# A Design Framework for Continuous-Time Systems Under Sampled-Data Control

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## Abstract

In this paper we give a compact derivation of new formulae for the synthesis of $\mathcal{H}_\infty$, sampled-data controllers for continuous-time, linear, time-invariant, generalised plants. The formulae are significantly more simple than those previously published and accommodate a direct feedthrough term from the disturbance inputs to the controlled outputs of the generalised plant. Furthermore, motivated by the $\mathcal{H}_\infty$, loop-shaping procedure of McFarlane and Glover, we propose a performance based framework for sampled-data controller design. A preliminary design example is presented.

## 1 Introduction

For the design and synthesis of useful sampled-data (SD) controllers for continuous-time, generalised plants, a framework which captures both the inter-sample behaviour and the behaviour at the sampling instants is clearly sensible. Given a continuous-time, linear, time-invariant (LTI), generalised plant, such a framework has been developed for $\mathcal{H}_\infty$, SD synthesis using the established technique of continuous lifting [1, 7, 10, 2, 5]. The main purpose of this paper is to introduce a new framework for SD, control-system development. This entails the derivation of new $\mathcal{H}_\infty$, SD synthesis formulae and the definition of a performance based design framework, modelled on the $\mathcal{H}_\infty$, loop-shaping procedure of McFarlane and Glover [6].

### 1.1 Notation

Most of the notation is standard. In addition we denote by $\mathcal{K}^n \equiv \mathcal{L}_2([0,h),\mathbb{R}^n)$, the Hilbert space of functions $f:[0,h)\to\mathbb{R}^n$, square (Lebesgue) integrable over the interval $[0,h)$. A lifted signal is a discrete-time signal that maps an interval of the integers to $\mathcal{K}^n$. $\ell_2(\mathbb{Z}_+,\mathcal{K}^n)$, denotes the Hilbert space of lifted signals, $f:\mathbb{Z}_+\to\mathcal{K}^n$, such that

$$< f, f >_{\ell_2(\mathbb{Z}_+,\mathcal{K}^n)} = \sum_{k=0}^{\infty}\int_0^h f[k]' f[k]dt < \infty.$$

For a stable, continuous-time, LTI operator $G : \mathcal{L}_2(\mathbb{R}_+,\mathbb{R}^m) \to \mathcal{L}_2(\mathbb{R}_+,\mathbb{R}^p)$, $\|G\|_\infty$ is equal to the $\mathcal{L}_2(\mathbb{R}_+)$-induced norm. The infinity norm is defined similarly for stable, discrete-time, LTI operators, be they mapping $\ell_2(\mathbb{Z}_+,\mathbb{R}^m) \to \ell_2(\mathbb{Z}_+,\mathbb{R}^p)$ or $\ell_2(\mathbb{Z}_+,\mathcal{K}^m) \to \ell_2(\mathbb{Z}_+,\mathcal{K}^p)$.

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$M'$ denotes the transpose of $M$. For a bounded linear operator $\mathcal{T}$, $\mathcal{T}^*$ denotes the adjoint operator. The conjugate transpose operator of an LTI operator $G(s)$, is denoted $G^\sim(s) = G(-s)'$.

The lower, fractional transformation $\mathcal{F}_\ell(\bullet,\bullet)$, is

$$\mathcal{F}_\ell(G,K) \stackrel{\Delta}{=} G_{11} + G_{12}K(I-G_{22}K)^{-1}G_{21},$$

where $G$ is partitioned appropriately, and the composition operator $\mathcal{C}_l(\bullet,\bullet)$, is

$$\mathcal{C}_l(\Theta,G) \stackrel{\Delta}{=} \begin{bmatrix} \mathcal{F}_\ell(\Theta,G_{11}) & \Theta_{12}\hat{S}^{-1}G_{12} \\ G_{21}\hat{R}^{-1}\Theta_{21} & G_{22}+G_{21}\Theta_{22}\hat{S}^{-1}G_{12} \end{bmatrix},$$

where $\hat{S} = (I - G_{11}\Theta_{22})$, $\hat{R} = (I - \Theta_{22}G_{11})$, and $\Theta$ is partitioned appropriately.

## 2 Direct SD Synthesis

### 2.1 Continuous Lifting

Although the following results hold in more generality, we concentrate on a lifting that is an isometric isomorphism between the Hilbert spaces $\mathcal{L}_2(\mathbb{R}_+,\mathbb{R}^n)$ and $\ell_2(\mathbb{Z}_+,\mathcal{K}^n)$. For more details see [1].

**Definition 2.1** *The lifting operator $L_h : \mathcal{L}_2(\mathbb{R}_+,\mathbb{R}^n) \to \ell_2(\mathbb{Z}_+,\mathcal{K}^n)$ is defined by*

$$\tilde{f}[k] = (L_h f)[k],\ \ \tilde{f}[k] \stackrel{\Delta}{=} f(kh+t),\ \text{ for } 0\le t<h.$$

**Definition 2.2** *The inverse lifting operator $L_h^{-1} : \ell_2(\mathbb{Z}_+,\mathcal{K}^n) \to \mathcal{L}_2(\mathbb{R}_+,\mathbb{R}^n)$ is defined by*

$$f(t) = (L_h^{-1}\tilde{f})(t),\ \ f(t) \stackrel{\Delta}{=} \tilde{f}[k],\ \text{ for } kh\le t<(k+1)h.$$

Recall the definition of the shift-operator $(\mathscr{U}_\tau x)(t) = x(t-\tau)$. An important property of $L_h$ is that it intertwines $\mathscr{U}_{kh}$ in continuous time and $\mathscr{U}_k$ in discrete time, for $k\in\mathbb{Z}$. That is,

$$L_h\mathscr{U}_{kh} = \mathscr{U}_k L_h \tag{1}$$

**Definition 2.3** *Given any linear system $G : \mathcal{L}_2(\mathbb{R}_+,\mathbb{R}^m) \to \mathcal{L}_2(\mathbb{R}_+,\mathbb{R}^p)$, its lifting is defined to be*

$$\tilde{G} \stackrel{\Delta}{=} L_h G L_h^{-1}.$$

*where $\tilde{G} : \ell_2(\mathbb{Z}_+,\mathcal{K}^m) \to \ell_2(\mathbb{Z}_+,\mathcal{K}^p)$.*

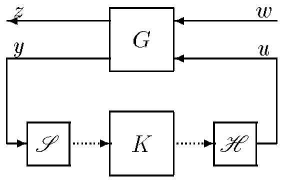


Figure 1: SD Control System

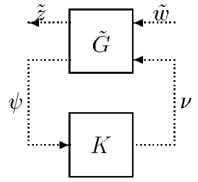


Figure 2: Lifted SD System

Consider the SD, control system shown in Figure 1, where $\mathscr{S}$ and $\mathscr{H}$ are the ideal sample and zero-order hold operators respectively, synchronised with period $h$, the generalised plant $G : \mathcal{L}_2(\mathbb{R}_+, \mathbb{R}^{m_1+m_2}) \to \mathcal{L}_2(\mathbb{R}_+, \mathbb{R}^{p_1+p_2})$ has transfer matrix

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & 0 & 0 \end{array}\right],$$

$K : \ell_2(\mathbb{Z}_+, \mathbb{R}^{p_2}) \to \ell_2(\mathbb{Z}_+, \mathbb{R}^{m_2})$ is a discrete-time, LTI controller, $w(t)$ are the disturbance inputs and $z(t)$ the controlled outputs. Note that $D_{21}$ and $D_{22}$ in the state-space realisation for $G$ are both zero. This is required to ensure boundedness of the ideal sampler [2]. Also consider the discrete-time system with infinite-dimensional input and output spaces, shown in Figure 2, where

$$\tilde{G} = \begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ \tilde{G}_{21} & \tilde{G}_{22} \end{bmatrix} = \begin{bmatrix} L_h G_{11} L_h^{-1} & L_h G_{12} \mathscr{H} \\ \mathscr{S} G_{21} L_h^{-1} & \mathscr{S} G_{22} \mathscr{H} \end{bmatrix},$$

has transfer operator

$$\tilde{G}(z) = \left[\begin{array}{c|cc} A_d & \tilde{B}_1 & B_{2d} \\ \hline \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ C_2 & 0 & 0 \end{array}\right], \tag{2}$$

$$A_d : \mathbb{R}^n \to \mathbb{R}^n, \qquad A_d x = e^{hA} x, \tag{3}$$

$$B_{2d} : \mathbb{R}^{m_2} \to \mathbb{R}^n, \qquad B_{2d} = \int_0^h e^{(h-t)A} dt B_2, \tag{4}$$

$$\tilde{B}_1 : \mathcal{K}^{m_1} \to \mathbb{R}^n, \qquad \tilde{B}_1 \tilde{w} = \int_0^h e^{(h-\tau)A} B_1 \tilde{w}(\tau)\, d\tau, \tag{5}$$

$$\tilde{C}_1 : \mathbb{R}^n \to \mathcal{K}^{p_1}, \qquad (\tilde{C}_1 x)(t) = C_1 e^{tA} x, \tag{6}$$

$$\tilde{D}_{11} : \mathcal{K}^{m_1} \to \mathcal{K}^{p_1}, \qquad (\tilde{D}_{11}\tilde{w})(t) = D_{11}\tilde{w}(t) + \int_0^t C_1 e^{(t-\tau)A} B_1 \tilde{w}(\tau)\, d\tau, \tag{7}$$

$$\tilde{D}_{12} : \mathbb{R}^{m_2} \to \mathcal{K}^{p_1}, \qquad (\tilde{D}_{12}\nu)(t) = D_{12}\nu + \int_0^t C_1 e^{(t-\tau)A}\, d\tau B_2 \nu, \tag{8}$$

the disturbance inputs $\tilde{w}[k] = (L_h w)[k]$, and the controlled outputs $\tilde{z}[k] = (L_h z)[k]$. It is well known, that if the sampling frequency is not pathological, then the SD control system shown in Figure 1 is internally stable iff the discrete-time, LTI system shown in Figure 2 is internally stable [2]. Furthermore, for a given controller $K$ the closed-loop induced norms of the two systems are equal. For details see [1].

The following lemma and theorem characterise a finite-dimensional (in terms of input, output and state spaces) representation of the plant $\tilde{G}$. Define

$$\Theta = \begin{bmatrix} -\tilde{D}_{11} & \gamma^{-1}(I - \gamma^{-2}\tilde{D}_{11}\tilde{D}_{11}^*)^{\frac{1}{2}} \\ \gamma(I - \gamma^{-2}\tilde{D}_{11}^*\tilde{D}_{11})^{\frac{1}{2}} & \gamma^{-2}\tilde{D}_{11}^* \end{bmatrix}.$$

**Lemma 2.4** *[1] Construct the LTI plant*

$$\underline{G} = \mathcal{C}_l(\Theta, \tilde{G}) = \left[\begin{array}{c|cc} \underline{A} & \underline{B}_1 & \underline{B}_2 \\ \hline \underline{C}_1 & 0 & \underline{D}_{12} \\ C_2 & 0 & 0 \end{array}\right] \tag{9}$$

$$\underline{A} = A_d + \gamma^{-2}\tilde{B}_1\tilde{D}_{11}^*\tilde{S}^{-1}\tilde{C}_1 \tag{10}$$

$$\underline{B}_1 = \gamma^{-1}\tilde{B}_1\tilde{R}^{-\frac{1}{2}} \tag{11}$$

$$\underline{B}_2 = \gamma^{-2}\tilde{B}_1\tilde{D}_{11}^*\tilde{S}^{-1}\tilde{D}_{12} + B_{2d} \tag{12}$$

$$\underline{C}_1 = \gamma\tilde{S}^{-\frac{1}{2}}\tilde{C}_1 \tag{13}$$

$$\underline{D}_{12} = \gamma\tilde{S}^{-\frac{1}{2}}\tilde{D}_{12} \tag{14}$$

*where $\tilde{R} = (I - \gamma^{-2}\tilde{D}_{11}^*\tilde{D}_{11})$ and $\tilde{S} = (I - \gamma^{-2}\tilde{D}_{11}\tilde{D}_{11}^*)$, then provided $\|\tilde{D}_{11}\| < \gamma$ and given a discrete-time, LTI controller $K$,*

*(i) $\mathcal{F}_\ell(\tilde{G}, K)$ is internally stable iff $\mathcal{F}_\ell(\underline{G}, K)$ is internally stable;*

*(ii) $\|\mathcal{F}_\ell(\tilde{G}, K)\|_\infty < \gamma$ iff $\|\mathcal{F}_\ell(\underline{G}, K)\|_\infty < \gamma$.*

**Theorem 2.5** *[1] Given the discrete-time, LTI plant $\underline{G}$ with infinite-dimensional input and output spaces defined in (9), obtain the following symmetric factorisations*

$$\underline{B}_1\underline{B}_1^* = T_{\underline{B}_1}^* \begin{bmatrix} \Sigma_{\underline{B}_1} & 0 \\ 0 & 0 \end{bmatrix} T_{\underline{B}_1},$$

*and*

$$\begin{bmatrix} \underline{C}_1^* \\ \underline{D}_{12}^* \end{bmatrix} \begin{bmatrix} \underline{C}_1 & \underline{D}_{12} \end{bmatrix} = T_{\underline{C}_1,\underline{D}_{12}}^* \begin{bmatrix} \Sigma_{\underline{C}_1,\underline{D}_{12}} & 0 \\ 0 & 0 \end{bmatrix} T_{\underline{C}_1,\underline{D}_{12}}.$$

*Define the discrete-time, finite-dimensional, LTI plant*

$$\grave{G} = \left[\begin{array}{c|cc} \underline{A} & \grave{B}_1 & \underline{B}_2 \\ \hline \grave{C}_1 & 0 & \grave{D}_{12} \\ C_2 & 0 & 0 \end{array}\right] \tag{15}$$

*where $\grave{B}_1 = T_{\underline{B}_1}^* \begin{bmatrix} \Sigma_{\underline{B}_1}^{\frac{1}{2}} & 0 \end{bmatrix}'$, and $\begin{bmatrix} \grave{C}_1 & \grave{D}_{12} \end{bmatrix} = \begin{bmatrix} \Sigma_{\underline{C}_1,\underline{D}_{12}}^{\frac{1}{2}} & 0 \end{bmatrix} T_{\underline{C}_1,\underline{D}_{12}}$. Then for a discrete-time, LTI controller $K$*

*(i) $\mathcal{F}_\ell(\underline{G}, K)$ is internally stable iff $\mathcal{F}_\ell(\grave{G}, K)$ is internally stable;*

*(ii) $\|\mathcal{F}_\ell(\underline{G}, K)\|_\infty$ iff $\|\mathcal{F}_\ell(\grave{G}, K)\|_\infty$.*

## 2.2 Formulae for $\grave{G}$

To calculate the state-space matrices in (15), we see from equations (9-15), that it is required to determine finite-dimensional expressions for the following operators:

$$\gamma^{-2}\tilde{B}_1(I-\gamma^{-2}\tilde{D}_{11}^*\tilde{D}_{11})^{-1}\tilde{B}_1^*, \tag{16}$$

$$\gamma^2\begin{bmatrix} \tilde{C}_1^* \\ \tilde{D}_{12}^* \end{bmatrix}(I-\gamma^{-2}\tilde{D}_{11}\tilde{D}_{11}^*)^{-1}\begin{bmatrix} \tilde{C}_1 & \tilde{D}_{12} \end{bmatrix}, \tag{17}$$

$$\gamma^{-2}\tilde{B}_1\tilde{D}_{11}^*(I-\gamma^{-2}\tilde{D}_{11}\tilde{D}_{11}^*)^{-1}\begin{bmatrix} \tilde{C}_1 & \tilde{D}_{12} \end{bmatrix}. \tag{18}$$

In the preceding section it was assumed that $\|\tilde{D}_{11}\| < \gamma$. For this to be a reasonable assumption we need a method for evaluating bounds on $\|\tilde{D}_{11}\|$. Since $\tilde{D}_{11} : \mathcal{K}^{m_1} \rightarrow \mathcal{K}^{p_1}$ is not a compact operator, we cannot simply look for the largest eigen-value of $\tilde{D}_{11}^*\tilde{D}_{11}$ to calculate $\|\tilde{D}_{11}\|$. However, using the principle of loop-shifting we can construct a compact operator $\check{D}_{11} : \mathcal{K}^{m_1} \rightarrow \mathcal{K}^{p_1}$ such that $\|\tilde{D}_{11}\| < \gamma \Leftrightarrow \|\check{D}_{11}\| < \gamma$. In this way we can evaluate bounds on $\|\tilde{D}_{11}\|$ in terms of the eigen-values of $\check{D}_{11}^*\check{D}_{11}$. Such an operator is characterised as follows: The action of $g(t) = (\check{D}_{11}f)(t)$ can be expressed in terms of the system of equations

$$\begin{aligned} \dot{x} &= \check{A}x + \check{B}f, \\ g &= \check{C}x; \quad x(0) = 0, \quad 0 \le t \le h, \end{aligned} \tag{19}$$

where $R = (I - \gamma^{-2}D_{11}^{'}D_{11})$, $S = (I-\gamma^{-2}D_{11}D_{11}^{'})$, $\check{A} = A + \gamma^{-2}B_1R^{-1}D_{11}^{'}C_1$, $\check{B} = \gamma^{-1}B_1R^{-\frac{1}{2}}$, and $\check{C} = \gamma S^{-\frac{1}{2}}C_1$. Note that since $\|D_{11}\| < \gamma$, $R^{-\frac{1}{2}}$ and $S^{-\frac{1}{2}}$ are well defined. The action of the adjoint operator $\hat{f}(t) = (\check{D}_{11}^*g)(t)$ can be expressed in terms of the system of equations

$$\dot{\lambda} = -\check{A}^{'}\lambda - \check{C}^{'}g, \tag{20}$$

$$\hat{f} = \check{B}^{'}\lambda; \quad \lambda(h) = 0, \quad 0 \le t \le h. \tag{21}$$

The following lemma characterises the eigen-values of $\check{D}_{11}^*\check{D}_{11}$. Define the matrix $E$ to be

$$\begin{bmatrix} -A^{'} - \gamma^{-2}C_1^{'}D_{11}R^{-1}B_1^{'} & -C_1^{'}S^{-1}C_1 \\ \gamma^{-2}B_1R^{-1}B_1^{'} & A + \gamma^{-2}B_1R^{-1}D_{11}^{'}C_1 \end{bmatrix}, \tag{22}$$

and

$$Q(t) = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{bmatrix} = e^{tE}. \tag{23}$$

**Lemma 2.6** *Assume $\gamma > 0$. Then, $\gamma^2$ is an eigen-value of $\check{D}_{11}^*\check{D}_{11}$ iff $det[Q_{11}(h)] = 0$.*

The proof is similar to that of Theorem 13.5.1 in Chen and Francis [2].

**Remark 2.7** *As a result of Lemma 2.6, under the assumption that $\|\tilde{D}_{11}\| < \gamma$, $Q_{11}(h)$ is invertible.*

We are now in a position to state the main synthesis result.

**Theorem 2.8** *Given $\gamma > \|\tilde{D}_{11}\|$, calculate the matrix exponential*

$$\hat{Q}(h) = e^{h\hat{E}} = \exp\left\{h\begin{bmatrix} -\hat{A}^{'} & -\hat{C}^{'}\hat{C} \\ \hat{B}\hat{B}^{'} & \hat{A} \end{bmatrix}\right\}, \tag{24}$$

*where*

$$\hat{A} = \bar{A} + \gamma^{-2}\begin{bmatrix} B_1 \\ 0 \end{bmatrix}R^{-1}D_{11}^{'}\begin{bmatrix} C_1 & D_{12} \end{bmatrix}, \tag{25}$$

$$\hat{C}^{'}\hat{C} = \begin{bmatrix} C_1^{'} \\ D_{12}^{'} \end{bmatrix}S^{-1}\begin{bmatrix} C_1 & D_{12} \end{bmatrix}, \tag{26}$$

$$\hat{B}\hat{B}^{'} = \gamma^{-2}\begin{bmatrix} B_1 \\ 0 \end{bmatrix}R^{-1}\begin{bmatrix} B_1^{'} & 0 \end{bmatrix}, \tag{27}$$

$$\bar{A} = \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \tag{28}$$

*and partition $\hat{Q}(h)$ conformably with $\hat{E}$. Then, the state-space matrices of the discrete-time, finite-dimensional, LTI plant $\grave{G}$ defined in equation (15), are given by:*

$$\underline{A} = \begin{bmatrix} I_n & 0 \end{bmatrix}(\hat{Q}_{11}(h)^{'})^{-1}\begin{bmatrix} I_n \\ 0 \end{bmatrix}, \tag{29}$$

$$\underline{B}_2 = \begin{bmatrix} I_n & 0 \end{bmatrix}(\hat{Q}_{11}(h)^{'})^{-1}\begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}, \tag{30}$$

*$\grave{B}_1$ a matrix satisfying*

$$\grave{B}_1\grave{B}_1^{'} = \begin{bmatrix} I_n & 0 \end{bmatrix}\hat{Q}_{21}\hat{Q}_{11}(h)^{-1}\begin{bmatrix} I_n \\ 0 \end{bmatrix}, \tag{31}$$

*and $\grave{C}_1$ and $\grave{D}_{12}$ matrices satisfying*

$$\begin{bmatrix} \grave{C}_1^{'} \\ \grave{D}_{12}^{'} \end{bmatrix}\begin{bmatrix} \grave{C}_1 & \grave{D}_{12} \end{bmatrix} = -\gamma^2\hat{Q}_{11}(h)^{-1}\hat{Q}_{12}(h). \tag{32}$$

**Proof :** From Theorem 2.5, it is required to determine finite-dimensional expressions for the operators given in equations (16), (17) and (18). Consider the first operator $\gamma^{-2}\tilde{B}_1(I-\gamma^{-2}\tilde{D}_{11}^*\tilde{D}_{11})^{-1}\tilde{B}_1^*$. Define

$$y = \gamma^{-1}\tilde{B}_1^*z; \quad z \in \mathbb{R}^n,$$

and

$$u = (I-\gamma^{-2}\tilde{D}_{11}^*\tilde{D}_{11})^{-1}y. \tag{33}$$

Let $r = \gamma^{-1}\tilde{D}_{11}u$, then from (7),

$$\begin{aligned} \dot{x}_1 &= Ax_1 + \gamma^{-1}B_1u; \quad x_1(0) = 0, \\ r &= C_1x_1 + \gamma^{-1}D_{11}u. \end{aligned} \tag{34}$$

Moreover, from (33), $u = \gamma^{-1}\tilde{D}_{11}^*r + \gamma^{-1}\tilde{B}_1^*z$. From (5) and (7), this can be expressed as

$$\begin{aligned} \dot{x}_2 &= -A^{'}x_2 - C_1^{'}r; \quad x_2(h) = z, \\ u &= \gamma^{-1}B_1^{'}x_2 + \gamma^{-1}D_{11}^{'}r. \end{aligned} \tag{35}$$

Note that the zero initial condition response gives $\gamma^{-1}\tilde{D}_{11}^* r$ and the initial condition response gives $\gamma^{-1}\tilde{B}_1^* z$. The systems (34) and (35) combined, form a TPBVP with solution,

$$\begin{bmatrix} x_2(t) \\ x_1(t) \end{bmatrix} = Q(t) \begin{bmatrix} x_2(0) \\ 0 \end{bmatrix}, \tag{36}$$

where $Q(t)$ is defined in equation (23). Now note that from equations (5) and (34)

$$\gamma^{-1}\tilde{B}_1 u = x_1(h). \tag{37}$$

From (36), $x_1(h) = Q_{21}(h)Q_{11}(h)^{-1}z$. Combining this with (37), characterises a finite-dimensional expression for the first operator. Note that $Q_{11}(h)$ is invertible by Remark 2.7 and that by the structure of $\hat{E}$ that $Q_{21}(h)Q_{11}(h)^{-1} = \begin{bmatrix} I_n & 0 \end{bmatrix} \hat{Q}_{21}\hat{Q}_{11}(h)^{-1} \begin{bmatrix} I_n \\ 0 \end{bmatrix}$.

Before continuing, we need to state a result from matrix theory [8]. Let $M_{11}$ and $M_{22}$ both be square, then,

$$\exp\left\{ t \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix} \right\} = \begin{bmatrix} e^{tM_{11}} & \hat{M}_{12} \\ 0 & e^{tM_{22}} \end{bmatrix}, \tag{38}$$

where $\hat{M}_{12} = \int_0^t e^{(t-\tau)M_{11}} M_{12} e^{\tau M_{22}}\, d\tau$, for $t \geq 0$. Now consider the second and third operators

$$\gamma^2 \begin{bmatrix} \tilde{C}_1^* \\ \tilde{D}_{12}^* \end{bmatrix} (I - \gamma^{-2}\tilde{D}_{11}\tilde{D}_{11}^*)^{-1} \begin{bmatrix} \tilde{C}_1 & \tilde{D}_{12} \end{bmatrix}$$

and $\gamma^{-2}\tilde{B}_1\tilde{D}_{11}^*(I - \gamma^{-2}\tilde{D}_{11}\tilde{D}_{11}^*)^{-1} \begin{bmatrix} \tilde{C}_1 & \tilde{D}_{12} \end{bmatrix}$, respectively. Define $f = \begin{bmatrix} \tilde{C}_1 & \tilde{D}_{12} \end{bmatrix} z;\ z \in \mathbb{R}^{n+m_2}$, which by (6), (8) and (38) can be expressed as

$$f = \begin{bmatrix} C_1 & D_{12} \end{bmatrix} e^{t\bar{A}} z,$$

where $\bar{A}$ is defined in equation (28). Also define

$$g = (I - \gamma^{-2}\tilde{D}_{11}D_{11}^*)^{-1} f. \tag{39}$$

Let $w = \gamma^{-1}\tilde{D}_{11}^* g$, then from (7),

$$\begin{aligned} \dot{\xi}_1 &= -\bar{A}'\xi_1 - \begin{bmatrix} C_1' \\ D_{12}' \end{bmatrix} g; \quad \xi_1(h) = 0, \\ w &= \begin{bmatrix} \gamma^{-1}B_1' & 0 \end{bmatrix} \xi_1 + \gamma^{-1}D_{11}' g. \end{aligned} \tag{40}$$

Moreover, from (39), $g = \begin{bmatrix} C_1 & D_{12} \end{bmatrix} e^{t\bar{A}} z + \gamma^{-1}\tilde{D}_{11} w$. Again from (7), this can be expressed as

$$\begin{aligned} \dot{\xi}_2 &= \bar{A}\xi_2 + \begin{bmatrix} \gamma^{-1}B_1 \\ 0 \end{bmatrix} w; \quad \xi_2(0) = z, \\ g &= \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \xi_2 + \gamma^{-1}D_{11} w. \end{aligned} \tag{41}$$

Note that the zero initial condition response gives $\gamma^{-1}\tilde{D}_{11} w$, and the initial condition response gives $\begin{bmatrix} C_1 & D_{12} \end{bmatrix} e^{t\bar{A}} z$. The systems (41) and (40) combined, form a TPBVP with solution

$$\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = \hat{Q}(t) \begin{bmatrix} \xi_1(0) \\ z \end{bmatrix}. \tag{42}$$

Now note that

$$\gamma^2 \begin{bmatrix} \tilde{C}_1^* \\ \tilde{D}_{12}^* \end{bmatrix} g = \gamma^2 \xi_1(0), \tag{43}$$

and

$$\gamma^{-2}\tilde{B}_1\tilde{D}_{11}^* g = \gamma^{-1}\tilde{B}_1 w = \begin{bmatrix} I_n & 0 \end{bmatrix} \{\xi_2(h) - e^{h\bar{A}} z\} \tag{44}$$

Recall the partitioning of $\hat{Q}(h)$. From (42), $\xi_1(0) = -\hat{Q}_{11}(h)^{-1}\hat{Q}_{12}(h)z$. Combining this with (43) characterises a finite-dimensional expression for the second operator. Similarly,

$$\xi_2(h) = \{\hat{Q}_{22}(h) - \hat{Q}_{21}(h)\hat{Q}_{11}(h)^{-1}\hat{Q}_{12}(h)\} z. \tag{45}$$

Define

$$J = \begin{bmatrix} 0 & -I_{n+m_2} \\ I_{n+m_2} & 0 \end{bmatrix}, \tag{46}$$

and note that $-\hat{E}' = J\hat{E}J^{-1}$. So,

$$(\hat{Q}(h)')^{-1} = e^{-h\hat{E}'} = J\hat{Q}(h)J^{-1}.$$

That is, $\hat{Q}(h)' J \hat{Q}(h) J^{-1} = J\hat{Q}(h)J^{-1}\hat{Q}(h)' = I$, which yields the relationship

$$\hat{Q}_{22}(h) - \hat{Q}_{21}(h)\hat{Q}_{11}(h)^{-1}\hat{Q}_{12}(h) = (\hat{Q}_{11}(h)')^{-1}. \tag{47}$$

Combining this with (44) and (45), characterises a finite-dimensional expression for the third operator. From (10),

$$\underline{A} = \begin{bmatrix} I_n & 0 \end{bmatrix} (\hat{Q}_{11}(h)')^{-1} \begin{bmatrix} I_n \\ 0 \end{bmatrix}.$$

Moreover, from (12),

$$\underline{B}_2 = \begin{bmatrix} I_n & 0 \end{bmatrix} (\hat{Q}_{11}(h)')^{-1} \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}.$$

Now, to see that $\hat{Q}_{11}(h)$ is invertible, note that from (42) and (38),

$$\hat{Q}_{11}(h) = \begin{bmatrix} Q_{11}(h) & 0 \\ ? & I_{m_2} \end{bmatrix},$$

where $Q_{11}(h)$ is defined as before and "?" denotes an irrelevant sub-matrix. Thus, $\hat{Q}_{11}(h)$ is invertible iff $Q_{11}(h)$ is, which is the case by Remark 2.7. This completes the proof.

**Remark 2.9** *In view of Theorems 2.5 and 2.8, with the assumptions that the sampling period is not pathological and that $(A, B_2)$ is stabilisable and $(A, C_2)$ detectable, the discrete-time, LTI component of a $\gamma$-suboptimal, SD controller can be synthesised for $G$ by synthesising a $\gamma$-suboptimal, discrete-time, LTI controller for $\grave{G}$ using well-known discrete-time, $\mathcal{H}_\infty$-synthesis techniques [3, 4].*

## 3 The Design Framework

Motivated by McFarlane and Glover's $\mathcal{H}_\infty$ loop-shaping procedure [6], in this section we propose a framework for SD, control-system design. This will be used in conjunction with the $\mathcal{H}_\infty$ SD synthesis results developed in Section 2.

The $\mathcal{H}_\infty$ loop-shaping procedure involves shaping the open-loop singular values by pre- and post-multiplication of the plant with stable (or marginally-stable to include the possibility of imaginary-axis poles), weighting functions $W_1$ and $W_2$, and synthesising a controller $K$ to minimise

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - G_s K)^{-1} \begin{bmatrix} G_s & I \end{bmatrix} \right\|_\infty , \tag{48}$$

where $G_s = W_2 G W_1$ is the shaped plant. The weights $W_1$ and $W_2$ are then absorbed into $K$ to give the final controller $\hat{K} = W_1 K W_2$. When designing an SD controller we cannot carry out this last step, since the weights cannot be simply absorbed across the sample and hold elements. However, for weights that are also minimum-phase,

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - G_s K)^{-1} \begin{bmatrix} G_s & I \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} W_1^{-1} & 0 \\ 0 & W_2 \end{bmatrix} \begin{bmatrix} \hat{K} \\ I \end{bmatrix} (I - G\hat{K})^{-1} \begin{bmatrix} G & I \end{bmatrix} \begin{bmatrix} W_1 & 0 \\ 0 & W_2^{-1} \end{bmatrix} \right\|_\infty .$$

This leads to a framework from which we can obtain $\hat{K}$ directly. On this basis we propose the new SD design framework. Consider the system interconnection shown in Figure 3, where $W_1$ and $W_2$ are weighting functions to be chosen by the designer.

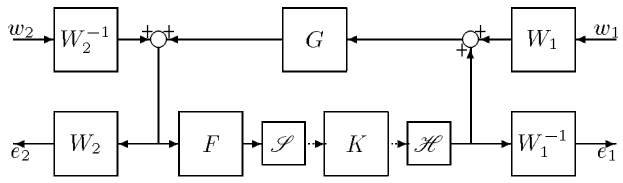


Figure 3: SD Performance Framework Interconnection

Note that the operator mapping $(w_1, w_2)$ to $(e_1, e_2)$ can be expressed as the LFT $\mathcal{F}_\ell(G_W, \mathscr{H}K\mathscr{S}) : \mathcal{K}^{p+m} \to \mathcal{K}^{p+m}$, where

$$G_W(s) = \begin{bmatrix} 0 & 0 & W_1^{-1} \\ W_2 G W_1 & I & W_2 G \\ F G W_1 & F W_2^{-1} & FG \end{bmatrix}, \tag{49}$$

is the generalised plant. Given suitable, stable, minimum-phase weights $W_1$ and $W_2$, synthesise an internally-stabilising, SD controller $\mathscr{H}K\mathscr{S}$, to minimise the $\mathcal{L}_2(\mathbb{R}_+)$-induced norm of the closed-loop operator $\mathcal{F}_\ell(G_W, \mathscr{H}K\mathscr{S})$, according to Theorems 2.5 and 2.8. Note that $G_W$ will have a $D_{11}$ term.

As in the standard $\mathcal{H}_\infty$ loop-shaping procedure, the structure of the proposed SD design framework ensures bounds on common performance objectives. For example, amongst others, bounds on the gain of the operator mapping disturbances at the plant input to the plant output, and bounds on the gain of the operator mapping sensor noise to the plant input. However, the following question has so far remained unanswered : What are the robust stability properties of a controller constructed using the framework described above? We answer this with the following lemma and theorem.

**Lemma 3.1** *Given a continuous-time, LTI, operator $G \in \mathcal{H}_\infty(\mathbb{C}_+)$ and a causal, LPTV operator $X$, if $G$ is co-inner, that is, $GG^\sim = I$, then $\|XG\| = \|X\|$, where $\|\bullet\|$ denotes the $\mathcal{L}_2(\mathbb{R}_+)$-induced norm.*

The proof follows an idea suggested in [9], but has been omitted for the sake of brevity.

**Theorem 3.2** *Given a continuous-time, LTI plant $G$, stable, minimum-phase weights $W_1$ and $W_2$ and an anti-aliasing filter $F$, the SD controller $K_{sd} = \mathscr{H}K\mathscr{S}F$, is internally stabilising for all plants in the model set*

$$G_\Delta(W_1, W_2) = \{(\tilde{M}_s W_2 + \Delta_M W_2)^{-1}(\tilde{N}_s W_1^{-1} + \Delta_N W_1^{-1}) : \|[\ \Delta_N \quad \Delta_M\ ]\|_\infty < \beta\},$$

*where $[\ \tilde{M}_s \quad \tilde{N}_s\ ]$ is a normalised, LCF of $G_s = W_2 G W_1$, if $K_{sd}$ is internally stabilising for $G_W$ (defined in (49)) and the $\mathcal{L}_2(\mathbb{R}_+)$-induced norm of the closed-loop operator $\mathcal{F}_\ell(G_W, \mathscr{H}K\mathscr{S})$, is less than $\beta^{-1}$.*

**Proof :** By Lemma 3.1 and that fact that $[\ \tilde{M}_s \quad \tilde{N}_s\ ]$ is co-inner, we note that the $\mathcal{L}_2(\mathbb{R}_+)$-induced norm of $\mathcal{F}_\ell(G_W, \mathscr{H}K\mathscr{S})$ is equal to the $\mathcal{L}_2(\mathbb{R}_+)$-induced norm of the operator

$$\begin{bmatrix} W_1^{-1} K_{sd} W_2^{-1} \\ I \end{bmatrix} (I - G_s W_1^{-1} K_{sd} W_2^{-1})^{-1} \tilde{M}_s^{-1}. \tag{50}$$

So, in view of the Small Gain Theorem and equation (50), since the $\mathcal{L}_2(\mathbb{R}_+)$-induced norm of the operator $\mathcal{F}_\ell(G_W, \mathscr{H}K\mathscr{S})$ is less than $\beta^{-1}$, it follows that $W_1^{-1} K_{sd} W_2^{-1}$ internally stabilises all plants in the model set

$$G_\Delta = \{(\tilde{M}_s + \Delta_M)^{-1}(\tilde{N}_s + \Delta_N) : \|[\ \Delta_N \quad \Delta_M\ ]\|_\infty < \beta\}.$$

Since the weights are minimum-phase, $W_1$, $W_2$, $W_1^{-1}$ and $W_2^{-1}$ are all stable, so moving them around the loop does not effect internal stability. So the statement above is equivalent to saying that $K_{sd}$ internally stabilises all plants in the model set

$$G_\Delta(W_1, W_2) = \{(\tilde{M}_s W_2 + \Delta_M W_2)^{-1}(\tilde{N}_s W_1^{-1} + \Delta_N W_1^{-1}) : \|[\ \Delta_N \quad \Delta_M\ ]\|_\infty < \beta\}.$$

This completes the proof.

**Remark 3.3** *Note that the uncertain model set in Theorem 3.2, has the same structure as that for the standard, continuous-time, $\mathcal{H}_\infty$, loop-shaping procedure.*

## 4 Design Example

The example presented concerns a satellite with two highly flexible solar arrays attached, and is taken from [6]. For details of the system refer to [6]. The control objectives considered are: Performance - Required pointing accuracy due to a $0.3Nm$ step torque disturbance of $y < 0.04^o$. Additionally we require a response time of less than 60 seconds. Robustness - A level of robust stability in the face of unstructured uncertainty is desirable. Control Effort - The following restriction is placed on the control effort due to a $0.3Nm$ step torque disturbances of $u < 0.5Nm$. As this is a preliminary design we simply use the weights used in [6], with the exception that any integral weighting is approximated. The bandwidth of the weighted frequency response is approximately $3 rads/sec$. In view of this, we use a sampling frequency of $1Hz$ (approx. $2\times$weighted open-loop bandwidth) and use a critically-damped, second-order, anti-aliasing filter with natural frequency at $10 rads/sec$, so as not to introduce excessive additional phase lag in the cross-over frequency region of the weighted plant.

The robust stability margin obtained with SD control was $\beta = 0.2029$, which as expected is less than that obtained with continuous-time control ($\beta = 0.426$). The plant output under SD control and continuous-time control resulting from a $0.3Nm$ step disturbance at the plant input are shown in Figure 4. The corresponding SD control effort is shown in Figure 5.

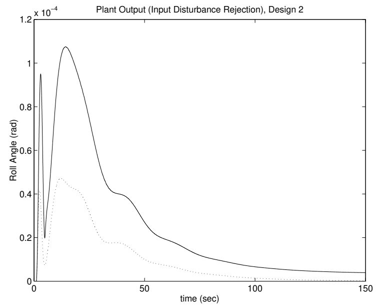


Figure 4: Plant Output For Highly Flexible Spacecraft Example. '—' System Under SD Control; '···' System Under Continuous-Time Control

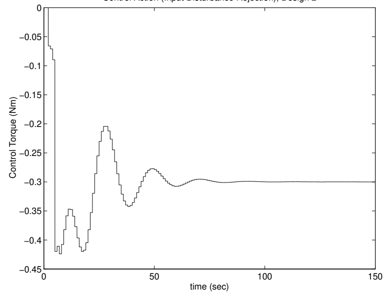


Figure 5: SD Control Effort For Highly Flexible Spacecraft Example

We note that the control objectives are met and that the introduction of approximate integral action in one of the weights gives zero steady-state plant output.

## 5 Conclusion

In this paper we have introduced a new framework for SD, control-system development. This involved two steps: The derivation of new formulae for SD controller synthesis, and the proposal of a design framework, modelled on the $\mathcal{H}_\infty$, loop-shaping procedure of McFarlane and Glover [6].

It can be seen from the preliminary design example presented that even for a relatively slow sampling frequency the performance of an SD controller developed in the new framework is not much worse that that of the full continuous-time controller. Further research is required to completely characterise the proposed framework, including, the derivation of sensible analytic bounds on achievable performance given a bound on the closed-loop operator $\mathcal{F}_\ell(G_W, \mathscr{H} K \mathscr{S})$. Such bounds would aid in the selection of weighting functions as do the frequency by frequency analytic bounds derived for the standard $\mathcal{H}_\infty$, loop-shaping procedure [6].

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