ROBUST RECEDING HORIZON CONTROL WITH AN EXPECTED VALUE COST

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Abstract: This paper is concerned with the control of constrained linear discrete-time systems subject to bounded state disturbances and arbitrary convex state and input constraints. The paper employs a class of finite horizon feedback control policies parameterized as affine functions of the system state, calculation of which has recently been shown to be tractable via a convex reparameterization. When minimizing the expected value of a finite horizon quadratic cost, we show that the value function in this finite horizon control problem is convex. When policies of this type are used in the synthesis of a robust receding horizon controller, we provide sufficient conditions under which the closed-loop system is input-to-state stable (ISS).

Keywords: Robust Control, Constraint Satisfaction, Robust Optimization, Predictive Control, Optimal Control

1. INTRODUCTION

This paper proposes a class of robust receding horizon control (RHC) laws for constrained linear discrete-time systems subject to bounded state disturbances. We consider a class of affine feedback control policies parameterized as affine functions of the system state, calculation of which has been shown to be tractable via a convex reparameterization (Goulart *et al.*, 2006; Ben-Tal *et al.*, 2006). When minimizing the expected value of a quadratic function of the states and inputs over a finite horizon, we show that the resulting value function is convex and provide sufficient conditions, when used in the design of a RHC law, to establish that the closed-loop system is input-to-state stable (ISS). This work is an extension of the results in (Goulart *et al.*, 2006) to the case where the state and input constraints are general convex sets, and is an abbreviated version of results appearing in (Goulart and Kerrigan, 2005), where proofs of the supporting results may be found.

In a recent publication (Goulart *et al.*, 2006), the authors demonstrate that the *non-convex* affine state feedback optimization problem can be reparameterized as an *equivalent* but *convex* problem by recasting the optimization problem in terms of affine disturbance or error feedback laws (see also (Ben-Tal *et al.*, 2006) for the output feedback case). They further demonstrate that, when implemented in a receding horizon fashion with a particular cost function, the closed-loop system is input-to-state stable (ISS) when the constraints and disturbance sets are polytopic.

Such a control parameterization is attractive because it serves as a computationally tractable alternative (though with a potentially reduced region of attraction) to those control schemes based on optimization over *arbitrary* control laws such as (Scokaert and Mayne, 1998). In general, such methods require, at each time step, the solution of an infinite dimensional optimization problem when the disturbance set is non-polytopic, or at best one whose dimension grows *expo-*

nentially with horizon length in the polytopic case. In contrast the method proposed here requires the solution of a convex problem whose dimension grows only *quadratically* with the horizon length, while potentially providing a much larger region of attraction than schemes that calculate a sequence of perturbations to a *fixed* pre-stabilizing control law, such as (Chisci *et al.*, 2001; Lee and Kouvaritakis, 1999; Mayne *et al.*, 2005).

In this paper we present a generalization of the results in (Goulart *et al.*, 2006), using the expected value of a quadratic cost. We demonstrate that, for systems with *arbitrary* convex state and input constraints and disturbance sets, the resulting value function is convex and lower semicontinuous when optimizing over state feedback policies, and provide conditions under which input-to-state stability can be established for such systems using convex Lyapunov functions. Since the constraints and disturbance sets we consider are arbitrary convex sets, the proofs differ substantially from those required in the case where these sets are polytopic, as in (Goulart *et al.*, 2006). This generalization is of particular interest, for example, in the case where the constraint set is polytopic and the disturbance set is $2-$norm bounded, so that the resulting optimization problem can be solved as a tractable second-order cone program (SOCP), but for which no proof of stability exists at present.

Notation: A continuous function $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a $\mathcal{K}$-function if it is strictly increasing and $\gamma(0) = 0$; it is a $\mathcal{K}_\infty$-function if, in addition, $\gamma(s) \rightarrow \infty$ as $s \rightarrow \infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}$-function if for all $k \geq 0$, the function $\beta(\cdot, k)$ is a $\mathcal{K}$-function and for each $s \geq 0$, $\beta(s, \cdot)$ is decreasing with $\beta(s, k) \rightarrow 0$ as $k \rightarrow \infty$. $\mathbb{E}$ is the expectation operator. Given sets X and Y, $X + Y := \{x + y \mid x \in X,\ y \in Y\}$, $\operatorname{int}X$ represents the interior of X and $\operatorname{lin}X$ its linear hull (i.e. the smallest subspace containing X). Given a vector x and matrices A and B, $A \otimes B$ is the Kronecker product of A and B, $\mathcal{N}(A)$ is the null space of A, $\operatorname{tr}(A)$ is the trace of A, $\operatorname{vec}(A)$ denotes the vector formed by stacking the columns of A into a vector, $A \succ 0$ ($A \succeq 0$) means that A is positive (semi)definite, $\|x\|_A^2 := x'Ax$ and $\|x\| := \sqrt{x'x}$.

2. DEFINITIONS AND STANDING ASSUMPTIONS

Consider the following discrete-time linear time-invariant system:

$$x^+ = Ax + Bu + w, \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state at the current time instant, x^+ is the state at the next time instant, $u \in \mathbb{R}^m$ is the control input and $w \in \mathbb{R}^n$ is the disturbance. It is assumed that (A, B) is stabilizable and that at each sample instant a measurement of the state is available. It is assumed that the current and future values of the disturbance are unknown and may change from one time instant to the next, but are contained in a compact set W containing the origin in its relative interior, and are independent and identically distributed with zero mean and covariance $\mathbb{E}[ww'] =: C_w \succeq 0$. Finally, we assume that the covariance C_w is sensibly defined with respect to the set W, i.e. we assume that $\mathcal{N}(C_w) \cap \operatorname{lin}W = \{0\}$.

The system is subject to mixed convex constraints on the state and input, so that the system must satisfy $(x, u) \in Z$ where $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ is a convex and compact set containing the origin in its interior. A design goal is to guarantee that the state and input of the closed-loop system remain in Z for all time and for all allowable disturbance sequences. We further assume that a target/terminal constraint set $X_f \subset \mathbb{R}^n$ is given, which is convex, compact and contains the origin in its interior.

In the sequel, predictions of the system's evolution over a finite control/planning horizon will be used to define a suitable control policy. Let the length N of this planning horizon be a positive integer and define stacked versions of the predicted input, state and disturbance vectors $\mathbf{u} \in \mathbb{R}^{mN}$, $\mathbf{x} \in \mathbb{R}^{n(N+1)}$ and $\mathbf{w} \in \mathbb{R}^{nN}$, respectively, as $\mathbf{x} := [x_0', \ldots, x_N']'$, $\mathbf{u} := [u_0', \ldots, u_{N-1}']'$ and $\mathbf{w} := [w_0', \ldots, w_{N-1}']'$, where $x_0 = x$ denotes the current measured value of the state and $x_{i+1} := Ax_i + Bu_i + w_i$, for all $i \in \{0, \ldots, N-1\}$ denotes the prediction of the state after i time instants. Let the se $\mathcal{W} := W^N := W \times \cdots \times W$, so that $\mathbf{w} \in \mathcal{W}$. Define the matrix $\mathbf{C}_w := I \otimes C_w$, so that $\mathbb{E}[\mathbf{w}\mathbf{w}'] = \mathbf{C}_w$ and $\mathcal{N}(\mathbf{C}_w) \cap \operatorname{lin}\mathcal{W} = \{0\}$. Define a convex and compact set $\mathcal{Y}$, appropriately constructed from Z and X_f, such that the constraints are equivalent to $(\mathbf{x}, \mathbf{u}) \in \mathcal{Y}$, i.e.

$$\mathcal{Y} := \left\{ (\mathbf{x}, \mathbf{u}) \;\middle|\; \begin{array}{c} (x_i, u_i) \in Z, \forall i \in \{0, \ldots, N-1\} \\ x_N \in X_f \end{array} \right\}. \tag{2}$$

Finally, using the relation (1), it is straightforward to define matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{E}$ such that $\mathbf{x} = \mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w}$.

3. AFFINE FEEDBACK CONTROL POLICIES

For reasons of computational tractability, we elect to work with the restricted class of finite horizon *affine* state feedback policies for control of the system (1), i.e. those where the control at each time is parameterized as $u_i = g_i + \sum_{j=0}^{i} L_{i,j} x_j$, rather than with *arbitrary* functions of the state as in (Scokaert and Mayne, 1998). Such an affine parameterization has been shown (Goulart *et al.*, 2006; Ben-Tal *et al.*, 2006) to be equivalent to the class of control policies parameterized as an affine function of the sequence of past *disturbances* (Ben-Tal *et al.*, 2004; Löfberg, 2003; van Hessem and Bosgra, 2002), so that

$$u_i = v_i + \sum_{j=0}^{i-1} M_{i,j} w_j, \quad \forall i \in \{0, \ldots, N-1\} \tag{3}$$

where each $M_{i,j} \in \mathbb{R}^{m\times n}$ and $v_i \in \mathbb{R}^m$. Note that the past disturbances are easily recovered using the relation $w_j = x_{j+1} - Ax_j - Bu_j$. Define the vector $\mathbf{v} \in \mathbb{R}^{mN}$ and the matrix $\mathbf{M} \in \mathbb{R}^{mN\times nN}$ such that

$$\mathbf{M} := \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-2} & 0 \end{bmatrix}, \ \mathbf{v} := \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix} \tag{4}$$

so that the control input sequence can be written in vector form as $\mathbf{u} = \mathbf{M}\mathbf{w} + \mathbf{v}$. Define the set of admissible $(\mathbf{M}, \mathbf{v})$, for which the constraints (2) are satisfied, as:

$$\Pi_N(x) := \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (4)} \\ \mathbf{x} = \mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w} \\ \mathbf{u} = \mathbf{M}\mathbf{w} + \mathbf{v} \\ (\mathbf{x}, \mathbf{u}) \in \mathcal{Y},\ \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}, \tag{5}$$

and define the set of states for which such an admissible control policy exists as

$$X_N := \{x \in \mathbb{R}^n \mid \Pi_N(x) \neq \emptyset\}. \tag{6}$$

The parameterization (4) is of particular interest because the set $\Pi_N(x)$ is convex (Goulart *et al.*, 2006), whereas the set of feasible state feedback parameters is non-convex, in general.

Remark 1. The method of proof for convexity of the set $\Pi_N(x)$ in (Goulart *et al.*, 2006) is insufficient for the general convex state and input constraints presented here. However, proof of convexity is straightforward by noting that the set

$$\mathcal{C}_N := \bigcap_{\mathbf{w}\in\mathcal{W}} \left\{ (\mathbf{M}, \mathbf{v}, x) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (4)} \\ \mathbf{x} = \mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w} \\ \mathbf{u}=\mathbf{M}\mathbf{w}+\mathbf{v},\ (\mathbf{x}, \mathbf{u})\in\mathcal{Y} \end{array} \right\}$$

is closed and convex, since it is the intersection of closed and convex sets.

We note that policies in this class can, in many cases, provide a larger region of attraction than policies based on perturbations to fixed linear feedback laws as in (Chisci *et al.*, 2001; Lee and Kouvaritakis, 1999; Mayne *et al.*, 2005), regardless of horizon length. Before proceeding, we demonstrate this via the following example:

Example 2. Consider the system

$$x^+ = 2x + 2u + w,$$

subject to the following input and terminal constraints:

$$u \in \{u \mid |u| \leq 0.7\}$$
$$X_f = \{x \mid |x| \leq 0.5\}$$

and subjected to bounded disturbances $|w| \leq 0.25$. We define a stabilizing controller K = -1.25, so that $(A + BK) = -\frac{1}{2}$ and the set X_f is robust positively invariant for the system $x^+ = (A + BK)x + w$. For increasing horizon length N, we consider the size of the set X_N in (6) for this system, as well as the size of the set of feasible initial conditions X_N^K when the control policy for the system is restricted to perturbations to the *fixed* linear feedback law $u = Kx$, i.e. those parameterized as $u_i = Kx_i + g_i$. The sizes of these sets with increasing horizon length are shown in Figure 1, where it is clear that $X_N^K \subset X_4$ for any N.

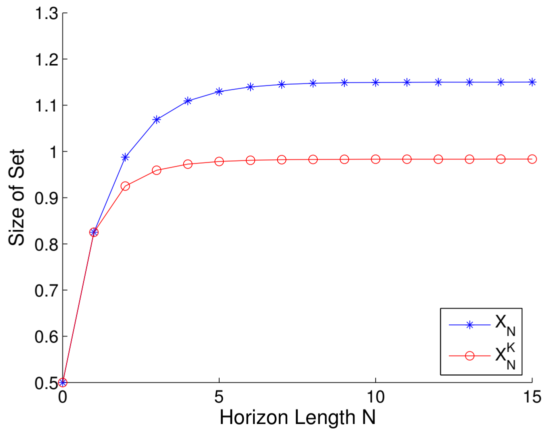

Fig. 1. Sizes of X_N and X_N^K with increasing N

4. AN EXPECTED VALUE COST FUNCTION

We consider a function which is quadratic in the state and control sequence, and seek a control policy that will minimize its expected value (over all disturbances) over the planning horizon. We define

$$V_N(x, \mathbf{M}, \mathbf{v}) := \mathbb{E}\left[\|x_N\|_P^2 + \sum_{i=0}^{N-1} (\|x_i\|_Q^2 + \|u_i\|_R^2)\right], \tag{7}$$

where, for all i, $x_{i+1} = Ax_i + Bu_i + w_i$, $\mathbf{u} := \mathbf{M}\mathbf{w} + \mathbf{v}$, and Q, R and P are positive definite. We define an optimal policy $(\mathbf{M}^*(x), \mathbf{v}^*(x))$ to be one which minimizes $V_N(x, \cdot, \cdot)$ over the set of feasible control policies, i.e.

$$(\mathbf{M}^*(x), \mathbf{v}^*(x)) := \underset{(\mathbf{M},\mathbf{v})\in\Pi_N(x)}{\operatorname{argmin}} V_N(x, \mathbf{M}, \mathbf{v}). \tag{8}$$

We also define the value function $V_N^* : X_N \to \mathbb{R}_{\geq 0}$ to be

$$V_N^*(x) := \min_{(\mathbf{M},\mathbf{v})\in\Pi_N(x)} V_N(x, \mathbf{M}, \mathbf{v}). \tag{9}$$

We assume for the moment that the minimizer in (8) exists and is well-defined. The receding horizon control policy $\mu_N : X_N \to \mathbb{R}^m$ is defined by the first part of the optimal affine feedback control policy, i.e. $\mu_N(x) := v_0^*(x)$. Note that the control law $\mu_N(\cdot)$ is time-invariant and is, in general, a nonlinear function of the current state. The closed-loop system becomes

$$x^+ = Ax + B\mu_N(x) + w. \tag{10}$$

We first demonstrate that $V_N(x, \cdot, \cdot)$ is convex, so that (9) represents a convex optimization problem.

Proposition 3. (Convex Cost Function). The function $(x, \mathbf{M}, \mathbf{v}) \mapsto V_N(x, \mathbf{M}, \mathbf{v})$ is convex and quadratic in the state x and parameter $\mathbf{M}$, and strictly convex and quadratic in the parameter $\mathbf{v}$.

PROOF. Since $\mathbb{E}[\mathbf{w}] = 0$ and $\mathbf{w}$ is independent of both $\mathbf{v}$ and $\mathbf{M}$, the cost function can be written as:

$$V_N(x, \mathbf{M}, \mathbf{v}) = \mathbb{E}\Big[\|\mathbf{v}\|_{\mathcal{S}}^2 + x' H_v \mathbf{v} + \mathbf{w}' H_M \mathbf{M}\mathbf{w} + \|\mathbf{M}\mathbf{w}\|_{\mathcal{S}}^2 + \|\mathbf{A}x\|_{\mathcal{Q}}^2 + \|\mathbf{E}\mathbf{w}\|_{\mathcal{Q}}^2\Big]$$

where $\mathcal{Q} := \left[\begin{smallmatrix} I \otimes Q & \\ & P \end{smallmatrix}\right]$, $\mathcal{R} := I \otimes R$, $H_v := 2\mathbf{A}'\mathcal{Q}\mathbf{B}$, $H_M := 2\mathbf{E}'\mathcal{Q}\mathbf{B}$, and $\mathcal{S} := \mathbf{B}'\mathcal{Q}\mathbf{B} + \mathcal{R}$. Recalling that $\mathbb{E}[\mathbf{w}'X\mathbf{w}] = \mathrm{tr}(X\mathbf{C}_w) = \mathrm{tr}(\mathbf{C}_w X)$ for any X yields

$$V_N(x, \mathbf{M}, \mathbf{v}) = \|\mathbf{v}\|_{\mathcal{S}}^2 + x' H_v \mathbf{v} + \mathrm{tr}(\mathbf{M}'\mathcal{S}\mathbf{M}\mathbf{C}_w) + \mathrm{tr}(\mathbf{C}_w H_M \mathbf{M}) + \gamma \quad (11)$$

$$= \|\mathbf{v}\|_{\mathcal{S}}^2 + x' H_v \mathbf{v} + \|\mathrm{vec}(\mathbf{M})\|_{(\mathbf{C}_w \otimes \mathcal{S})}^2 + [\mathrm{vec}(H_M' \mathbf{C}_w)]'\mathrm{vec}(\mathbf{M}) + \gamma. \quad (12)$$

where $\gamma := \mathrm{tr}(\mathbf{E}'\mathcal{Q}\mathbf{E}\mathbf{C}_w) + \|\mathbf{A}x\|_{\mathcal{Q}}^2$. The result follows immediately, since $\mathbf{C}_w \succeq 0$ implies $(\mathbf{C}_w \otimes \mathcal{S}) \succeq 0$ (Horn and Johnson, 1991, Thm. 4.2.12), and $\mathcal{S} \succ 0$ since $\mathcal{R} \succ 0$ by assumption. □

Proposition 4. The function $V_N(x, \cdot, \cdot)$ attains its minimum on the set $\Pi_N(x)$.

Remark 5. Since the cost function $V_N(x, \cdot, \cdot)$ and set of feasible policies $\Pi_N(x)$ are convex, the problem (9) can be solved using standard methods in convex optimization. In particular, if the constraints $\mathcal{Y}$ are polytopic, then (9) can be expressed as a second-order cone program (SOCP) when W is ellipsoidal or $2-$norm bounded, and as a quadratic program (QP) when W is polytopic (Goulart *et al.*, 2006), both in a polynomial number of variables and constraints.

5. PRELIMINARY RESULTS

We wish to find conditions under which the closed-loop system (10) is input-to-state stable (ISS). To do this, we requires some preliminary results related to the convexity of the value function $V_N^*(\cdot)$ in (9), and to input-to-state stability for systems with convex Lyapunov functions. We first consider convexity and continuity of the value function $V_N^*(\cdot)$ in (9); this property will prove useful in our subsequent consideration of stability for the closed loop system (10). Complete proofs of the results in this section may be found in (Goulart and Kerrigan, 2005).

Proposition 6. If X_N has non-empty interior, then the receding horizon control law $\mu_N(\cdot)$ is unique on X_N and continuous on $\mathrm{int}X_N$. The value function $V_N^*(\cdot)$ is convex on X_N, continuous on $\mathrm{int}X_N$ and lower semicontinuous everywhere on X_N.

We next consider the input-to-state stability of systems with convex value functions. We can then exploit the convexity of the value function $V_N^*(\cdot)$ to provide conditions in which the closed-loop system (10) is input-to-state stable (ISS) when implemented in a receding horizon fashion.

Consider a nonlinear, time-invariant, discrete-time system of the form

$$x^+ = f(x, w), \quad (13)$$

where $x \in \mathbb{R}^n$ is the state and $w \in \mathbb{R}^l$ is a disturbance that takes on values in a compact set $W \subset \mathbb{R}^l$ containing the origin. It is assumed that the state is measured at each time instant, that $f : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$ is continuous at the origin and that $f(0, 0) = 0$. Given an initial state x and a disturbance sequence $w(\cdot)$, let the solution to (13) at time k be denoted by $\phi(k, x, w(\cdot))$, where $w(\cdot)$ is taken from $\mathcal{M}_W$, the set of infinite disturbance sequences drawing values from W. For systems of this type, a useful definition of stability is input-to-state stability:

Definition 7. (ISS). The system (13) is *input-to-state stable (ISS)* in $\mathcal{X} \subseteq \mathbb{R}^n$ if there exist a $\mathcal{KL}$-function $\beta(\cdot)$ and a $\mathcal{K}$-function $\gamma(\cdot)$ such that for all initial states $x \in \mathcal{X}$ and disturbance sequences $w(\cdot) \in \mathcal{M}_W$, the solution of the system satisfies $\phi(k, x, w(\cdot)) \in \mathcal{X}$ and for all $k \in \mathbb{N}$,

$$\|\phi(k, x, w(\cdot))\| \leq \beta(\|x\|, k) + \gamma\left(\sup\{\|w(\tau)\| \mid \tau \in \{0, \ldots, k-1\}\}\right) \quad (14)$$

Lemma 8. (ISS-Lyapunov function). (Jiang and Wang, 2001, Lem. 3.5): The system (13) is ISS in $\mathcal{X} \subseteq \mathbb{R}^n$ if the following conditions are satisfied:

- $\mathcal{X}$ contains the origin in its interior and is robust positively invariant for (13), i.e. $f(x, w) \in \mathcal{X}$ for all $x \in \mathcal{X}$ and all $w \in W$.
- There exist $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and $\alpha_3(\cdot)$, a $\mathcal{K}$-function $\sigma(\cdot)$, and a function $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ such that for all $x \in \mathcal{X}$,

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$
$$V(f(x, w)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|w\|)$$

A function $V(\cdot)$ that satisfies these conditions is called an *ISS-Lyapunov function*.

Proposition 9. Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a compact robust positively invariant set for (13) containing the origin in its interior. Furthermore, let there exist $\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and $\alpha_3(\cdot)$ and a function $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ that is convex on $\mathcal{X}$ such that for all $x \in \mathcal{X}$,

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \quad (16a)$$
$$V(f(x, 0)) - V(x) \leq -\alpha_3(\|x\|) \quad (16b)$$

The function $V(\cdot)$ is an ISS-Lyapunov function and the origin is ISS for the system (13) with region of attraction $\mathcal{X}$ if $f(\cdot)$ can be written as $f(x, w) :=$

$g(x) + w$, and W is compact and convex, containing the origin in its relative interior.

Remark 10. Note that unlike in (Goulart *et al.*, 2006), which requires Lipschitz continuity of $V(\cdot)$, Proposition 9 requires that $V(\cdot)$ be convex on $\mathcal{X}$; recall that, in general, convex functions are *not* guaranteed to be continuous. It is important to note that continuity of the function $V(\cdot)$ is *not* required in the proof of (Jiang and Wang, 2001, Lem. 3.5).

6. STABILITY OF RHC LAW

Given the results of the previous sections, we can now provide conditions under which the closed-loop system (10) is guaranteed to be ISS. We first make the following assumption:

A1 (Terminal Cost and Constraint) The terminal constraint set X_f is both constraint admissible and robust positively invariant for the system (1) under the control $u = Kx$, i.e.

$$X_f \subseteq \{x \mid (x, Kx) \in Z\} \tag{17a}$$
$$(A + BK)x + w \in X_f,\ \forall x \in X_f, \forall w \in W \tag{17b}$$

We further assume that the feedback matrix K and terminal cost function P are derived from the solution to the discrete algebraic Riccati equation:

$$P := Q + A'PA - A'PB(R + B'PB)^{-1}B'PA$$
$$K := -(R + B'PB)^{-1}B'PA$$

Remark 11. The reader is referred to (Blanchini, 1999; Kolmanovsky and Gilbert, 1998; Lee and Kouvaritakis, 1999) and the references therein for details on how to compute a set X_f that satisfies (17). The terminal cost $F(x) := x'Px$ is a Lyapunov function in the terminal set X_f for the undisturbed closed loop system $x^+ = (A + BK)x$, so that, for all $x \in X_f$,

$$F((A + BK)x) - F(x) \leq -x'(Q + K'RK)x$$

Remark 12. In the absence of constraints, the proposed control policy minimizes *both* the expected value function $V_N(x, , \cdot, \cdot)$ (assuming $\mathbb{E}[\mathbf{w}] = 0$), *and* the value of the deterministic or certainty-equivalent cost (van de Water and Willems, 1981). This certainty equivalence property does *not* hold in the more general constrained case considered here. However, it is still true that $v_0^*(x) = Kx$ for all $x \in X_f$, since in this case the conditions (17) guarantee that the optimal *unconstrained* state feedback gain K is also constraint admissible.

Theorem 13. (ISS for RHC). If **A1** holds, then the origin is ISS for the closed-loop system (10) with region of attraction X_N. Furthermore, the input and state constraints (2) are satisfied for all time and all allowable disturbance sequences if and only if the initial state is in X_N.

PROOF. For the system of interest, we select $V(\cdot) := V_N^*(\cdot) - V_N^*(0)$, and let $f(x, w) := Ax + B\mu_N(x) + w$. The set X_N is robust positively invariant for system (10), with $0 \in \text{int}X_N$ if **A1** holds (Goulart *et al.*, 2006, Prop. 13). X_N is compact since it is closed (cf. Remark 1) and bounded because Z is assumed bounded. Since $0 \in X_f$, it is easy to show that $f(0, 0) = 0$ and V is lower bounded by $\alpha_1(\|x\|) := \|x\|_Q^2$ if **A1** holds by exploiting equivalence between affine disturbance and state feedback policies (Goulart *et al.*, 2006, Thm. 9). Since $0 \in \text{int}X_N$ and $V_N(x, \cdot, \cdot)$ has a finite upper bound on its domain, one can also construct a function $\alpha_2(\cdot)$ satisfying (16a). Using standard techniques (Mayne *et al.*, 2000), one can show that $V(\cdot)$ is a Lyapunov function for the *undisturbed* system $x^+ = Ax + B\mu_N(x)$ and, in particular, that (16b) holds with $\alpha_3(z) := \lambda_{\min}(Q)z^2$. Finally, recall from Proposition 6 that $V_N^*(\cdot)$ is convex on X_N and continuous on $\text{int}X_N$. By combining all of the above, it follows from Proposition 9 that $V_N^*(\cdot)$ is an ISS-Lyapunov function for system (10). □

7. CONCLUSIONS

Using a finite horizon affine feedback policy parameterization and exploiting the results in (Goulart *et al.*, 2006), we have shown that receding horizon control laws can be constructed that guarantee input-to-state stability for systems with general convex state and input constraints, given appropriate terminal conditions. The method is based on minimization of the expected value of a finite horizon quadratic cost at each time instant.

This result represents an important generalization of the results in (Goulart *et al.*, 2006), as it establishes stability for a broad class of RHC problems using this framework with non-polytopic convex disturbance sets (e.g. ellipsoidal or 2-norm bounded disturbances), or for problems with general convex constraints on the states and inputs.

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