Closed-loop Model Validation: Relinearizing Problems to deal with Non-zero Initial States and Improve Approximations

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***Abstract*— There are several methods for the non-invalidation of systems under feedback control based on tangential Carathéodory-Fejér interpolation techniques. These account for differences between observed behaviour and modelled behaviour through model uncertainty and exogenous noise. Methods published to date generally assume a system initially at equilibrium. This paper proposes a technique for modelling an initial state using a hypothetical 'pre-record sequence'. Owing to the approximations needed to accurately accommodate noise sequences, it is necessary to linearize problems about some nominal solution; this paper proposes a least-squares solution to a zero-uncertainty non-invalidation problem. The application of this is demonstrated through a numerical example. A similar technique is proposed for improving approximate solutions to non-convex model validation problems; this is also demonstrated using a numerical example.**

I. Introduction

A. Notation

Notation in this document is generally standard, and symbols will be introduced as they appear in the text. The notation

$$\mathcal{S}_k^p := \{w : w = \{w_0, w_1, \ldots, w_{k-1}\}, w \in \mathcal{R}^p\}$$

is used to denote finite-length sequences of real vectors and the corresponding *lower block Toeplitz operator*

$$T_w := \begin{pmatrix} w_0 & 0 & \cdots & 0 \\ w_1 & w_0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ w_{k-1} & w_{k-2} & \cdots & w_0 \end{pmatrix}.$$

Note that for $w \in \mathcal{S}_k^p$, $T_w \in \mathcal{R}^{kp\times k}$. z^{-1} denotes the inverse of the unit delay operator.

B. Motivation

Model validation is the process of determining whether a given mathematical model accurately reflects real data. Strictly speaking, one can never truly 'validate' a model: one can show that a model is consistent with all available data, but it is always possible that scenarios not yet considered would invalidate the model. This problem is particularly important in the field of robust control. Typical robust design methods provide guarantee a satisfactory level of stability or performance provided that the true system under control belongs to a pre-defined model set based on some nominal model. Traditional mathematical techniques based on tangential Carathéodory-Fejér interpolation (see section II-B below) implicitly assume that the system providing the validation data is initially at equilibrium. In many practical applications, e.g. flight control, it is impossible to guarantee this. This paper proposes a method of accommodating non-zero initial states using synthetic 'pre-record' sequences to build up an initial state.

Another feature of many tangential Carathéodory-Fejér methods is that to solve problems where exogenous noise can act on the plant input and output, it is necessary to approximate validation constraints, linearizing about the zero-noise solution. A by-product of the method developed above is a technique for improving these approximations by successively re-linearizing the problem about intermediate solutions.

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C. General Framework

Figure 1 shows a generic framework for robust model validation with input-output noise: the generalized plant transfer function

$$G(z) = \begin{pmatrix} G_{11}(z) & G_{12}(z) \\ G_{21}(z) & G_{22}(z) \end{pmatrix}$$

is known, as are the recorded input-output sequences (u, y). The exogenous noise signals (w_u, w_y) are unknown, as is the perturbation operator Δ. In this work, it will be assumed that the inverses G_{12}^{-1} and G_{21}^{-1} exist, giving the relationship

$$\begin{pmatrix} \hat{s} \\ \hat{t} \end{pmatrix} = M_\Delta(G) \begin{pmatrix} u \\ y \end{pmatrix} + M_\Delta(G) \begin{pmatrix} w_u \\ w_y \end{pmatrix} \tag{1}$$

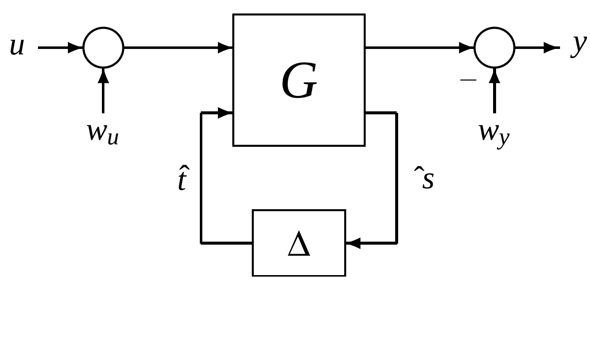


Fig. 1. Block Diagram for Model Validation.

where

$$M_\Delta(G) := \begin{pmatrix} G_{21} - G_{22}G_{21}^{-1}G_{11} & G_{22}G_{12}^{-1} \\ -G_{12}^{-1}G_{11} & G_{12}^{-1} \end{pmatrix}. \quad (2)$$

Note that where G_{21}^{-1} exists, $(M_\Delta(G))^{-1}$ also exists. Note that Δ represents a *stable* norm-bounded operator; separate apply for linear time-invariant (LTI) perturbations and linear time-varying (LTV) perturbations.

D. Noise Minimization Validation Problem

In the context of the framework of Section I-C, a *Model Validation Noise Minimization Problem (MVNMP)* can be defined as follows: given a generalized plant G, a positive scalar γ_Δ and recorded input-output data (u,y), what is the smallest value of γ_w given by

$$\gamma_w = ||w||_2 := \sqrt{||w_u||_2^2 + ||w_y||_2^2} \quad (3)$$

such that there exist (Δ, w_u, w_y) simultaneously satisfying

$$\begin{pmatrix} y + w_y \\ \hat{s} \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} u + w_u \\ \hat{t} \end{pmatrix}, \quad (4)$$

$$\hat{t} = \Delta \hat{s} \quad (5)$$

and

$$||\Delta||_\infty < \gamma_\Delta \text{ ?} \quad (6)$$

There are of course many variations on this. [Dav96] describes *Model Validation Decision Problems* (MVDPs) and *Model Validation Optimization Problems* (MVOPs); the latter is similar to the MVNMP described above except that it is γ_Δ that is minimized rather than γ_w. The MVNMP has been chosen because it is computationally convenient and always has a solution.

II. Model Validation in the ν-gap Metric

A. The ν-gap Metric and its Properties

In the context of closed-loop control, the ν-gap metric defined in, e.g. [Vin01] is useful. Given a plant P and a controller C, the robust stability margin $b(P,C)$ may be defined as

$$b(P,C) = \left\{ \left\| \begin{pmatrix} I \\ C \end{pmatrix} (I - PC)^{-1} \begin{pmatrix} I & -P \end{pmatrix} \right\|_\infty \right\}^{-1} \quad (7)$$

if C stabilizes P and 0 otherwise. Given two systems P_1, P_2, the ν-gap between them is defined as

$$\delta_\nu(P_1,P_2) = \begin{cases} \|\tilde{G}_2 G_1\|_\infty & \text{if } \det G_2^* G_1(j\Omega) \neq 0 \, \forall \Omega \in (-\pi,\pi) \\ & \text{and } \mathrm{wno}, \det, (G_2^* G_1) = 0, \\ 1 & \text{otherwise.} \end{cases} \quad (8)$$

where $G_i := \begin{pmatrix} M_i \\ Ni \end{pmatrix}$ and $\tilde{G}_i := \begin{pmatrix} -N_i & Mi \end{pmatrix}$, with $P_i = N_i M_i^{-1} = \tilde{M}_i^{-1} \tilde{N}_i$ being normalized coprime factorizations. The ν-gap gives some powerful results:

1) Given a plant P_1 and a compensator C, then $[P_2, C]$ is stable for all P_2 satisfying $\delta_\nu(P_1,P_2) \leq \beta$ if and only if $b(P_1,C) > \beta$.
2) Given two plants (P_1,P_2) and an value β such that $\exists C_1$ s.t. $b(P_1,C_1) > \beta$ then $[P_2,C]$ is stable for all compensators C satisfying $b(P_1,C) > \beta$ if and only if $\delta_\nu(P_1,P_2) \leq \beta$.

It is possible to find a parameterization of a 'ball' of systems in the ν-gap: this was first done in [VG94] and presentations may be found in [Dav96] and [SV01] and a discrete-time algorithm is presented in [Can01]. Essentially, the idea is given a nominal plant P, one can compute a 'central controller' K_{cent} such that

$$b(P,C) > \beta \text{ for all } C \in \mathcal{C}$$

where

$$\mathcal{C} := \left\{ C_1 : C_1 = \mathcal{F}_\ell(K_{\text{cent}}, Q), Q \in \mathcal{RH}_\infty, ||Q||_\infty \leq 1 \right\}$$

In the framework of Section I-C, G is given by K_{cent}^{-1}. Note that the existence of the required inverses is presented in [GGLD90].

For convenience, the notation $G_\nu(P,\beta)$ shall be used to denote the value of G computed above for a nominal system P.

B. Validation in the ν-gap Metric

The published literature, e.g. [Dav96], [SV01] contains conditions for model validation in the ν-gap metric. In this section, let

$$\begin{pmatrix} s \\ t \end{pmatrix} := M_\Delta(G) \begin{pmatrix} u \\ y \end{pmatrix}$$

Theorem 1 (LTI Validation, Nec. and Suff. Condition) *Given a nominal system P, recorded data (u,y) and scalars $\beta, \gamma_w > 0$, the following conditions are equivalent:*

1) *There exist noise sequences w_u and w_y and a linear time-invariant Δ satisfying (4)–(6) with $G = G_\nu(P,\beta)$ and also*

$$||w||_2 := \sqrt{\|w_u\|_2^2 + \|w_y\|_2^2} < \gamma_w$$

2) *There exist noise sequences (w_s, w_t) simultaneously satisfying*

$$\begin{pmatrix} (T_s + T_{ws})^*(T_s + T_{ws}) & T_t^* + T_{wt}^* \\ T_t + T_{wt} & I \end{pmatrix} \geq 0 \quad (9)$$

and

$$\sqrt{\|w_{\hat{u}}\|_2^2 + \|w_{\hat{y}}\|_2^2} < \gamma_w \quad (10)$$

where

$$\begin{pmatrix} w_{\hat{u}} \\ w_{\hat{y}} \end{pmatrix} = (M_\Delta(G))^{-1} \begin{pmatrix} w_s \\ w_t \end{pmatrix}.$$

Moreover, (w_u, w_y) satisfying the first condition are given by $(w_{\hat{u}}, w_{\hat{y}})$ satisfying the second.

Proof: See [SV01]. In essence, this is a straightforward application of the tangential Carathéodory-Fejér interpolation method of, e.g. [P$^+$92] and the Schur complement. ■

There is an analagous condition for LTV validation; this has not been included for the sake of brevity, but the interested reader will find it in [SV01].

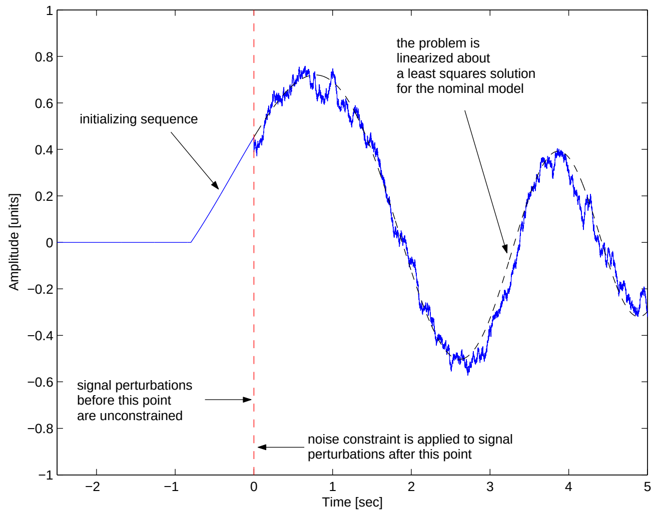


Fig. 2. Linearization about a Nominal Solution.

A difficulty with this condition is that its solution does not amount to a convex problem: there is a nonlinear $T_{ws}^* T_{ws}$ term that makes the problem nonlinear. In [SV01] an approximation was proposed giving a sufficient condition for non-invalidation:

Theorem 2 (LTI Validation, Sufficient Condition) *Given a nominal system P, recorded data (u,y) and scalars $\beta, \gamma_w > 0$, then there exist noise sequences w_u and w_y and a linear time-invariant Δ satisfying (4)–(6) with $G = G_\nu(P,\beta)$ and also*

$$||w||_2 := \sqrt{\|w_u\|_2^2 + \|w_y\|_2^2} < \gamma_w$$

***if** (but not only if!) there exist noise sequences (w_s, w_t) simultaneously satisfying*

$$\begin{pmatrix} T_s^* T_s + T_{ws}^* T_s + T_s^* T_{ws} & T_t^* + T_{wt}^* \\ T_t + T_{wt} & I \end{pmatrix} \geq 0 \qquad (11)$$

and

$$\sqrt{\|\tilde{w}_u\|_2^2 + \|\tilde{w}_y\|_2^2} < \gamma_w \qquad (12)$$

where

$$\begin{pmatrix} \tilde{w}_u \\ \tilde{w}_y \end{pmatrix} = (M_\Delta(G))^{-1} \begin{pmatrix} w_s \\ w_t \end{pmatrix}.$$

Moreover, (w_u, w_y) satisfying the first condition are given by (w_u, w_y) satisfying the second.

Proof: Again, the proof is available in [SV01], relying on the Schur complement. ■

III. Initial States and Pre-Record Sequences

A. The Essential Idea

In the work discussed so far, it is assumed that the system under consideration is initially at equilibrium. In [Ste01], the author suggests that an initial state might be built up by allowing the noise record to start before the origin, though the idea is not explored in detail.

To ease readability, define the *pre-record zero-padding* operator $p_n(z)$ for a sequence $z = \{z(0), z(1), \ldots, z(k-1)\}$,

$$p_n(z) := \begin{cases} 0, & i = 0, 1, \ldots, (n-1), \\ z(i-n), & i = n, (n+1), \ldots, (n+k-1). \end{cases}$$

Define also the reverse operator, $q_n(z)$ for a sequence $z = \{z(0), z(1), \ldots, z(k+n-1)\}$

$$q_n(x) := z(i+n), \; i = 0, 1, \ldots, (k-1)$$

This allows us to consider an interpolation problem starting before the zero instant. This may be described by the following equation:

$$\begin{pmatrix} p_n(y) + w_y \\ \hat{s} \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} p_n(u) + w_u \\ \hat{t} \end{pmatrix}. \qquad (13)$$

This can be used to formulate a trivial re-expression of Theorem 1, allowing a pre-record noise sequence of some prechosen length to be present in the tangential Carathéodory-Fejér interpolation constraint but not in the exogenous noise constraint.

Theorem 3 (LTI Validation, Nec. and Suff. Condition) *Given a nominal system P, recorded data (u,y) and scalars $\beta, \gamma_w > 0$ and some scalar n, the following conditions are equivalent:*

1) *There exist noise sequences w_u and w_y and a linear time-invariant Δ satisfying (5,6,13) with $G = G_\nu(P,\beta)$ and also*

$$||w||_2 := \sqrt{\|q_n(w_u)\|_2^2 + \|q_n(w_y)\|_2^2} < \gamma_w$$

2) *There exist noise sequences (w_s, w_t) simultaneously satisfying*

$$\begin{pmatrix} (T_s + T_{ws})^*(T_s + T_{ws}) & T_t^* + T_{wt}^* \\ T_t + T_{wt} & I \end{pmatrix} \geq 0 \qquad (14)$$

where

$$\begin{pmatrix} s \\ t \end{pmatrix} := M_\Delta(G) \begin{pmatrix} p_n(u) \\ p_n(y) \end{pmatrix}$$

and

$$\sqrt{\|q_n(\tilde{w}_u)\|_2^2 + \|q_n(\tilde{w}_y)\|_2^2} < \gamma_w \qquad (15)$$

where

$$\begin{pmatrix} \tilde{w}_u \\ \tilde{w}_y \end{pmatrix} = (M_\Delta(G))^{-1} \begin{pmatrix} w_s \\ w_t \end{pmatrix}.$$

Moreover, (w_u, w_y) satisfying the first condition are given by $(\tilde{w}_u, \tilde{w}_y)$ satisfying the second.

Proof: This is essentially re-stating Theorem 1 with a different noise set. Proof follows exactly the same lines. ■
As always, there is an analagous LTV version.

B. Why a Naive Approach Will Not Work

Having reached this stage, a natural thing to do would be to apply the approximation technique used to generate Theorem 2. Sadly, this does not work as illustrated by the following:

Theorem 4 (LTI Equivalence Theorem) *Given sequences $s \in \mathcal{S}_\ell^q$, $t \in \mathcal{S}_\ell^p$ with $t_0 \neq 0$, define $\sigma \in \mathcal{S}_\ell^q$ such that $\sigma_i = 0$ and $\tau \in \mathcal{S}_\ell^p$ such that $\tau_i = 0$, with $\hat{s}$ and $\hat{t}$ given by*

$$\hat{s} = \{\sigma_0, \sigma_1, \dots \sigma_{\ell-1}, s_0, s_1, \dots s_{k-1}\} \quad (16)$$

$$\hat{t} = \{\tau_0, \tau_1, \dots \tau_{\ell-1}, t_0, t_1, \dots t_{k-1}\} \quad (17)$$

Then there exist sequences $w_\sigma \in \mathcal{S}_\ell^q$, $w_\tau \in \mathcal{S}_\ell^p$ and $w_s, w_t \in \mathbf{W}_s \times \mathbf{W}_t$ such that

$$\begin{bmatrix} T_{\hat{s}}^* T_{\hat{s}} + T_{w_{\hat{s}}}^* T_{\hat{s}} + T_{\hat{s}}^* T_{w_{\hat{s}}} & T_{\hat{t}}^* + T_{w_{\hat{t}}}^* \\ T_{\hat{t}} + T_{w_{\hat{t}}} & I \end{bmatrix} \geq 0 \quad (18)$$

where $w_{\hat{s}} = \{w_\sigma, w_s\}$ and $w_{\hat{t}} = \{w_\tau, w_t\}$, if and only if there exist sequences $w_s, w_t \in \mathbf{W}_s \times \mathbf{W}_t$ such that

$$\begin{bmatrix} T_s^* T_s + T_{w_s}^* T_s + T_s^* T_{w_s} & T_t^* + T_{w_t}^* \\ T_t + T_{w_t} & I \end{bmatrix} \geq 0 \quad (19)$$

and the only w_σ and w_τ satisfying (18) are $w_\sigma = \{0, 0, \dots, 0\}$ and and $w_\tau = \{0, 0, \dots, 0\}$.

A rather tedious proof is given in [Aug04, App. A]. There is a similar though slightly weaker result for LTV conditions. Because of the approximation, used to remove the awkward quadratic unknown term, there is nothing to be gained by naively zero-padding the front ends of the sequences and more imagination is needed to get a useful improvement.

C. Linearization about a Least-Squares Solution

Work to date has considered a problem linearized about the 'zero noise' case. There is no particular reason why this needs to be so.

Given $\tilde{u} \in \mathcal{S}_k^{n_u}$, $\tilde{y} \in \mathcal{S}_k^{n_y}$, and some integer n define $u = p_n(\tilde{u})$ and $y = p_n(\tilde{y})$.

Theorem 5 (LTI Validation, Nec. and Suff. Condition) *Given a nominal system P, recorded data $(u,y) \in \mathcal{S}_k^q \times \mathcal{S}_k^p$ and scalars $\beta, \gamma_w > 0$, a non-negative integer n, and sequences $(w_{u0}, w_{y0}) \in \mathcal{S}_k^q \times \mathcal{S}_k^p$ the following conditions are equivalent:*

1) *There exist noise sequences w_u and w_y and a linear time-invariant Δ satisfying (5,6,13) with $G = G_\nu(P, \beta)$ and also*

$$||w||_2 := \sqrt{\|q_n(w_u)\|_2^2 + \|q_n(w_y)\|_2^2} < \gamma_w$$

2) *There exist noise sequences (w_s, w_t) simultaneously satisfying*

$$\begin{pmatrix} (T_s + T_{ws})^*(T_s + T_{ws}) & T_t^* + T_{wt}^* \\ T_t + T_{wt} & I \end{pmatrix} \geq 0 \quad (20)$$

where

$$\begin{pmatrix} s \\ t \end{pmatrix} := M_\Delta(G) \begin{pmatrix} p_n(u) + w_{u0} \\ p_n(y) + w_{y0} \end{pmatrix}$$

and

$$\sqrt{\|q_n(w_{\hat{u}})\|_2^2 + \|q_n(w_{\hat{y}})\|_2^2} < \gamma_w \quad (21)$$

where

$$\begin{pmatrix} w_{u0} + p_n(w_{\hat{u}}) \\ w_{y0} + p_n(w_{\hat{y}}) \end{pmatrix} = (M_\Delta(G))^{-1} \begin{pmatrix} w_s \\ w_t \end{pmatrix}.$$

Moreover, (w_u, w_y) satisfying the first condition are given by $(w_{\hat{u}}, w_{\hat{y}})$ satisfying the second.

This is a trivial extension of the earlier theorems and proof follows easily, and it is easy to approximate this by neglecting the quadratic term. This gives the following useful result:

Theorem 6 (LTI Validation, Sufficient Condition) *Given a nominal system P, recorded data $(u,y) \in \mathcal{S}_k^q \times \mathcal{S}_k^p$ and scalars $\beta, \gamma_w > 0$, a non-negative integer n, and sequences $(w_{u0}, w_{y0}) \in \mathcal{S}_k^q \times \mathcal{S}_k^p$ then there exist noise sequences w_u and w_y and a linear time-invariant Δ satisfying (5,6,13) with $G = G_\nu(P, \beta)$ and also*

$$||w||_2 := \sqrt{\|q_n(w_u)\|_2^2 + \|q_n(w_y)\|_2^2} < \gamma_w$$

if *there exist noise sequences (w_s, w_t) simultaneously satisfying*

$$\begin{pmatrix} (T_s + T_{ws})^*(T_s + T_{ws}) & T_t^* + T_{wt}^* \\ T_t + T_{wt} & I \end{pmatrix} \geq 0 \quad (22)$$

where

$$\begin{pmatrix} s \\ t \end{pmatrix} := M_\Delta(G) \begin{pmatrix} p_n(u) + w_{u0} \\ p_n(y) + w_{y0} \end{pmatrix}$$

and

$$\sqrt{\|q_n(w_{\hat{u}})\|_2^2 + \|q_n(w_{\hat{y}})\|_2^2} < \gamma_w \quad (23)$$

where

$$\begin{pmatrix} w_{u0} + p_n(w_{\hat{u}}) \\ w_{y0} + p_n(w_{\hat{y}}) \end{pmatrix} = (M_\Delta(G))^{-1} \begin{pmatrix} w_s \\ w_t \end{pmatrix}.$$

Moreover, (w_u, w_y) satisfying the first condition are given by $(w_{\hat{u}}, w_{\hat{y}})$ satisfying the second.

Again, proof follows trivially along the lines of that of earlier theorems. There is an analagous LTV result.

It is possible to choose any sensible starting point. In [Aug04, Chapter 5], a least-squares method is used to linearize the problem about the nominal model.

D. First Numerical Example

Synthetic data were obtained by finding first 94 elements of the response of the discrete time system

$$P_{\text{true}}(z) = \frac{0.02247z + 0.02093}{z^2 - 1.764z + 0.8075}$$

to the chirp signal $u_{\text{true}}(k) = \sin\frac{\pi}{30}\left(k + 0.025k^2\right)$. The first 14 samples of the data record were discarded, giving the validation data $(\tilde{u}, \tilde{y})$ shown in Figure 3(a).

Assuming a nominal model

$$P = \frac{0.01867z + 0.01746}{z^2 - 1.783z + 0.8187}$$

and taking $n = 2$, w_{u0} and w_{y0} were found using a least squares technique assuming zero modle perturbation, as

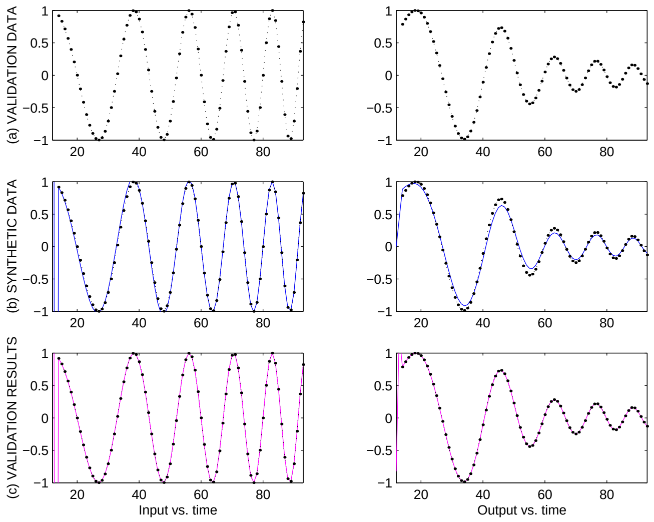


Fig. 3. First Numerical Example

described in the previous section and detailed in [Aug04, Chap. 5] (Figure 3(b)).

The method described in the previous section was used to find the sequences giving the smallest value of $\gamma = \{||w_u||_2^2 + ||w_y||_2^2\}^{1/2}$ consistent with a system $\hat{P} \in \mathcal{B}_\nu^{\mathrm{LTI}}(P, 0.12)$.[1] The minimum was found to be zero, corresponding to zero noise sequences (Figure 3(c)). Since there was no noise on the original data, and $\delta_\nu(P_{\mathrm{true}}, P) = 0.12$, these results are as expected.

IV. Relinearizing to Improve Approximations

A. Outline Algorithm

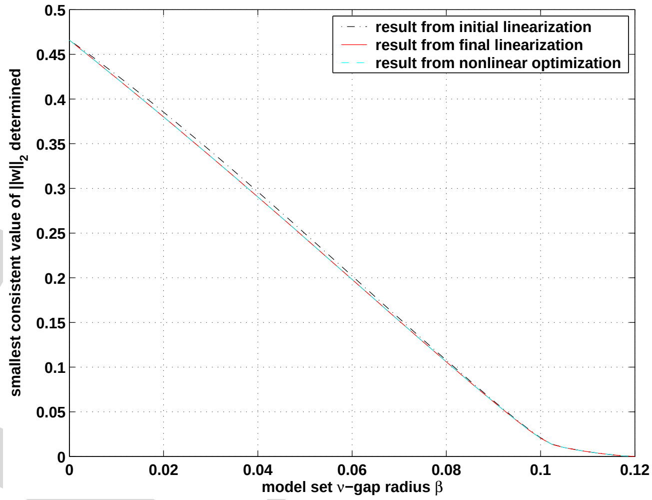
The techniques described in the previous section can also be used to improve initial approximations by iteratively relinearizing about successive solutions.

Algorithm

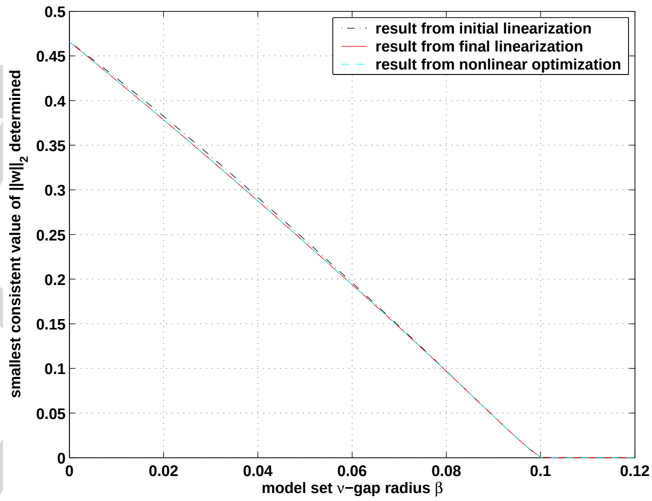
Inputs: a $(p \times q)$ model set $\mathcal{P}$ and input-output sequences $(u \in \mathcal{S}_\ell^q, y \in \mathcal{S}_\ell^p)$ and an objective function $c(w_u, w_y)$ and the required tolerance for the result τ_γ. BEGIN.

1) Define counter i and let $i = 0$.
2) Linearize the full non-invalidation problem, and use the linear approximation to find sequences (w_{u0}, w_{y0}) consistent with the problem such that $c(w_{u0}, w_{y0})$ is minimized; denote the minimum γ_0
3) Increment the counter i by 1.
4) Re-linearize the full non-invalidation problem about $(w_{u(i-1)}, w_{y(i-1)})$, and and use the linear approximation to find sequences $(w_{ui} \in \mathcal{S}_\ell^q, w_{yi})$ consistent with the problem such that $c(w_{ui}, w_{yi})$ is minimized; denote the minimum γ_i.
5) If $\gamma_{i-1} - \gamma_i > \tau_\gamma$, go to step 3; otherwise continue.
6) Let $\hat{\gamma} = \gamma_i$, $\hat{w}_u = w_{ui}$ and $\hat{w}_y = w_{yi}$.

END.



(a) LTI



(b) LTV

Fig. 4. Numerical Example: Iterative Refinement

B. Second Numerical Example

As a numerical example, we shall consider the problem addressed in [SV01]: invalidation data of length $k = 60$ is generated from the impulse response of

$$P_{\mathrm{true}}(z) = \frac{-0.1157z^2 + 0.2671z + 0.002967}{z^2 - 1.893z + 0.905}$$

and this is compared to a nominal model

$$P(z) = \frac{0.0625z^2 + 0.125z + 0.0625}{z^2 - 2z + 1}$$

The following optimization problem is considered: minimize

$$\gamma_w := \left\{||w_u||_2^2 + ||w_y||_2^2\right\}^{1/2}$$

[1]Full details are presented in [Aug04, Chap. 5].

subject to the constraint

$$(y+w_y)=\hat{P}(u+w_u)$$

for some $\hat{P}\in\mathcal{B}_\nu(P,\beta)$, considering the LTI and LTV cases separately.

For each of the LTI and LTV cases, three quantities are calculated for a range of β values:

(i) the upper bound on the smallest consistent $\|w\|_2$ from the initial relinearization; the same quantity as that investigated in [SV01]
(ii) the upper bound on the smallest consistent $\|w\|_2$ obtained from two further relinearizations
(iii) a final upper bound obtained by applying MATLAB's `fmincon` function to the full non-invalidation problem using the results of (ii) as a starting point. (This uses a 'brute force' optimization method, fully documented in [Mat00].)

The results are shown in Figures 4(a) and 4(b). In both cases we observe that the three-iteration relinearization method provides a slight improvement on the initial linearization, and that the nonlinear optimization is not able to do noticeably better.

V. Conclusions and Future Work

A. Conclusions

The following conclusions can be drawn in respect of the type of model validation problem under consideration:

1) It is possible to account for a non-zero initial state through use of a pre-record noise sequence. A numerical example demonstrates that this need not be lengthy.
2) Linearization of the problem is important: a poor choice of linearization can make the pre-record sequences ineffective.
3) It is possible to refine approximated invalidation conditions. In a numerical example, it was demonstrated that a small improvement can be achieved. However, it is not guaranteed that the solution will converge to the 'true' solution to the non-approximated problem.

B. Future Work

The techniques described in this paper have been applied to a challenging flight control problem in [Aug04, Chap. 7]. The authors hope to publish this in a wider arena in due course.

C. Acknowledgements

The author gratefully thanks the UK's Engineering and Physical Sciences Research Council (EPSRC) for funding this research and Dr G. Vinnicombe who supervised his doctoral studies. Thanks are also due to Qinetiq and to Alenia Marconi Systems of Chelmsford, Essex. The YALMIP LMI parser [Löf04] and Jos F. Sturm's SeDuMi optimizer.

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