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Validated numerical computation of the $\mathcal{L}_\infty$-norm for linear dynamical systems[☆]

Masaaki Kanno[a], Malcolm C. Smith[b,*]

[a] CREST, Japan Science and Technology Agency, 4-1-8, Honcho, Kawaguchi-shi, Saitama, 332-0012, Japan
[b] Department of Engineering, University of Cambridge, Trumpington Street, Cambridge, CB2 1PZ, United Kingdom

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Abstract

This paper develops a validated numerical algorithm to compute the $\mathcal{L}_\infty$-norm, a norm which plays an important role in modern control. The method reduces the $\mathcal{L}_\infty$-norm computation problem to real root localization of polynomials and some Sturm chain tests, both of which can be executed in a manner which guarantees accuracy. A computational complexity estimate is also given.

Keywords: Validated numerical methods; Guaranteed accuracy; Control systems

1. Introduction

The most important characteristic of a control system is stability. Nevertheless this is by no means the only objective and a system may be expected to satisfy several additional performance criteria. In modern control it has become common practice to use norms of signals and systems to evaluate a performance level. The $\mathcal{H}_2$-norm and the $\mathcal{H}_\infty$-norm in particular have become popular since they arise naturally in engineering problems and also these norms are comparatively easy to calculate in an analytical sense. Furthermore, explicit solutions for controller synthesis problems involving these norms have been derived; see Zhou et al. (1996).

Computer aided design tools are pervasive in modern control. Current approaches for computation are based almost exclusively on standard linear algebra routines and high speed

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* Corresponding author. Tel.: +44 1223332745; fax: +44 1223332662.
E-mail addresses: M.Kanno.99@cantab.net (M. Kanno), mcs@eng.cam.ac.uk (M.C. Smith).

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floating point arithmetic. While the search for numerically reliable routines has made good progress (e.g., Golub and Van Loan, 1996), the problem remains that numerical difficulties such as ill-conditioning may arise in some cases (Higham et al., 2004) and the development of routines with a guarantee is desired (Kanno and Smith, 2003).

The idea of *validated numerical methods* has become established in the computer science field (Krandick and Rump, 1997a) and has also found its way in some application areas in science and engineering (Adams and Kulisch, 1993). In the systems and control area, the use of interval methods in order to improve the reliability of the results is seen (Jaulin et al., 2001; Weinhofer and Haas, 1997). However, it appears that little has been done in this area to achieve the property of guaranteed accuracy defined in the next section.

It is not usually straightforward to adapt an existing algorithm to give a validated numerical one (Corliss, 1990). The purpose of this paper is to develop a validated numerical $\mathcal{L}_\infty$-norm computation algorithm for the transfer function matrix of a linear dynamical system.

The paper is organized as follows. The meaning of guaranteed accuracy is formally defined in Section 2 and the definitions of the $\mathcal{L}_\infty$ and $\mathcal{H}_\infty$ spaces and their corresponding norms, the $\mathcal{L}_\infty$ and $\mathcal{H}_\infty$-norms, are reviewed in Section 3. Section 4 develops the guaranteed accuracy algorithm for the $\mathcal{L}_\infty$-norm computation and Section 5 provides a computational complexity estimation of the determinant of a polynomial matrix required in the guaranteed accuracy $\mathcal{L}_\infty$-norm computation. Two numerical examples are presented in Section 6. Some concluding remarks are made in Section 7.

2. The meaning of guaranteed accuracy

Krandick and Rump (1997b) elucidate the idea of validated numerical methods as a search for algorithms with a rigorous specification, e.g., methods that can never produce an answer which deviates from the true solution by more than a pre-specified tolerance. When solving a problem which finds a single real number, such an algorithm has to use a computer representable number system and also produce an interval, which is a pair of elements in the number system used, to bound the true answer. The following formal definition for guaranteed accuracy is thus suggested.

Definition 1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be well-defined (not necessarily continuous). Let A be some given algorithm taking the form of an executable procedure, which generates a well-defined function $A : (\mathbb{F}^n, \mathbb{F}_{>0}) \to \mathbb{F}^2$, where $\mathbb{F} \subset \mathbb{R}$ is a set of computer representable numbers, $\mathbb{F}_{>0}$ is a set of strictly positive elements in $\mathbb{F}$ and $A(\mathbf{P}, \epsilon) = (f_\ell, f_r)$ where $f_\ell \le f_r$. Then, A is said to be a guaranteed accuracy algorithm for f over $\mathbb{F}$ if, for any $\mathbf{P} \in \mathbb{F}^n$ and any $\epsilon \in \mathbb{F}_{>0}$, (the true) $f(\mathbf{P})$ is contained in the closed interval $[f_\ell, f_r]$ and $f_r - f_\ell \le \epsilon$.

Floating point arithmetic is unsuitable as the choice for $\mathbb{F}$ in the above definition partly because of its lack of closure under ordinary arithmetic operations. Following the practice in many computer algebra packages, we choose to employ the rational number system in the present work, i.e., $\mathbb{F}$ in the definition will be taken to be $\mathbb{Q}$.

3. $\mathcal{L}_\infty/\mathcal{H}_\infty$-norm

Two classes of complex matrix valued functions bounded on the imaginary axis, namely, the $\mathcal{L}_\infty$ and $\mathcal{H}_\infty$ spaces, are reviewed. In the following, $\overline{\sigma}\{\cdot\}$ denotes the largest singular value of a matrix.

Definition 2. $\mathcal{L}_\infty$ is the Lebesgue space of matrix valued functions G that are (essentially) bounded on the imaginary axis, with norm defined by

$$\|G\|_\infty := \operatorname*{ess\,sup}_{\omega\in\mathbb{R}} \overline{\sigma}\{G(j\omega)\}. \tag{1}$$

Definition 3. $\mathcal{H}_\infty$ is the space of matrix valued functions G that are analytic and bounded in the open right half plane. The corresponding norm is defined by

$$\|G\|_\infty := \sup_{\mathrm{Re}(s)>0} \overline{\sigma}\{G(s)\}. \tag{2}$$

It can be shown that, for $G \in \mathcal{H}_\infty$, there is an analytic continuation of G onto the imaginary axis which provides a boundary value function $G(j\omega) \in \mathcal{L}_\infty$. Moreover, the right-hand side of (2) agrees with the right-hand side of (1). Thus, $\mathcal{H}_\infty$ can be considered to be a subspace of $\mathcal{L}_\infty$. Hence, in this paper, the same notation $\|\cdot\|_\infty$ is used for both the $\mathcal{L}_\infty$ and $\mathcal{H}_\infty$-norms.

The prefix $\mathcal{R}$ is used to denote a subspace consisting of real rational functions. For example, $\mathcal{RL}_\infty$ (resp., $\mathcal{RH}_\infty$) is the subspace of $\mathcal{L}_\infty$ (resp., $\mathcal{H}_\infty$) consisting of all matrices whose elements are rational functions with real coefficients and no poles on the imaginary axis or at infinity.

4. Guaranteed accuracy $\mathcal{L}_\infty$-norm computation algorithm

Current approaches for the computation of the $\mathcal{L}_\infty$-norm using ordinary numerical methods involve an existence test for imaginary axis eigenvalues of a Hamiltonian matrix (Boyd and Balakrishnan, 1990; Boyd et al., 1989; Bruinsma and Steinbuch, 1990; Robel, 1989). In the context of floating point arithmetic, this step is particularly prone to numerical difficulties since it essentially tries to compute nearly multiple eigenvalues near the imaginary axis. Although the possibility of using Sturm chains (an algebraic means to count the number of real roots of a real polynomial in an interval; see Gantmacher (1960)) for this eigenvalue test has been mentioned in the literature (Boyd et al., 1989), there does not seem to have been any attempt to exploit this idea in a practical algorithm.

In this paper we will develop a method for guaranteed accuracy computation of the $\mathcal{L}_\infty$-norm which requires a finite number of real roots of certain polynomials to be localized plus (possibly) some additional Sturm chain tests. We note here that, for polynomials with rational coefficients, the Sturm chain test allows a validated numerical algorithm to be established and that real root localization can be implemented in a guaranteed accuracy manner by means of Descartes' rule of signs (Collins and Akritas, 1976). We further note that validated numerical algorithms for the Sturm chain test and real root localization are readily available, for instance, in Maple, under the names of `sturm` and `realroot`, respectively.

Our method will be derived by a series of propositions leading to the main result in Theorem 8. In contrast to Boyd and Balakrishnan (1990), Boyd et al. (1989), Bruinsma and Steinbuch (1990) and Robel (1989) where a state space technique is used, our method uses only the transfer function matrix, i.e., we will assume that a $G \in \mathcal{RL}_\infty$ is given. The derivation begins with a characterization of the $\mathcal{L}_\infty$-norm in terms of the existence of zeros on the imaginary axis. This is a standard result (see Zhou et al. (1996, Section 4.7) and the references therein). We define the conjugate system $G^\sim(s)$ of $G(s)$ as $G^\sim(s) := G^T(-s)$.

Proposition 4. *Let $\gamma > 0$, $G \in \mathcal{RL}_\infty$ and $\Phi_\gamma(s) := \gamma^2 I - G^\sim(s)G(s)$. Then, $\gamma > \|G\|_\infty$ if and only if $\gamma > \overline{\sigma}\{G(j\infty)\}$ and* $\det \Phi_\gamma(j\omega) \neq 0$ *for all $\omega \in \mathbb{R}$.*

Note that $\det \Phi_\gamma(s)$ is a real rational function in s^2 since

$$\begin{aligned}\det \Phi_\gamma(-s) &= \det(\gamma^2 I - G^\sim(-s)G(-s)) = \det(\gamma^2 I - G^\sim(-s)G(-s))^T \\ &= \det(\gamma^2 I - G^T(-s)(G^\sim(-s))^T) = \det(\gamma^2 I - G^\sim(s)G(s)) \\ &= \det \Phi_\gamma(s).\end{aligned}$$

We now substitute x for s^2 in $\det \Phi_\gamma(s)$ and write as $g_\gamma(x)$, i.e., $g_\gamma(s^2) = \det \Phi_\gamma(s)$.

Proposition 5. *Let $\gamma > 0$, $G \in \mathcal{RL}_\infty$ and $\Phi_\gamma(s)$, $g_\gamma(x)$ be defined as above. Write $g_\gamma(x) = \frac{n_\gamma(x)}{d_\gamma(x)}$, where $n_\gamma(x)$ and $d_\gamma(x)$ are polynomials in x whose coefficients are polynomials in γ (γ^2, in fact) and, when seen as polynomials in x and γ, are coprime over $\mathbb{R}[x, \gamma]$. Then, $\gamma > \|G\|_\infty$ if and only if $\gamma > \overline{\sigma}\{G(j\infty)\}$ and $n_\gamma(x)$ does not have roots in $-\infty < x \le 0$.*

Proof. It is immediate that $\det \Phi_\gamma(j\omega) \neq 0$ for all $\omega \in \mathbb{R}$ is equivalent to $g_\gamma(x)$ does not have zeros in $-\infty < x \le 0$.

The denominator of each element in $\Phi_\gamma(s)$ can be written as a polynomial in s only. Let $\ell(s)$ be the least common multiple of all the denominators in $\Phi_\gamma(s)$, which is still a polynomial in s only and in fact a polynomial in s^2. We substitute x for s^2 and write as $\ell'(x)$, i.e., $\ell'(s^2) = \ell(s)$. Also, since $G \in \mathcal{RL}_\infty$, $\ell(s)$ has no imaginary axis root and hence there is no root of $\ell'(x)$ in $-\infty < x \le 0$. Suppose that the size of $\Phi_\gamma(s)$ is $k \times k$. Then we can check that, since $n_\gamma(x)$ and $d_\gamma(x)$ are coprime, $d_\gamma(x)$ divides $\left\{\ell'(x)\right\}^k$ over $\mathbb{R}[x, \gamma]$ (in fact over $\mathbb{R}[x]$). Since there is no root of $\ell'(x)$ in $-\infty < x \le 0$, then $d_\gamma(x) \neq 0$ for $-\infty < x \le 0$. Therefore, $g_\gamma(x) \neq 0$ for $-\infty < x \le 0$ is equivalent to $n_\gamma(x) \neq 0$ for $-\infty < x \le 0$. □

We point out that, since $n_\gamma(x)$ is a real polynomial (for real γ), the question of whether $n_\gamma(x) \neq 0$ for $-\infty < x \le 0$ can be examined using Sturm chains. Hence bisection search in γ using Sturm chains is now directly applicable to calculate the $\mathcal{L}_\infty$-norm. However, it will be advantageous to develop the condition further. Firstly, we give a condition that $g_\gamma(x)$ exhibits when $\gamma = \|G\|_\infty$.

Proposition 6. *Let $G \in \mathcal{RL}_\infty$ and $g_\gamma(x)$, $n_\gamma(x)$, $d_\gamma(x)$ be defined as previously. Suppose that $\gamma_\infty := \|G\|_\infty$ is not achieved at $s = j\infty$ ($\omega = \infty$), i.e., $\gamma_\infty > \overline{\sigma}\{G(j\infty)\}$. Then either*

(i) $g_{\gamma_\infty}(0) = 0$, *or*
(ii) $g_{\gamma_\infty}(x_0) = \frac{\mathrm{d}}{\mathrm{d}x} g_{\gamma_\infty}(x_0) = 0$ *for some* x_0, $-\infty < x_0 < 0$.

Equivalently, either

(i′) $n_{\gamma_\infty}(0) = 0$, *or*
(ii′) $n_{\gamma_\infty}(x)$ *has a root of multiplicity at least two in* $-\infty < x < 0$.

Proof. From the assumption that $\gamma_\infty > \overline{\sigma}\{G(j\infty)\}$, the $\mathcal{L}_\infty$-norm is achieved for some $s = j\omega, \omega \in [0, \infty)$. If $\gamma_\infty = \overline{\sigma}\{G(0)\}$, then $\det \Phi_{\gamma_\infty}(0) = 0$, thus (i) holds. Otherwise let $\omega_0 \in (0, \infty)$ be the frequency where the $\mathcal{L}_\infty$-norm is achieved, i.e., $\gamma_\infty = \overline{\sigma}\{G(j\omega_0)\}$, and write $x_0 = -\omega_0^2$. Then, $\det \Phi_{\gamma_\infty}(j\omega_0) = 0$, which implies that $g_{\gamma_\infty}(x_0) = 0$. However, for any real ω and, in particular, in the vicinity of ω_0, $\Phi_{\gamma_\infty}(j\omega)$ is a positive semi-definite matrix and, thus, $\det \Phi_{\gamma_\infty}(j\omega) \ge 0$, that is, $g_{\gamma_\infty}(x) \ge 0$ for $-\infty < x \le 0$. Since $g_{\gamma_\infty}(x)$ is a rational function, it follows that $\frac{\mathrm{d}}{\mathrm{d}x} g_{\gamma_\infty}(x_0) = 0$. Hence, (ii) holds.

Since $d_\gamma(x) \neq 0$ for any $\gamma > 0$ and $-\infty < x \leq 0$, as is shown in the proof of Proposition 5, it is immediate that (i) is equivalent to (i′). In the case of (ii), it is also immediate that $g_{\gamma_\infty}(x_0) = 0$ if and only if $n_{\gamma_\infty}(x_0) = 0$. Since

$$\frac{\mathrm{d}}{\mathrm{d}x} g_{\gamma_\infty}(x) = \frac{\frac{\mathrm{d}}{\mathrm{d}x} n_{\gamma_\infty}(x) d_{\gamma_\infty}(x) - n_{\gamma_\infty}(x) \frac{\mathrm{d}}{\mathrm{d}x} d_{\gamma_\infty}(x)}{\{d_{\gamma_\infty}(x)\}^2},$$

if $\frac{\mathrm{d}}{\mathrm{d}x} g_{\gamma_\infty}(x_0) = 0$ in addition, then $\frac{\mathrm{d}}{\mathrm{d}x} n_{\gamma_\infty}(x_0) = 0$, noting that $d_{\gamma_\infty}(x_0) \neq 0$. Conversely, $n_{\gamma_\infty}(x_0) = \frac{\mathrm{d}}{\mathrm{d}x} n_{\gamma_\infty}(x_0) = 0$ implies that $g_{\gamma_\infty}(x_0) = \frac{\mathrm{d}}{\mathrm{d}x} g_{\gamma_\infty}(x_0) = 0$. Thus, (ii) is equivalent to (ii′). □

We can factorize $n_\gamma(x)$ over $\mathbb{R}[x, \gamma]$ as

$$n_\gamma(x) = \prod_i \left\{n_i^p(x, \gamma)\right\}^i \prod_k \left\{n_k^c(\gamma)\right\}^k,$$

where $n_i^p(x, \gamma)$ are real polynomials in x and γ, $n_k^c(\gamma)$ are real polynomials in γ and free of x, $n_i^p(x, \gamma)$, $n_k^c(\gamma)$ are relatively prime, and each $n_i^p(x, \gamma)$, $n_k^c(\gamma)$ is free of multiple factors. The square-free part $h_\gamma(x)$ of $n_\gamma(x)$ is

$$h_\gamma(x) := \prod_i n_i^p(x, \gamma) \prod_k n_k^c(\gamma).$$

We have the formula (Cox et al., 1996)

$$h_\gamma(x) = \frac{n_\gamma(x)}{\mathrm{GCD}\left(n_\gamma(x), \frac{\partial}{\partial x} n_\gamma(x), \frac{\partial}{\partial \gamma} n_\gamma(x)\right)}.$$

For our purposes we introduce the following notation for a square-free divisor of $h_\gamma(x)$:

$$h_\gamma^s(x) := \prod_i n_i^p(x, \gamma) = \frac{n_\gamma(x)}{\mathrm{GCD}\left(n_\gamma(x), \frac{\partial}{\partial x} n_\gamma(x)\right)},$$

where the greatest common divisor is that of the polynomials in x and γ. The above proposition is now stated in a stronger way in terms of $h_\gamma^s(x)$.

Proposition 7. *Let $G \in \mathcal{RL}_\infty$ and $n_\gamma(x)$, $h_\gamma^s(x)$, γ_∞ be defined as above. Then, $\gamma > \gamma_\infty$ if and only if $\gamma > \overline{\sigma}\{G(j\infty)\}$ and $h_\gamma^s(x)$ has no roots in $-\infty < x \leq 0$. Further, if γ_∞ is achieved in $0 < \omega < \infty$, then $h_{\gamma_\infty}^s(x)$ has a multiple root in $-\infty < x < 0$.*

Proof. The roots of $\prod_k \left\{n_k^c(\gamma)\right\}^k$ are singular values of $G(j\omega)$ independent of frequency ω and naturally these roots are singular values of $G(j\infty)$. Therefore, $n_k^c(\gamma) \neq 0$ for $\gamma > \overline{\sigma}\{G(j\infty)\}$ (and naturally for $\gamma > \gamma_\infty$). When $\prod_k \left\{n_k^c(\gamma)\right\}^k \neq 0$, because of the Nullstellensatz (Cox et al., 1996), $n_\gamma(x) \neq 0$ for $-\infty < x \leq 0$ is equivalent to $h_\gamma^s(x) \neq 0$ for $-\infty < x \leq 0$. Hence the first claim follows from Proposition 5.

We now consider the case where γ_∞ is achieved in $0 < \omega < \infty$. The assumption, $n_{\gamma_\infty}(x)$ has a root at some $-\infty < x_0 < 0$, along with the Nullstellensatz, implies that

$$h_{\gamma_\infty}^s(x_0) = 0. \tag{3}$$

We then observe that, if $h_\gamma^s(x) > 0$ (resp., $h_\gamma^s(x) < 0$) for some $-\infty < x < 0$ and some $\gamma > \gamma_\infty$, then $h_\gamma^s(x) > 0$ (resp., $h_\gamma^s(x) < 0$) for all $-\infty < x < 0$ and all $\gamma > \gamma_\infty$. To see this, suppose that $h^s_{\gamma_1}(x_1) > 0$ for some pair (x_1, γ_1) and $h^s_{\gamma_2}(x_2) < 0$ for another pair (x_2, γ_2). Taking a continuous path within the admissible region from (x_1, γ_1) to (x_2, γ_2), we will have a point (x_0, γ_0) such that $h^s_{\gamma_0}(x_0) = 0$ for some $\gamma_0 > \gamma_\infty$. This is a contradiction to the first claim of this proposition, which establishes the claim.

Now we prove that, if $h^s_\gamma(x) > 0$ (resp., $h^s_\gamma(x) < 0$) for all $-\infty < x < 0$ and all $\gamma > \gamma_\infty$, then

$$h^s_{\gamma_\infty}(x) \geq 0 \text{ (resp., } h^s_{\gamma_\infty}(x) \leq 0) \qquad \text{for all } -\infty < x < 0. \tag{4}$$

Suppose that $h^s_\gamma(x) > 0$ but $h^s_{\gamma_\infty}(x_1) < 0$ for some $-\infty < x_1 < 0$. By continuity, $h^s_\gamma(x_1)$ tends to $h^s_{\gamma_\infty}(x_1)$ as γ tends to γ_∞, which contradicts $h^s_\gamma(x) > 0$. Hence, $h^s_{\gamma_\infty}(x) \geq 0$. Similarly we can deduce that, if $h^s_\gamma(x) < 0$, then $h^s_{\gamma_\infty}(x) \leq 0$.

From (3) and (4) and analyticity of $h^s_\gamma(x)$, it is concluded that

$$h^s_{\gamma_\infty}(x_0) = \frac{\mathrm{d}}{\mathrm{d}x} h^s_{\gamma_\infty}(x_0) = 0,$$

that is, $h^s_{\gamma_\infty}(x)$ has a multiple root at $x = x_0$. □

Finally, the following theorem provides a way of constructing real polynomials one of which contains the $\mathcal{L}_\infty$-norm as a real root.

Theorem 8. *Suppose that $G \in \mathcal{RL}_\infty$ and let $h^s_\gamma(x)$, γ_∞ be defined as previously. Then, γ_∞ is one of the following quantities:*

(i) $\overline{\sigma}\{G(0)\}$,
(ii) $\overline{\sigma}\{G(j\infty)\}$,
(iii) *a real root of the discriminant of* $h^s_\gamma(x)$.

Moreover, each of the above quantities is a (real) root of a real polynomial.

Proof. When γ_∞ is achieved at $\omega = 0$ $(s = 0)$ (resp., $\omega = \infty$ $(s = j\infty)$), then $\gamma_\infty = \overline{\sigma}\{G(0)\}$ (resp., $\gamma_\infty = \overline{\sigma}\{G(j\infty)\}$), which corresponds to case (i) (resp., (ii)). Furthermore, γ_∞ is then the largest (real) root of $\det(\lambda^2 I - G^*(0)G(0)) = 0$ (resp., $\det(\lambda^2 I - G^*(j\infty)G(j\infty)) = 0$).

Now suppose that γ_∞ is achieved in $0 < \omega < \infty$. Notice that $h^s_\gamma(x)$ is square-free and its discriminant with respect to x is not identically zero, and Proposition 7 says that $h^s_{\gamma_\infty}(x)$ has a multiple root when seen as a polynomial in x. Hence, γ_∞ is a root of the discriminant of $h^s_\gamma(x)$. We note that $h^s_\gamma(x)$ is a real polynomial and therefore the discriminant is a real polynomial as well. This concludes the proof. □

Theorem 8 provides a finite set of candidate real numbers, one of which is the $\mathcal{L}_\infty$-norm of G. If either $\overline{\sigma}\{G(0)\}$ or $\overline{\sigma}\{G(j\infty)\}$ is the maximum in this set, then this is obviously equal to $\|G\|_\infty$. Otherwise we need to test the candidate real numbers to determine which is the true $\mathcal{L}_\infty$-norm. This can be achieved by using the first claim in Proposition 7, which can be implemented using Sturm chains.

This approach allows the $\mathcal{L}_\infty$-norm to be calculated with guaranteed accuracy in the sense of Definition 1. If the coefficients in the transfer function matrix G are rational numbers, then $h^s_\gamma(x)$ is a polynomial in x and γ with rational number coefficients. Moreover the discriminant of $h^s_\gamma(x)$ is a polynomial in γ with rational coefficients. Also, $\det(\lambda^2 I - G^*(0)G(0)) = 0$ and $\det(\lambda^2 I - G^*(j\infty)G(j\infty)) = 0$ yield polynomials in λ^2 with rational coefficients. The problem is thus reduced to real root localization and some Sturm chain tests of polynomials with rational coefficients, both of which can be carried out with guaranteed accuracy, as is mentioned at the beginning of the section.

We can summarize the whole procedure to compute the $\mathcal{L}_\infty$-norm with guaranteed accuracy. Let ϵ be a pre-specified tolerance.

(1) Create a set of intervals of width $\epsilon/2$ which contain $\overline{\sigma}\{G(0)\}$, $\overline{\sigma}\{G(j\infty)\}$ and real roots of the discriminant of $h_\gamma^s(x)$.
(2) If the maximum of the lower bounds for $\overline{\sigma}\{G(0)\}$ and $\overline{\sigma}\{G(j\infty)\}$ is larger than the upper bound of any candidate from the discriminant of $h_\gamma^s(x)$, then the $\mathcal{L}_\infty$-norm is achieved at $\omega = 0$ or ∞. In this case, let the lower (resp., upper) bound for $\|G\|_\infty$ be the maximum of the lower (resp., upper) bounds for $\overline{\sigma}\{G(0)\}$ and $\overline{\sigma}\{G(j\infty)\}$ and the algorithm is terminated.
(3) Select a candidate from the set defined in Step (1), let its lower bound be γ and check if $\gamma \le \|G\|_\infty < \gamma + \epsilon$ using $\overline{\sigma}\{G(j\infty)\}$, $h_\gamma^s(x)$ and Sturm chains. If so, set the bound to $[\gamma, \gamma + \epsilon]$ and terminate the algorithm. Otherwise discard the candidate from the list and repeat this step for another candidate.

5. Complexity of the determinant computation

Empirical data show that the computation of det $\Phi_\gamma(s)$ is potentially the most time-consuming part of the whole $\mathcal{L}_\infty$-norm computation. Elements of $\Phi_\gamma(s)$ are rational functions, but, by clearing denominators, we can compute det $\Phi_\gamma(s)$ via computation of the determinant of a (bivariate) polynomial matrix.

For the case of matrices with integer elements it is a relatively straightforward fact that the determinant may be computed in polynomial time in *arithmetic* operations. It is also true that determinant computation is possible which is polynomial in terms of *word* operations, although this is more difficult to prove. An expression is provided for the required number of word operations in, e.g., von zur Gathen and Gerhard (2003). For determinants of matrices with univariate entries, sophisticated methods, e.g., Giorgi et al. (2003), are available whose computational complexity is bounded by polynomials in arithmetic operations. For multivariate polynomial matrix determinants, the computation cost is shown to be polynomial in terms of arithmetic operations in Marco and Martínez (2004). A result in terms of word operations for (bivariate) polynomial matrices, namely, that the determinant computation cost is polynomial in terms of *word* operations, does not seem to be readily available in the literature. A proof of this fact, in a form relevant for the present algorithm, has been provided in Kanno (2004). Below we briefly summarize this result.

After clearing denominators in rational number coefficients, we may assume $G(s) = \left(g_{ij}(s)\right) \in \mathbb{Z}(s)^{m\times n}$. Since $\|G\|_\infty = \left\|G^T\right\|_\infty$ (here, assuming that $G \in \mathcal{L}_\infty$), we can assume that $m \le n$ without loss of generality. Write $g_{ij}(s) = \frac{n_{ij}(s)}{d_{ij}(s)}$, where $n_{ij}(s), d_{ij}(s) \in \mathbb{Z}[s]$. Suppose that $\deg n_{ij}(s), \deg d_{ij}(s) \le d$. Further suppose that $\left\|n_{ij}\right\|_\infty, \left\|d_{ij}\right\|_\infty \le A$, where $\|\cdot\|_\infty$ is the max-norm of a polynomial, namely, if $f(x) = \sum_{0\le i\le n} a_i x^i \in \mathbb{Z}[x]$, then $\|f\|_\infty := \max_{0\le i\le n} |a_i|$. Define

$$\Phi(\gamma, s) := \gamma^2 I - G^\sim(s)G(s) \in \mathbb{Z}(\gamma, s)^{n\times n}.$$

We actually consider the computation cost of the determinant of

$$\Phi'(\gamma, s) := T^\sim(s)\Phi(\gamma, s)T(s) \in \mathbb{Z}[\gamma, s]^{n\times n}$$

where

$$T(s) = \text{diag}\left(\prod_{1\le k\le m} d_{k1}(s), \prod_{1\le k\le m} d_{k2}(s), \ldots, \prod_{1\le k\le m} d_{kn}(s)\right) \in \mathbb{Z}[s]^{n\times n}.$$

In Kanno (2004), it is shown that the computation of det $\Phi'(\gamma, s)$ requires at most

$$O^{\sim}\left(m^2 n^5 d\left\{m^2 d^3 + nd + n\log A + m^2 d \log^2 A\right\}\right)$$

word operations where the 'soft Oh' notation is used to 'swallow' all the log-factors (von zur Gathen and Gerhard, 2003). In the case of a square system, namely, when $m = n$, the computation cost of det $\Phi'(\gamma, s)$ is bounded by

$$O^{\sim}\left(n^9 d^2\left(d^2 + \log^2 A\right)\right)$$

word operations.

The approach makes naïve use of multivariate Lagrange polynomial interpolation. It is believed that the upper bound can be improved by means of the upper bound of the coefficients of the determinant (Lossers, 1974) and the Chinese remainder algorithm (von zur Gathen and Gerhard, 2003), since all the numbers appearing in the calculation are integers. Nevertheless the analysis seems fairly complicated and it is not attempted here.

6. Numerical examples for guaranteed accuracy $\mathcal{L}_\infty$-norm computation

Two numerical examples are used to demonstrate the algorithm developed in Section 4. In the following, numbers are displayed as finite decimals for convenience, but rational numbers are used in the actual algorithm. The first example is

$$G(s) = \begin{bmatrix} \frac{s^2+s+1}{s^2+0.1s+1} & 0 & 0 & 0 \\ 0 & \frac{4s}{s^2-1} & 0 & 0 \\ 0 & 0 & \frac{4s}{s^2-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

A straightforward hand computation is possible here which shows that the exact $\mathcal{L}_\infty$-norm is

$$\|G\|_\infty = 10.$$

Following the algorithm in Section 4, we first compute the largest singular values at $\omega = 0, \infty$:

$$\overline{\sigma}\{G(0)\} = \overline{\sigma}\{G(j\infty)\} = 1.$$

Again these are written down exactly in this case, but in general we would only expect to bound them to arbitrary (but guaranteed) accuracy. Other candidates for the $\mathcal{L}_\infty$-norm are found from the discriminant of $h_\gamma^s(x)$. We compute det $\Phi_\gamma(s) = \det\left(\gamma^2 I - G^{\sim}(s)G(s)\right)$, take its numerator and then substitute s^2 with x to get

$$\begin{aligned} n_\gamma(x) = {} & (\gamma^8 - 2\gamma^6 + \gamma^4)x^6 + (-2.01\gamma^8 + 37.01\gamma^6 - 67\gamma^4 + 32\gamma^2)x^5 \\ & + (-0.96\gamma^8 - 2.36\gamma^6 + 291.32\gamma^4 - 544\gamma^2 + 256)x^4 \\ & + (3.94\gamma^8 - 65.3\gamma^6 + 570.8\gamma^4 - 765.44\gamma^2 + 256)x^3 \\ & + (-0.96\gamma^8 - 2.36\gamma^6 + 291.32\gamma^4 - 544\gamma^2 + 256)x^2 \\ & + (-2.01\gamma^8 + 37.01\gamma^6 - 67\gamma^4 + 32\gamma^2)x + \gamma^8 - 2\gamma^6 + \gamma^4 \end{aligned}$$

$$= (\gamma^2 - 1)\left\{(\gamma^2 - 1)x^2 + (1.99\gamma^2 - 1)x + \gamma^2 - 1\right\} \times \left\{\gamma^2 x^2 + (16 - 2\gamma^2)x + \gamma^2\right\}^2.$$

The factor $(\gamma^2 - 1)$ appears since $\gamma = 1$ is a singular value of $G(j\omega)$ throughout the entire frequency (because of the $(4, 4)$-element of G) and the power 2 of $\left\{\gamma^2 x^2 + (16 - 2\gamma^2)x + \gamma^2\right\}$ arises since there is always a double singular value due to the $(2, 2)$ and $(3, 3)$-elements of G. Then,

$$\mathrm{GCD}\left(n_\gamma(x), \frac{\partial}{\partial x} n_\gamma(x)\right) = (\gamma^2 - 1)\left\{\gamma^2 x^2 + (16 - 2\gamma^2)x + \gamma^2\right\}$$

and we obtain

$$h^s_\gamma(x) = \frac{n_\gamma(x)}{\mathrm{GCD}\left(n_\gamma(x), \frac{\partial}{\partial x} n_\gamma(x)\right)} = \left\{(\gamma^2 - 1)x^2 + (1.99\gamma^2 - 1)x + \gamma^2 - 1\right\}\left\{\gamma^2 x^2 + (16 - 2\gamma^2)x + \gamma^2\right\}.$$

Notice that the factor $(\gamma^2 - 1)$ is removed and that the power 2 of $\left\{\gamma^2 x^2 + (16 - 2\gamma^2)x + \gamma^2\right\}$ is reduced to 1. The discriminant of $h^s_\gamma(x)$ with respect to x is

$$\frac{3}{15625000000}(\gamma^2 - 4)(\gamma^2 - 100)(133\gamma^2 - 100)(399\gamma^4 - 1900\gamma^2 + 1600)^4.$$

In this case we can write down exact expressions for the positive real roots:

$$10,\ 2,\ \frac{10}{\sqrt{133}}\ (\simeq 0.867110),\ \frac{1}{399}\sqrt{379050 + 3990\sqrt{2641}}\ (\simeq 1.91545),$$
$$\frac{1}{399}\sqrt{379050 - 3990\sqrt{2641}}\ (\simeq 1.04545).$$

By Theorem 8, the actual $\mathcal{L}_\infty$-norm is either 1 or one of the above. Using a guaranteed accuracy polynomial real root computation algorithm and Sturm chains, we can choose the right one, i.e., 10, from the candidates and thus find $\|G\|_\infty$ with guaranteed accuracy.

We further illustrate the algorithm on the plant in Example 4.2 in Zhou and Doyle (1998):

$$G(s) = \begin{bmatrix} \frac{s^2+0.15s+2.5}{s^4+0.35s^3+3.51s^2+0.45s+2.0} & \frac{0.1s+0.5}{s^4+0.35s^3+3.51s^2+0.45s+2.0} \\ \frac{0.1s+0.5}{s^4+0.35s^3+3.51s^2+0.45s+2.0} & \frac{0.5s^2+0.1s+0.5}{s^4+0.35s^3+3.51s^2+0.45s+2.0} \end{bmatrix}.$$

The largest singular values at $\omega = 0, \infty$ are

$$\overline{\sigma}\{G(0)\} \simeq 1.3090169944,\ \overline{\sigma}\{G(j\infty)\} = 0.$$

The discriminant of $h^s_\gamma(x)$ yields the following 12th order polynomial in γ (or 6th order in γ^2):

$$\begin{aligned} &15405834505989388373\,\gamma^{12} - 207008808434667878 1094\,\gamma^{10} \\ &+ 570723795377730975 5325\,\gamma^8 - 408294833968356609 7500\,\gamma^6 \\ &+ 89020094992965000000\,\gamma^4 - 2628051175000000000 0\,\gamma^2 \\ &+ 3240000000000000000. \end{aligned} \qquad (5)$$

By Proposition 7, the $\mathcal{L}_\infty$-norm of G is found to be one of the real roots of (5). This cannot be expressed explicitly but, by specifying $\epsilon = 10^{-10}$, say, we can get the following bound for the $\mathcal{L}_\infty$-norm using Descartes' rule of signs:

$$\|G\|_\infty \in [11.47039654321, 11.47039654328].$$

7. Conclusion

In this paper we have developed an algorithm for the computation of the $\mathcal{L}_\infty$-norm of a rational function matrix which is suitable for a computer algebra implementation. The motivation is to provide an algorithm which can achieve the property of guaranteed accuracy. The coefficients of the rational functions are assumed to be rational numbers and the algorithm provides an interval of arbitrarily small width which contains the true $\mathcal{L}_\infty$-norm. The method developed in the paper reduces the problem to finding real roots of three (univariate) polynomials with rational coefficients and (possibly) some additional Sturm chain tests.

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