a(s)$ and $b(s)$ are coprime.

Further investigations could possibly reveal more about the number of critical points of the H_2-criterion function for systems with various characteristics. It is important to note that the criterion takes on a global minimum over the relevant manifold of systems and that at that point the criterion function has a critical point (cf. [1]).

An algorithm to obtain all the critical points of the criterion function of H_2 model reduction by one can now be constructed as follows.

1. Construct the matrix M

2. Construct the matrices $A_{X_1}, \ldots, A_{X_N}$.

3. Calculate the eigenvalues of these matrices that correspond to a common eigenvector of all these matrices. The result is a vector $\xi \in \mathbf{C}^N$. All nonzero vectors ξ obtained in this way form the (finite) set Ξ.

4. For each element of Ξ solve the equations (13) and (4) for a, b and select those a, b that correspond to transfer functions $\frac{b}{a} \in \Sigma S_{N-1}$.

The global approximant is found by selecting from this finite set the transfer function that minimizes the criterion function.

Note that 1) and 2) can be done purely symbolically. (In the Appendix this is partly worked out in a specific case). Apart from considerations of memory storage and perhaps calculation time, it is not necessary to specify the original system; one can present it symbolically by its poles and the (non-zero) values of the numerator polynomial in these poles.

If the original system is specified numerically then step 3) can be worked out by either constructive algebra algorithms (using e.g. methods of isolation of zeros of polynomials) or by numerical algorithms that admit round-off errors. In the section 6 we present some results obtained by calculations of the latter type.

6 Examples

6.1 General

This section presents two examples on solving the H_2 model reduction problem and discusses several computational issues.

The following is an outline of the algorithm implemented:

1. For the given N-th order transfer function to be reduced, construct the N-by-N matrix M (see equation (13)).
2. For $i = 1, ..., N$, construct the 2^N-by-2^N matrix A_{X_i} from M (see Theorem 4.3 and the paragraph following its proof, and note that the reduction procedure of section 3 is crucial in enabling this to be done).
3. Compute the eigenvalues and eigenvectors of all the A_{X_i}'s. Assume, for simplicity, that each A_{X_i} has a simple Jordan structure. Arrange these eigenvalues and eigenvectors such that the j-th eigenvector of $A_{X_{i_1}}$ corresponds to the j-th eigenvector of $A_{X_{i_2}}$ for all $j = 1, ..., 2^N$ and $i_1, i_2 = 1, ..., N$. Letting $\xi_{i,j}$ denote the j-th eigenvalue of A_{X_i}, form the N-tuples $(\xi_{1,j}, \ldots, \xi_{N,j})$ for $j = 1, ..., 2^N$. Now each of these N-tuples contains the eigenvalues that correspond to one of the common eigenvectors of the set $\{A_{X_i}\}$. Our current implementation of this step uses numerical methods, so there are potential problems which can arise if eigenvalues and/or eigenvectors are repeated, or nearly so. We have not attempted to cope with all such eventualities.
4. Solve for $\tilde{a}_i$, using equation (10), by making the association

   $$[\tilde{a}(-\delta_1), \ldots, \tilde{a}(-\delta_N)] = [\xi_{1,j}, \ldots, \xi_{N,j}].$$

   Normalise the coefficients such that $a_{N-1} = 1$ to obtain a_i. Eliminate those polynomials $a(s) = s^{N-1} + a_{N-2}s^{N-2} + \ldots + a_0$ which are not admissible pole polynomials of an approximating system, because they are not real Hurwitz.
5. For each admissible pole polynomial $a(s)$, obtain the zero polynomial $b(s)$ from equation (4). In practice the equation does not hold exactly, so a least-squares solution is found.

All the above steps except that of computing eigenvalues and eigenvectors can in principle be performed symbolically. Two different implementations have been attempted and they differ only in whether step 2 is performed symbolically or numerically; note that steps 3 and 4 are done numerically here. For the symbolic implementation of step 2, the A_{X_i}'s are computed from a symbolic definition of $M = [m_{jk}]$ using computer algebra software[1] and the resulting symbolic expressions

[1] In our case, Maple.

for the A_{X_i}'s (see the Appendix) are stored in a file to be read in by numerical software[2] later. This has the advantage that the symbolic computation only has to be performed *once* for a given model order. Unfortunately, the length of these symbolic expressions soon becomes very large as the model order increases; the size of the file storing these expressions comes to more than 5 Mbytes for model order equal to 7 and this thus presents a practical limit to this implementation. The alternative is to implement step 2 numerically. In this case, the highest model order that we could reduce is 9, which involves storing 9 512×512 matrices, and we ran into memory problems for model orders higher than this.

There are a number of numerical issues pertaining to this algorithm. Some of these issues are well known, e.g. possible ill-conditioning of Vandermonde matrices and the computation of eigenvalues and eigenvectors. These numerical problems will also cause difficulty in later steps of the algorithm. For example, numerical error may cause us to regard a real polynomial as complex in step 4 and as a result, a true local minimum of the problem may be mistakenly considered as inadmissible. The current implementation of this algorithm does not strive to overcome nor detect these problems. It is also beyond the scope of this paper to give full numerical analysis of the proposed algorithm of this paper. A rudimentary check that we have employed is to examine the least-squares error in step 5; however, this error must be interpreted with care as a small residual error does not necessarily indicate an accurate solution [11]. Moreover, this check will not be able to tell us whether a correct solution has been rejected. We have applied our algorithm to the three third order systems that were investigated in [12] where a symbolic algorithm was used to reduce them to second order systems. In this case, symbolic computation ensures that all stationary points of the problem are computed and we find that the algorithm of this paper is able to find the same sets of critical points as those reported in [12]. This comparison may indicate that our algorithm is likely to return the entire set of stationary points when the model order is small.

6.2 Example 1: An easily reduced system

The system to be reduced is a 9th order transfer function and it is the highest order model that we could reduce thus far. This system has Hankel singular values $9, 8, \ldots, 2, 1$ and its transfer function is

$$\frac{8.4800s^8-2.5942s^7+153.5350s^6+38.8803s^5+599.3205s^4+196.3752s^3+315.3021s^2+6.4558s+9.4478\times10^{-5}}{s^9+2.1179s^8+16.1278s^7+25.6052s^6+62.7884s^5+79.1895s^4+42.6617s^3+32.5279s^2+0.2514s+2.2495\times10^{-6}}$$

The algorithm finds 8 admissible stationary points altogether. The best approximant is

$$\frac{8.4799s^7-2.5955s^6+153.5327s^5+38.8546s^4+599.3039s^3+196.2798s^2+315.2701s+6.4351}{s^8+2.1176s^7+16.1275s^6+25.6013s^5+62.7850s^4+79.1756s^3+42.6527s^2+32.5215s+0.2499}$$

and it gives H_2 model reduction error of 0.0344. Note that the coefficients of this

[2] In our case, Matlab.

approximant are very similar to those of the original transfer function and this can be accounted for as follows: the original transfer function has a pole at -8.9582×10^{-6} and a zero at -1.4645×10^{-5}. The model reduction algorithm appears to have removed this very closely spaced pole-zero pair and to have left the other poles and zeros nearly unchanged. The other seven approximants give errors of 0.8703, 0.8707, 1.6463, 1.6466, 1.6536, 1.6538 and 1.6650. Provided that all the stationary points of this optimisation problem have been computed, then the solution that gives the minimum error is in fact the global minimum of the problem. The other stationary points may correspond to local minima, saddle points or even local maxima.

6.3 Example 2: A relaxation system

The system to be reduced is taken from p.162 of [24] and is given by

$$G(s) \quad = \quad \sum_{j=1}^{n} \frac{\alpha^{2j}}{s + \alpha^{2j}} \quad \text{with } \alpha > 0. \tag{16}$$

It is shown in [24] that all the Hankel singular values of this system tend to $\frac{1}{2}$ as $\alpha \to \infty$. On the other hand, when $\alpha \approx 1$ and $n > 1$, the system is close to non-minimality as $\alpha = 1$ gives rise to a first order system. Our algorithm has numerical difficulty when α is chosen either too large or too close to 1. In both cases, the Vandermonde matrix becomes ill-conditioned: the rows contain entries of drastically different magnitude in the first case and the poles are too close to each other in the second.

Since the poles of this system are all real, this gives rise to a real M matrix and in turn real A_{X_i}'s. Due to the form of Gröbner basis defined by M, zero is always an eigenvalue of A_{X_i} (independent of whether M is real or complex). Since the dimension of A_{X_i} is 2^N — an even number — and A_{X_i} is real, A_{X_i} must have at least one other non-zero real eigenvalue. For α close to zero or unity, we find this real eigenvalue is approximately zero and the eigenvectors corresponding to this eigenvalue and zero are almost parallel to each other. This gives rise to difficulty in matching the eigenvectors.

For model order $n = 5$, our algorithm succeeded in finding an approximant for systems with α in the interval $[0.38, 0.79]$ but failed in the intervals $(0, 0.38)$ and $(0.79, 1)$. For α in the intervals $(0, 0.38)$ and $(0.84, 1)$, our algorithm returns no solution as it either has difficulty in matching the eigenvectors or has rejected the admissible solutions because they are not real Hurwitz. Our algorithm does return a solution for $\alpha \in (0.79, 0.84]$ but a closer examination of the obtained approximant shows that it is not a relaxation system. Since the system in equation (16) is a relaxation system and it is proved in [3] that H_2 approximants of relaxation systems are also relaxation systems, it implies that the solution given by our algorithm for this range of α is unacceptable.

It is also shown in [3] that any stable relaxation system, whose poles all have modulus smaller than $\frac{1}{\sqrt{2}} \approx 0.707$, has only one admissible solution of the first-order optimality conditions. For $\alpha = 0.78$, the largest pole is 0.6084 and there should therefore be only one such solution. For this case our algorithm returns precisely one admissible solution, in accordance with this theory. It has error 0.0334 and transfer function

$$\frac{1.4240s^3+1.0946s^2+0.2371s+0.0134}{s^4+1.1781s^3+0.4457s^2+0.0627s+0.0028}.$$

which can be shown to be a relaxation system.

Appendix: Symbolic computation of A_{X_i}

In this appendix we demonstrate how the matrices A_{X_i} can be derived symbolically given the set of polynomial equations (15) with $\mu = 0$. Recall that the dimension of the matrix A_{X_i} is 2^N by 2^N, so this naturally restricts our illustration to the case where $N = 3$. We have implemented the computation described below using the computer algebra package *Maple* and we have successfully computed these matrices for N from 2 up to 7. The reason why we stopped at $N = 7$ has already been mentioned in section 6. This is because the size of the file that acts as an interface between the numerical and symbolic packages becomes too large. This means that it may be possible to use this procedure to compute A_{X_i} for N larger than 7 but we have not attempted this.

The set of polynomial equations (15) is shown in section 3 to constitute a Gröbner basis with respect to total degree ordering. Let G and I denote respectively this Gröbner basis and the ideal generated by it. It can be verified that any monomial that is irreducible modulo I takes the form $x_1^{\alpha_1}...x_i^{\alpha_i}...x_N^{\alpha_N}$ where $\alpha_i = 0$ or 1 for all $i = 1, ..., N$. As argued in section 3, the set of all these irreducible monomials is a basis for the finite dimensional vector space $\mathbf{C}[x_1, \ldots, x_N]/I$. For $N = 3$, there are eight such irreducible monomials and we have ordered them in a vector as follows:

$$x_B \;=\; \left[\begin{array}{cccccccc} 1 & x_1 & x_2 & x_3 & x_1x_2 & x_2x_3 & x_1x_3 & x_1x_2x_3 \end{array}\right]^T \tag{17}$$

Note that the way in which these monomials are ordered is not important, a different ordering simply corresponds to a permutation of rows and columns of A_{X_i}. Let the matrix M in equation (15) be parameterised by $[m_{ij}]$ and let $\mu = 0$. With this

monomial ordering, the matrix A_{X_1} is:

$$A_{X_1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & m_{11} & 0 & 0 & m_{21}m_{12} & 0 & m_{31}m_{13} & m_{21}m_{32}m_{13} + m_{31}m_{23}m_{12} \\ 0 & m_{12} & 0 & 0 & m_{22}m_{12} & 0 & m_{32}m_{13} & m_{22}m_{32}m_{13} + m_{32}m_{23}m_{12} \\ 0 & m_{13} & 0 & 0 & m_{23}m_{12} & 0 & m_{33}m_{13} & m_{23}m_{32}m_{13} + m_{33}m_{23}m_{12} \\ 0 & 0 & 1 & 0 & m_{11} & 0 & 0 & m_{31}m_{13} \\ 0 & 0 & 0 & 0 & m_{13} & 0 & m_{12} & m_{22}m_{12} + m_{33}m_{13} \\ 0 & 0 & 0 & 1 & 0 & 0 & m_{11} & m_{21}m_{12} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & m_{11} \end{bmatrix}$$

Before explaining how this matrix may be computed, it may be instructive to point out the following interpretation of this matrix. Let

$$c = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 & c_{12} & c_{23} & c_{13} & c_{123} \end{bmatrix}^T \in \mathbf{C}^8 \tag{18}$$

then the following property holds

$$\overline{x_1 x_B^T c}^G = x_B^T A_{X_1} c. \tag{19}$$

In other words, the normal form of the polynomial $x_1 x_B^T c$ is given by $x_B^T A_{X_1} c$. This also demonstrates how each column of A_{X_1} may be computed: let c be e_j where e_j is the unit vector whose j-th element is 1, then the j-th column of A_{X_1} may be read off from the normal form of $x_1 x_B^T e_j$ modulo I.

For completeness, the expressions for A_{X_2} and A_{X_3} are included below.

$$A_{X_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{21} & 0 & m_{11}m_{21} & m_{31}m_{23} & 0 & m_{11}m_{31}m_{23} + m_{31}m_{13}m_{21} \\ 1 & 0 & m_{22} & 0 & m_{21}m_{12} & m_{32}m_{23} & 0 & m_{21}m_{32}m_{13} + m_{31}m_{23}m_{12} \\ 0 & 0 & m_{23} & 0 & m_{13}m_{21} & m_{33}m_{23} & 0 & m_{13}m_{31}m_{23} + m_{33}m_{13}m_{21} \\ 0 & 1 & 0 & 0 & m_{22} & 0 & 0 & m_{32}m_{23} \\ 0 & 0 & 0 & 1 & 0 & m_{22} & 0 & m_{21}m_{12} \\ 0 & 0 & 0 & 0 & m_{23} & m_{21} & 0 & m_{11}m_{21} + m_{33}m_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & m_{22} \end{bmatrix}$$

$$A_{X_3} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{31} & 0 & m_{21}m_{32} & m_{11}m_{31} & m_{11}m_{21}m_{32} + m_{21}m_{31}m_{12} \\ 0 & 0 & 0 & m_{32} & 0 & m_{22}m_{32} & m_{31}m_{12} & m_{12}m_{21}m_{32} + m_{22}m_{31}m_{12} \\ 1 & 0 & 0 & m_{33} & 0 & m_{32}m_{23} & m_{31}m_{13} & m_{21}m_{32}m_{13} + m_{31}m_{23}m_{12} \\ 0 & 0 & 0 & 0 & 0 & m_{31} & m_{32} & m_{11}m_{31} + m_{22}m_{32} \\ 0 & 0 & 1 & 0 & 0 & m_{33} & 0 & m_{31}m_{13} \\ 0 & 1 & 0 & 0 & 0 & 0 & m_{33} & m_{32}m_{23} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & m_{33} \end{bmatrix}$$

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